Application of He’s variational iteration method for solving the Cauchy reaction–diffusion problem

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Abstract

In this paper, the solution of Cauchy reaction–diffusion problem is presented by means of variational iteration method. Reaction–diffusion equations have special importance in engineering and sciences and constitute a good model for many systems in various fields. Application of variational iteration technique to this problem shows the rapid convergence of the sequence constructed by this method to the exact solution. Moreover, this technique does not require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computations.

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1. Introduction

By a reaction–diffusion we mean an equation of the following form:

$$\frac{\partial u}{\partial t} = \Delta u + f(u, \nabla u; x, t).$$

The term $\Delta u$ is diffusion term and $f(u, \nabla u; x, t)$ is the reaction term. More generally the diffusion term may be of type $A(u)$, where $A$ is a second-order elliptic operator, which may be nonlinear and degenerate.

In this paper, we consider the one-dimensional reaction–diffusion equation

$$\frac{\partial^2 u}{\partial t^2}(x, t) = D \frac{\partial^2 u}{\partial x^2}(x, t) + p(x, t)u(x, t),$$

(1.1)
where \( u \) is the concentration, \( p \) is the reaction parameter and \( D > 0 \) is the diffusion coefficient, subject to the initial or boundary conditions

\[
\begin{align*}
  u(x, 0) &= g(x), \quad x \in \mathbb{R}, \\
  u(0, t) &= f_0(t), \quad \frac{\partial u}{\partial x}(0, t) = f_1(t), \quad t \in \mathbb{R}.
\end{align*}
\]

(1.2)

(1.3)

The problem given by Eqs. (1.1) and (1.2) is called the characteristic Cauchy problem in the domain \( \Omega = \mathbb{R} \times \mathbb{R}^+ \) and the problem given by Eqs. (1.1) and (1.3) is called the non-characteristic problem in the domain \( \Omega = \mathbb{R}^+ \times \mathbb{R} \). In [24], this equation is solved by Adomian decomposition method.

For simplicity in illustrating the procedure of variational iteration method, we rewrite \( p(x, t) \) as \( p(x, t) = p_1 + p_2(x, t) \), where \( p_1 \) is the constant part of \( p(x, t) \) if there exists and \( p_2(x, t) \) is the remainder part of \( p(x, t) \).

Reaction–diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [7,8,13,28].

This paper is organized in the following way:

The variational iteration method is introduced in Section 2. In Section 3 the described technique is applied on several test problems to show the efficiency of the proposed approach. Section 4 ends this work with a brief conclusion.

### 2. Variational iteration method

Variational iteration method (VIM) was first proposed by the Chinese mathematician He [14,16,19]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions.

This technique is used in [12] for solving nonlinear Jaulent–Miodek, coupled KdV and coupled MKdV equations. In [32] the applications of the present method to Shock-peakon and shock-compacton solutions for \( K(p, q) \) equation are provided. The variational iteration technique is employed to solve the nonlinear dispersive equation, a nonlinear partial differential equation which arise in the process of understanding the role of nonlinear dispersion and in the forming of structures like liquid drops and exhibits compactons [22]. Also this method is applied in [25] for solving two-point boundary value problems and in [30] for solving cubic nonlinear Schrödinger equation in one and two space variables.

Author of [33] employed VIM for determining rational solutions for the KdV, the \( K(2, 2) \), the Burgers, and the cubic Boussinesq equations. This technique is also employed in [9] to solve the Fokker–Planck equation. The linear and nonlinear cases are discussed in their work and several test examples are given to show the efficiency of this procedure. In 1998 [17] applied VIM to a fractional differential equation arising in seepage flow. Following the idea of the above reference, Draganescu [11] applied VIM to nonlinear oscillator with fractional damping and then [10] to nonlinear viscoelastic models with fractional derivatives. Odibat and Momani [27] applied the method to nonlinear differential equations of fractional order with great success, see [6,26]. In [1], VIM is studied for solving nonlinear fractional differential equation with Riemann–Liouville fractional derivatives. This approach is used to solve numerically the harmonic wave generation in a nonlinear, one-dimensional elastic half-space model subjected initially to a prescribed harmonic displacement [29]. This method is successfully and effectively applied to delay differential equations [15,21] autonomous ordinary differential equations [20], Blasius equation [18], generalized Burgers–Huxley equation [5], generalized Zakharov equation [23], etc. The convergence of He’s variational iteration method is investigated in [31]. Author of [2] employed VIM for solving the quadratic Riccati differential equation. In [34] VIM is compared with Adomian decomposition method.

It is shown in [4] that the application of VIM to a special kind of nonlinear differential equations leads to calculation of unneeded terms and more time consuming in repeated calculations for series solutions. A modified VIM is introduced to eliminate the shortcomings and in [3] the Padé technique was successfully linked with this modification.

The idea of VIM is constructing a correction functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial function) and is the best within the flexibility of trial function, accordingly we can identify the multiplier by variational theory. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions.
To illustrate the procedure of this approach, we consider the following general differential equation:

\[ Lu + Nu = g, \]  
(2.1)

where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( g(t) \) is an inhomogeneous term.

According to the VIM, the terms of a sequence \( \{ u_n \} \) are constructed such that this sequence converges to the exact solution. \( u_n \) are calculated by a correction functional as follows:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left( L u_n(s) + N \tilde{u}_n(s) - g(s) \right) \, ds, \]  
(2.2)

where \( \lambda \) is the general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript \( n \) denotes the \( n \)th approximation and \( \tilde{u}_n \) is considered as a restricted variation, i.e. \( \delta \tilde{u}_n = 0 \). For linear problems, the exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In nonlinear problems, in order to determine the Lagrange multiplier in a simple manner, the nonlinear terms have to be considered as restricted variations.

We apply this procedure to Eq. (1.1):

(a) regarding initial condition (1.2) and with respect to the variable \( t \),
(b) regarding boundary conditions (1.3) and with respect to the variable \( x \).

The recursive formula (2.2) in the cases of (a) and (b) reduces to:

\[ u_{n+1}(x, t) = u_n(x, t) + \mu_n(x, t), \]  
(2.3)

where \( \mu_n \) is respectively in the following forms:

\[ \mu_n(x, t) = \int_0^t \lambda(s) \left\{ \frac{\partial u_n}{\partial s}(x, s) - p_1 u_n(x, s) - D \frac{\partial^2 u_n}{\partial x^2}(x, s) - p_2(x, s) \tilde{u}_n(x, s) \right\} \, ds, \]  
(2.4)

\[ \mu_n(x, t) = \int_0^x \lambda(r) \left\{ \frac{\partial^2 u_n}{\partial r^2}(r, t) + p_1 u_n(r, t) - \frac{\partial \tilde{u}_n}{\partial t}(r, t) + p_2(r, t) \tilde{u}_n(r, t) \right\} \, dr. \]  
(2.5)

By taking variation of (2.3) with respect to the independent variable \( u_n \) and making the correction functional stationary, \( \lambda \), the Lagrange multiplier, will be specified. Then starting with an initial approximation, we can identify the next approximations successively.

3. Test examples

In this section, we present some examples to show the efficiency of VIM for solving Eq. (1.1). Examples have been chosen so that their analytical solutions exist.

3.1. Example 1

Consider Eq. (1.1) with \( D = 1 \), \( p(x, t) = -1 + \cos(x) - \sin^2(x) \) and the following initial and boundary conditions:

\[ u(x, 0) = g(x) = \frac{1}{10} e^{\cos(x) - 11}, \quad x \in R, \]  
(3.1)

\[ u(0, t) = f_0(t) = \frac{1}{10} e^{-t - 10}, \quad t \in R, \]  
(3.2)

\[ \frac{\partial u}{\partial x}(0, t) = f_1(t) = 0, \quad t \in R. \]  
(3.3)

The exact solution of this problem is \( u(x, t) = \frac{1}{10} e^{\cos(x) - t - 11} \).
First we consider the Eq. (1.1) with initial condition (3.1) and solve this equation with respect to the variable \( t \). To apply VIM to this equation, according to (2.3) and (2.4), we have

\[
un_{n+1}(x, t) = un(x, t) + \mu_n(x, t), \tag{3.4}
\]

where

\[
\mu_n(x, y, t) = \int_{0}^{t} \lambda(s) \left\{ \frac{\partial u_n}{\partial s}(x, s) + un(x, s) - \frac{\partial^2 u_n}{\partial x^2}(x, s) - \cos(x)\overset{\sim}{u}_n(x, s) + \sin^2(x)\overset{\sim}{u}_n(x, s) \right\} \, ds. \tag{3.5}
\]

Taking variation with respect to the independent variable \( u_n \) and making the correction functional stationary, we obtain

\[
\delta u_{n+1}(x, t) = 0,
\]

and therefore we have

\[
\delta u_n(x, t) + \lambda(s)\delta u_n(x, s) \bigg|_{s=t} - \int_{0}^{t} (\lambda' - \lambda)\delta u_n(x, s) \, ds = 0.
\]

This condition implies the following stationary conditions:

\[
\delta u_n : 1 + \lambda(t) = 0,
\]

\[
\delta u_n : (\lambda' - \lambda)(s) = 0.
\]

Therefore, the Lagrange multiplier can be readily identified as

\[
\lambda(s) = -e^{s-t}.
\]

As a result we have the variational iteration formula (3.4) where

\[
\mu_n(x, t) = -\int_{0}^{t} e^{s-t} \left\{ \frac{\partial u_n}{\partial s}(x, s) + un(x, s) - \frac{\partial^2 u_n}{\partial x^2}(x, s) - (\cos(x) - \sin^2(x))u_n(x, s) \right\} \, ds.
\]

We start with the initial approximation \( u_0(x, t) = u(x, 0) = \frac{1}{10} e^{\cos(x) - 11} \). By the iteration formula (3.4), we have

\[
u_1(x, t) = u_0(x, t) + \mu_0(x, t),
\]

where

\[
\mu_0(x, t) = -\int_{0}^{t} e^{s-t} \left\{ \frac{\partial u_0}{\partial s}(x, s) + u_0(x, s) - \frac{\partial^2 u_0}{\partial x^2}(x, s) - (\cos(x) - \sin^2(x))u_0(x, s) \right\} \, ds.
\]

\[
= -\frac{1}{10} e^{t+\cos(x) - 11} \bigg|_{s=0} = \frac{1}{10} e^{\cos(x) - t - 11} - \frac{1}{10} e^{\cos(x) - 11},
\]

and therefore

\[
u_1(x, t) = \frac{1}{10} e^{\cos(x) - 11} + \frac{1}{10} e^{\cos(x) - t - 11} - \frac{1}{10} e^{\cos(x) - 11} = \frac{1}{10} e^{\cos(x) - t - 11},
\]

which is the exact solution.

Now, we solve this example regarding boundary conditions (3.2) and (3.3) and with respect to the variable \( x \). Using (2.3) and (2.5), we obtain

\[
u_{n+1}(x, t) = \nu_n(x, t) + \mu_n(x, t), \tag{3.6}
\]

where \( \mu_n \) is as follows:

\[
\mu_n(x, t) = \int_{0}^{x} \lambda(r) \left\{ \frac{\partial^2 u_n}{\partial r^2}(r, t) - \nu_n(r, t) - \frac{\partial \nu_n}{\partial t}(r, t) + \cos(r)\overset{\sim}{u}_n(r, t) - \sin^2(r)\overset{\sim}{u}_n(r, t) \right\} \, dr. \tag{3.7}
\]
By the same manipulation as in the previous part, the following stationary conditions are obtained:
\[
\delta u_n : 1 - \lambda' = 0,
\]
\[
\delta u_n : \frac{\partial u_n}{\partial x} = 0,
\]
\[
\delta u_n : (\lambda - \lambda') = 0.
\]
and therefore we get
\[
\lambda (r) = -\frac{1}{2} (e^{x-r} - e^{x-r}).
\]

Consider \(u_0(x,t) = a + bx\), where \(a\) and \(b\) are unknown constants with respect to the variable \(x\). By the recurrent formula (3.6), \(u_1\) is obtained in the following form:
\[
u_1(x,t) = u_0(x,t) + \mu_0(x,t),
\]
where
\[
\mu_0(x,t) = -\int_0^x \left(\frac{1}{2} (e^{x-r} - e^{x-r}) \right) \left\{ \frac{\partial^2 u_0}{\partial r^2}(r,t) - u_0(r,t) - \frac{\partial u_0}{\partial r}(r,t) + \cos(r)u_0(r,t) - \sin^2(r)u_0(r,t) \right\} \, dr
\]
\[
= \frac{1}{2} \cos(x)(bx + a) - \frac{1}{4} a \sin^2(x) - b \sin(x) (\frac{4}{25} \cos(x) + \frac{2}{5}) + 9 \frac{a}{20} (e^{x} + e^{-x}) - 9 \frac{b}{20} (e^{-x} - e^{x}) + \frac{1}{2} bx(\cos^2(x) - 8) - \frac{7}{5} a,
\]
and therefore
\[
u_1(x,t) = a + bx + \frac{1}{2} \cos(x)(bx + a) - \frac{1}{4} a \sin^2(x) - b \sin(x) (\frac{4}{25} \cos(x) + \frac{2}{5}) + 9 \frac{a}{20} (e^{x} + e^{-x}) - 9 \frac{b}{20} (e^{-x} - e^{x}) + \frac{1}{2} bx(\cos^2(x) - 8) - \frac{7}{5} a.
\]
By imposing the boundary conditions (3.2) and (3.3), we have
\[
a = \frac{1}{10} e^{-t-10}, \quad b = 0.
\]
Substituting these values of \(a\) and \(b\) in \(u_1\) gives
\[
u_1(x,t) = \frac{1}{500} e^{-10-t}(-10 + 2 \cos(2x) + 10 \cos(x) + 9e^{-x} + 9e^{x}).
\]

We calculate the next \(u_n\)'s for \(n = 2, \ldots, 5\) and consider \(u_5\) as an approximation of the exact solution. Numerical results by this approximation are summarized in Table 1 and the absolute error function \(|u_5(x,t) - u(x,t)|\) is plotted in Fig. 1. These results show the high accuracy of this approximation.

<table>
<thead>
<tr>
<th>((x,t))</th>
<th>Exact value</th>
<th>Approximate value by VIM</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5, -5)</td>
<td>3.291743831895160 \times 10^{-4}</td>
<td>3.296122776678473 \times 10^{-4}</td>
<td>4.37894478312938 \times 10^{-7}</td>
</tr>
<tr>
<td>(4, -4)</td>
<td>4.743127608418425 \times 10^{-5}</td>
<td>4.734715018929820 \times 10^{-5}</td>
<td>8.41258984869387 \times 10^{-8}</td>
</tr>
<tr>
<td>(3, -3)</td>
<td>1.246510285405382 \times 10^{-5}</td>
<td>1.245520638900535 \times 10^{-5}</td>
<td>9.89465048492351 \times 10^{-9}</td>
</tr>
<tr>
<td>(2, -2)</td>
<td>8.13990597089405 \times 10^{-6}</td>
<td>8.14158342978239 \times 10^{-6}</td>
<td>1.67746238830920 \times 10^{-9}</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>7.793014619502044 \times 10^{-6}</td>
<td>7.793033516658417 \times 10^{-6}</td>
<td>1.889715637683246 \times 10^{-11}</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>4.539992976248485 \times 10^{-6}</td>
<td>4.539992976248485 \times 10^{-6}</td>
<td>0</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>1.054669840797373 \times 10^{-6}</td>
<td>1.054672398249383 \times 10^{-6}</td>
<td>2.55745201055414 \times 10^{-12}</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>1.490075738455668 \times 10^{-7}</td>
<td>1.491183021409301 \times 10^{-7}</td>
<td>3.072379536326035 \times 10^{-11}</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>3.087990083185595 \times 10^{-8}</td>
<td>3.08733694757570 \times 10^{-8}</td>
<td>2.453088428021881 \times 10^{-11}</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>1.591142051997001 \times 10^{-8}</td>
<td>1.588319942619686 \times 10^{-8}</td>
<td>2.822109377314891 \times 10^{-11}</td>
</tr>
<tr>
<td>(5, 5)</td>
<td>1.494449387641330 \times 10^{-8}</td>
<td>1.496437425497293 \times 10^{-8}</td>
<td>1.988037855962217 \times 10^{-11}</td>
</tr>
</tbody>
</table>
3.2. Example 2

In this example we solve Eq. (1.1) with \( p(x, t) = -16t \) and \( D = 1 \). The initial and boundary conditions are as follows:

\[
\begin{align*}
  u(x, 0) &= g(x) = e^{-x-4}, \quad x \in R, \quad (3.8) \\
  u(0, t) &= f_0(t) = e^{-t(8t-1)-4}, \quad t \in R, \quad (3.9) \\
  \frac{\partial u}{\partial x}(0, t) &= f_1(t) = -e^{-t(8t-1)-4}, \quad t \in R \quad (3.10)
\end{align*}
\]

\( u(x, t) = e^{-x-t(8t-1)-4} \) is the exact solution of this problem. To solve this equation by VIM regarding the initial condition (3.8), we use (2.3) and (2.4)

\[
u_{n+1}(x, t) = u_n(x, t) + \mu_n(x, t), \quad (3.11)
\]

where

\[
\mu_n(x, t) = \int_0^t \lambda(s) \left\{ \frac{\partial u_n}{\partial s}(x, s) - \frac{\partial^2 u_n}{\partial x^2}(x, s) + 16s u_n(x, s) \right\} ds \quad (3.12)
\]

By the same manipulation as in the previous example, the stationary conditions of the above correction functional can be expressed as follows:

\[
\delta u_n : 1 + \lambda(t) = 0, \\
\delta u_n : \lambda'(s) = 0,
\]

and therefore we get

\[
\lambda(s) = -1.
\]

Consider \( u_0(x, t) = u(x, 0) = e^{-x-4} \). By the recurrent formula (3.11), the terms of the sequence \( \{u_n\} \) are constructed as follows:

\[
u_1(x, t) = u_0(x, t) + \mu_0(x, t),
\]
where
\[
\mu_0(x, t) = - \int_0^t \left\{ \frac{\partial u_0}{\partial s}(x, s) - \frac{\partial^2 u_0}{\partial x^2}(x, s) + 16su_0(x, s) \right\} \, ds
\]
\[= (t - 8t^2)e^{-x - 4},\]
and therefore
\[u_1(x, t) = (1 + t - 8t^2)e^{-x - 4}.\]

Also
\[u_2(x, t) = u_1(x, t) + \mu_1(x, t),\]
where
\[
\mu_1(x, t) = - \int_0^t \left\{ \frac{\partial u_1}{\partial s}(x, s) - \frac{\partial^2 u_1}{\partial x^2}(x, s) + 16su_1(x, s) \right\} \, ds = \frac{(t - 8t^2)^2}{2} e^{-x - 4},
\]
and therefore
\[u_2(x, t) = \left( 1 + t - 8t^2 + \frac{(t - 8t^2)^2}{2} \right) e^{-x - 4} = \left( 1 + t - 8t^2 + \frac{(t - 8t^2)^2}{2} \right) e^{-x - 4}.
\]

The next terms of \(u_n\)'s can be determined in a similar way and we can construct the \(n\)th approximation of \(u\) as
\[u_n(x, t) = \sum_{k=0}^{n} \frac{(t - 8t^2)^k}{k!} e^{-x - 4},\]
and since
\[\lim_{n \to \infty} u_n = e^{-x - 4 + t - 8t^2},\]
the approximation obtained by this procedure converges to the exact solution.

Now we solve Eq. (1.1) with the boundary conditions (3.9) and (3.10). By using (2.3) and (2.5), we have
\[u_{n+1}(x, t) = u_n(x, t) + \mu_n(x, t),\]  \hspace{1cm} (3.13)

where
\[
\mu_n(x, t) = \int_0^x \hat{\lambda}(r) \left\{ \frac{\partial^2 u_n}{\partial r^2}(r, t) - \frac{\partial^2 u_n}{\partial t^2}(r, t) - 16tu_n(r, t) \right\} \, dr.
\]  \hspace{1cm} (3.14)

In this case the following stationary conditions result in
\[
\delta u_n : 1 - \hat{\lambda}'(x) = 0,
\]
\[
\delta^2 u_n : \hat{\lambda}(x) = 0,
\]
\[
\delta u_n : \hat{\lambda}''(r) = 0,
\]
and therefore \(\hat{\lambda}(r) = r - x\).

Starting with the initial approximation \(u_0(x, t) = a + bx\) with unknown constants \(a\) and \(b\) (with respect to the variable \(x\)) and applying (3.13), we get
\[u_1(x, t) = u_0(x, t) + \mu_0(x, t),\]
where

\[\mu_0(x, t) = \int_0^x (r - x) \left\{ \frac{\partial^2 u_0(r, t)}{\partial r^2} - u_0(r, t) - \frac{\partial u_0(r, t)}{\partial t} - 16tu_0(r, t) \right\} \, dr.\]

\[= \frac{8}{3}x^2t(bx + 3a).\]

and then

\[u_1(x, t) = a + bx + \frac{8}{3}bx^3t + 8ax^2t.\]

Imposing the boundary conditions (3.9) and (3.10) yields

\[a = e^{-t(8t-1)-4}, \quad b = -e^{-t(8t-1)-4},\]

and therefore

\[u_1(x, t) = -\frac{1}{4}e^{t(8t-4)}(-3 + 3x + 8tx^3 - 24x^2t),\]

and continuing we get \(u_n\) for \(n = 2, \ldots, 5\). Fig. 2 presents the absolute error function \(|u(x, t) - u_5(x, t)|\) and Table 2 shows the numerical results obtained by this approximation.

### 3.3. Example 3

Consider Eq. (1.1) with \(p(x, t) = -\frac{1}{4}, D = 1\) and the following initial and boundary conditions

\[u(x, 0) = \frac{1}{2}x + e^{-x/2}, x \in R, \tag{3.15}\]

\[u(0, t) = 1, t \in R, \tag{3.16}\]

\[\frac{\partial u}{\partial x}(0, t) = \frac{1}{2}e^{-t/4} - \frac{1}{2}, t \in R. \tag{3.17}\]
The exact solution of this problem is
\[ u(x, t) = \frac{1}{2} x e^{-t/4} + e^{-x/2}. \]

To apply VIM to this equation with initial condition (3.15), according to (2.3) and (2.4), we have
\[ u_{n+1}(x, t) = u_n(x, t) + \mu_n(x, t), \]
(3.18)
where
\[ \mu_n(x, t) = \int_0^t \lambda(s) \left\{ \frac{\partial u_n(x, s)}{\partial s} + \frac{1}{4} u_n(x, s) - \frac{\partial^2 u_n}{\partial x^2}(x, s) \right\} ds. \]
(3.19)

Such as previous examples, we can obtain the following stationary conditions:
\[ \delta u_n : 1 + \lambda(t) = 0, \]
\[ \delta u_n : \lambda' - \frac{1}{4} \lambda(s) = 0, \]
which yield
\[ \lambda(s) = -e^{(1/4)(s-t)}. \]

By this formula and considering \( u_0(x, t) = u(x, 0) = \frac{1}{2} x + e^{-x/2}, \) \( u_1 \) is obtained in the following form:
\[ u_1(x, t) = u_0(x, t) + \mu_0(x, t), \]
where
\[ \mu_0(x, t) = \int_0^t -e^{1/4(s-t)} \left\{ \frac{\partial u_0(x, s)}{\partial s} (x, s) - \frac{\partial^2 u_0}{\partial x^2}(x, s) + \frac{1}{4} u_0(x, s) \right\} ds = -\frac{1}{2} x + \frac{1}{4} x e^{-(1/4)t}, \]
and therefore
\[ u_1(x, t) = \frac{1}{2} x + e^{-x/2} - \frac{1}{2} x + \frac{1}{4} x e^{-(1/4)t} = e^{-x/2} + \frac{1}{4} x e^{-(1/4)t}, \]
which is the exact solution.

Now we solve Eq. (1.1) regarding the boundary conditions (3.16) and (3.17). Applying (2.3) and (2.5), results
\[ u_{n+1}(x, t) = u_n(x, t) + \mu_n(x, t), \]
(3.20)
Table 3
Comparison of the exact and approximate values by VIM in Example 3

<table>
<thead>
<tr>
<th>((x, t))</th>
<th>Exact value</th>
<th>Approximate value by VIM</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3, -3))</td>
<td>3.39863018506744</td>
<td>3.39863511415814</td>
<td>4.929090697380900 \times 10^{-6}</td>
</tr>
<tr>
<td>((2, -2))</td>
<td>2.01660071187157</td>
<td>2.01660075451843</td>
<td>4.264686337407397 \times 10^{-8}</td>
</tr>
<tr>
<td>((1, -1))</td>
<td>1.24854336805650</td>
<td>1.24854336807234</td>
<td>1.583333464338921 \times 10^{-11}</td>
</tr>
<tr>
<td>((0,0))</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>((1,1))</td>
<td>0.99593105124834</td>
<td>0.99593105125794</td>
<td>9.603207118402679 \times 10^{-12}</td>
</tr>
<tr>
<td>((2,2))</td>
<td>0.97441010088408</td>
<td>0.97441011657298</td>
<td>1.568890434233694 \times 10^{-8}</td>
</tr>
<tr>
<td>((3,3))</td>
<td>0.93167998925995</td>
<td>0.93168108908875</td>
<td>1.099828796702340 \times 10^{-6}</td>
</tr>
</tbody>
</table>

Fig. 3. Plot of absolute error in Example 3.

where

\[
\lambda_n(x, t) = \int_0^x \hat{\lambda}(r) \left\{ \frac{\partial^2 u_n(r, t)}{\partial r^2} - \frac{1}{4} u_n(r, t) - \frac{\partial^2 u_n}{\partial t} (r, t) \right\} dr. \tag{3.21}
\]

Taking variation of (3.20) with respect to the independent variable \(u_n\) and making the correction functional stationary, yield the following stationary conditions:

\[
\begin{align*}
\delta \frac{\partial u_n}{\partial x} : \lambda(x) &= 0, \\
\delta u_n : 1 - \lambda'(x) &= 0, \\
\delta u_n : (\lambda'' - \frac{1}{4} \lambda)(r) &= 0,
\end{align*}
\]

and therefore \(\lambda\) can be identified as follows

\[
\lambda(r) = e^{1/2(r-x)} - e^{1/2(x-r)}.
\]
Consider $u_0(x, t) = a + bx$ with unknown constants $a$ and $b$ with respect to the variable $x$. By (3.20) and (3.21), we get $u_1$ as follows:

$$u_1(x, t) = u_0(x, t) + \mu_0(x, t),$$  \hspace{1cm} (3.22)

where

$$\mu_0(x, t) = \int_0^x \left( e^{1/2(r-x)} - e^{1/2(x-r)} \right) \left\{ \frac{\partial^2 u_0}{\partial r^2}(r, t) - \frac{1}{2} \frac{\partial u_0}{\partial t}(r, t) \right\} \, dr$$

$$= -a - bx + \frac{1}{2} e^{(1/2)x} a + e^{(1/2)x} b + \frac{1}{2} e^{-(1/2)x} a - e^{-(1/2)x} b,$$

and then

$$u_1(x, t) = \frac{1}{2} e^{(1/2)x} a + e^{(1/2)x} b + \frac{1}{2} e^{-(1/2)x} a - e^{-(1/2)x} b.$$

By imposing the boundary conditions (3.16) and (3.17), $a$ and $b$ are specified as follows

$$a = 1, \quad b = \frac{1}{2} e^{-t/4} - \frac{1}{2}.$$

We substitute these values of $a$ and $b$ in $u_1$ and calculate the next $u_n$s for $n = 2, \ldots, 5$. Numerical results and error function $|u(x, t) - u_5(x, t)|$ are presented in Table 3 and Fig. 3, respectively.

4. Conclusion

In this work, the variational iteration method has been successfully applied to time-dependent reaction–diffusion equation. This technique produces the terms of a sequence using the iteration of the correction functional which converges to the exact solution rapidly. Application of this method is easy and calculation of successive approximations is direct and straightforward. Also the variational iteration technique provides the solution of the problem without calculating Adomian’s polynomials which is an important advantage over the Adomian decomposition method. Furthermore this approach unlike the mesh points schemes [35] does not provide any linear or nonlinear system of equations and reduces the volume of calculations by not requiring discretization of the variables.

References


[23] M. Javidi, A. Golbabai, Exact and numerical solitary wave solutions of generalized Zakharov equation by the variational iteration method, Chaos, Solitons and Fractals, in press.


