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Non-vanishing Gram determinants for cyclotomic Nazarov–Wenzl and Birman–Murakami–Wenzl algebras ☆

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ABSTRACT

In this paper, we use the method in Rui and Si (2011) [35] to give a necessary and sufficient condition on non-vanishing Gram determinants for cyclotomic NW and cyclotomic BMW algebras over an arbitrary field. Equivalently, we give a necessary and sufficient condition for each cell module of such algebras being equal to its simple head over an arbitrary field.

1. Introduction

In [31], Nazarov introduced a class of infinite dimensional algebras called affine Wenzl algebras when he studied the action of the Jucys–Murphy elements on the irreducible modules of Brauer algebras. These algebras can be considered as the degenerate affine Birman–Murakami–Wenzl algebras [22]. In order to study their finite dimensional representations, Ariki, Mathas and Rui [4] considered the cyclotomic quotients of affine Wenzl algebras, called the cyclotomic Nazarov–Wenzl algebras or cyclotomic NW algebras for brevity. They have proved that the cyclotomic Nazarov–Wenzl algebras are cellular algebras in the sense of [19]. The representations of these algebras have been studied in [4,32].

In [22], Häring-Oldenburg introduced the cyclotomic Birman–Murakami–Wenzl algebras or cyclotomic BMW algebras for brevity in order to study the link invariants. Recently, such algebras have

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been studied extensively by Goodman and Hauschild, Wilcox and Yu, Xu and the authors in [12,17, 13,18,33,36,38–42]. In particular, it has been proved in [38] that the cyclotomic BMW algebras are cellular algebras in the sense of [19].

A fundamental problem in the representation theory of cellular algebras is to determine whether a cell module is equal to its simple head or not. Such a problem has been solved for Hecke algebras [24], Ariki–Koike algebras [25] and Birman–Murakami–Wenzl algebras [35], etc. In this paper, we address this problem for cyclotomic NW algebras and cyclotomic BMW algebras over an arbitrary field. Equivalently, we need to determine whether the Gram determinant associated to a cell module of such algebras is equal to zero or not.

The Birman–Murakami–Wenzl algebras (or BMW algebras) are introduced by Birman–Wenzl [5] and independently Murakami [30] in order to study the link invariants. Such algebras can be considered as the q-analog of the Brauer algebras in [6]. Motivated by Cox, De Visscher and Martin's work on the blocks of Brauer algebras in characteristic zero [7], we have classified the blocks of BMW algebras over the field κ which contains invertible parameters r and q such that $o(q^2)$, the order of q^2, is large enough and char(κ), the characteristic of the field κ, is not equal to 2 [35]. As a by-product, we determine certain zero divisors of the Gram determinant associated to each cell module for BMW algebras. We have proved the remaining zero divisors of such a Gram determinant can be determined by those of the corresponding Gram determinant associated to the cell module of Hecke algebra of type A. This enables us to use James–Mathas's result in [24] to give a necessary and sufficient condition for each Gram determinant of BMW algebras being not equal to zero. The main purpose of this paper is to use this method to settle the same problems for cyclotomic NW algebras and cyclotomic BMW algebras over an arbitrary field.

The paper is organized as follows. In Section 2, we recall some results on cellular algebras with a JM-basis and JM-elements [29]. We give a criterion to verify whether an element of a cell module is in its radical or not. In Section 3, we recall the notion of cyclotomic NW algebras and cyclotomic BMW algebras. We also recall some results for such algebras which will be used later on. In Sections 4–5, we keep Assumption 4.1, which is equivalent to saying that the degenerate cyclotomic Hecke algebras [27] (resp. Ariki–Koike algebras [2]) are semisimple. Further, in the later case, we assume char$(\kappa) \neq 2$. We determine the structure of the cell modules for cyclotomic NW and cyclotomic BMW algebras with respect to $(1, \lambda)$, where λ ranges over all multipartitions of $n-2$. In particular, we give explicit formulae to compute the dimensions of the simple heads of such cell modules. This generalizes Doran–Wales–Hanlon's work on the Brauer algebras [10] and our work on the BMW algebras [35]. Such results suggest the definition of (f, λ)-admissible partitions for all r-partitions λ of $n-2f$ in Section 5. Via it, we classify the blocks of cyclotomic NW and cyclotomic BMW algebras over the field κ under the additional Assumption 4.1. This generalizes Cox–De Visscher–Martin's work on Brauer algebras and our work on BMW algebras in [7,35]. Via such results, we determine certain zero factors of the Gram determinant associated to each cell module for cyclotomic NW algebras and cyclotomic BMW algebras. In Section 6, we prove the main result of this paper, which gives a necessary and sufficient condition for each cell module of cyclotomic NW and cyclotomic BMW algebras being equal to its simple head over an arbitrary field. We remark that we will give such a result for Brauer algebras by using Cox–De Visscher–Martin's work on the blocks of Brauer algebras in characteristic zero together with the corresponding results for symmetric groups in [26]. In this case, we leave the details of the proof to the readers.

2. Cellular algebras with JM-basis and JM-elements

Throughout this section, we assume that R is a commutative ring with multiplicative identity 1. First, we recall the notion of cellular algebras in [19].

Definition 2.1. (See [19].) Let A be an R-algebra. Fix a partially ordered set $\Lambda = (\Lambda, \trianglerighteq)$ and for each $\lambda \in \Lambda$ let $T(\lambda)$ be a finite set. Finally, fix $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \in A$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$.

Then the triple (Λ, T, C) is a cell datum for A if:

a) $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is an R-basis for A;
b) the R-linear map $*: A \to A$ determined by $(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}})^* = \mathfrak{m}_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \Lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an anti-isomorphism of A;
c) for all $\lambda \in \Lambda$, $\mathfrak{s} \in T(\lambda)$ and $a \in A$ there exist scalars $r_{\mathfrak{t}\mathfrak{u}}(a) \in R$ such that

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} a = \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{t}\mathfrak{u}}(a) \mathfrak{m}_{\mathfrak{s}\mathfrak{u}} \quad \left(\operatorname{mod} A^{\rhd \lambda}\right),$$

where $A^{\rhd\lambda} = R\text{-span}\{\mathfrak{m}_{\mathfrak{u}\mathfrak{v}} \mid \mu \rhd \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu)\}$. Furthermore, each scalar $r_{\mathfrak{t}\mathfrak{u}}(a)$ is independent of $\mathfrak{s}$.

An algebra A is a cellular algebra if it has a cell datum and in this case we call $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda),\ \lambda \in \Lambda\}$ a cellular basis of A.

Now, we briefly recall the representation theory of cellular algebras over a field in [19]. We remark that all modules considered in this paper are right modules.

Every irreducible A-module arises in a unique way as the simple head of some cell module. For each $\lambda \in \Lambda$ fix $\mathfrak{s} \in T(\lambda)$ and let $\mathfrak{m}_{\mathfrak{t}} = \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} + A^{\rhd\lambda}$. The cell modules of A can be considered as the modules $\Delta(\lambda)$ which are the free R-modules with basis $\{\mathfrak{m}_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$. The cell module $\Delta(\lambda)$ comes equipped with a natural symmetric bilinear form ϕ_λ which is determined by the equation

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mathfrak{m}_{\mathfrak{t}'\mathfrak{s}} \equiv \phi_\lambda(\mathfrak{m}_{\mathfrak{t}}, \mathfrak{m}_{\mathfrak{t}'}) \cdot \mathfrak{m}_{\mathfrak{s}\mathfrak{s}} \quad \left(\operatorname{mod} A^{\rhd\lambda}\right).$$

Note that ϕ_λ is independent of $\mathfrak{s}$. The bilinear form ϕ_λ is A-invariant in the sense that $\phi_\lambda(xa, y) = \phi_\lambda(x, ya^*)$, for $x, y \in \Delta(\lambda)$ and $a \in A$. Consequently,

$$\operatorname{Rad}\Delta(\lambda) = \left\{x \in \Delta(\lambda) \;\middle|\; \phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda)\right\}$$

is an A-submodule of $\Delta(\lambda)$ and $D^\lambda = \Delta(\lambda)/\operatorname{Rad}\Delta(\lambda)$ is either zero or absolutely irreducible. Graham and Lehrer [19] have proved that (a) $\{D^\lambda \mid D^\lambda \neq 0\}$ consists of a complete set of pairwise non-isomorphic irreducible A-modules; (b) a cellular algebra is (split) semisimple if and only if $\operatorname{Rad}\Delta(\lambda) = 0$ for all $\lambda \in \Lambda$ [19].

We are going to recall some results on cellular algebras with JM-basis and JM-elements in [29, 2.4].

Definition 2.2. Suppose that $\{L_i \mid 1 \leqslant i \leqslant n\}$ is a set of commutative elements of A. $\{L_i \mid 1 \leqslant i \leqslant n\}$ are called JM-elements of A with respect to the cellular basis $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda),\ \lambda \in \Lambda\}$ if:

a) for each positive integer $i \leqslant n$, there is a finite subset $\mathcal{C}_\lambda(i) := \{c_{\mathfrak{t}}(i) \mid \mathfrak{t} \in T(\lambda),\ \lambda \in \Lambda\}$ of R;
b) $L_i^* = L_i$ for all positive integers $i \leqslant n$;
c) for each $\lambda \in \Lambda$, there is a linear order $\succ$ on $T(\lambda)$;
d) for any $\lambda \in \Lambda$ and any $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$,

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} L_i \equiv c_{\mathfrak{t}}(i) \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{u} \succ \mathfrak{t}} a_{\mathfrak{u}} \mathfrak{m}_{\mathfrak{s}\mathfrak{u}} \quad \left(\operatorname{mod} A^{\rhd\lambda}\right), \tag{2.3}$$

where $a_{\mathfrak{u}}$'s are scalars in R.

In this case, $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is called the JM-basis with respect to the JM-elements $\{L_i \mid 1 \leqslant i \leqslant n\}$. We remark that Definition 2.2 is the strong version of [29, 2.4] since we use linear order $\succ$ instead of the partial order in [29, 2.4]. Let $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in T(\lambda)$, write $(\mathfrak{s}, \mathfrak{t}) \succ (\mathfrak{u}, \mathfrak{v})$ if $\mathfrak{s} \succcurlyeq \mathfrak{u}$, $\mathfrak{t} \succcurlyeq \mathfrak{v}$ and $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$.

Suppose that $(\mathcal{O}, K, \kappa)$ is a modular system for the cellular algebra A where $\mathcal{O}$ is a discrete valuation ring, K is the field of the fractions of $\mathcal{O}$ and κ is the residue field of $\mathcal{O}$. Further, for $\mathfrak{s} \in T(\lambda)$, $\mathfrak{t} \in T(\mu)$ and $\lambda, \mu \in \Lambda$, assume that $\mathfrak{s} = \mathfrak{t}$ if and only if $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in K for all positive integers $i \leqslant n$. In other words, the "separation condition" in [29] holds over the field K. In this case, Mathas [29] has proved that A_K is semisimple. We will use A_x, $x \in \{\mathcal{O}, K, \kappa\}$ to emphasis A over the ground ring x.

It is possible that $\mathfrak{s} \neq \mathfrak{t}$ although $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in κ for all positive integers $i \leqslant n$. In this case, we need the notion of residue classes in [29].

Definition 2.4. (See [29, 4.1].) Given two $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, $\mathfrak{s}$ and $\mathfrak{t}$ are said to be in the same residue class and write $\mathfrak{s} \approx \mathfrak{t}$ if $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in κ for all positive integers $i \leqslant n$.

Proposition 2.5. *(See [29].) Given the residue class $\mathbb{T}$ in $T(\lambda)$ which contains $\mathfrak{t} \in T(\lambda)$, let $F_\mathbb{T} = \sum_{\mathfrak{s}\in\mathbb{T}} F_\mathfrak{s}$ where*

$$F_\mathfrak{t} = \prod_{i=1}^{n} \prod_{\substack{c\in\mathcal{C}_\lambda(i) \\ c\neq c_\mathfrak{t}(i)}} \frac{L_i - c}{c_\mathfrak{t}(i) - c} \in A_K.$$

We have

a) *$F_\mathfrak{s} F_\mathfrak{t} = \delta_{\mathfrak{s}\mathfrak{t}} F_\mathfrak{s}$ for any $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$;*
b) *$F_\mathbb{T} \in A_\mathcal{O}$ and $F_\mathbb{T} F_\mathbb{S} = \delta_{\mathbb{T},\mathbb{S}} F_\mathbb{T}$ where $\mathbb{T}$, $\mathbb{S}$ are two residue classes which contain $\mathfrak{t}$ and $\mathfrak{s}$, respectively.*

Proposition 2.6. *Let $\mathbb{S}$, $\mathbb{T}$ be two residue classes which contain $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, respectively. Define $\tilde{g}_{\mathfrak{s}\mathfrak{t}} = F_\mathbb{S} \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} F_\mathbb{T}$, $f_{\mathfrak{s}\mathfrak{t}} = F_\mathfrak{s} \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} F_\mathfrak{t}$ and $g_{\mathfrak{s}\mathfrak{t}} = \tilde{g}_{\mathfrak{s}\mathfrak{t}} \otimes 1_\kappa$.*

a) *$\{\tilde{g}_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ is a cellular basis of $A_\mathcal{O}$. Further, it is the JM-basis of $A_\mathcal{O}$ with respect to the JM-elements $L_1, L_2, \ldots, L_n$;*
b) [29, 3.7] *$\{f_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ is a cellular basis of A_K;*
c) [29, 4.5] *$\{g_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ is a cellular basis of A_κ.*

Proof. By (2.3), for $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$,

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \equiv f_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u},\mathfrak{v}\in T(\lambda) \\ (\mathfrak{u},\mathfrak{v})\succ(\mathfrak{s},\mathfrak{t})}} a_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}} \quad \left(\mathrm{mod}\ A^{\rhd\lambda}\right) \tag{2.7}$$

for some scalars $a_{\mathfrak{u}\mathfrak{v}} \in K$. Let $\mathbb{T}$, $\mathbb{S}$ be the residue classes in $T(\lambda)$ which contain $\mathfrak{t}$, $\mathfrak{s}$, respectively. Acting $F_\mathbb{T}$, $F_\mathbb{S}$ on both sides of (2.7) and using Proposition 2.5 yields

$$\tilde{g}_{\mathfrak{s}\mathfrak{t}} \equiv f_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u},\mathfrak{v}\in T(\lambda) \\ (\mathfrak{u},\mathfrak{v})\succ(\mathfrak{s},\mathfrak{t}) \\ \mathfrak{u}\approx\mathfrak{s},\,\mathfrak{v}\approx\mathfrak{t}}} a_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}} \quad \left(\mathrm{mod}\ A^{\rhd\lambda}\right). \tag{2.8}$$

This proves that the transition matrix between $\{f_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ and $\{\tilde{g}_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ is upper unitriangular over K. So is the transition matrix between the JM-basis $\{\mathfrak{m}_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ and $\{\tilde{g}_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ over K. Since both bases are well defined over $\mathcal{O}$, the result is available over $\mathcal{O}$. This proves that $\{\tilde{g}_{\mathfrak{t}\mathfrak{s}} \mid \mathfrak{t}, \mathfrak{s} \in T(\lambda)\}$ is an $\mathcal{O}$ basis of $A_\mathcal{O}$. It is routine to check that such a basis is a cellular basis. Further, the second part of Proposition 2.6(a) follows from (2.3). (b)–(c) have already been proved in [29]. $\square$

We remark that the cell module $\Delta(\lambda)$ which is defined via the JM-basis $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda),\ \lambda \in \Lambda\}$ of A_x is the same as that defined via the cellular basis of A_x in Proposition 2.6. The following result follows from the definition directly.

Lemma 2.9. *If $\{\mathfrak{t}\}$ is a residue class in $T(\lambda)$, then $\tilde{g}_{\mathfrak{t}} = f_{\mathfrak{t}}$ and $\tilde{g}_{\mathfrak{t}} L_i = c_{\mathfrak{t}}(i)\tilde{g}_{\mathfrak{t}}$ for all positive integers $i \leqslant n$.*

The following result has been proved for Birman–Murakami–Wenzl algebras in [35]. In general, it can be proved by similar arguments.

Lemma 2.10. $g_{\mathfrak{t}} \in \operatorname{Rad}\Delta(\lambda)$ *if $\{\mathfrak{t}\}$ is a residue class in $T(\lambda)$ and $\phi_\lambda(g_{\mathfrak{t}}, g_{\mathfrak{t}}) = 0$.*

At the end of this section, we remark that the notion of weakly cellular algebras were introduced in [12], which can be obtained from Definition 2.1 by using $(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}})^* \equiv \mathfrak{m}_{\mathfrak{t}\mathfrak{s}} \pmod{A^{\rhd\lambda}}$ instead of Definition 2.1(b). Note that both cellular algebras and weakly cellular algebras are standardly based algebras in the sense of [9]. From this, one can see that cellular algebras and weakly cellular algebras share the similar results on representation theory. All the statements on cellular algebras above remain valid for weakly cellular algebras. In this case, we also use the notion of "cell modules", etc.

3. Cyclotomic NW algebras and cyclotomic BMW algebras

In this section, we recall the definitions of cyclotomic NW algebras and cyclotomic BMW algebras. We also state some results which we will need later on.

Definition 3.1. (See [4].) Fix two positive integers $r, n \geqslant 2$. Let R be a commutative ring which contains 1_R, $u_1, u_2, \dots, u_r$ and the invertible element 2. Let $\Omega = \{\omega_a \mid a \geqslant 0\} \subset R$. The cyclotomic NW algebra $\mathscr{W}_{r,n}$ [4] is the unital associative algebra with generators $\{S_i, E_i, X_j \mid 1 \leqslant i < n \text{ and } 1 \leqslant j \leqslant n\}$ and relations

a) (Involutions)
 $S_i^2 = 1$, for $1 \leqslant i < n$.
b) (Affine braid relations)
 (i) $S_i S_j = S_j S_i$, if $|i - j| > 1$,
 (ii) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$, for $1 \leqslant i < n-1$,
 (iii) $S_i X_j = X_j S_i$, if $j \neq i, i+1$.
c) (Idempotent relations)
 $E_i^2 = \omega_0 E_i$, for $1 \leqslant i < n$.
d) (Commutation relations)
 (i) $S_i E_j = E_j S_i$, if $|i - j| > 1$,
 (ii) $E_i E_j = E_j E_i$, if $|i - j| > 1$,
 (iii) $E_i X_j = X_j E_i$, if $j \neq i, i+1$,
 (iv) $X_i X_j = X_j X_i$, for $1 \leqslant i, j \leqslant n$.
e) (Skein relations)
 $S_i X_i - X_{i+1} S_i = E_i - 1$ and $X_i S_i - S_i X_{i+1} = E_i - 1$, for $1 \leqslant i < n$.
f) (Unwrapping relations)
 $E_1 X_1^a E_1 = \omega_a E_1$, for $a > 0$.
g) (Tangle relations)
 (i) $E_i S_i = E_i = S_i E_i$, for $1 \leqslant i \leqslant n-1$,
 (ii) $S_i E_{i+1} E_i = S_{i+1} E_i$, for $1 \leqslant i \leqslant n-2$,
 (iii) $E_{i+1} E_i S_{i+1} = E_{i+1} S_i$, for $1 \leqslant i \leqslant n-2$.
h) (Untwisting relations)
 $E_{i+1} E_i E_{i+1} = E_{i+1}$ and $E_i E_{i+1} E_i = E_i$, for $1 \leqslant i \leqslant n-2$.
i) (Anti-symmetry relations)
 $E_i(X_i + X_{i+1}) = 0$ and $(X_i + X_{i+1})E_i = 0$, for $1 \leqslant i < n$.
j) $(X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_r) = 0$.

$\mathscr{W}_{r,n}$'s are cyclotomic quotient of affine Wenzl algebras which were introduced by Nazarov [31] when he studied the action of "Jucys–Murphy" elements on the irreducible modules of Brauer algebras. If we denote by I the two-sided ideal of $\mathscr{W}_{r,n}$ generated by E_1, then $\mathscr{W}_{r,n}/I$ is isomorphic to the degenerate cyclotomic Hecke algebras $\mathcal{H}_{r,n}$ of type $G(r,1,n)$ [27]. It is proved in [4] that both $\mathscr{W}_{r,n}$ and $\mathcal{H}_{r,n}$ are cellular over R.

We need some combinatorics as follows.

A partition of m is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i \geqslant \lambda_{i+1}$ for all positive integers i and $|\lambda| := \lambda_1 + \lambda_2 + \cdots = m$. We call $l(\lambda)$, the number of positive integers λ_i, the length of λ. Similarly, an r-partition of m is an ordered r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions $\lambda^{(s)}$ with $1 \leqslant s \leqslant r$ such that $|\lambda| := |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = m$. Let $\Lambda_r^+(n)$ be the set of all r-partitions of n. We say that μ dominates λ and write $\lambda \trianglelefteq \mu$ if

$$\sum_{j=1}^{i-1} |\lambda^{(j)}| + \sum_{k=1}^{l} \lambda_k^{(i)} \leqslant \sum_{j=1}^{i-1} |\mu^{(j)}| + \sum_{k=1}^{l} \mu_k^{(i)}$$

for $1 \leqslant i \leqslant r$ and $l \geqslant 0$. Then $(\Lambda_r^+(n), \trianglelefteq)$ is a poset. If $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$, we write $\lambda \triangleleft \mu$.

Fix two positive integers r, n. Define

$$\Lambda_{r,n} = \bigl\{(f, \lambda) \bigm| 0 \leqslant f \leqslant \lfloor n/2 \rfloor,\ \lambda \in \Lambda_r^+(n - 2f)\bigr\}.$$

Given $(f, \lambda), (\ell, \mu) \in \Lambda_{r,n}$, we say that (f, λ) dominates (ℓ, μ) and write $(f, \lambda) \geqslant (\ell, \mu)$ if $f > \ell$ in the usual sense or $f = \ell$ and $\lambda \trianglerighteq \mu$. Then $(\Lambda_{r,n}, \geqslant)$ is a poset. Write $(f, \lambda) > (\ell, \mu)$ if $(f, \lambda) \geqslant (\ell, \mu)$ and $(f, \lambda) \neq (\ell, \mu)$.

Suppose that λ and μ are two r-partitions. We say that μ is obtained from λ by adding a box (or node) and write $\lambda \to \mu$ if there exists a pair (s, i) such that $\mu_i^{(s)} = \lambda_i^{(s)} + 1$ and $\mu_j^{(t)} = \lambda_j^{(t)}$ for $(t, j) \neq (s, i)$. In this case, we will also say that λ is obtained from μ by removing a box (or node). In the remaining part of this paper, we denote by $\mathscr{A}(\lambda)$ (resp. $\mathscr{R}(\lambda)$) the set of all addable (resp. removable) nodes of λ.

Following [4], we say that a sequence of r-partitions $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_n)$ is an up–down λ-tableau if $\mathfrak{t}_0 = \emptyset$, $\mathfrak{t}_n = \lambda$, and either $\mathfrak{t}_i \to \mathfrak{t}_{i-1}$ or $\mathfrak{t}_{i-1} \to \mathfrak{t}_i$, $1 \leqslant i \leqslant n$. Let $\mathscr{T}_n^{ud}(\lambda)$ be the set of all up–down λ-tableaux. When $\lambda \in \Lambda_r^+(n)$, any $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ can be identified with the corresponding standard λ-tableau $\mathbf{t}$ such that the entry i, $1 \leqslant i \leqslant n$, is in the node $\mathfrak{t}_i \setminus \mathfrak{t}_{i-1}$. In this case, we will use $\mathscr{T}_n^{std}(\lambda)$ instead of $\mathscr{T}_n^{ud}(\lambda)$.

The Young diagram $[\lambda]$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a collection of boxes arranged in left-justified rows with λ_i boxes in the i-th row of $[\lambda]$. For any $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \Lambda_r^+(n)$, the Young diagram $[\lambda]$ is an r-tuple Young diagrams $([\lambda^{(1)}], \ldots, [\lambda^{(r)}])$.

It is necessary to impose conditions on the parameters of the ground ring in order that the cyclotomic NW algebras have a well-behaved representation theory. The suitable condition, called $\mathbf{u}$-admissibility was found by Ariki, Mathas and Rui in [4, 3.6]. The explicit conditions will not be needed here. It was shown in [14, 5.2] that $\mathbf{u}$-admissibility is equivalent to $\mathscr{W}_{r,2}$ being free of rank $3r^2$.

In the remaining part of this paper, when we discuss the cyclotomic NW algebras, we always keep the $\mathbf{u}$-admissible conditions.

The following result, which is motivated by Enyang's work on Brauer algebras and Birman–Murakami–Wenzl algebras in [11], has been proved in [32]. We remark that the linear order $\prec$ for $\mathscr{W}_{r,n}$ needed in Definition 2.2 is defined in [32]. In this paper, we do not need the explicit definition.

Theorem 3.2. *(See [32].) Fix two positive integers $r, n \geqslant 2$. Let R be a commutative ring which contains 1_R, $u_1, u_2, \ldots, u_r$ and invertible 2. $\mathscr{W}_{r,n}$ has a JM-basis $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda),\ (f, \lambda) \in \Lambda_{r,n}\}$ and JM-elements $X_1, X_2, \ldots, X_n$. The scalars $c_{\mathfrak{t}}(k)$, $1 \leqslant k \leqslant n$ which are needed in Definition 2.2 are defined as*

$$c_{\mathfrak{t}}(k) = \begin{cases} u_s + j - i, & \text{if } \mathfrak{t}_k = \mathfrak{t}_{k-1} \cup (s, i, j), \\ -u_s - j + i, & \text{if } \mathfrak{t}_{k-1} = \mathfrak{t}_k \cup (s, i, j), \end{cases}$$

where $\mathfrak{t}_{k-1} \cup (s, i, j)$ is the r-partition obtained from $\mathfrak{t}_{k-1}$ by adding the node (s, i, j) which is in the i-th row, j-th column of s-component of $\mathfrak{t}_{k-1} \cup (s, i, j)$.

If $p=(s,i,j)\in[\lambda]$, we define

$$c_\lambda(p)=u_s+j-i. \tag{3.3}$$

Since $c_\lambda(p)$ depends only on (s,i,j), we will use $c(p)$ instead of $c_\lambda(p)$ in the remaining part of this paper.

The irreducible $\mathscr{W}_{r,n,\kappa}$-modules have been classified in [4, 8.5] for $\omega_0\neq 0$ and [32, 3.12] for the remaining cases. Let $e=\mathrm{char}(\kappa)$ (resp. $+\infty$) if $\mathrm{char}(\kappa)>0$ (resp. $\mathrm{char}(\kappa)=0$). The following result, which will be needed in Section 5, is the special case of such results.

Theorem 3.4. *(See [4,32].) Fix the positive integers r,n with $r,n\geqslant 2$. Suppose that κ is a field which contains $1,u_1,\dots,u_r$ such that $e>n$ and $|d|\geqslant n$ whenever $u_i-u_j=d\cdot 1_\kappa$ for some $d\in\mathbb{Z}$.*

a) *If either $2\nmid n$ or $2\mid n$ with $\omega_i\neq 0$ for some i, $0\leqslant i\leqslant r-1$, then the irreducible $\mathscr{W}_{r,n,\kappa}$-modules are indexed by $\Lambda_{r,n}$.*
b) *If $2\mid n$ and $\omega_i=0$ for all i, $0\leqslant i\leqslant r-1$, then the irreducible $\mathscr{W}_{r,n,\kappa}$-modules are indexed by $\Lambda_{r,n}\setminus\{(n/2,\emptyset)\}$.*

The following result gives a criterion for $\mathscr{W}_{r,n,\kappa}$ being semisimple over a field κ.

Theorem 3.5. *(See [32, 7.9].) Fix two positive integers $r,n\geqslant 2$.*

a) *If $(2u_i-(-1)^r)(u_i+u_j)=0$ for some i, j, with $1\leqslant i,j\leqslant r$ and $i\neq j$, then $\mathscr{W}_{r,n,\kappa}$ is not semisimple.*
b) *If $(2u_i-(-1)^r)(u_i+u_j)\neq 0$ for any i, j, with $1\leqslant i,j\leqslant r$ and $i\neq j$,*
   1) *$\mathscr{W}_{r,2,\kappa}$ is semisimple if and only if $e>n$ and $|d|\geqslant n$ whenever $u_i-u_j=d\cdot 1_\kappa$ for any $1\leqslant i<j\leqslant r$ and $d\in\mathbb{Z}$.*
   2) *Suppose $n\geqslant 3$ and $2\nmid r$. Then $\mathscr{W}_{r,n,\kappa}$ is semisimple if and only if*
      (a) *$e>n$,*
      (b) *$|d|\geqslant n$ whenever $u_i-u_j=d\cdot 1_\kappa$ for any $1\leqslant i<j\leqslant r$ and $d\in\mathbb{Z}$,*
      (c) *$2u_i\notin\bigcup_{k=3}^n\{3-k,3-2k,k-3\}$,*
      (d) *$u_i+u_j\notin\bigcup_{k=3}^n\{2-k,k-2\}$.*
   3) *Suppose $n\geqslant 3$ and $2\mid r$. Then $\mathscr{W}_{r,n}$ is semisimple if and only if*
      (a) *$e>n$,*
      (b) *$|d|\geqslant n$ whenever $u_i-u_j=d\cdot 1_\kappa$ for any $1\leqslant i<j\leqslant r$ and $d\in\mathbb{Z}$,*
      (c) *$2u_i\notin\bigcup_{k=3}^n\{3-k,2k-3,k-3\}$,*
      (d) *$u_i+u_j\notin\bigcup_{k=3}^n\{2-k,k-2\}$.*

Now, we recall the recursive formulae of Gram determinants for all cell modules of the cyclotomic Nazarov–Wenzl algebras in [32].

Let (a,b) and (c,d) be two pairs of positive integers. We write $(a,b)\geqslant(c,d)$ if $a>c$ or $a=c$ and $b\geqslant d$. Write $(a,b)>(c,d)$ if $(a,b)\geqslant(c,d)$ and $(a,b)\neq(c,d)$.

Definition 3.6. Given a $\lambda\in\Lambda_r^+(n-2f)$. For any removable (resp. an addable) node $p=(s,k,\lambda_k)$ (resp. (s,k,λ_k+1)) of λ, define

a) $\mathscr{R}(\lambda)^{<p}=\{(h,l,\lambda_l)\in\mathscr{R}(\lambda)\mid (h,l)>(s,k)\}$;
b) $\mathscr{A}(\lambda)^{<p}=\{(h,l,\lambda_l+1)\in\mathscr{A}(\lambda)\mid (h,l)>(s,k)\}$;
c) $\mathscr{R}(\lambda)^{\geqslant p}=\{(h,l,\lambda_l)\in\mathscr{R}(\lambda)\mid (h,l)\leqslant(s,k)\}$;
d) $\mathscr{A}(\lambda)^{\geqslant p}=\{(h,l,\lambda_l+1)\in\mathscr{A}(\lambda)\mid (h,l)\leqslant(s,k)\}$.

Unlike our previous definition of $c_\lambda(p)$ for $p\in[\lambda]$, we define

$$c_\lambda(p)=-u_s+i-j,\quad \text{if } p=(s,i,j)\in\mathscr{R}(\lambda),$$

in [32]. In order to obtain Propositions 3.8, 3.10 and 3.12 from Propositions 6.17, 6.22 and 6.34 in [32], respectively, we have to use $-c_\lambda(p)$ instead of $c_\lambda(p)$ in [32] when p is a removable node of $[\lambda]$. This is the key difference between Propositions 3.8–3.12 and those in [32]. We remark that we will use same wording for the cyclotomic Birman–Murakami–Wenzl algebras on page 198.

Suppose that $(f,\lambda)\in\Lambda_{r,n}$. Let $\mathfrak{t}^\lambda\in\mathscr{T}_n^{ud}(\lambda)$ be such that $\mathfrak{t}^\lambda_{2i}=\emptyset$ and $\mathfrak{t}^\lambda_{2i-1}=((1),\emptyset,\dots,\emptyset)$ for $1\leqslant i\leqslant f$ and $\mathfrak{t}^\lambda_j=\mathbf{t}^\lambda_{j-2f}$ where $\mathbf{t}^\lambda$ is defined below Proposition 5.2. Given $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{n-1}=\mu$, define $\mathfrak{u}\in\mathscr{T}_{n-1}^{ud}(\mu)$ such that $\mathfrak{u}_i=\mathfrak{t}_i$, $1\leqslant i\leqslant n-1$, and $\mathfrak{v}\in\mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{v}_j=\mathfrak{t}^\mu_j$ for $1\leqslant j\leqslant n-1$ and $\mathfrak{v}_n=\mathfrak{t}_n=\lambda$. In the remaining part of this paper, we also denote the previous $\mathfrak{u}$ by $\hat{\mathfrak{t}}$.

Proposition 3.7. *(See [32, 6.15].) Suppose $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda)\in\Lambda_{r,n}$. If $\mathfrak{t}_{n-1}=\mu$ with $(\ell,\mu)\in\Lambda_{r,n-1}$, then $\langle f_\mathfrak{t},f_\mathfrak{t}\rangle=\langle f_\mathfrak{u},f_\mathfrak{u}\rangle\frac{\langle f_\mathfrak{v},f_\mathfrak{v}\rangle}{\langle f_{\mathfrak{t}^\mu},f_{\mathfrak{t}^\mu}\rangle}$.*

Proposition 3.8. *(See [32, 6.17].) Let $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda)\in\Lambda_{r,n}$. If $\hat{\mathfrak{t}}=\mathfrak{t}^\mu$ with $\mathfrak{t}_n=\mathfrak{t}_{n-1}\cup\{p\}$ and $p=(m,k,\lambda^{(m)}_k)$, then*

$$\frac{\langle f_\mathfrak{t},f_\mathfrak{t}\rangle}{\langle f_{\mathfrak{t}^\mu},f_{\mathfrak{t}^\mu}\rangle}=(-1)^{r-m+1}\frac{\prod_{q\in\mathscr{A}(\lambda)^{<p}}(-c(p)+c(q))}{\prod_{r\in\mathscr{R}(\lambda)^{<p}}(-c(p)+c(r))}. \tag{3.9}$$

Proposition 3.10. *(See [32, 6.22].) Given a $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}^\mu=\hat{\mathfrak{t}}$. If $\mathfrak{t}_{n-1}=\mathfrak{t}_n\cup p$ with $p=(s,k,\mu^{(s)}_k)$ such that $\mu^{(j)}=\emptyset$ for all integers j, $s<j\leqslant r$ and $l(\mu^{(s)})=k$, then*

$$\frac{\langle f_\mathfrak{t},f_\mathfrak{t}\rangle}{\langle f_{\mathfrak{t}^\mu},f_{\mathfrak{t}^\mu}\rangle}=\prod_{j=s+1}^{r}\left(u_s+\mu^{(s)}_k-k+u_j\right)\prod_{q\in\mathscr{R}(\lambda)^{\geqslant p}}\frac{c(p)-c(q)}{c(p)+c(q)}\prod_{\substack{q\neq p\\ q\in\mathscr{A}(\lambda)^{\geqslant p}}}\frac{c(p)+c(q)}{c(p)-c(q)}A, \tag{3.11}$$

where

$$A=\begin{cases}(2u_s+2\mu^{(s)}_k-2k-(-1)^r), & \text{if } \lambda^{(s)}_k=0,\\ (2u_s+\mu^{(s)}_k-2k)(2u_s+2\mu^{(s)}_k-2k-(-1)^r), & \text{if } \lambda^{(s)}_k>0.\end{cases}$$

Proposition 3.12. *(See [32, 6.34].) Let $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$ with $(f,\lambda)\in\Lambda^+_{r,n}$, $\lambda=(\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(s)},\emptyset,\dots,\emptyset)$ and $l(\lambda^{(s)})=l$. Suppose $\hat{\mathfrak{t}}=\mathfrak{t}^\mu$, and $\mathfrak{t}_{n-1}=\mathfrak{t}_n\cup p$ with $p=(m,k,\mu^{(m)}_k)$ such that $(m,k)<(s,l)$. Then*

$$\frac{\langle f_\mathfrak{t},f_\mathfrak{t}\rangle}{\langle f_{\mathfrak{t}^\mu},f_{\mathfrak{t}^\mu}\rangle}=(-1)^{r-m+1}\prod_{\substack{q\neq p\\ q\in\mathscr{A}(\lambda)^{\geqslant p}}}\frac{c(p)+c(q)}{c(p)-c(q)}\prod_{q\in\mathscr{R}(\lambda)^{\geqslant p}}\frac{c(p)-c(q)}{c(p)+c(q)}$$
$$\times\frac{\prod_{q\in\mathscr{A}(\mu)^{<p}}(-c(p)-c(q))}{\prod_{r\in\mathscr{R}(\mu)^{<p}}(-c(p)-c(r))}B$$

where

$$B=\begin{cases}2u_m+2\mu^{(m)}_k-2k-(-1)^r, & \text{if } (m,k,\lambda^{(m)}_k)\in\mathscr{R}(\lambda),\\ \frac{2u_m+2\mu^{(m)}_k-2k-(-1)^r}{2u_m-2k+2\lambda^{(m)}_k+1}, & \text{if } (m,k,\lambda^{(m)}_k)\notin\mathscr{R}(\lambda).\end{cases}$$

From now on, we recall the notion of cyclotomic BMW algebras in [22].

Definition 3.13. Fix two positive integers $r, n \geqslant 2$. Let R be an integral domain which contains the multiplicative identity 1_R, ω_a, $a \in \mathbb{Z}$ and invertible elements q, $u_1, \dots, u_r$, ϱ, δ such that $\delta = q - q^{-1}$ and $\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1})$. The cyclotomic BMW algebra $\mathscr{B}_{r,n}$ [22] is the unital associative R-algebra generated by $\{T_i, E_i, X_j, X_j^{-1} \mid 1 \leqslant i < n \text{ and } 1 \leqslant j \leqslant n\}$ subject to the following relations:

a) $X_i X_i^{-1} = X_i^{-1} X_i = 1$, for $1 \leqslant i \leqslant n$.
b) (Kauffman skein relation) $1 = T_i^2 - \delta T_i + \delta \varrho E_i$, for $1 \leqslant i < n$.
c) (Braid relations)
  (i) $T_i T_j = T_j T_i$, if $|i - j| > 1$,
  (ii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for $1 \leqslant i < n - 1$,
  (iii) $T_i X_j = X_j T_i$, if $j \neq i, i + 1$.
d) (Idempotent relations) $E_i^2 = \omega_0 E_i$, for $1 \leqslant i < n$.
e) (Commutation relations) $X_i X_j = X_j X_i$, for $1 \leqslant i, j \leqslant n$.
f) (Skein relations)
  (i) $T_i X_i - X_{i+1} T_i = \delta X_{i+1}(E_i - 1)$, for $1 \leqslant i < n$,
  (ii) $X_i T_i - T_i X_{i+1} = \delta(E_i - 1) X_{i+1}$, for $1 \leqslant i < n$.
g) (Unwrapping relations) $E_1 X_1^a E_1 = \omega_a E_1$, for $a \in \mathbb{Z}$.
h) (Tangle relations)
  (i) $E_i T_i = \varrho E_i = T_i E_i$, for $1 \leqslant i \leqslant n - 1$,
  (ii) $E_{i+1} E_i = E_{i+1} T_i T_{i+1} = T_i T_{i+1} E_i$, for $1 \leqslant i \leqslant n - 2$.
i) (Untwisting relations)
  (i) $E_{i+1} E_i E_{i+1} = E_{i+1}$, for $1 \leqslant i \leqslant n - 2$,
  (ii) $E_i E_{i+1} E_i = E_i$, for $1 \leqslant i \leqslant n - 2$.
j) (Anti-symmetry relations) $E_i X_i X_{i+1} = E_i = X_i X_{i+1} E_i$, for $1 \leqslant i < n$.
k) (Cyclotomic relation) $(X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0$.

We remark that there is a slight difference between Definition 3.13 and that given in [22]. It has been pointed in [36] that such two definitions are equivalent when δ is invertible. Note that Häring-Oldenburg did not assume that δ is invertible. However, he did assume that ω_0 is invertible. Recently, $\mathscr{B}_{r,n}$ have been studying extensively by three groups of mathematicians in [12,17,18,13,36,33,41,38–40,42], etc.

If we denote by I the two-sided ideal of $\mathscr{B}_{r,n}$ generated by E_1, then $\mathscr{B}_{r,n}/I$ is isomorphic to the cyclotomic Hecke algebras (or Ariki–Koike algebras) $\mathscr{H}_{r,n}$ of type $G(r, 1, n)$ [2]. It is proved in [8] (resp. [38]) that $\mathscr{H}_{r,n}$ (resp. $\mathscr{B}_{r,n}$) is cellular over R. We remark that Goodman and Graber [15] give a new proof of the cellularity of $\mathscr{B}_{r,n}$. In this paper, we will make use of JM-basis for $\mathscr{B}_{r,n}$ [33] which is a weakly cellular basis. Goodman and Graber [16] constructed JM-basis by using different method, which recovers our results on JM-basis.

It is necessary to impose conditions on the parameters of the ground ring in order that the cyclotomic BMW algebras have a well-behaved representation theory. One suitable condition, called **u**-admissibility was found by Rui and Xu in [36, 2.27] and another condition, called admissibility was found by Wilcox and Yu in [38]. The explicit conditions are somewhat complicated and will not be needed here. It was shown in [13, 4.4] that **u**-admissibility and admissibility are equivalent, under the assumptions on the ground ring R adopted in Definition 3.13. Moreover, by [39], both conditions are equivalent to $\mathscr{B}_{r,2}$ being free of rank $3r^2$. In this case, $\varrho^{-1} = \varepsilon u_1 u_2 \cdots u_r$ and $\varepsilon \in \{-1, 1\}$ (resp. $\varrho^{-1} = \varepsilon q^{-\varepsilon} u_1 u_2 \cdots u_r$) if $2 \nmid r$ (resp. $2 \mid r$). For the simplification of notation, we use $u_{1,r}$ instead of $u_1 u_2 \cdots u_r$ later on.

In the remaining part of this paper, we always assume the **u**-admissible conditions when we discuss the cyclotomic BMW algebras. The following result gives the JM-basis for $\mathscr{B}_{r,n}$. We remark that the linear order for $\mathscr{B}_{r,n}$ which is needed in Definition 2.2 is given in [33]. In this paper, we do not need its explicit definition.

Theorem 3.14. *(See [33].) Fix two positive integers $r, n \geqslant 2$. Let R be an integral domain which contains the multiplicative identity 1_R, ω_a, $a \in \mathbb{Z}$ and invertible elements $q, u_1, \dots, u_r, \varrho, \delta$ such that $\delta = q - q^{-1}$ and $\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1})$. Then $\mathscr{B}_{r,n}$ has the JM-basis $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(f, \lambda), (f, \lambda) \in \Lambda_{r,n}\}$ and JM-elements $X_1, X_2, \dots, X_n$. The scalars $c_{\mathfrak{t}}(k)$, $1 \leqslant k \leqslant n$ which are needed in Definition 2.2 are defined as*

$$c_{\mathfrak{t}}(k) = \begin{cases} u_s q^{2(j-i)}, & \text{if } \mathfrak{t}_k = \mathfrak{t}_{k-1} \cup (s, i, j), \\ u_s^{-1} q^{-2(j-i)}, & \text{if } \mathfrak{t}_{k-1} = \mathfrak{t}_k \cup (s, i, j). \end{cases}$$

We remark that the JM-basis mentioned in Theorem 3.14 is a weakly cellular basis. Note that JM-basis and JM-elements for Birman–Murakami–Wenzl algebras have been constructed by Enyang in [11]. This JM-basis is a cellular basis.

If $p = (s, i, j) \in [\lambda]$, we define

$$c_\lambda(p) = u_s q^{2(j-i)}. \tag{3.15}$$

Since $c_\lambda(p)$ for both algebras with different definitions depend only on (s, i, j), we will use $c(p)$ instead of $c_\lambda(p)$ in the remaining part of this paper.

We set $o(q^2) = \infty$ if q^2 is not a root of unity. The following result, which will be needed in Section 5, is a special case of [36, 5.3].

Theorem 3.16. *Suppose that κ is a field which contains non-zero elements $q, u_1, \dots, u_r$ and $q - q^{-1}$ such that $o(q^2) > n$ and $|d| \geqslant n$ whenever $u_i u_j^{-1} = q^{2d}$ for some $d \in \mathbb{Z}$.*

a) *If either $2 \nmid n$ or $2 \mid n$ and $\omega_i \neq 0$ for some integers $i \leqslant r$, then the set of all pairwise non-isomorphic irreducible $\mathscr{B}_{r,n,\kappa}$-modules are indexed by $\Lambda_{r,n}$.*
b) *If $2 \mid n$ and $\omega_i = 0$ for all positive integers $i \leqslant r$, then the irreducible $\mathscr{B}_{r,n,\kappa}$-modules are indexed by $\Lambda_{r,n} \setminus \{(n/2, \emptyset)\}$.*

For convenience, we define

$$Q_{r,\varrho} = \begin{cases} \{-\varepsilon q, \varepsilon q^{-1}\}, & \text{if } 2 \nmid r,\ \varrho^{-1} = \varepsilon u_{1,r}, \\ \{-q^{\varepsilon}, q^{\varepsilon}\}, & \text{if } 2 \mid r,\ \varrho^{-1} = \varepsilon q^{-\varepsilon} u_{1,r}, \end{cases} \tag{3.17}$$

and

$$S_{r,\varrho} = \begin{cases} \bigcup_{k=3}^{n} \{\pm q^{3-k}, \pm q^{k-3}, \varepsilon q^{3-2k}, -\varepsilon q^{2k-3}\}, & \text{if } 2 \nmid r,\ \varrho^{-1} = \varepsilon u_{1,r}, \\ \bigcup_{k=3}^{n} \{\pm q^{3-k}, \pm q^{k-3}, \pm q^{(2k-3)\varepsilon}\}, & \text{if } 2 \mid r,\ \varrho^{-1} = \varepsilon q^{-\varepsilon} u_{1,r}. \end{cases} \tag{3.18}$$

The following result is the generalization of Theorem 5.9 in [34].

Theorem 3.19. *(See [33, 6.5].) Fix two positive integers $n, r \geqslant 2$. Let κ be a field which contains ω_i, $i \in \mathbb{Z}$ and non-zero u_i, $1 \leqslant i \leqslant r$, $q, q - q^{-1}$. Suppose that $\Omega \cup \{\varrho\}$ is **u**-admissible.*

a) *If either $u_i - u_j^{-1} = 0$ for different positive integers $i, j \leqslant r$ or $u_i \in Q_{r,\varrho}$ for some positive integer $i \leqslant r$, then $\mathscr{B}_{r,n,\kappa}$ is not semisimple.*
b) *Assume $u_i - u_j^{-1} \neq 0$ for all different positive integers $i, j \leqslant r$ and $u_i \notin Q_{r,\varrho}$ for all positive integers $i \leqslant r$.*
   (1) *$\mathscr{B}_{r,2,\kappa}$ is semisimple if and only if $o(q^2) > 2$ and $|d| \geqslant 2$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 \leqslant i < j \leqslant r$ and $d \in \mathbb{Z}$.*
   (2) *Suppose $n \geqslant 3$. Then $\mathscr{B}_{r,n,\kappa}$ is semisimple if and only if*
      (a) *$o(q^2) > n$,*
      (b) *$|d| \geqslant n$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 \leqslant i < j \leqslant r$ and $d \in \mathbb{Z}$,*

(c) $u_i \notin S_{r,\varrho}$,
(d) $u_i u_j \notin \bigcup_{k=3}^{n}\{q^{4-2k}, q^{2k-4}\}$ *for all different positive integers* $i, j \leqslant r$.

We recall the recursive formulae on Gram determinants for cyclotomic BMW algebras in [33], which will be used later on. We remark that we defined

$$c_\lambda(p) = u_s^{-1} q^{-2(j-i)}, \quad \text{if } (s,i,j) \in \mathscr{R}(\lambda)$$

in [33]. In order to obtain Propositions 3.23, 3.25 and 3.27, we have to use $c_\lambda(p)^{-1}$ instead of $c_\lambda(p)$ in Propositions 4.9, 4.10 and 4.11 in [33] when p is a removable node of $[\lambda]$.

Assume $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$. If r is odd, we define

$$E_{\mathfrak{s}\mathfrak{s}}(k) = \begin{cases} \frac{1}{\varrho c_{\mathfrak{s}}(k)}\left(\frac{c_{\mathfrak{s}}(k) - c_{\mathfrak{s}}(k)^{-1}}{\delta} + 1\right) A_1 B_1, & \text{if } \varrho^{-1} = u_{1,r}, \\ \frac{1}{\varrho c_{\mathfrak{s}}(k)}\left(\frac{c_{\mathfrak{s}}(k) - c_{\mathfrak{s}}(k)^{-1}}{\delta} - 1\right) A_1 B_1, & \text{if } \varrho^{-1} = -u_{1,r}, \end{cases} \tag{3.20}$$

where

$$A_1 = \prod_{\alpha \in \mathscr{A}(\mathfrak{s}_{k-1})} \frac{c_{\mathfrak{s}}(k) - c(\alpha)^{-1}}{c_{\mathfrak{s}}(k) - c(\alpha)} \quad \text{and} \quad B_1 = \prod_{\alpha \in \mathscr{R}(\mathfrak{s}_{k-1})} \frac{c_{\mathfrak{s}}(k) - c(\alpha)}{c_{\mathfrak{s}}(k) - c(\alpha)^{-1}}$$

and $\alpha \neq \mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$. Here $\mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ is the set difference between $\mathfrak{s}_k$ and $\mathfrak{s}_{k-1}$.

If r is even, then we define

$$E_{\mathfrak{s}\mathfrak{s}}(k) = \begin{cases} \frac{1}{\varrho\delta}\left(1 - \frac{q^2}{c_{\mathfrak{s}}(k)^2}\right) A_1 B_1, & \text{if } \varrho^{-1} = q^{-1} u_{1,r}, \\ \frac{1}{\varrho\delta}\left(1 - \frac{1}{q^2 c_{\mathfrak{s}}(k)^2}\right) A_1 B_1, & \text{if } \varrho^{-1} = -q u_{1,r}, \end{cases} \tag{3.21}$$

where A_1 and B_1 are defined above.

Proposition 3.22. *(See [33, 4.8].) Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $(f,\lambda) \in \Lambda_{r,n}$. *If* $\mathfrak{t}_{n-1} = \mu$ *with* $(\ell,\mu) \in \Lambda_{r,n-1}$, *then* $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle = \langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle \frac{\langle f_{\mathfrak{v}}, f_{\mathfrak{v}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle}$.

Proposition 3.23. *(See [33, 4.9].) Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $(f,\lambda) \in \Lambda_{r,n}$. *If* $\hat{\mathfrak{t}} = \mathfrak{t}^\mu$ *with* $\mathfrak{t}_n = \mathfrak{t}_{n-1} \cup \{p\}$ *and* $p = (m, k, \lambda_k^{(m)})$, *then*

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = \frac{(-1)^{r-m} q^{2k}}{u_m(1-q^2)} \frac{\prod_{a \in \mathscr{A}(\lambda)^{<p}} (c(a) - c(p))}{\prod_{b \in \mathscr{R}(\lambda)^{<p}} (c(b) - c(p))}. \tag{3.24}$$

For each positive integer a, define the q-integer $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$.

Proposition 3.25. *(See [33, 4.10].) Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\lambda \in \Lambda_r^+(n-2f)$ *and* $\mathfrak{t}^\mu = \hat{\mathfrak{t}}$. *If* $\mathfrak{t}_{n-1} = \mathfrak{t}_n \cup \{p\}$ *with* $p = (s, k, \mu_k^{(s)})$ *such that* $\mu^{(j)} = \emptyset$ *for all integers* j, $s < j \leqslant r$ *and* $l((\mu^{(s)})) = k$, *then*

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = q^{a-1}[a] E_{\mathfrak{t}\mathfrak{t}}(n-1) \prod_{j=s+1}^{r} \left(u_s q^{2(\mu_k^{(s)} - k)} - u_j\right), \tag{3.26}$$

where $a = \mu_k^{(s)}$.

For any $\mu \in \Lambda_r^+(n)$, write $[\mu] = [b_1, b_2, \dots, b_r]$ such that $b_i = \sum_{j \leqslant i} |\lambda^{(j)}|$ for all positive integers $i \leqslant r$.

Let $\mathfrak{S}_n$ be the symmetric group in n letters. Then $\mathfrak{S}_n$ is a Coxeter group with distinguished generators $\{s_1, s_2, \dots, s_{n-1}\}$ where s_i is the basic transposition $(i, i+1)$. We write $s_{i,j} = s_{i-1}s_{i-2}\cdots s_j$ if $i < j$.

Suppose $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and $s_k \in \mathfrak{S}_n$. If $\mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ and $\mathfrak{s}_k \ominus \mathfrak{s}_{k+1}$ are in different rows and in different columns, we define

$$\mathfrak{s}s_k = (\mathfrak{s}_1, \dots, \mathfrak{s}_{k-1}, \mathfrak{t}_k, \mathfrak{s}_{k+1}, \dots, \mathfrak{s}_n),$$

where $\mathfrak{t}_k$ is the multipartition which is uniquely determined by the conditions $\mathfrak{t}_k \ominus \mathfrak{s}_{k+1} = \mathfrak{s}_{k-1} \ominus \mathfrak{s}_k$ and $s_{k-1} \ominus \mathfrak{t}_k = \mathfrak{s}_k \ominus \mathfrak{s}_{k+1}$. If the nodes $s_k \ominus \mathfrak{s}_{k-1}$ and $\mathfrak{s}_{k+1} \ominus \mathfrak{s}_k$ are both in the same row, or both in the same column, then $\mathfrak{s}s_k$ is not defined.

Proposition 3.27. *(See [33, 4.11].) Suppose that $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(s)}, \emptyset, \dots, \emptyset) \in \Lambda_r^+(n-2f)$ and $l(\lambda^{(s)}) = l$. Let $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}$ such that $\hat{\mathfrak{t}} = \mathfrak{t}^\mu$, and $\mathfrak{t}_{n-1} = \mathfrak{t}_n \cup \{p\}$ with $p = (m, k, \mu_k^{(m)})$ and $(m,k) < (s,l)$. Let $[\mu] = [b_1, b_2, \dots, b_r]$. We define $\mathfrak{u} = \mathfrak{t}s_{n,a+1}$ with $a = 2(f-1) + b_{m-1} + \sum_{j=1}^{k} \mu_j^{(m)}$ and $\mathfrak{v} = (\mathfrak{u}_1, \dots, \mathfrak{u}_{a+1})$. Then*

$$\frac{\langle f_\mathfrak{t}, f_\mathfrak{t}\rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu}\rangle} = q^{c-1}[c]E_{\mathfrak{v}\mathfrak{v}}(a)\big(u_m q^{-2k} - u_m^{-1} q^{-2(\mu_k^{(m)}-k)}\big)^{-1} A \tag{3.28}$$

where $c = \mu_k^{(m)}$ and

$$A = \prod_{j=m+1}^{r} \frac{(u_m q^{2(\mu_k^{(m)}-k)} - u_j)}{(u_j - u_m^{-1} q^{-2(\mu_k^{(m)}-k)})} \frac{\prod_{a \in \mathscr{A}(\mu)^{<p}} (c(a) - c(p)^{-1})}{\prod_{b \in \mathscr{R}(\mu)^{<p}} (c(b) - c(p)^{-1})}.$$

At the end of this section, we recall some results on two functors, which will play an important role in our work. Such results have been proved for the Brauer and BMW algebras in [10,35].

Let $A_{r,n,\kappa}$-mod be the category of right $A_{r,n,\kappa}$-modules where $A_{r,n,\kappa} \in \{\mathscr{W}_{r,n,\kappa}, \mathscr{B}_{r,n,\kappa}\}$. It has been proved in [4,36] that

$$E_{n-1} A_{r,n,\kappa} E_{n-1} = E_{n-1} A_{r,n-2,\kappa}.$$

By standard arguments in Section 6 in [21], one can define the exact functor $\mathcal{F}_{r,n} : A_{r,n,\kappa}\text{-mod} \to A_{r,n-2,\kappa}\text{-mod}$ and right exact functor $\mathcal{G}_{r,n-2} : A_{r,n-2,\kappa}\text{-mod} \to A_{r,n,\kappa}\text{-mod}$. For the simplification of notation, we will use $\mathcal{F}$ and $\mathcal{G}$ instead of $\mathcal{F}_{r,n}$, $\mathcal{G}_{r,n-2}$ respectively. The following results have been proved for $\mathscr{B}_{r,n,\kappa}$ in [33]. Further, one can prove them for $\mathscr{W}_{r,n,\kappa}$ by similar arguments in [33]. We leave the details to the reader.

Proposition 3.29. *For each $(f,\lambda), (\ell,\mu) \in \Lambda_{r,n}$, let $\Delta(f,\lambda)$ and $\Delta(\ell,\mu)$ be the cell modules of $A_{r,n,\kappa}$ where $A_{r,n,\kappa} \in \{\mathscr{W}_{r,n,\kappa}, \mathscr{B}_{r,n,\kappa}\}$. We have:*

a) $\mathcal{F}\mathcal{G} = 1$;
b) $\mathcal{G}(\Delta(f,\lambda)) = \Delta(f+1,\lambda)$;
c) $\mathcal{F}(\Delta(f,\lambda)) = \Delta(f-1,\lambda)$;
d) $\mathrm{Hom}_{A_{r,n+2,\kappa}}(\mathcal{G}(\Delta(f,\lambda)), \Delta(\ell,\mu)) \cong \mathrm{Hom}_{A_{r,n,\kappa}}(\Delta(f,\lambda), \mathcal{F}(\Delta(\ell,\mu)))$ *as κ-modules*;
e) *If $A_{r,n,\kappa}$-homomorphism $\phi : \Delta(f,\lambda) \to N$ is non-trivial for $0 < f \leqslant \lfloor n/2 \rfloor$, then $\mathcal{F}(\phi) \neq 0$.*

4. The structure of $\Delta(1, \lambda)$

In this section, we keep the **u**-admissible conditions with different definitions over the field κ when we deal with $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$. Further, we keep the additional constraint, Assumption 4.1. The main purpose of this section is to study the structure of the cell modules $\Delta(1, \lambda)$ for $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$ with respect to $(1, \lambda) \in \Lambda_{r,n}$.

Assumption 4.1. *Let κ be a field. We set $e = \mathrm{char}(\kappa)$ if $\mathrm{char}(\kappa) > 0$ and $e = \infty$ if $\mathrm{char}(\kappa) = 0$.*

a) *For $\mathscr{W}_{r,n,\kappa}$, $e > n$ and $|d| \geqslant n$ if $u_i - u_j = d \cdot 1_\kappa$ for some $d \in \mathbb{Z}$ and $1 \leqslant i < j \leqslant r$.*
b) *For $\mathscr{B}_{r,n,\kappa}$, $e \neq 2$, $o(q^2) > n$, and $|d| \geqslant n$ if $u_i u_j^{-1} = q^{2d}$ for some $d \in \mathbb{Z}$ and $1 \leqslant i < j \leqslant r$.*

Under Assumption 4.1(a) (resp. Assumption 4.1(b) without $\mathrm{char}(\kappa) \neq 2$), the degenerate cyclotomic Hecke algebra $\mathcal{H}_{r,n}$ (resp. cyclotomic Hecke algebra $\mathscr{H}_{r,n}$ [2]) is split semisimple over κ [4] (resp. [1]).

The following result is well known. The key point is that Assumption 4.1 on κ implies that the residues of the addable and removable nodes of $[\lambda]$ are distinct so an $\mathfrak{s} \in \mathscr{T}_n^{std}(\lambda)$ is uniquely determined by the sequence of residues $c_\mathfrak{s}(i)$, for $i = 1, \dots, n$. We remark that cyclotomic Nazarov–Wenzl algebras and cyclotomic Birman–Murakami–Wenzl algebras are being considered simultaneously in Lemmas 4.2–4.4.

Lemma 4.2. *For any $\mathfrak{s} \in \mathscr{T}_n^{std}(\lambda)$ and $\mathfrak{t} \in \mathscr{T}_n^{std}(\mu)$, and $\lambda, \mu \in \Lambda_r^+(n)$, we have $\mathfrak{s} = \mathfrak{t}$ if and only if $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in κ, for all positive integers $i \leqslant n$.*

Lemma 4.3. *Suppose that $\lambda \in \Lambda_r^+(n)$ and $p, \tilde{p} \in [\lambda]$. $c(p) = c(\tilde{p})$ if and only if p and $\tilde{p}$ are in the same diagonal of some component of λ.*

Lemma 4.4. *For each $\lambda \in \Lambda_r^+(n)$ and $\mu \in \Lambda_r^+(n-2)$, $\Delta(0, \lambda)$ is a composition factor of $\Delta(1, \mu)$ if*

a) *there is an $\mathfrak{s} \in \mathscr{T}_n^{std}(\lambda)$ and a unique $\mathfrak{t} \in \mathscr{T}_n^{ud}(\mu)$, such that $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ for all positive integers $i \leqslant n$,*
b) *$g_\mathfrak{t} \in \mathrm{Rad}\, \Delta(1, \mu)$.*

Proof. This result can be proved by arguments similar to those for BMW algebras in [35]. We give the details as follows.

Let $A_{r,n,\kappa} \in \{\mathscr{W}_{r,n,\kappa}, \mathscr{B}_{r,n,\kappa}\}$ and let M be the cyclic $A_{r,n,\kappa}$-module generated by $g_\mathfrak{t}$. Since $g_\mathfrak{t} \in \mathrm{Rad}\, \Delta(1, \mu)$, $M \subset \mathrm{Rad}\, \Delta(1, \mu)$. So, there is a proper submodule N of M such that $M/N \cong D^{\ell,\nu}$ with $(\ell, \nu) < (1, \mu)$. Therefore, either $\ell = 1$, $\nu \lhd \mu$ or $\ell = 0$. In the first case, we apply the exact functor $\mathcal{F}$ on both $\Delta(1, \nu)$ and $\Delta(1, \mu)/N$ and use Proposition 3.29(e) to obtain a non-trivial homomorphism from $\Delta(0, \nu)$ to $\Delta(0, \mu)/\mathcal{F}(N)$. Under Assumption 4.1(a) (resp. (b)), $\mathcal{H}_{r,n,\kappa}$ (resp. $\mathscr{H}_{r,n,\kappa}$) is semisimple. So, $\Delta(0, \mu)$, which can be considered as the cell module for either $\mathcal{H}_{r,n-2,\kappa}$ or $\mathscr{H}_{r,n-2,\kappa}$, is irreducible, forcing $\mathcal{F}(N) = 0$ and $\nu = \mu$, a contradiction. In the later case, since N is a proper submodule of M, $g_\mathfrak{t} \notin N$. By Lemma 2.9, $(g_\mathfrak{t} + N)L_i = c_\mathfrak{t}(i)(g_\mathfrak{t} + N)$. Note that $\{f_\mathfrak{u} \mid \mathfrak{u} \in \mathscr{T}_n^{std}(\nu)\}$ is a basis of $\Delta(0, \nu)$ and $f_\mathfrak{u} L_i = c_\mathfrak{u}(i) f_\mathfrak{u}$ for all $\mathfrak{u} \in \mathscr{T}_n^{std}(\nu)$ and all positive integers $i \leqslant n$. So, there is a unique $\mathfrak{u} \in \mathscr{T}_n^{std}(\nu)$ such that $c_\mathfrak{u}(i) = c_\mathfrak{t}(i)$ for all positive integers $i \leqslant n$. Since we are assuming $c_\mathfrak{t}(i) = c_\mathfrak{s}(i)$ for any $1 \leqslant i \leqslant n$, by Lemma 4.2, $\mathfrak{u} = \mathfrak{s}$. In particular, $\nu = \lambda$. This proves that $\Delta(0, \lambda)$ is a composition factor of $\Delta(1, \mu)$. $\square$

We choose a modular system for either $\mathscr{W}_{r,n,\kappa}$ or $\mathscr{B}_{r,n,\kappa}$ as follows. Note that we are keeping Assumption 4.1.

Let $\kappa[t]$ be the ring of polynomials in indeterminate t. Let $\mathcal{O}$ be the localization of $\kappa[t]$ at $t - 1$. Then $\mathcal{O}$ is a discrete valuation ring with the unique maximal ideal π generated by $1 - t$ in $\mathcal{O}$. Moreover, $\kappa \cong \mathcal{O}/\pi$. Let K be the field of fractions of $\mathcal{O}$. When we consider $\mathscr{W}_{r,n,\mathcal{O}}$ or $\mathscr{W}_{r,n,K}$, we need to use $u_i t$ instead of u_i for $1 \leqslant i \leqslant r$. In this case, the residue $c_\mathfrak{t}(i)$ for $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ can be obtained from that in Theorem 3.2 by using u_it instead of u_i. It is easy to check that $c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i)$ in K for all $1 \leqslant i \leqslant n$ if and only if $\mathfrak{s} = \mathfrak{t}$.

Let $\kappa[t, t^{-1}]$ be the ring of Laurent polynomials in indeterminate t. Let $\mathcal{O}$ be the localization of $\kappa[t, t^{-1}]$ at the maximal ideal generated by $1 - t$. Then $\mathcal{O}$ is a discrete valuation ring with maximal ideal π generated by $1 - t$ in $\mathcal{O}$. Moreover, $\kappa \cong \mathcal{O}/\pi$. Let K be the field of fractions of $\mathcal{O}$. When we consider $\mathscr{B}_{r,n,\mathcal{O}}$ or $B_{r,n,K}$, we need to use u_it instead of u_i for $1 \leqslant i \leqslant r$. In this case, the residue $c_{\mathfrak{t}}(i)$ for $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ can be obtained from that in Theorem 3.14 by using u_it instead of u_i. It is easy to check that $c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i)$ in K for all $1 \leqslant i \leqslant n$ if and only if $\mathfrak{s} = \mathfrak{t}$.

For $\mathscr{W}_{r,n,\mathcal{O}}$ or $\mathscr{W}_{r,n,K}$, the residue $c(p)$ is defined to be $u_st + (j - i)$ if $p = (s, i, j) \in [\lambda]$. Similarly, for $\mathscr{B}_{r,n,\mathcal{O}}$ or $\mathscr{B}_{r,n,K}$, the residue $c(p)$ is defined to be $u_stq^{2(j-i)}$ if $p = (s, i, j) \in [\lambda]$.

We define $p^+ = (k, i, j + 1)$ and $p^- = (k, i + 1, j)$ if $p = (k, i, j)$. We say that two nodes p and $\tilde{p}$ consist of an admissible pair if $c(p)c(\tilde{p}) = 1$ (resp. $c(p) + c(\tilde{p}) = 0$) for $\mathscr{B}_{r,n,\kappa}$ (resp. $\mathscr{W}_{r,n,\kappa}$).

The following definition is motivated by our formulae on Gram determinants for both $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$.

Definition 4.5. Suppose that $\lambda \in \Lambda_r^+(n)$ and $\mu \in \Lambda_r^+(n - 2)$. We say that λ is $(1, \mu)$-admissible over κ if:

a) For $\mathscr{W}_{r,n,\kappa}$, $\lambda \supset \mu$ and $[\lambda/\mu]$ consists of an admissible pair such that $[\lambda/\mu] \neq \{p, p^-\}$ (resp. $[\lambda/\mu] \neq \{p, p^+\}$) if $2 \nmid r$ (resp. $2 \mid r$).
b) For $\mathscr{B}_{r,n,\kappa}$, $\lambda \supset \mu$ and $[\lambda/\mu]$ consists of an admissible pair. Furthermore,
   (i) if $2 \mid r$ and $\rho^{-1} = q^{-1}u_{1,r}$, then $[\lambda/\mu] \neq \{p, p^+\}$;
   (ii) if $2 \mid r$ and $\rho^{-1} = -qu_{1,r}$, then $[\lambda/\mu] \neq \{p, p^-\}$;
   (iii) if $2 \nmid r$ and $\varrho^{-1} = \varepsilon u_{1,r}$, then $[\lambda/\mu] \neq \{p, p^-\}$ (resp. $[\lambda/\mu] \neq \{p, p^+\}$) if $r \in \varepsilon q^{\mathbb{Z}}$ (resp. $r \in -\varepsilon q^{\mathbb{Z}}$), and $q^{\mathbb{Z}} = \{q^\ell \mid \ell \in \mathbb{Z}\}$.

Since we are keeping Assumption 4.1, we have $\varepsilon q^{-1} \notin \{-\varepsilon q^{-1}, -\varepsilon q\}$ and $q^{-1}u_{1,r} \neq -qu_{1,r}$. So, Definition 4.5 is well defined.

Lemma 4.6. *Let $(1, \mu) \in \Lambda_{r,n}$. If λ is $(1, \mu)$-admissible, then $\Delta(0, \lambda)$ is a composition factor of $\Delta(1, \mu)$.*

Proof. By assumption, $\lambda = \mu \cup \{p, \tilde{p}\}$ for some $p \in \mathscr{A}(\mu)$. We consider the $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ such that $\mathfrak{t}^\mu = (\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_{n-2})$ and $\mathfrak{s}_{n-1} = \mu \cup p$ and $\mathfrak{s}_n = \mu$. Then $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n) \in \mathscr{T}_n^{std}(\lambda)$ if $\mathbf{t}_i = \mathfrak{s}_i$, $1 \leqslant i \leqslant n - 1$ and $\mathfrak{t}_n = \mathfrak{s}_{n-1} \cup \{\tilde{p}\}$. Further, $c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i)$, $1 \leqslant i \leqslant n$. By Lemma 4.3, the residue class which contains $\mathfrak{s}$ is $\{\mathfrak{s}\}$.

We want to compute $\phi_{1,\mu}(\tilde{g}_{\mathfrak{s}}, \tilde{g}_{\mathfrak{s}})$ over $\mathcal{O}$. First, we deal with $\mathscr{W}_{r,n,\mathcal{O}}$. Let $\mathfrak{u} = (\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1})$. By Propositions 3.7–3.12,

$$\phi_{1,\mu}(\tilde{g}_{\mathfrak{s}}, \tilde{g}_{\mathfrak{s}}) = \gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} \phi_{0,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}})$$

where

$$\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} = b_1\big(2c(p) - (-1)^r\big) \prod_{\substack{p_1 \in \mathscr{A}(\mu)\\ p \neq p_1}} \big(c(p_1) + c(p)\big) \prod_{\tilde{p}_1 \in \mathscr{R}(\mu)} \big(c(p) + c(\tilde{p}_1)\big)^{-1}$$

for some $b_1 \in \mathcal{O}$ whose image in κ is invertible. Further, by Proposition 3.8, $\phi_{0,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}}) \in \mathcal{O}$ whose image in κ is invertible. There are two cases we have to discuss.

First, we assume $2 \nmid r$ and $\tilde{p} \neq p^-$. If $\tilde{p} = p^+$, by Lemma 4.3, neither $p_1 \in \mathscr{A}(\mu)$ nor $\tilde{p}_1 \in \mathscr{R}(\mu)$ such that the image of $(c(p_1) + c(p))(c(p) + c(\tilde{p}_1))$ in κ is zero. However, since $\tilde{p} = p^+$, we have

$2c(p)+1=0$, forcing $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}=0$ in κ. If $\tilde{p}\notin\{p^-,p^+\}$, then $\tilde{p}\in\mathscr{A}(\mu)$. By Lemma 4.3, there is no $\tilde{p}_1\in\mathscr{R}(\mu)$ such that the image of $(c(p)+c(\tilde{p}_1))$ in κ is zero. Since

$$\prod_{\substack{p_1\in\mathscr{A}(\mu)\\ p\neq p_1}}\big(c(p_1)+c(p)\big)=0,$$

we have $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}=0$

Second, we assume $2\mid r$ and $\tilde{p}\neq p^+$. If $\tilde{p}=p^-$, then there is neither $p_1\in\mathscr{A}(\mu)$ nor $\tilde{p}_1\in\mathscr{R}(\mu)$ such that the image of $(c(p_1)+c(p))(c(p)+c(\tilde{p}_1))$ in κ is zero. However, since $2c(p)-1=0$, we have $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}=0$ in κ. If $\tilde{p}\notin\{p^-,p^+\}$, then $\tilde{p}\in\mathscr{A}(\mu)$. By Lemma 4.3, there is no $\tilde{p}_1\in\mathscr{R}(\mu)$ such that the image of $(c(p)+c(\tilde{p}_1))$ in κ is zero. So $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}=0$ in κ.

Now, we deal with $\mathscr{B}_{r,n,\kappa}$. We use Propositions 3.22–3.27 to obtain the following equality

$$\phi_{1,\mu}(\tilde{g}_{\mathfrak{s}},\tilde{g}_{\mathfrak{s}})=\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}\phi_{0,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}},\tilde{g}_{\mathfrak{u}})$$

where

$$\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}=b_1B\prod_{\substack{p_1\in\mathscr{A}(\mu)\\ p_1\neq p}}\big(c(p_1)c(p)-1\big)\prod_{\tilde{p}_1\in\mathscr{R}(\mu)}\big(c(p)c(\tilde{p}_1)-1\big)^{-1}$$

for some $b_1\in\mathcal{O}$ whose image in κ is invertible, and

$$B=\begin{cases}(c(p)-\varepsilon q^{-1})(c(p)+\varepsilon q), & \text{if } 2\nmid r \text{ and } \varrho^{-1}=\varepsilon u_{1,r};\\ (c(p)^2-q^{-2\varepsilon}), & \text{if } 2\mid r \text{ and } \varrho^{-1}=-\varepsilon q^{\varepsilon}u_{1,r}.\end{cases}$$

Further, by Proposition 3.23, $\phi_{0,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}},\tilde{g}_{\mathfrak{u}})\in\mathcal{O}$ whose image in κ is not zero.

In order to show $\phi_{1,\mu}(\tilde{g}_{\mathfrak{s}},\tilde{g}_{\mathfrak{s}})=0$, we have to deal with three cases as follows: (1) $\tilde{p}\notin\{p^-,p^+\}$; (2) $\tilde{p}=p^+$; (3) $\tilde{p}=p^-$. We remark that one can use similar arguments for $\mathscr{W}_{r,n,\kappa}$ to verify $\phi_{1,\mu}(\tilde{g}_{\mathfrak{s}},\tilde{g}_{\mathfrak{s}})=0$. We leave the details to the reader. By Lemma 2.10, $g_{\mathfrak{s}}\in\operatorname{Rad}\Delta(1,\mu)$. Using Lemma 4.4 yields the result, as required. □

Let $r_{\lambda,p,\tilde{p}}=\dim\Delta(0,\lambda\cup p\cup\tilde{p})$ if $\lambda\cup p\cup\tilde{p}$ is an r-partition. For each $(f,\lambda)\in\Lambda_{r,n}$, let $\det G_{f,\lambda}$ be the Gram determinant associated to the cell module $\Delta(f,\lambda)$ with respect to the invariant form $\phi_{f,\lambda}$, which is defined via its JM-basis.

Proposition 4.7. *(See [33, 4.14].) Suppose $\lambda\in\Lambda_r^+(n-2)$ and $r\geqslant 2$ and $n\geqslant 2$. If $2\nmid r$ and $\varrho^{-1}=\varepsilon u_{1,r}t^r$, we define*

$$B=\prod_{\lambda\cup p\cup p^+\in\Lambda_r^+(n)}\big(c(p)-\varepsilon q^{-1}\big)^{r_{\lambda,p,p^+}}\prod_{\lambda\cup p\cup p^-\in\Lambda_r^+(n)}\big(c(p)+\varepsilon q\big)^{r_{\lambda,p,p^-}}.$$

Otherwise, we define

$$B=\begin{cases}\prod_{\lambda\cup p\cup p^-\in\Lambda_r^+(n)}(c(p)^2-q^2)^{r_{\lambda,p,p^-}}, & \text{if } 2\mid r,\ \varrho^{-1}=q^{-1}u_{1,r}t^r,\\ \prod_{\lambda\cup p\cup p^+\in\Lambda_r^+(n)}(c(p)^2-q^{-2})^{r_{\lambda,p,p^+}}, & \text{if } 2\mid r,\ \varrho^{-1}=-qu_{1,r}t^r.\end{cases}$$

Then there is an $A\in\mathcal{O}$ whose image in κ is invertible such that

$$\det G_{1,\lambda}=AB\prod_{\substack{p,\tilde{p}\in\mathscr{A}(\lambda)\\ \tilde{p}\neq p}}\big(c(p)c(\tilde{p})-1\big)^{r_{\lambda,p,\tilde{p}}}. \tag{4.8}$$

The following result has been proved in [37] by induction on n together with arguments similar to those for $\mathscr{B}_{r,n,\kappa}$ in [33].

Proposition 4.9. *Let $\Delta(1,\lambda)$ be the cell module of $\mathscr{W}_{r,n,\mathcal{O}}$, which is defined by its JM-basis. We have*

$$\det G_{1,\lambda} = AB \prod_{\substack{p,\tilde{p}\in\mathscr{A}(\lambda)\\ p\neq\tilde{p}}} \big(c(p)+c(\tilde{p})\big)^{r_{\lambda,p,\tilde{p}}}$$

where $A \in \mathcal{O}$ whose image in κ is invertible and

$$B = \begin{cases} \prod_{\lambda\cup p\cup p^+\in\Lambda_r^+(n)} (c(p)+c(p^+))^{r_{\lambda,p,p^+}}, & \text{if } r \text{ is odd},\\ \prod_{\lambda\cup p\cup p^-\in\Lambda_r^+(n)} (c(p)+c(p^-))^{r_{\lambda,p,p^-}}, & \text{if } r \text{ is even}. \end{cases}$$

Theorem 4.10. *For each $(1,\mu)\in\Lambda_{r,n}$, let $\Delta(1,\mu)$ be the cell module for either $\mathscr{W}_{r,n,\kappa}$ or $\mathscr{B}_{r,n,\kappa}$.*

a) *$\Delta(0,\lambda)$ is a composition factor of $\Delta(1,\mu)$ if and only if λ is $(1,\mu)$-admissible. Further, $\Delta(1,\mu)$ is multiplicity free.*
b) *$\dim_\kappa D^{1,\mu} = \dim_\kappa \Delta(1,\mu) - \sum_\lambda \dim_\kappa \Delta(0,\lambda)$ where λ ranges over all $(1,\mu)$-admissible r-partitions of n.*

Proof. By Lemma 4.6, $\Delta(0,\lambda)$ is a composition factor of $\Delta(1,\mu)$ if λ is $(1,\mu)$-admissible. Motivated by the proof of Theorem 5.1 in [20], we use $mult(\det G_{1,\mu})$ to denote the multiplicity of $t-1$ in $\det G_{1,\mu}$. We have

$$mult(\det G_{1,\mu}) = \sum_{i>0} \dim rad^i \tag{4.11}$$

where rad^i denote the image under $\alpha:\Delta(1,\mu)_{\mathcal{O}} \to \Delta(1,\mu)_\kappa : v \to v\otimes 1$ of the $\mathcal{O}$-submodule $\{v\in\Delta(1,\mu),\ \phi_{1,\mu}(v,w)\in(t-1)^i\kappa^* \text{ for any } w\in\Delta(1,\mu)\}$. By Proposition 4 in [23], $\mathcal{O}$ is a principal ideal domain, row and column operations can be used to reduce the Gram matrix $G_{1,\mu}$ as a diagonal matrix. Therefore, $mult(\det G_{1,\mu})$ is no less than the dimension of the radical of $\Delta(1,\mu)$. When we discuss $\mathscr{B}_{r,n,\mathcal{O}}$, we need the fact that $\mathrm{char}(\kappa)\neq 2$. In this case, $t+1\neq t-1$. By Lemma 4.6, $\dim \operatorname{Rad}\Delta(1,\mu) \geqslant \sum_\lambda \dim\Delta(0,\lambda)$, where λ ranges over all $(1,\mu)$-admissible r-partitions. Now, everything follows from Propositions 4.7, 4.9. □

Corollary 4.12. *If either $n>2$ or $n=2$ and $\omega_i\neq 0$ for some $0\leqslant i\leqslant r-1$, then $\Delta(1,\mu)$ is irreducible if and only if there is no $(1,\mu)$-admissible r-partition λ.*

Proof. It follows from Theorems 3.4 and 3.16 that each cell module $\Delta(1,\mu)$ has a simple head under our assumption. Now, everything follows from Theorem 4.10. □

Corollary 4.13. *Suppose $r\geqslant 2$, $n=2$ and $\omega_i=0$ for all $0\leqslant i\leqslant r-1$. Then:*

a) *If $r>2$, then $\Delta(1,\emptyset)$ is always reducible;*
b) *If $r=2$, then $\mathscr{W}_{r,2,\kappa}$-module $\Delta(1,\emptyset)$ is irreducible if and only if $u_1+u_2=0$;*
c) *If $r=2$, then $\mathscr{B}_{r,2,\kappa}$-module $\Delta(1,\emptyset)$ is irreducible if and only if $u_1u_2=1$.*

Proof. By Theorem 4.10, any composition factor of $\Delta(1,\emptyset)$ is of form $\Delta(0,\lambda)$ for some $(1,\emptyset)$-admissible r-partition λ of n. Since $\dim\Delta(1,\emptyset)=r$, and the dimension of $\Delta(0,\lambda)$ is at most 2, $\Delta(1,\emptyset)$ is reducible when $r>2$. If $r=2$, $\Delta(1,\emptyset)$ is irreducible if and only if $\Delta(0,((1),(1)))$ is the unique

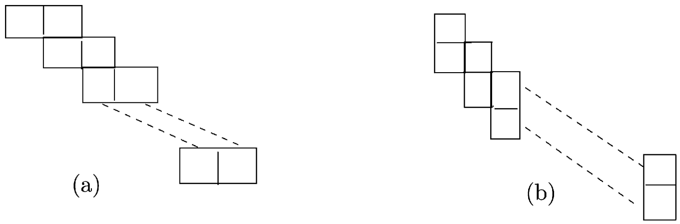

Fig. 1.

composition factor of $\Delta(1, \emptyset)$. This is equivalent to $u_1 + u_2 = 0$ in κ for $\mathscr{W}_{r,2,\kappa}$ and $u_1 u_2 = 1$ in κ for $\mathscr{B}_{r,2,\kappa}$. $\square$

5. Blocks of $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$

In this section, we classify the blocks of cyclotomic NW algebras and cyclotomic BMW algebras over the field κ under Assumption 4.1. This will give some descriptions about the non-invertible factors of the Gram determinant associated to each cell module of such algebras.

First, we introduce the notion of (f, μ)-admissible partitions which can be considered as the generalization of those for BMW algebras in [35]. We remark that all of our definitions for (f, μ)-admissible partitions are motivated by Cox–De Visscher–Martin's work on the blocks of the Brauer algebras in characteristic zero in [7].

Definition 5.1. Let $\lambda \in \Lambda_r^+(n)$ and $\mu \in \Lambda_r^+(n-2f)$, $0 < f \leqslant \lfloor n/2 \rfloor$.

a) For $\mathscr{W}_{r,n,\kappa}$, we say that λ is (f, μ)-admissible over κ if $\lambda \supset \mu$ and $[\lambda/\mu]$ consists of admissible pairs $\{p_i, \tilde{p}_i\}$, $1 \leqslant i \leqslant f$, such that the following conditions hold.
   (i) If $2 \nmid r$ and $\{p, p^-\}$ is an admissible pair in the i-th component of $[\lambda/\mu]$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ has to be in the i-th component of $[\lambda/\mu]$.
   There are two possible configurations of paired boxes with residues $c(p)$ and $c(p^-)$ in the i-th component of $[\lambda/\mu]$ as follows. Further, the number of columns in Fig. 1(b) is even.
   (ii) If $2 \mid r$ and $\{p, p^+\}$ is an admissible pair in the i-th component of $[\lambda/\mu]$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ has to be in the i-th component of $[\lambda/\mu]$. Further, the number of rows in Fig. 1(a) is even.

b) For $\mathscr{B}_{r,n,\kappa}$, we say that λ is (f, μ)-admissible over κ if $\lambda \supset \mu$ and $[\lambda/\mu]$ consists of admissible pairs $\{p_i, \tilde{p}_i\}$, $1 \leqslant i \leqslant f$, such that the following conditions hold.
   (i) Suppose that $2 \nmid r$ and $\varrho^{-1} = \varepsilon u_{1,r}$.
      (1) If $\{p, p^-\}$ is an admissible pair in the i-th component of $[\lambda/\mu]$ with $c(p) = \varepsilon q$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ and $c(p_1) \in \{\varepsilon q, \varepsilon q^{-1}\}$ has to be in the i-th component of $[\lambda/\mu]$. Further, the number of columns in Fig. 1(b) is even;
      (2) If $\{p, p^+\}$ is an admissible pair in the j-th component of $[\lambda/\mu]$ with $c(p) = -\varepsilon q^{-1}$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ and $c(p_1) \in \{-\varepsilon q, -\varepsilon q^{-1}\}$ has to be in the j-th component of $[\lambda/\mu]$. Further, the number of rows in Fig. 1(a) is even;
      Note that (1) and (2) may occur simultaneously when $i \neq j$.
   (ii) Suppose that $2 \mid r$ and $\varrho^{-1} = q^{-1} u_{1,r}$ and $\alpha \in \{1, -1\}$. If there is an admissible pair $\{p, p^+\}$ in the i-th component of $[\lambda/\mu]$ with $c(p) = \alpha q^{-1}$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ and $c(p_1) \in \{\alpha q^{-1}, \alpha q\}$ has to be in the i-th component of $[\lambda/\mu]$. Further, the number of rows in Fig. 1(a) is even.
   (iii) Suppose that $2 \mid r$ and $\varrho^{-1} = -q u_{1,r}$ and $\alpha \in \{1, -1\}$. If there is an admissible pair $\{p, p^-\}$ in the i-th component of $[\lambda/\mu]$ with $c(p) = \alpha q$, then any admissible pair $\{p_1, \tilde{p}_1\}$ with $\tilde{p}_1 \in \{p_1^-, p_1^+\}$ and $c(p_1) \in \{\alpha q, \alpha q^{-1}\}$ has to be in the i-th component of $[\lambda/\mu]$. Further, the number of columns in Fig. 1(b) is even.

Since we are keeping Assumption 4.1, we have $q^{-1}u_{1,r} \neq -qu_{1,r}$ and $\varepsilon q^{-1} \neq -\varepsilon q^{-1}$. Therefore, Definition 5.1 is well defined. If we allow $r = 1$, then Definition 5.1 for $\mathscr{W}_{1,n,\mathbb{C}}$ (resp. $\mathscr{B}_{1,n,\kappa}$) is the same as the definition of balanced pair for Brauer algebras in [7] (resp. that for BMW algebras in [35]). Finally, we remark that Definition 4.5 is a special case of Definition 5.1.

Given $(\ell,\mu) \in \Lambda_{r,n-1}$, $(f,\lambda) \in \Lambda_{r,n}$ with $f > 0$, we write $(\ell,\mu) \to (f,\lambda)$ if either $\ell = f$ and $\mu \to \lambda$ or $\ell = f - 1$ and $\lambda \to \mu$.

Let $\{(f_i,\mu^{(i)}) \mid 1 \leqslant i \leqslant m\}$ be all elements in $\Lambda_{r,n-1}$ such that $(f_i,\mu^{(i)}) \to (f,\lambda)$. The partial order $\geqslant$ defined on $\Lambda_{r,n-1}$ gives rise to a linear order on the set $\{(f_i,\mu^{(i)}) \mid 1 \leqslant i \leqslant m\}$. We arrange such elements as

$$\left(f_1,\mu^{(1)}\right) > \left(f_2,\mu^{(2)}\right) > \cdots > \left(f_m,\mu^{(m)}\right).$$

For each $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$, we identify $\mathfrak{t}_i$ with $(\frac{i-|\mathfrak{t}_i|}{2},\mathfrak{t}_i)$. We write $\mathfrak{t}_i \geqslant \mathfrak{u}_i$ if $(\frac{i-|\mathfrak{t}_i|}{2},\mathfrak{t}_i) \geqslant (\frac{i-|\mathfrak{u}_i|}{2},\mathfrak{u}_i)$. Note that $(\frac{i-|\mathfrak{t}_i|}{2},\mathfrak{t}_i), (\frac{i-|\mathfrak{u}_i|}{2},\mathfrak{u}_i) \in \Lambda_{r,i}$.

In Proposition 5.2, we assume that R is the commutative ring either in Definition 3.1 for $\mathscr{W}_{r,n}$ or in Definition 3.13 for $\mathscr{B}_{r,n}$. We also keep the $\mathbf{u}$-admissible conditions.

Proposition 5.2. *(See [32,35].) Let $\Delta(f,\lambda)$ be the cell module for $A_{r,n,R}$ where $A_{r,n,R} \in \{\mathscr{W}_{r,n,R}, \mathscr{B}_{r,n,R}\}$. Let*

$$M_i = R\text{-span}\left\{\mathfrak{m}_\mathfrak{t} \;\middle|\; \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda),\ \mathfrak{t}_{n-1} \geqslant \mu^{(i)}\right\}.$$

The cell module $\Delta(f,\lambda)$ has the $A_{r,n-1,R}$-filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = \Delta(f,\lambda)$ such that $M_i/M_{i-1} \cong \Delta(f_i,\mu^{(i)})$. In particular, this result is available over an arbitrary field.

Given a $\lambda \in \Lambda_r^+(n)$, let $\mathbf{t}^\lambda$ (resp. $\mathbf{t}_\lambda$) be the λ-tableau obtained from $[\lambda]$ by adding $1,2,\ldots,n$ from left to right along the rows of $[\lambda^{(1)}]$, $[\lambda^{(2)}]$, etc. (resp. from top to bottom down columns of $[\lambda^{(r)}]$, $[\lambda^{(r-1)}]$, etc.).

For example, if $\lambda = ((3,2),(2,1),(1,1)) \in \Lambda_3^+(10)$, then

$$\mathbf{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|}\hline 6&7\\\hline 8\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 9\\\hline 10\\\hline\end{array}\right), \qquad \mathbf{t}_\lambda = \left(\begin{array}{|c|c|c|}\hline 6&8&10\\\hline 7&9\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right).$$

In this case, $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda$) can be identified with $(\mathfrak{t}_0,\mathfrak{t}_1,\mathfrak{t}_2,\ldots,\mathfrak{t}_n) \in \mathscr{T}_n^{ud}(\lambda)$ such that $(\mathbf{t}^\lambda)_i$ (resp. $(\mathbf{t}_\lambda)_i$) is the $\mathfrak{t}_i$-tableau, where $(\mathbf{t}^\lambda)_i$ (resp. $(\mathbf{t}_\lambda)_i$) is obtained from $\mathbf{t}^\lambda$ (resp. $\mathbf{t}_\lambda$) by removing its entries which are strictly greater than i.

Given two positive integers a, b, let (a^b) be the partition $(\lambda_1,\lambda_2,\ldots,\lambda_b)$ of ab such that $\lambda_i = a$ for $1 \leqslant i \leqslant b$.

Lemma 5.3. *Let λ be (f,μ)-admissible where $\mu \in \Lambda_r^+(n-2f)$ with $0 < f \leqslant \lfloor n/2 \rfloor$ such that $\lambda^{(i)} = (a^b)$ with $ab = n$ for some i, $1 \leqslant i \leqslant r$. Let $p_0 = (i,b,a)$, $p_1 = (i,b-1,a)$ and $p_2 = (i,b,a-1)$. If $\{p_0,p_1\}$ (resp. $\{p_0,p_2\}$) is an admissible pair in $[\lambda/\mu]$, we define $\mathfrak{t} := \mathfrak{t}_\lambda$ (resp. $\mathfrak{t} = \mathfrak{t}^\lambda$). Then there is a unique $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ such that $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in κ for any $1 \leqslant i \leqslant n$.*

Proof. First, we prove the result for $\mathscr{B}_{r,n,\kappa}$. Suppose $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ such that $c_\mathfrak{s}(i) = c_\mathfrak{t}(i)$ in κ for all positive integers $i \leqslant n$. If there is a $j \leqslant n$ such that the ℓ-component of $\mathfrak{s}_j$ is not empty and $\ell \neq i$, we pick the minimal j. So, $c_\mathfrak{s}(j) = u_\ell$. Note that $c_\mathfrak{t}(j) = c_\mathfrak{s}(j)$. By the definition of $\mathfrak{t}$, we have $u_\ell = u_i q^{2d}$ for some $d \in \mathbb{Z}$ with $|d| < n$. This contradicts our Assumption 4.1. In other words, if such an $\mathfrak{s}$ exists, the ℓ-th component of each $\mathfrak{s}_j$ has to be empty provided $\ell \neq i$. Thus, it suffices to prove our result for $r = 1$. We have done it for BMW algebras in [35, 4.3]. This completes the proof for $\mathscr{B}_{r,n,\kappa}$.

Finally, one can construct $\mathfrak{s}$ for $\mathscr{W}_{r,n,\kappa}$ by similar arguments in the proof of [35, 4.3]. We leave the details to the reader. $\square$

Recall that $\{g_\mathfrak{s} \mid \mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)\}$ (resp. $\{f_\mathfrak{s} \mid \mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)\}$) is a κ-basis (resp. K-basis) of the cell module $\Delta(f,\lambda)$ either for $\mathscr{W}_{r,n,\kappa}$ or for $\mathscr{B}_{r,n,\kappa}$ (resp. either for $\mathscr{W}_{r,n,K}$ or for $\mathscr{B}_{r,n,K}$). Write

$$g_\mathfrak{s} E_i = \sum_{\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)} \tilde{E}_{\mathfrak{s}\mathfrak{t}}(i) g_\mathfrak{t} \quad \text{and} \quad f_\mathfrak{s} E_i = \sum_{\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)} E_{\mathfrak{s}\mathfrak{t}}(i) f_\mathfrak{t}. \tag{5.4}$$

Lemma 5.5. *Let $\mathfrak{s}$, p_0, p_1 and p_2 be defined in Lemma* 5.3. *For $\mathscr{W}_{r,n,\kappa}$, $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) \neq 0$ if $\mathfrak{s}_{n-2} = \mathfrak{s}_n$ and one of the following conditions holds*:

a) $2 \nmid r$, *and* $c(p_0) + c(p_1) = 0$;
b) $2 \mid r$ *and* $c(p_0) + c(p_2) = 0$.

Proof. For any $\mathfrak{t} \in \mathscr{T}_n^{ud}(\mu)$, by (2.8)

$$f_\mathfrak{t} = \tilde{g}_\mathfrak{t} + \sum_{\substack{\mathfrak{u} \in \mathscr{T}_n^{ud}(\mu) \\ \mathfrak{u} \succ \mathfrak{t} \\ \mathfrak{u} \approx \mathfrak{t}}} c_\mathfrak{u} \tilde{g}_\mathfrak{u} \tag{5.6}$$

for some scalars $c_\mathfrak{u} \in K$. By Lemma 5.3, the residue class in $\mathscr{T}_n^{ud}(\mu)$ which contains $\mathfrak{s}$ is $\{\mathfrak{s}\}$ forcing $\tilde{g}_\mathfrak{s} = f_\mathfrak{s}$. So, $\tilde{g}_\mathfrak{s}$ can not appear on the right-hand side of (5.6) with non-zero coefficient in K whenever $\mathfrak{s} \neq \mathfrak{t}$. Write $\mathfrak{u} \overset{k}{\sim} \mathfrak{v}$ if $\mathfrak{u}_j = \mathfrak{v}_j$ for any $j \neq k$ and $\mathfrak{u}, \mathfrak{v} \in \mathscr{T}_n^{ud}(\mu)$. By [32, 6.9] for $\mathscr{W}_{r,n,\kappa}$,

$$f_\mathfrak{s} E_{n-1} = E_{\mathfrak{s}\mathfrak{s}}(n-1) f_\mathfrak{s} + \sum_{\substack{\mathfrak{u} \overset{n-1}{\sim} \mathfrak{s} \\ \mathfrak{u} \neq \mathfrak{s}}} E_{\mathfrak{s}\mathfrak{u}}(n-1) f_\mathfrak{u} = E_{\mathfrak{s}\mathfrak{s}}(n-1) \tilde{g}_\mathfrak{s} + \sum_{\mathfrak{v} \neq \mathfrak{s}} b_\mathfrak{v} \tilde{g}_\mathfrak{v}, \tag{5.7}$$

for some scalars $b_\mathfrak{v} \in \mathcal{O}$. So, $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1)$ is equal to the image of $E_{\mathfrak{s}\mathfrak{s}}(n-1)$ in κ. By [4, 4.8],

$$E_{\mathfrak{s}\mathfrak{s}}(n-1) = \begin{cases} b_1 B_1(-2c(\alpha_1) - (-1)^r), & \text{if } \mathfrak{s}_{n-1} = \mu/\alpha_1, \\ b_2 B_2(2c(\alpha_2) - (-1)^r), & \text{if } \mathfrak{s}_{n-1} = \mu \cup \alpha_2, \end{cases}$$

where $b_1, b_2 \in \mathcal{O}$ such that the images of b_1, b_2 in κ are invertible and

$$B_1 = \prod_{\substack{\tilde{p} \in \mathscr{R}(\mu) \\ \tilde{p} \neq \alpha_1}} \big(c(\alpha_1) + c(\tilde{p})\big) \prod_{\tilde{p} \in \mathscr{A}(\mu)} \big(c(\alpha_1) + c(\tilde{p})\big)^{-1},$$

$$B_2 = \prod_{\substack{\tilde{p} \in \mathscr{A}(\mu) \\ \tilde{p} \neq \alpha_2}} \big(c(\alpha_2) + c(\tilde{p})\big) \prod_{\tilde{p} \in \mathscr{R}(\mu)} \big(c(\alpha_2) + c(\tilde{p})\big)^{-1}.$$

By Lemma 4.3, neither removable node $\tilde{p}$ with $\tilde{p} \neq \alpha_1$ nor addable node $\tilde{p}$ with $\tilde{p} \neq \alpha_2$ of μ has residue $-c(\alpha_1)$ (resp. $-c(\alpha_2)$) where $\alpha_1 \in \mathscr{R}(\mu)$ (resp. $\alpha_2 \in \mathscr{A}(\mu)$) with $c(\alpha_1) = c(p_0)$ (resp. $c(\alpha_2) = -c(p_0)$), forcing the image of B_1 (resp. B_2) in κ is not equal to zero.

Note that we are assuming that $\operatorname{char}(\kappa) \neq 2$. It is routine to check that $2c(\alpha_1) + (-1)^r \neq 0$ in κ if $\mathfrak{s}_{n-1} = \mu/\alpha_1$ and $2c(\alpha_2) - (-1)^r \neq 0$ in κ if $\mathfrak{s}_{n-1} = \mu \cup \alpha_2$. In any case, $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) \neq 0$. $\square$

Lemma 5.8. *Let $\mathfrak{s}$, p_0, p_1 and p_2 be defined in Lemma* 5.3. *For $\mathscr{B}_{r,n,\kappa}$, $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) \neq 0$ if $\mathfrak{s}_{n-2} = \mathfrak{s}_n$ and one of the following conditions holds*:

a) $2 \nmid r$, $\varrho^{-1} = \varepsilon u_{1,r}$, *and either* $c(p_1) = \varepsilon q$ *with* $c(p_0)c(p_1) = 1$ *or* $c(p_2) = -\varepsilon q^{-1}$ *with* $c(p_0)c(p_2) = 1$;
b) $2 \mid r$ *and* $\varrho^{-1} = q^{-1}u_{1,r}$ *and* $c(p_0)c(p_2) = 1$;
c) $2 \mid r$ *and* $\varrho^{-1} = -qu_{1,r}$, *and* $c(p_0)c(p_1) = 1$.

Proof. We write $f_{\mathfrak{s}}E_{n-1} = \sum_{\mathfrak{t} \overset{n-1}{\sim} \mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(n-1)f_{\mathfrak{t}}$ over K, where $(\mathcal{O}, K, \kappa)$ is the modular system for $\mathscr{B}_{r,n,\kappa}$ which we constructed in Section 4. We remark that we have to use $u_i t$ instead of u_i, $1 \leqslant i \leqslant r$ in the definition of $\mathscr{B}_{r,n,x}$ with $x \in \{\mathcal{O}, K\}$.

If $2 \nmid r$, $\varrho^{-1} = \varepsilon u_{1,r}t^r$, by [33, 4.20],

$$E_{\mathfrak{s}\mathfrak{s}}(n-1) = \begin{cases} b_1 B_1 (c(\alpha_1)^{-1} - \varepsilon q^{-1})(c(\alpha_1)^{-1} + \varepsilon q), & \text{if } \mathfrak{s}_{n-1} = \mu/\alpha_1, \\ b_2 B_2 (c(\alpha_2) - \varepsilon q^{-1})(c(\alpha_2) + \varepsilon q), & \text{if } \mathfrak{s}_{n-1} = \mu \cup \alpha_2, \end{cases}$$

where $b_1, b_2 \in \mathcal{O}$ such that the images of b_1, b_2 are in κ^* and

$$\begin{cases} B_1 = \prod\limits_{\substack{\tilde{p} \in \mathscr{R}(\mu) \\ \tilde{p} \neq \alpha_1}} \big(c(\alpha_1)c(\tilde{p}) - 1\big) \prod\limits_{\tilde{p} \in \mathscr{A}(\mu)} \big(c(\alpha_1)c(\tilde{p}) - 1\big)^{-1}, \\ B_2 = \prod\limits_{\substack{\tilde{p} \in \mathscr{A}(\mu) \\ \tilde{p} \neq \alpha_2}} \big(c(\alpha_2)c(\tilde{p}) - 1\big) \prod\limits_{\tilde{p} \in \mathscr{R}(\mu)} \big(c(\alpha_2)c(\tilde{p}) - 1\big)^{-1}. \end{cases} \tag{5.9}$$

If $2 \mid r$ and $\varrho^{-1} = -\varepsilon q^{\varepsilon} u_{1,r}t^r$, by [33, 4.21],

$$E_{\mathfrak{s}\mathfrak{s}}(n-1) = \begin{cases} b_1 B_1 (c(\alpha_1)^2 - q^{2\varepsilon}), & \text{if } \mathfrak{s}_{n-1} = \mu/\alpha_1, \\ b_2 B_2 (c(\alpha_2)^2 - q^{-2\varepsilon}), & \text{if } \mathfrak{s}_{n-1} = \mu \cup \alpha_2, \end{cases}$$

where $b_1, b_2 \in \mathcal{O}$ such that the images of b_1, b_2 are in κ^* and B_1, B_2 are given in (5.9).

By arguments similar to those in the proof of Lemma 5.5, one can verify $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) \neq 0$ in κ. We leave the details to the reader. $\square$

For each $A_{r,n,\kappa}$-module M and each irreducible $A_{r,n,\kappa}$-module N, define $[M : N]$ to be the multiplicity of N in M. The following result is motivated by Cox–De Visscher–Martin's work on the blocks of the Brauer algebras in characteristic zero in [7]. Such a result has been proved for the BMW algebras in [35].

Proposition 5.10. *If* $[\Delta(f,\mu) : \Delta(0,\lambda)] \neq 0$ *for* $(0,\lambda), (f,\mu) \in \Lambda_{r,n}$ *with* $0 < f \leqslant \lfloor n/2 \rfloor$, *then* λ *is* (f,μ)-*admissible.*

Proof. We prove our result by induction on n. The result for $n = 2$ follows from Theorem 4.10. Now, we assume $n > 2$.

By our Assumption 4.1(a) (resp. (b)) and [4, 6.11] (resp. [1]), $\mathcal{H}_{r,n,\kappa}$ (resp. $\mathscr{H}_{r,n,\kappa}$) for $r \geqslant 2$ is split semisimple over κ. Further, $\Delta(0,\lambda) = D^{0,\lambda}$ since $\Delta(0,\lambda)$ can be considered as the cell module for either $\mathcal{H}_{r,n,\kappa}$ or $\mathscr{H}_{r,n,\kappa}$. In the following, we use $A_{r,n,\kappa}$ instead of either $\mathscr{W}_{r,n,\kappa}$ or $\mathscr{B}_{r,n,\kappa}$.

If $[\Delta(f,\mu) : \Delta(0,\lambda)] \neq 0$, then there is a non-zero monomorphism

$$\phi : \Delta(0,\lambda) \to \Delta(f,\mu)/M$$

for some $A_{r,n,\kappa}$-submodule M of $\Delta(f,\mu)$. Acting $\sum_{i=1}^n X_i$ (resp. $\prod_{i=1}^n X_i$) on $\mathscr{W}_{r,n,\kappa}$-modules (resp. $\mathscr{B}_{r,n,\kappa}$-modules) $\Delta(0,\lambda)$ and $\Delta(f,\mu)$, and using [32, 5.9] (resp. [33, 3.5b]) yields

$$\sum_{p \in [\lambda]} c(p) = \sum_{p \in [\mu]} c(p) \tag{5.11}$$

respectively

$$\prod_{p\in[\lambda]} c(p) = \prod_{p\in[\mu]} c(p). \tag{5.12}$$

For each $p \in \mathscr{R}(\lambda)$, by Frobenius reciprocity and Proposition 5.2,

$$\mathrm{Hom}_{A_{r,n,\kappa}}\bigl(\mathrm{Ind}_{A_{r,n-1,\kappa}}^{A_{r,n,\kappa}} \Delta\bigl(0, \lambda/\{p\}\bigr), \Delta(0,\lambda)\bigr) \neq 0.$$

Since ϕ is injective and $\Delta(0,\lambda)$ is irreducible,

$$\mathrm{Hom}_{A_{r,n,\kappa}}\bigl(\mathrm{Ind}_{A_{r,n-1,\kappa}}^{A_{r,n,\kappa}} \Delta\bigl(0, \lambda/\{p\}\bigr), \Delta(f,\mu)/M\bigr) \neq 0.$$

By Frobenius reciprocity, Proposition 5.2 and (5.11)–(5.12) again, either

$$\bigl[\Delta\bigl(f, \mu/\{\tilde{p}\}\bigr) : \Delta\bigl(0, \lambda/\{p\}\bigr)\bigr] \neq 0$$

for some $\tilde{p} \in \mathscr{R}(\mu)$ with $c(p) = c(\tilde{p})$ or

$$\bigl[\Delta\bigl(f-1, \mu \cup \{p_1\}\bigr) : \Delta\bigl(0, \lambda/\{p\}\bigr)\bigr] \neq 0$$

for some $p_1 \in \mathscr{A}(\mu)$ such that p, p_1 is an admissible pair.

In the first case, we use induction assumption on $n-1$ to get $\lambda/\{p\} \supset \mu/\{\tilde{p}\}$. Since $c(p) = c(\tilde{p})$, by Lemma 4.3, p and $\tilde{p}$ have to be in the same diagonal of some component of λ. So, $\lambda \supset \mu$. By induction assumption on $n-1$, the skew Young diagram $[\lambda/\{p\}/\mu/\{\tilde{p}\}]$ consists of admissible pairs. So is $[\lambda/\mu]$.

In the second case, $\lambda \supset \lambda/\{p\} \supset \mu \cup p_1 \supset \mu$ and $\{p, p_1\}$ is an admissible pair. In any case, we have proved that $\lambda \supset \mu$ and $[\lambda/\mu]$ consists of admissible pairs.

We can assume that there is a unique component of λ, say $\lambda^{(i)}$, such that $|\lambda^{(i)}| \neq 0$. Otherwise, we can find two removable nodes of $[\lambda]$, which are in different components of $[\lambda]$. In this case, the results follow from induction assumption on $n-1$ together with Frobenius reciprocity.

If there is at least one $p \in \mathscr{R}(\lambda)$ such that the admissible pair p, $\tilde{p}$ is neither of the form in Definition 5.1(a)(i)–(ii) nor of the form in Definition 5.1(b)(i)–(iii), we can use Frobenius reciprocity and induction assumption on $n-1$ to get the result as required.

So, we can assume that $\lambda^{(i)} = (a^b) \in \Lambda_1^+(n)$ for some $1 \leqslant i \leqslant r$. In this case, λ has a unique removable node say $p_1 = (i, b, a)$. We remark that p_1 is denoted by p_0 in Lemma 5.3.

Since $\lambda \supset \mu$, we have $p_1 \notin [\mu]$. Note that λ is (f,μ)-admissible, we can find p_2 such that

a) $p_1 = p_2^-$ if p_1, p_2 is the admissible pair given in Definition 5.1(a)(i) or Definition 5.1(b)(i)(1) or Definition 5.1(b)(iii);
b) $p_1 = p_2^+$ if p_1, p_2 is the admissible pair given in Definition 5.1(a)(ii) or Definition 5.1(b)(i)(2) or Definition 5.1(b)(ii).

Using Frobenius reciprocity and induction assumption on $n-1$, we have the results except the case when

$$\bigl[\Delta(f-1, \mu) : \Delta\bigl(0, \lambda/\{p_1, p_2\}\bigr)\bigr] \neq 0.$$

We complete the proof by showing that

$$\bigl[\Delta(f-1, \mu) : \Delta\bigl(0, \lambda/\{p_1, p_2\}\bigr)\bigr] = 0.$$

In fact, we have $\Delta(0,\lambda) = D^{0,\lambda}$ and $\{g_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)\}$ is a κ-basis of $\Delta(0,\lambda)$. Since $[\Delta(f,\mu) : \Delta(0,\lambda)] \neq 0$ and $\Delta(0,\lambda)$ is irreducible, there is a non-trivial $A_{r,n,\kappa}$-homomorphism $\phi : \Delta(0,\lambda) \to \Delta(f,\mu)/M$ such that

$$0 \neq \phi(g_{\mathfrak{t}}) \equiv \sum_{\mathfrak{u} \in \mathscr{T}_n^{ud}(\mu)} a_{\mathfrak{u}} g_{\mathfrak{u}} \pmod M \tag{5.13}$$

for any $\mathfrak{t} \in \mathscr{T}_n^{std}(\lambda)$. Let $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ be given in Lemma 5.3. By Lemma 5.3, we find a unique $\mathfrak{s} \in \mathscr{T}_n^{ud}(\mu)$ such that $c_{\mathfrak{s}}(j) = c_{\mathfrak{t}}(j)$ for all positive integers $j \leqslant n$. Further, $g_{\mathfrak{t}} F_{\mathfrak{s}} = g_{\mathfrak{t}}$ and $g_{\mathfrak{s}}$ appears on the right-hand side of (5.13) with non-zero coefficient. Otherwise, acting $F_{\mathfrak{s}}$ on both sides of (5.13) and using Proposition 2.5 yields $\phi(g_{\mathfrak{t}}) = 0$, a contradiction. So, $\phi(g_{\mathfrak{t}}) \equiv a_{\mathfrak{s}} g_{\mathfrak{s}} \pmod M$ with $a_{\mathfrak{s}} \neq 0$ and $g_{\mathfrak{s}} \notin M$.

By the uniqueness of $\mathfrak{s}$ in Lemma 5.3, $\mathfrak{s}_n = \mathfrak{s}_{n-2}$. Since E_{n-1} acts trivially on $\Delta(0,\lambda)$, $\phi(g_{\mathfrak{t}})E_{n-1} = 0$, forcing $g_{\mathfrak{s}} E_{n-1} \in M$. By (5.7),

$$f_{\mathfrak{s}} E_{n-1} = E_{\mathfrak{s}\mathfrak{s}}(n-1)\tilde{g}_{\mathfrak{s}} + \sum_{\mathfrak{v} \neq \mathfrak{s}} b_{\mathfrak{v}} \tilde{g}_{\mathfrak{v}}$$

for some scalars $b_{\mathfrak{v}} \in \mathcal{O}$. By base change,

$$g_{\mathfrak{s}} E_{n-1} = \tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) g_{\mathfrak{s}} + \sum_{\mathfrak{v} \neq \mathfrak{s}} c_{\mathfrak{v}} g_{\mathfrak{v}}, \tag{5.14}$$

for some scalars $c_{\mathfrak{v}}$'s in κ, where $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1)$ is the image of $E_{\mathfrak{s}\mathfrak{s}}(n-1)$ in κ. By Lemma 5.5 for $\mathscr{W}_{r,n,\kappa}$ (resp. Lemma 5.8 for $\mathscr{B}_{r,n,\kappa}$), $\tilde{E}_{\mathfrak{s}\mathfrak{s}}(n-1) \in \kappa^*$, forcing $g_{\mathfrak{s}} \in M$, and $\phi(g_{\mathfrak{t}}) = 0$, a contradiction. $\square$

We give the following lemma which will be used in Definition 5.16.

Lemma 5.15. *Assume* $(0,\lambda), (f,\mu) \in \Lambda_{r,n}$ *with* $0 < f \leqslant \lfloor n/2 \rfloor$ *such that* $\lambda \supset \mu$.

a) *If $p, \tilde{p}$ is an admissible pair in the i-th component of $[\lambda/\mu]$, then any node p_1 in the i-th component of $[\lambda/\mu]$ has to be paired with $\tilde{p}_1$ which is in the i-th component of $[\lambda/\mu]$, too.*
b) *If $p, \tilde{p}$ is an admissible pair such that p (resp. $\tilde{p}$) is in the i-th (resp. j-th) component of $[\lambda/\mu]$, then any node p_1 in the i-th component of $[\lambda/\mu]$ has to be paired with $\tilde{p}_1$ which is in the j-th component of $[\lambda/\mu]$.*

Proof. Suppose that $\tilde{p}_1$ is in the j-th component of $[\lambda/\mu]$ with $j \neq i$. Note that two nodes in a Young diagram $[\lambda]$ have the same residues if they are in the same diagonal of a component of $[\lambda]$. So, we need only to deal with the case when $\lambda^{(i)}$ is a hook and $\lambda^{(j)}$ is either (a) or (1^a) where $a = |\lambda^{(j)}|$. It is routine to check that it will result in a contradiction since we are assuming Assumption 4.1. This proves (a). (b) can be verified similarly. $\square$

We need Definition 5.16 to describe some composition factors of cell modules for either $\mathscr{W}_{r,n,\kappa}$ or $\mathscr{B}_{r,n,\kappa}$. This is motivated by Cox–De Visscher–Martin's work on the blocks of Brauer algebras in characteristic zero [7] together with our Theorem 4.10. Since our definition is similar to that for Brauer algebras in [7], we copy the arguments in [7] and make some modification.

Definition 5.16. Let $\mu \in \Lambda_r^+(n-2f)$ for some $0 < f \leqslant \lfloor n/2 \rfloor$. If λ is (f,μ)-admissible, we want to define an r-partition $\nu_{\lambda,\mu}$ such that λ is $(\frac{n-|\nu_{\lambda,\mu}|}{2}, \nu_{\lambda,\mu})$-admissible, $\lambda \supset \nu_{\lambda,\mu} \supseteq \mu$ and $|\nu_{\lambda,\mu}|$ is maximal.

Pick a $p \in \mathscr{R}(\lambda) \cap [\lambda/\mu]$. Since λ is (f,μ)-admissible, there is a node, say $\tilde{p} \in [\lambda/\mu]$ such that $\{p, \tilde{p}\}$ is an admissible pair.

Case 1. p **(resp.** $\tilde{p}$**) is in the** i**-th (resp.** j**-th) component of** λ **and** $i \neq j$**.**

By Lemma 5.15, all nodes which can be paired with p have to be in the j-th component of $[\lambda]$. Further, by Lemma 5.15(a), we cannot find an admissible pair $\{p, p^+\}$ or $\{p, p^-\}$ either in i-th or j-th component of $[\lambda/\mu]$. Pick $\tilde{p}$ in the j-th component of λ such that the row index of $\tilde{p}$ is maximal. Let $(\lambda/\mu)_0 = \{p, \tilde{p}\}$. Given $(\lambda/\mu)_m$, set

$$(\lambda/\mu)_{m+1} = (\lambda/\mu)_m \cup A_{m+1} \cup \tilde{A}_{m+1}$$

where A_{m+1} is the set of nodes p in the i-th (resp. j-th) component of λ such that p is to the right of or below a node which is in the i-th (resp. j-th) component of $(\lambda/\mu)_m$, and $\tilde{A}_{m+1}$ is the set of nodes $\tilde{p}$ in the j-th (resp. i-th) component of $[\lambda/\mu]$ such that $p, \tilde{p}$ is an admissible pair for some $p \in A_{m+1}$, and the row index of $\tilde{p}$ is maximal with such residue among the nodes of $[\lambda/\mu]$ not already in $(\lambda/\mu)_m$.

This iterative process eventually stabilizes, and we obtain $(\lambda/\mu)_t$ which may be a disconnected subset of the edge of $[\lambda/\mu]$, having width one. We define

$$\lambda/\nu_{\lambda,\mu^{(p)}} = (\lambda/\mu)_t.$$

Case 2. p **and** $\tilde{p}$ **are in the** i**-th component of** λ**.**

By Lemma 4.3, all nodes which can be paired with p have to be in the i-th component of $[\lambda]$. We pick $\tilde{p}$ such that the row index of $\tilde{p}$ is maximal among all nodes with such residues in $[\lambda/\mu]$ and $p \neq \tilde{p}$. Let $(\lambda/\mu)_0 = \{p, \tilde{p}\}$. Given $(\lambda/\mu)_m$, set

$$(\lambda/\mu)_{m+1} = (\lambda/\mu)_m \cup A_{m+1} \cup \tilde{A}_{m+1}$$

where A_{m+1} is the set of nodes p in the i-th component of λ such that p is to the right of or below a node in $(\lambda/\mu)_m$, and $\tilde{A}_{m+1}$ is the set of nodes $\tilde{p}$ with $p \neq \tilde{p}$ in $[\lambda/\mu]$ such that $p, \tilde{p}$ is an admissible pair for some $p \in A_{m+1}$, and the row index of $\tilde{p}$ is maximal with such residue among the nodes of $[\lambda/\mu]$ not already in $(\lambda/\mu)_m$. By Lemma 5.15, all nodes in $\tilde{A}_{m+1}$ have to be in the i-th component of $[\lambda/\mu]$.

This iterative process eventually stabilizes, and we obtain $(\lambda/\mu)_t$ which is a possibly disconnected subset of the edge of $[\lambda/\mu]$, having width at most two. We remark that $(\lambda/\mu)_t$ has width two only when $(\lambda/\mu)_t$ contains an admissible pair $p, \tilde{p}$ such that $c(p) = c(\tilde{p})$. Since we are assuming Assumption 4.1, we cannot find two admissible pairs with forms either $\{p_l, p_l^+\}$ or $\{p_l, p_l^-\}$, $l = 1, 2$, in the i-th component of $[\lambda/\mu]$. In this case, we define

$$\lambda/\nu_{\lambda,\mu^{(p)}} = (\lambda/\mu)_t.$$

Suppose that the width of $(\lambda/\mu)_t$ is one. If there is no pair $\{p, p^-\}$ (resp. (p, p^+)) in $(\lambda/\mu)_t$ satisfying the conditions in Definition 5.1(a)(i), (b)(i)(1) and (b)(iii) (resp. Definition 5.1(a)(ii), (b)(i)(2) and (b)(ii)), then we define

$$\lambda/\nu_{\lambda,\mu^{(p)}} = (\lambda/\mu)_t.$$

Otherwise, we define $\lambda/\nu_{\lambda,\mu^{(p)}}$ for $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$ in Cases 3–4.

Case 3. $\mathscr{B}_{r,n,\kappa}$.

(a) Suppose that either $2 \nmid r$, $\rho^{-1} = \varepsilon u_{1,r}$ and $c(p) = \varepsilon q$ or $2 \mid r$ and $\rho^{-1} = -q u_{1,r}$. If $\{p, p^-\}$ is an admissible pair in $(\lambda/\mu)_t$, we set $(\lambda/\mu)_{t+1} = (\lambda/\mu)_t \cup \{x, x^-\}$ where $\{x, x^-\}$ is an admissible pair in $[\lambda/\mu]$ with residue $c(x) = c(p)$ and $x \notin (\lambda/\mu)_t$ such that the row index of x is equal to the row index of p minus 1.

(b) Suppose that either $2 \nmid r$, $\rho^{-1} = \varepsilon u_{1,r}$ and $c(p) = -\varepsilon q^{-1}$ or $2 \mid r$ and $\rho^{-1} = q^{-1}u_{1,r}$. If $\{p, p^{+}\}$ is an admissible pair in $(\lambda/\mu)_t$, we set $(\lambda/\mu)_{t+1} = (\lambda/\mu)_t \cup \{x, x^{+}\}$ where $\{x, x^{+}\}$ is an admissible pair in $[\lambda/\mu]$ with $c(x) = c(p)$ and $x \notin (\lambda/\mu)_t$ such that the row index of x is equal to the row index of p minus 1.

Since we are keeping Assumption 4.1, the cases (a) and (b) cannot occur simultaneously.

Case 4. $\mathscr{W}_{r,n,\kappa}$.

(a) Suppose that $2 \nmid r$. If $\{p, p^{-}\}$ is an admissible pair in $(\lambda/\mu)_t$, we set $(\lambda/\mu)_{t+1} = (\lambda/\mu)_t \cup \{x, x^{-}\}$ where x, x^{-} are nodes in $[\lambda/\mu]$ with residue $c(x) = c(p)$ and $x \notin (\lambda/\mu)_t$ such that the row index of x is equal to the row of index of p minus 1.
(b) Suppose that $2 \mid r$. If $\{p, p^{+}\}$ is an admissible pair in $(\lambda/\mu)_t$, we set $(\lambda/\mu)_{t+1} = (\lambda/\mu)_t \cup \{x, x^{+}\}$ where x, x^{+} are nodes in $[\lambda/\mu]$ with residue $c(x) = c(p)$ and $x \notin (\lambda/\mu)_t$ such that the row index of x is equal to the row index of p minus 1.

Then $(\lambda/\mu)_{t+1}$ may not be stable under the process we have used in Cases 2. We apply the process repeatedly until the skew r-partition eventually stabilizes at some step, say s. We set

$$\lambda/\nu_{\lambda,\mu^{(p)}} = (\lambda/\mu)_s.$$

We remark that $\lambda/\nu_{\lambda,\mu^{(p)}}$ is a subset of λ/μ having width two.

Let $\nu_{\lambda,\mu}$ be maximal in the sense that $\eta = \lambda$ if η is an r-partition such that η is $(f_1, \nu_{\lambda,\mu})$-admissible, and $\eta \supsetneq \nu_{\lambda,\mu}$. This is well defined since $|\lambda| < \infty$ and $\lambda \supset \nu_{\lambda,\mu^{(p)}} \supseteq \mu$ for each $p \in \mathscr{R}(\lambda) \cap [\lambda/\mu]$, we will get $\nu_{\lambda,\mu}$ after a finite number of previous processes. We remark that $\nu_{\lambda,\mu}$ may not be unique.

Lemma 5.17. *Let $(f, \mu) \in \Lambda_{r,n}$ with $0 < f \leqslant \lfloor n/2 \rfloor$. If λ is (f, μ)-admissible, we write $2\ell = |\lambda| - |\nu_{\lambda,\mu}|$ where $\nu_{\lambda,\mu}$ is an r-partition defined in Definition* 5.16. *Define $\mathfrak{s} \in \mathscr{T}_n^{ud}(\nu_{\lambda,\mu})$ be such that*:

a) *$\mathfrak{s}_i$, $n - \ell \leqslant i \leqslant n$, is obtained from $\mathfrak{s}_{i+1}$ by adding the node p_i such that $\mathfrak{s}_i$ is maximal in the set $\{\mathfrak{s}_{i+1} \cup p \mid p \in [\lambda/\nu_{\lambda,\mu}]\}$;*
b) *$\mathfrak{s}_i = (\mathfrak{t}^{\eta})_i$, for all positive integers $i \leqslant n - \ell$, where $\mathfrak{s}_{n-\ell} = \eta$ and $|\eta| = n - \ell$.*

We have:

(1) *there exists a $\mathfrak{t} \in \mathscr{T}_n^{std}(\lambda)$ such that $c_{\mathfrak{t}}(i) = c_{\mathfrak{s}}(i)$ for any $1 \leqslant i \leqslant n$;*
(2) *If $\mathfrak{u} \in \mathscr{T}_n^{ud}(\nu_{\lambda,\mu})$ with $\mathfrak{u} \approx \mathfrak{s}$, then $\mathfrak{u} = \mathfrak{s}$.*

Proof. Let $\mathfrak{t} \in \mathscr{T}_n^{std}(\lambda)$ such that $\mathfrak{t}_i = \mathfrak{s}_i$, $1 \leqslant i \leqslant n - \ell$ and $\mathfrak{t}_j$ is maximal in the set $\{\mathfrak{t}_{j-1} \cup p \mid p \in [\lambda/\nu_{\lambda,\mu}]\}$, $n - \ell + 1 \leqslant j \leqslant n$. We want to prove $c_{\mathfrak{t}}(i) = c_{\mathfrak{s}}(i)$ for $1 \leqslant i \leqslant n$.

Suppose $\mathfrak{t}_n = \mathfrak{t}_{n-1} \cup \{p\}$, where $p = (i, j, k)$. Then $p \in \mathscr{R}(\lambda) \cap [\lambda/\nu_{\lambda,\mu}]$ and no node $(i_1, j_1, k_1) \in \mathscr{R}(\lambda) \cap [\lambda/\nu_{\lambda,\mu}]$ such that either $i_1 > i$ or $i_1 = i$ and $j_1 > j$. Otherwise, $\mathfrak{t}_{n-1}$ is not maximal among the set $\{\mathfrak{t}_{n-2} \cup \{p\} \mid p \in [\lambda/\nu_{\lambda,\mu}]\}$. Since $[\lambda/\nu_{\lambda,\mu}]$ consists of admissible pairs, there is a $\tilde{p} \in [\lambda/\nu_{\lambda,\mu}]$ such that $\{p, \tilde{p}\}$ is an admissible pair. By Lemma 4.3, all such $\tilde{p}$ are in the same diagonal of a component of $[\lambda]$. So, we pick the $\tilde{p}$ with minimal row index. We claim:

a) $\tilde{p} \in \mathscr{A}(\nu_{\lambda,\mu})$.
b) $\nu_{\lambda,\mu} \cup \{\tilde{p}\}$ is maximal in the set $\{\nu_{\lambda,\mu} \cup \{a\} \mid a \in \mathscr{A}(\nu_{\lambda,\mu}) \cap [\lambda]\}$.

If so, by the definition of $\mathfrak{s}$ in Lemma 5.17, $\mathfrak{s}_{n-1} = \mathfrak{s}_n \cup \{\tilde{p}\}$ and $c_{\mathfrak{s}}(n) = c_{\mathfrak{t}}(n)$. Now, we use induction assumption on $n - 1$ to obtain $c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i)$, $1 \leqslant i \leqslant n - 1$.

$[\lambda/\nu_{\lambda,\mu}] =$ 

Fig. 2.

Fig. 3.

Recall that $p = (i, j, k)$. We write $\alpha = (i, j-1, k-1)$ and $p_1 = (i_1, j_1+1, k_1+1)$ if $\tilde{p} = (i_1, j_1, k_1)$. Let $\beta^- = \tilde{p}$ and $\gamma^+ = \tilde{p}$. In order to prove (a)–(b), we have to discuss the four cases as follows.

Case 1. $\alpha^+, \alpha^- \notin [\lambda/\nu_{\lambda,\mu}]$.

We have $\tilde{p} \in \mathscr{R}(\lambda) \cap [\lambda/\nu_{\lambda,\mu}]$ and $[\lambda/\nu_{\lambda,\mu}] = \{p, \tilde{p}\}$. This proves (a)–(b).

Case 2. $\alpha^+ \notin [\lambda/\nu_{\lambda,\mu}]$, $\alpha^- \in [\lambda/\nu_{\lambda,\mu}]$.

If p and $\tilde{p}$ are in the same row, then $[\lambda/\nu_{\lambda,\mu}]$ is given in Fig. 2. This proves (a)–(b).

In the rest case, since $\alpha^+ \notin [\lambda/\nu_{\lambda,\mu}]$, we have $\tilde{p}^-, \gamma \notin [\lambda/\nu_{\lambda,\mu}]$. Note that $\alpha^- \in [\lambda/\nu_{\lambda,\mu}]$ and the scalar $c(\alpha^-)$ appears once among the residues of the nodes in $[\lambda/\nu_{\lambda,\mu}]$, either $\tilde{p}^+$ or β is paired with α^- in $[\lambda/\nu_{\lambda,\mu}]$. In the later case, $\tilde{p} \in \mathscr{R}(\lambda)$ and $[\lambda/\nu_{\lambda,\mu}] = \{p, \tilde{p}\}$, a contradiction. So $\tilde{p}^+$ is paired with α^- in $[\lambda/\nu_{\lambda,\mu}]$ and $\beta \notin [\lambda/\nu_{\lambda,\mu}]$, forcing $\tilde{p} \in \mathscr{A}(\nu_{\lambda,\mu})$.

If $\alpha_1 \in [\lambda]$ such that α_1 and $\tilde{p}$ are in the same row and $\alpha_1^+ \notin [\lambda]$, then α_1 must be paired with one node $\tilde{\alpha}_1$ in $[\lambda/\nu_{\lambda,\mu}]$. Since $\alpha \notin [\lambda/\nu_{\lambda,\mu}]$, $\tilde{\alpha}_1$ and p are in the same row and $\tilde{\alpha}_1 \in \mathscr{A}(\nu_{\lambda,\mu})$. This shows that

$$[\lambda/\nu_{\lambda,\mu}] = \{r_0, r_1, \ldots, r_\ell\} \cup \{\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_\ell\},$$

where $r_0 = p$, $\tilde{r}_0 = \tilde{p}$ and $r_{i+1}^+ = r_i$ and $\tilde{r}_i^+ = \tilde{r}_{i+1}$, $0 \leqslant i \leqslant \ell - 1$, and $\alpha_1 = \tilde{r}_\ell$. This proves (b).

Case 3. $\alpha^+ \in [\lambda/\nu_{\lambda,\mu}]$, $\alpha^- \notin [\lambda/\nu_{\lambda,\mu}]$.

If p and $\tilde{p}$ are in the same column, by the construction of $\nu_{\lambda,\mu}$, the transpose of $[\lambda/\nu_{\lambda,\mu}]$ is given as Fig. 2. So, both (a) and (b) follow.

In the rest case, $\beta \notin [\lambda/\nu_{\lambda,\mu}]$ and $\tilde{p}^+ \notin [\lambda]$ since $\alpha^- \notin [\lambda/\nu_{\lambda,\mu}]$. Note that $\alpha^+ \in [\lambda/\nu_{\lambda,\mu}]$ and $c(\alpha^+)$ appears once among the residues of the nodes in $[\lambda/\nu_{\lambda,\mu}]$. Either $\tilde{p}^-$ or γ is paired with α^+ in $[\lambda/\nu_{\lambda,\mu}]$. If γ is paired with α^+, then $\tilde{p} \in \mathscr{R}(\lambda)$ and $[\lambda/\nu_{\lambda,\mu}] = \{p, \tilde{p}\}$, a contradiction. So $\tilde{p}^-$ is paired with α^+ in $[\lambda/\nu_{\lambda,\mu}]$ and $\gamma \notin [\lambda/\nu_{\lambda,\mu}]$, forcing $\tilde{p} \in \mathscr{A}(\nu_{\lambda,\mu})$.

If there is a node with row index strictly less than that of $\tilde{p}$ in $[\lambda/\nu_{\lambda,\mu}]$, then this node must be paired with the node to the left of p. This is a contradiction since $\alpha^- \notin [\lambda/\nu_{\lambda,\mu}]$. This proves (b).

Case 4. $\alpha^+, \alpha^- \in [\lambda/\nu_{\lambda,\mu}]$.

There are two cases we have to discuss.

Subcase 4.1. $\alpha \notin [\lambda/\nu_{\lambda,\mu}]$.

Then p and $\tilde{p}$ cannot be in the same row. Otherwise, since the multiplicity of $c(p)$ is one, $[\lambda/\nu_{\lambda,\mu}]$ is given in Fig. 2, a contradiction. When the row number of $\tilde{p}$ is equal to that of p minus one, $[\lambda/\nu_{\lambda,\mu}]$ is one of skew Young diagram in Fig. 3. In the second diagram of Fig. 3, the nodes in dark are in $[\nu_{\lambda,\mu}]$. It is impossible because $\alpha^+ \in [\lambda/\nu_{\lambda,\mu}]$. This proves (a)–(b).

In the rest case, we have $\beta \notin [\lambda/\nu_{\lambda,\mu}]$. Otherwise, β must be paired with α^-. Since $c(\alpha^-)$ appears once among the residues of the nodes in $[\lambda/\nu_{\lambda,\mu}]$, $\tilde{p}^+ \notin [\lambda/\nu_{\lambda,\mu}]$. By the definition of $\nu_{\lambda,\mu}$, $\alpha^- \notin [\lambda/\nu_{\lambda,\mu}]$, a contradiction.

Fig. 4.

If $\gamma \in [\lambda/\nu_{\lambda,\mu}]$, then γ must be paired with α^+. Since $c(\alpha^+)$ appears once among the residues of the nodes in $[\lambda/\nu_{\lambda,\mu}]$, $\tilde{p}^- \notin [\lambda/\nu_{\lambda,\mu}]$. By the definition of $\nu_{\lambda,\mu}$, $\{\gamma, \alpha^+\} \notin [\lambda/\nu_{\lambda,\mu}]$, a contradiction. This proves $\gamma \notin [\lambda/\nu_{\lambda,\mu}]$, forcing $\tilde{p} \in \mathscr{A}(\nu_{\lambda,\mu})$.

We cannot find any node in $[\lambda/\nu_{\lambda,\mu}]$ whose row index is strictly less than that of $\tilde{p}$. Otherwise, we can find $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k\}$ such that $\{a_i, \tilde{a}_i\}$ is an admissible pair for any $1 \leqslant i \leqslant k$ and λ is $(\ell - k, \nu_{\lambda,\mu} \cup \{a_1, a_2, \dots, a_k\} \cup \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k\})$-admissible, a contradiction. So (b) is proved in this case.

Subcase 4.2. $\alpha \in [\lambda/\nu_{\lambda,\mu}]$.

The width of $[\lambda/\nu_{\lambda,\mu}]$ is 2. So, all the nodes in $[\lambda/\nu_{\lambda,\mu}]$ have to be in the same component of $[\lambda]$. In $[\lambda/\nu_{\lambda,\mu}]$, there exist two nodes with the residue $c(\tilde{p})$. Since the row number of $\tilde{p}$ is minimal, another node in $[\lambda/\nu_{\lambda,\mu}]$ which can be paired with p has to be p_1 where $p_1 = (\tilde{p}^+)^-$. We remark that p and $\tilde{p}$ cannot be in the same row. Otherwise, the row index of p_1 is equal to that of p plus one. This is a contradiction since the row index of p in $[\lambda^{(i)}]$ is maximal.

Note that $\alpha \in [\lambda/\nu_{\lambda,\mu}]$. When the row index of $\tilde{p}$ is equal to that of p minus one, $[\lambda/\nu_{\lambda,\mu}]$ is one of skew Young diagrams in Fig. 4. In each case, $\{r_1, r_2\}$ is an admissible pair. This proves (a)–(b).

In the rest case, since the width of $[\lambda/\nu_{\lambda,\mu}]$ is 2, by the definition of $\nu_{\lambda,\mu}$, we have $p_1^- \in [\lambda/\nu_{\lambda,\mu}]$. Further, since the scalar $c(p_1^-)$ appears twice, we have $\gamma \notin [\lambda/\nu_{\lambda,\mu}]$.

We want to prove that there is no node in $[\lambda/\nu_{\lambda,\mu}]$ whose row index is strictly less than that of $\tilde{p}$. Otherwise, let $\{a_1, a_2, \dots, a_k\}$ be all nodes which are in the first row of the skew Young diagram $[\lambda/\nu_{\lambda,\mu}]$. Since $[\lambda/\nu_{\lambda,\mu}]$ consists of admissible pairs, we can find $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k\}$ such that each $\{a_i, \tilde{a}_i\}$ is an admissible pair. Here the residues of $c(a_i)$, $1 \leqslant i \leqslant k$ are different. Further, we can choose $\tilde{a}_i$ such that it is either in the same row of p or in the same row with α. We take such $\tilde{a}_i$'s such that $\nu_{\lambda,\mu} \cup \{\tilde{a}_1, \dots, \tilde{a}_k\} \cup \{a_1, \dots, a_k\}$ is an r-partition. In this case, λ is $(\ell - k, \nu_{\lambda,\mu} \cup \{a_1, a_2, \dots, a_k\} \cup \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k\})$-admissible, which contradicts the fact that $|\nu_{\lambda,\mu}|$ is maximal. This proves (a) and (b).

We have proved (1) of this lemma. Suppose $\mathfrak{u}$ is given in (2) of this lemma. By the definition of $\mathfrak{s}$, $\mathfrak{s}_{n-1} = \mathfrak{s}_n \cup \{p\} = \nu_{\lambda,\mu} \cup \{p\}$ for some $p \in \mathscr{A}(\nu_{\lambda,\mu})$. We claim that $\mathfrak{u}_{n-1} \neq \nu_{\lambda,\mu}/\{p_1\}$ for any $p_1 \in \mathscr{R}(\nu_{\lambda,\mu})$. Otherwise, since we are assuming $\mathfrak{u} \approx \mathfrak{s}$, $\{p_1, p\}$ is an admissible pair. Note that $p_1 \notin [\lambda/\nu_{\lambda,\mu}]$. The node which is paired with p is $(p_1^-)^+$. Write $p_2 := p_1^-$. By Lemma 5.15, $\tilde{p}_2$ and p are in the same component of $[\lambda]$. Since $p \in \mathscr{A}(\nu_{\lambda,\mu})$, by the construction of $\nu_{\lambda,\mu}$, $\tilde{p}_2 = p^+ \in [\lambda/\nu_{\lambda,\mu}]$ and $\{p_2, \tilde{p}_2\}$ is an admissible pair, too. Since $\mathfrak{u} \approx \mathfrak{s}$, $\mathfrak{u}_{n-2}$ can be obtained from $\mathfrak{u}_{n-1}$ by removing the node p_3 with $p_3^+ = p_1$. Note that the column index of p_3 is strictly less than that of p_1. We remark that p_3 and p_1 are in the same component, the above statement makes sense. After a finite number of steps, we will not find such a removable node which is similar to p_3. This is a contradiction. So $\mathfrak{s}_{n-1} = \nu_{\lambda,\mu} \cup \{p_1\}$. Since $\mathfrak{s} \approx \mathfrak{u}$, by Lemma 4.3, $p = p_1$. So, $\mathfrak{u}_{n-1} = \mathfrak{s}_{n-1}$. By induction assumption on $n-1$, $\mathfrak{s}_i = \mathfrak{u}_i$ for all $1 \leqslant i \leqslant n-1$, forcing $\mathfrak{s} = \mathfrak{u}$. □

The following result is for both $\mathscr{W}_{r,n,\kappa}$ and $\mathscr{B}_{r,n,\kappa}$.

Lemma 5.18. *Let $\mu \in \Lambda_r^+(n-2f)$ with $0 < f \leqslant \lfloor n/2 \rfloor$. If λ is (f,μ)-admissible and if $\nu_{\lambda,\mu}$ is one of r-partitions defined in Definition* 5.16, *we have $\phi_{\ell,\nu_{\lambda,\mu}}(g_\mathfrak{s}, g_\mathfrak{s}) = 0$ where $\ell = \frac{n-|\nu_{\lambda,\mu}|}{2}$ and $\mathfrak{s} \in \mathscr{T}_n^{ud}(\nu_{\lambda,\mu})$ which is defined in Lemma* 5.17.

Proof. We prove the result for $\mathscr{W}_{r,n,\kappa}$. One can prove it for $\mathscr{B}_{r,n,\kappa}$ by similar arguments.

Suppose $\mathfrak{s}_{n-1} = \mathfrak{s}_n \cup p_1$ for the $p_1 \in \mathscr{A}(\nu_{\lambda,\mu}) \cap [\lambda/\nu_{\lambda,\mu}]$ such that the row index of p_1 is minimal in the skew Young diagram $[\lambda/\nu_{\lambda,\mu}]$. Since $[\lambda/\nu_{\lambda,\mu}]$ consists of admissible pairs, there exists $\tilde{p}_1 \in [\lambda/\nu_{\lambda,\mu}]$ such that $c(p_1) + c(\tilde{p}_1) = 0$. We are going to compute $\phi_{\ell,\nu_{\lambda_{\lambda,\mu}}}(\tilde{g}_\mathfrak{s}, \tilde{g}_\mathfrak{s})$ over the ground ring $\mathcal{O}$, where $(\mathcal{O}, K, \kappa)$ is the modular system for $\mathscr{W}_{r,n,\kappa}$ that we have constructed in Section 4. We write $\langle \tilde{g}_{\mathfrak{s}}, \tilde{g}_{\mathfrak{s}} \rangle$ instead of $\phi_{\ell,\nu_{\lambda,\mu}}(\tilde{g}_{\mathfrak{s}}, \tilde{g}_{\mathfrak{s}})$.

Write $\mathfrak{u} = (\mathfrak{s}_0, \mathfrak{s}_1, \dots, \mathfrak{s}_{n-1})$, then $f_{\mathfrak{s}} = \tilde{g}_{\mathfrak{s}}$ and $f_{\mathfrak{u}} = \tilde{g}_{\mathfrak{u}}$. By Propositions 3.7–3.12,

$$\langle \tilde{g}_{\mathfrak{s}}, \tilde{g}_{\mathfrak{s}} \rangle = \gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} \phi_{\ell-1,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}})$$

where

$$\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} = b_1\bigl(2c(p_1) - (-1)^r\bigr) \frac{\prod_{p \in \mathscr{A}(\nu_{\lambda,\mu}),\, p \neq p_1} (c(p_1) + c(p))}{\prod_{\tilde{p} \in \mathscr{R}(\nu_{\lambda,\mu})} (c(p_1) + c(\tilde{p}))}$$

for some $b_1 \in \mathcal{O}$ whose image in κ is invertible. Note that $\phi_{\ell-1,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}}) \in \mathcal{O}$.

By the definition of $\mathfrak{s}$, there is an $a \in \mathscr{R}(\lambda)$ such that $\{p_1, a\}$ is an admissible pair. If $[\lambda/\nu_{\lambda,\mu}] = \{p_1, a\}$, then the result follows from the arguments in the proof of Lemma 4.6. Otherwise, cardinality of the set of the nodes in $[\lambda/\nu_{\lambda.\mu}]$ is strictly greater than 2.

If $a = p_1^-$, then $2 \mid r$. If $a = p_1^+$, then $2 \nmid r$. In any case, $2c(p_1) - (-1)^r = 0$. Note that $\lambda/\{a\}$ is $(\frac{1}{2}(n - 1 - |\mathfrak{s}_{n-1}|), \mathfrak{s}_{n-1})$-admissible. By induction and the fact that $\ell - 1 > 0$, $\phi_{\ell-1,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}}) = 0$. Note that the multiplicity of zero divisor in $\prod_{\tilde{p} \in \mathscr{R}(\nu_{\lambda,\mu})} (c(p_1) + c(\tilde{p}))$ is at most one. We have $\langle g_{\mathfrak{s}}, g_{\mathfrak{s}} \rangle = 0$.

Now, we assume that $a \notin \{p_1^-, p_1^+\}$. We need only discuss the case when the image of $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}$ in κ is not well defined. Otherwise, since $\lambda/\{a\}$ is $(\frac{1}{2}(n - 1 - |\mathfrak{s}_{n-1}|), \mathfrak{s}_{n-1})$-admissible and $\frac{1}{2}(n - 1 - |\mathfrak{s}_{n-1}|) > 0$, by the previous arguments, we still have $\phi_{\ell-1,\mathfrak{s}_{n-1}}(\tilde{g}_{\mathfrak{u}}, \tilde{g}_{\mathfrak{u}}) = 0$, forcing $\langle g_{\mathfrak{s}}, g_{\mathfrak{s}} \rangle = 0$.

If $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}}$ in κ is not well defined, then $\{p_1, p_1^+\}$ (resp. $\{p_1, p_1^-\}$) is not an admissible pair in $[\lambda/\nu_{\lambda,\mu}]$ when r is odd (resp. even). In this case, there is a $\tilde{p} \in \mathscr{R}(\nu_{\lambda,\mu})$ such that $c(p_1) + c(\tilde{p}) = 0$.

Let $\{p_1, p_2, \dots, p_b\} \in [\lambda/\nu_{\lambda,\mu}]$ such that $p_i = p_{i-1}^+$ and $p_b^+ \notin [\lambda]$. We can find a minimal integer $i \leqslant b$ such that $\tilde{p}_i \in \mathscr{A}(\nu_{\lambda,\mu})$ and $c(p_i) + c(\tilde{p}_i) = 0$. Further, the row indexes of $\tilde{p}_j$ for $1 \leqslant j \leqslant i$ are the same.

By Propositions 3.10 and 3.12, we have, for $1 \leqslant k \leqslant i$,

$$\gamma_{\mathfrak{s}_{n-k+1}/\mathfrak{s}_{n-k}} = b_2\bigl(2c(p_k) - (-1)^r\bigr) \frac{\prod_{p \in \mathscr{A}(\mathfrak{s}_{n-k+1}),\, p \neq p_k} (c(p_k) + c(p))}{\prod_{p \in \mathscr{R}(\mathfrak{s}_{n-k+1})} (c(p_k) + c(p))}. \tag{5.19}$$

Hence

$$\begin{aligned}\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} \gamma_{\mathfrak{s}_{n-i+1}/\mathfrak{s}_{n-i}} = {} & b_1 b_2\bigl(2c(p_1) - (-1)^r\bigr)\bigl(2c(p_i) - (-1)^r\bigr) \\ & \times \frac{\prod_{p \in \mathscr{A}(\nu_{\lambda,\mu}),\, p \neq p_1} (c(p_1) + c(p))}{\prod_{p \in \mathscr{R}(\nu_{\lambda,\mu}),\, c(p) \neq -c(p_1)} (c(p_1) + c(p))} \\ & \times \frac{\prod_{p \in \mathscr{A}(\mathfrak{s}_{n-i+1}),\, p \notin \{p_i, \tilde{p}_i\}} (c(p_i) + c(p))}{\prod_{p \in \mathscr{R}(\mathfrak{s}_{n-i+1})} (c(p_i) + c(p))}. \end{aligned} \tag{5.20}$$

So, $\gamma_{\mathfrak{s}_n/\mathfrak{s}_{n-1}} \gamma_{\mathfrak{s}_{n-i+1}/\mathfrak{s}_{n-i}} \neq 0$.

If $k < i$, we cannot find $p \in \mathscr{A}(\mathfrak{s}_{n-k+1})$ (resp. $p \in \mathscr{R}(\mathfrak{s}_{n-k+1})$) such that $p \neq p_k$ and $c(p) + c(p_k) = 0$. So,

$$\gamma_{\mathfrak{s}_{n-1}/\mathfrak{s}_{n-2}} \cdots \gamma_{\mathfrak{s}_{n-i+1}/\mathfrak{s}_{n-i}} \neq 0.$$

If there is another minimal integer, say i' with $b \geqslant i' > i$ such that $\tilde{p}_{i'} \in \mathscr{A}(\nu_{\lambda,\mu})$ and $c(\tilde{p}_{i'}) + c(p_{i'}) = 0$, then we use previous arguments to verify

$$\gamma_{\mathfrak{s}_{n-i}/\mathfrak{s}_{n-i-1}} \cdots \gamma_{\mathfrak{s}_{n-i'+1}/\mathfrak{s}_{n-i'}} \neq 0$$

in κ. After a finite number of steps, we cannot find such an addable node. Let $\tilde{p}_i$ be the addable node with i maximal, then

$$\gamma_{\mathfrak{s}_{n-1}/\mathfrak{s}_{n-2}} \cdots \gamma_{\mathfrak{s}_{n-i+1}/\mathfrak{s}_{n-i}} \neq 0$$

in κ. By (5.19) for $i+1 \leqslant k \leqslant b$, we have $\gamma_{\mathfrak{s}_{n-k+1}/\mathfrak{s}_{n-k}} \neq 0$. So, $\gamma_{\mathfrak{s}_{n-1}/\mathfrak{s}_{n-2}} \cdots \gamma_{\mathfrak{s}_{n-b+1}/\mathfrak{s}_{n-b}} \neq 0$. By Lemma 5.17, we can find a sequence of nodes $a_1, a_2, \ldots, a_b \in [\lambda]$ such that

$$\nu_{\lambda/\{a_1,\ldots,a_b\}, \nu_{\lambda,\mu} \cup \{p_1,\ldots,p_b\}} = \nu_{\lambda,\mu} \cup \{p_1, \ldots, p_b\}.$$

By induction assumption $\phi_{\ell_1,\mathfrak{v}}(g_\mathfrak{v}, g_\mathfrak{v}) = 0$ in κ where $2\ell_1 = n - b - |\mathfrak{s}_{n-b}|$ and $\mathfrak{v} = (\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_{n-b})$. This proves $\phi_{\ell,\nu_{\lambda,\mu}}(g_\mathfrak{s}, g_\mathfrak{s}) = 0$ in κ, as required. □

The following result is motivated by Cox–De Visscher–Martin's work on the blocks of Brauer algebras in characteristic 0 [7]. However, the proof is different from that given in [7]. Note that we have already generalized it for BMW algebras in [35].

Proposition 5.21. *Let $\lambda \in \Lambda_r^+(n)$ and $(f, \mu) \in \Lambda_{r,n}$ with $0 < f \leqslant [n/2]$. If λ is (f, μ)-admissible and if $\nu_{\lambda,\mu}$ is one of r-partitions for $A_{r,n,\kappa} \in \{\mathscr{W}_{r,n,\kappa}, \mathscr{B}_{r,n,\kappa}\}$ in Definition 5.16, then*

$$\operatorname{Hom}_{A_{r,n,\kappa}}\big(\Delta(0,\lambda), \Delta(\ell, \nu_{\lambda,\mu})\big) \neq 0,$$

where $\ell = \frac{n-|\nu_{\lambda,\mu}|}{2}$.

Proof. Let M be the cyclic $A_{r,n,\kappa}$-submodule of $\Delta(\ell, \nu_{\lambda,\mu})$ generated by $g_\mathfrak{s}$, where $\mathfrak{s}$ is defined in Lemma 5.17. By Lemmas 5.18 and 2.10, $g_\mathfrak{s} \in \operatorname{Rad}\Delta(\ell, \nu_{\lambda,\mu})$. So $0 \subsetneq M \subset \operatorname{Rad}\Delta(\ell, \nu_{\lambda,\mu})$. Let M_1 be a maximal $A_{r,n,\kappa}$-submodule of M. Then $M/M_1 \cong D^{(\ell_1,\eta)}$, for some $(\ell_1, \eta) \in \Lambda_{r,n}$ with $(\ell_1, \eta) < (\ell, \nu_{\lambda,\mu})$. Further, $\ell \neq \ell_1$. Otherwise, there is a non-trivial homomorphism from $\Delta(\ell_1, \eta)$ to $\Delta(\ell, \nu_{\lambda,\mu})/M_1$. Using the exact functor $\mathcal{F}$ repeatedly yields a non-trivial homomorphism from $\Delta(0, \eta)$ to $\Delta(0, \nu_{\lambda,\mu})$, forcing $\eta = \nu_{\lambda,\mu}$, a contradiction.

Obviously, there is a non-zero epimorphism $\phi : M \to D^{(\ell_1,\eta)}$. Note that $D^{(\ell_1,\eta)} = \Delta(\ell_1, \eta)/\operatorname{Rad}\Delta(\ell_1, \eta)$. We can write

$$0 \neq \phi(g_\mathfrak{s}) = \sum_{\mathfrak{u} \in \mathscr{T}_n^{ud}(\eta)} a_\mathfrak{u} g_\mathfrak{u} + \operatorname{Rad}\Delta(\ell_1, \eta).$$

We have

$$\phi(g_\mathfrak{s})L_i = \sum_{\mathfrak{u} \in \mathscr{T}_n^{ud}(\eta)} a_\mathfrak{u} g_\mathfrak{u} L_i + \operatorname{Rad}\Delta(\ell_1, \eta). \tag{5.22}$$

By Lemma 2.9, $g_\mathfrak{s} L_i = c_\mathfrak{s}(i) g_\mathfrak{s}$. Let $\sigma \in \mathscr{T}_n^{ud}(\eta)$ be minimal with respect to $\succ$ such that $a_\sigma \neq 0$, where $\succ$ is the linear order defined on the set $\mathscr{T}_n^{ud}(\eta)$ (see the statements above Theorems 3.2 and 3.14).

Comparing the coefficients of g_σ on both sides of (5.22) yields $c_\mathfrak{s}(i) = c_\sigma(i)$ for any $1 \leqslant i \leqslant n$. If $\eta \nsubseteq \lambda$, then there is an i such that the multiplicities of $c_\sigma(i)$ in $\{c_\sigma(i) \mid 1 \leqslant i \leqslant n\}$ and $\{c_{\mathfrak{t}^\lambda}(i) \mid 1 \leqslant i \leqslant n\}$ are not equal, a contradiction. So $\eta \subseteq \lambda$.

Since $[\Delta(\ell, \nu_{\lambda,\mu}) : D^{(\ell_1,\eta)}] \neq 0$, we use the functor $\mathcal{F}$ for $A_{r,n,\kappa}$ to obtain $[\Delta(\ell - \ell_1, \nu_{\lambda,\mu}) : D^{(0,\eta)}] \neq 0$. Since $\ell - \ell_1 > 0$, by Proposition 5.10, $\eta \supset \nu_{\lambda,\mu}$ and η is $(\ell - \ell_1, \nu_{\lambda,\mu})$-admissible. Note that $|\nu_{\lambda,\mu}|$ is maximal. We have $\lambda = \eta$, as required. □

Definition 5.23. Suppose that $(f,\lambda),(\ell,\mu)\in\Lambda_{r,n}$. We say that (f,λ) and (ℓ,μ) is **b**-equivalent and write $(f,\lambda)\overset{\mathbf{b}}{\sim}(\ell,\mu)$ if both λ is $(\ell_1,\lambda\cap\mu)$-admissible and μ is $(\ell_2,\lambda\cap\mu)$-admissible where $2\ell_1=|\lambda|-|\lambda\cap\mu|$ and $2\ell_2=|\mu|-|\lambda\cap\mu|$.

Note that $\lambda\cap\mu$ is an r-partition if both λ and μ are r-partitions. So, the above definition is well defined. Further, $\overset{\mathbf{b}}{\sim}$ is an equivalence relation.

By Theorems 3.4 and 3.16, we have $D^{(f,\lambda)}\neq 0$ for any $(f,\lambda)\in\Lambda_{r,n}$ if either $\omega_i\in\kappa^*$, for some i, $0\leqslant i\leqslant r-1$ or $\omega_i=0$, for $0\leqslant i\leqslant r-1$ and $2\nmid r$. In the rest case, $(f,\lambda)\neq(n/2,\emptyset)$. When we write $D^{(f,\lambda)}$ in Theorem 5.24, we always assume that $D^{(f,\lambda)}\neq 0$.

Theorem 5.24. *Let $A_{r,n,\kappa}\in\{\mathscr{W}_{r,n,\kappa},\mathscr{B}_{r,n,\kappa}\}$ where κ is the field given in Assumption 4.1. Two irreducible $A_{r,n,\kappa}$-modules $D^{(f,\lambda)}$ and $D^{(\ell,\mu)}$ are in the same block if and only if $(f,\lambda)\overset{\mathbf{b}}{\sim}(\ell,\mu)$.*

Proof. In [19], Graham and Lehrer have proved that the cell blocks of a cellular algebra can be determined by the notion of cell linked. In our case, (f,λ) and (ℓ,μ) are said to be cell linked if there is a sequence of elements $(f_i,\lambda(i))\in\Lambda_{r,n}$, $1\leqslant i\leqslant k$ such that $(f_1,\lambda(1))=(f,\lambda)$, $(f_k,\lambda(k))=(\ell,\mu)$ and either $D^{(f_i,\lambda(i))}$ is a composition factor of $\Delta(f_{i-1},\lambda(i-1))$ or $D^{(f_{i-1},\lambda(i-1))}$ is a composition factor of $\Delta(f_i,\lambda(i))$ for all possible i. Now, the result follows immediately from Propositions 5.10 and 5.21. $\square$

6. Non-vanishing Gram determinants

In this section, we prove the main result of this paper, which is a necessary and sufficient condition for the Gram determinant associated to each cell module of cyclotomic NW and cyclotomic Birman–Murakami–Wenzl algebras being not equal to zero over an arbitrary field κ. This will give a necessary and sufficient condition for each cell module of such algebras being equal to its simple head over an arbitrary field.

First, we consider the cyclotomic Birman–Murakami–Wenzl algebras $\mathscr{B}_{r,n,\kappa}$ over an arbitrary field κ. Let $R=\mathbb{Z}[u_1^{\pm},u_2^{\pm},\dots,u_r^{\pm},q^{\pm},(q-q^{-1})^{-1}]$. By Theorem 3.19, any non-invertible factor which appears in $\det G_{f,\lambda}$ is one of the following forms:

a) q-integer $[k]:=\frac{q^k-q^{-k}}{q-q^{-1}}$;
b) $u_i-u_jq^{2a}$ for some $a\in\mathbb{Z}$ and $1\leqslant i<j$;
c) $u_iu_jq^{2a}-1$ for some $a\in\mathbb{Z}$ and $1\leqslant i<j$;
d) $u_iq^a+\varepsilon$ where $\varepsilon\in\{1,-1\}$ and $a\in\mathbb{Z}$.

First, we explain how to use Theorem 5.24 to determine the non-invertible factors in (c) for fix i, j with $i\neq j$ and the non-invertible factors in (d) for fix i. We consider $\mathscr{B}_{r,n,\mathbb{C}}$ where $\mathbb{C}$ is the complex field. In order to determine $u_iu_jq^{2a}-1$ for some $a\in\mathbb{Z}$ and fixed i, j with $i\neq j$, we assume that $u_i=bq^c$ and $u_j=b^{-1}q^d$ where $b,q\in\mathbb{C}^*$ are not roots of unity and $c,d\in\mathbb{Z}$ such that $b\notin\pm q^{\mathbb{Z}}$ and $b^2\notin q^{\mathbb{Z}}$ where $q^{\mathbb{Z}}=\{q^\ell\mid \ell\in\mathbb{Z}\}$. In this case, we also assume $u_l\in\mathbb{C}^*$, $l\notin\{i,j\}$ such that both u_i, u_l's and u_j, u_l's are algebraically independent. It is easy to see that Assumption 4.1 holds. Further, the factors in (d) are invertible in $\mathbb{C}$.

If there is an $(f-\ell,\lambda)$-admissible partition with $0\leqslant\ell\leqslant f-1$, it implies that $\det G_{f,\lambda}=0$ when $u_iu_jq^{2a}-1=0$ for some $a\in\mathbb{Z}$. Since we are assuming that $o(q^2)=\infty$, the integer a has to appear in the formula $\det G_{f,\lambda}\in R$. Conversely, if $u_iu_jq^{2a}-1$ is a non-invertible factor of $\det G_{f,\lambda}\in R$, then $\det G_{f,\lambda}=0$ in $\mathbb{C}$ if $u_iu_jq^{2a}=1$. Therefore, we can find a composition factor $\Delta(\ell,\mu)$ such that $(\ell,\mu)\lhd(f,\lambda)$. Further, $\ell\neq f$. Otherwise, we use the exact functor $\mathcal{F}$ to show that $\lambda=\mu$, a contradiction. Acting the exact functor $\mathcal{F}$ repeatedly shows that $\Delta(0,\mu)$ is a composition factor of $\Delta(f-\ell,\lambda)$. By Proposition 5.10, μ is an $(f-\ell,\lambda)$-admissible partition. In other words, the non-invertible factors $u_iu_jq^{2a}-1$ of $\det G_{f,\lambda}$ can be determined completely by $(f-\ell,\lambda)$-admissible partitions with $0\leqslant\ell\leqslant f-1$ over the field $\mathbb{C}$.

In order to determine the non-invertible factor $u_iq^a+\varepsilon$, we assume that $u_j\in\mathbb{C}^*$ for $j\neq i$ such that all such u_j are algebraically independent. Further, we assume that $u_i=\alpha q^b$ for some $b\in\mathbb{Z}$ and $o(q^2)=\infty$ and $\alpha\in\{1,-1\}$. It is easy to see that Assumption 4.1 holds. Further, the factors in (c) are invertible in $\mathbb{C}$. By the similar arguments as above, we see that the non-invertible factors $u_iq^a+\varepsilon$ of $\det G_{f,\lambda}$ can be determined by $(f-\ell,\lambda)$-admissible partitions with $0\leqslant\ell\leqslant f-1$ over the field $\mathbb{C}$.

Similarly, for $\mathscr{W}_{r,n,\kappa}$, by Theorem 3.5, any non-invertible factor which appears in $\det G_{f,\lambda}$ is one of the following forms:

a) the integers k;
b) u_i-u_j-a for some $a\in\mathbb{Z}$ and $i\neq j$;
c) u_i+u_j-a for some $a\in\mathbb{Z}$ and $i\neq j$;
d) $2u_i-a$ where $a\in\mathbb{Z}$.

By similar arguments for $\mathscr{B}_{r,n,\kappa}$ as above, we can determine the non-invertible factors of $\det G_{f,\lambda}$, which appear in (c)–(d), by using $(f-\ell,\lambda)$-admissible partitions over the complex field $\mathbb{C}$. Here $0\leqslant\ell\leqslant f-1$.

We want to determine the non-invertible factors of $\det G_{f,\lambda}$ given in (a)–(b). In this case, we consider $\mathscr{W}_{r,n,\mathbb{C}}$ and $\mathscr{B}_{r,n,\mathbb{C}}$. For $\mathscr{B}_{r,n,\mathbb{C}}$, we assume that $u_i=mq^{b_i}$, $1\leqslant i\leqslant r$ for some $b_i\in\mathbb{Z}$ and $m\in\mathbb{C}^*$ is not a root of unity. Further, q is a root of unity. For $\mathscr{W}_{r,n,\mathbb{C}}$, we assume that $u_i=m+b_i$ for some $b_i\in\mathbb{Z}$ and $m\in\mathbb{C}\setminus\mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. It is easy to see that all factors in (c)–(d) are invertible. By standard arguments (see the proof of Proposition 4.13 in [35]), we have $\det G_{f,\lambda}=0$ if and only if $\det G_{0,\lambda}=0$. This implies that the factors of $\det G_{f,\lambda}$ which appear in (a)–(b) can be determined completely by those of $\det G_{0,\lambda}$.

Lyle and Mathas [28] have given an explicit condition for the Gram determinant associated to each cell module of Ariki–Koike algebras being not equal to zero over an arbitrary field. This enables us to give an explicit condition for each $\det G_{f,\lambda}$ of $\mathscr{B}_{r,n,\kappa}$ being not equal to zero over an arbitrary field, too. However, for degenerate cyclotomic Hecke algebras, there is no explicit condition for each Gram determinant not being equal to zero over an arbitrary field. Mathas has told one of the authors that such a result can be obtained by arguments similar to those for Ariki–Koike algebras in [28].

Remark 6.1. Recently, X. Xu, a student of Rui in East China Normal University, is working on Gram determinants for degenerate cyclotomic Hecke algebras. In order to determine the non-invertible factors u_i-u_j-a which appears in any Gram determinant, he uses the previous ideas given by Mathas. In order to determine the prime integers which appear in the Gram determinants, he uses Morita equivalence between degenerate cyclotomic Hecke algebras and the group algebras of certain symmetric groups. We remark that the result for $\mathscr{W}_{r,n,\kappa}$ which is similar to Theorem 6.3 will be formulated elsewhere.

In order to formulate Theorem 6.3, we need some notions in [28] as follows. Suppose κ is a field with characteristic p and $q,u_1,u_2,\dots,u_r\in\kappa^*$. Let t be an indeterminate over κ and let $\mathcal{O}=\kappa[t,t^{-1}]_\pi$ be the localization of $\kappa[t,t^{-1}]$ at the prime ideal π generated by $t-1$. Suppose $\mathbf{X}=(X_1,\dots,X_r)$ such that $X_i=u_it^{ni}$. Note that we assume u_i is invertible when we define $\mathscr{B}_{r,n}$. Let ν_π be the π-adic evaluation map on $\mathcal{O}^*$, i.e. $\nu_\pi(f(t))=k$, where $k\geqslant0$ is maximal such that $(t-1)^k$ dividing $f(t)\in\kappa[t,t^{-1}]$.

Suppose $\lambda\in\Lambda_r^+(n)$ and $p=(a,i,j)\in[\lambda]$. Define a rim hook

$$r_x^\lambda=\bigl\{(a,k,l)\in[\lambda]\bigm| k\geqslant i,\ l\geqslant j\text{ and }(a,k+1,l+1)\notin[\lambda]\bigr\}.$$

Let i' be maximal such that $(a,i',j)\in[\lambda]$. Define $f_x^\lambda=(i',j,a)\in[\lambda]$ and r_x^λ has leg length $\ell(r_x^\lambda)=i'-i$. If $x\in[\lambda]$, let $\lambda\setminus r_x^\lambda$ be the multipartition with diagram $[\lambda]\setminus r_x^\lambda$. We say that $\lambda\setminus r_x^\lambda$ is the multipartition obtained by unwrapping the rim hook r_x^λ from λ, and that λ is the multipartition obtained from $\lambda \setminus r_x^\lambda$ by wrapping on the rim hook r_x^λ.

Define the $\mathcal{O}$-residue of the node $x = (a, i, j)$ to be $res_{\mathcal{O}}(x) = (qt)^{j-i} X_a$. The following definition is due to Lyle and Mathas [28].

Definition 6.2. (See [28, 2.5].) Suppose $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ are r-multipartitions of n. The Jantzen coefficient $J_{\lambda\mu}$ is the integer, which can be computed by the following formula

$$J_{\lambda\mu} = \begin{cases} \sum_{x \in [\lambda]} \sum_{y \in [\mu],\, [\mu]\setminus r_y^\mu = [\lambda]\setminus r_x^\lambda} (-1)^{\ell(r_x^\lambda)+\ell(r_y^\mu)} \nu_\pi(res_{\mathcal{O}}(f_x^\lambda) - res_{\mathcal{O}}(f_y^\mu)), & \text{if } \lambda \rhd \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $J_{\lambda\mu}$ can be computed explicitly by [28, 3.6, 4.2, 4.5] over an arbitrary field.

Theorem 6.3. *Suppose that κ is a field with characteristic p. Let $(f, \lambda) \in \Lambda_{r,n}$ and let* $\det G_{f,\lambda}$ *be the Gram determinants associated to the cell modules $\Delta(f, \lambda)$ of $\mathscr{B}_{r,n,\kappa}$. Then* $\det G_{f,\lambda} \neq 0$ *if and only if $\Delta(f, \lambda) = D^{f,\lambda}$ if and only if the following conditions hold*:

a) *$u_i u_j q^{2a} \neq 1$ for $1 \leqslant i, j \leqslant r$, $i \neq j$ and $a \in \mathbb{Z}$, which are determined by $(f - \ell, \lambda)$-admissible partition* (*see our arguments in page* 216);
b) *$u_i \neq \alpha q^a$ where $\alpha \in \{1, -1\}$ and $a \in \mathbb{Z}$ are determined by $(f - \ell, \lambda)$-admissible partitions with $0 \leqslant \ell \leqslant f - 1$* (*see our arguments in page* 217);
c) *λ is a Kleshchev-multipartition in* [3];
d) *$J_{\lambda\nu} = 0$ for any $\nu \in \Lambda_r^+(n - 2f)$.*

Proof. By [28, 2.5], (c)–(d) is equivalent to $\det G_{0,\lambda} \neq 0$. So, Theorem 6.3 follows from Theorem 5.24 and the arguments in pages 216–217. □

The Brauer algebras [6] can be considered as $\mathscr{W}_{1,n}$. We formulate the result for the Brauer algebras which is similar to Theorem 6.3 as follows. This result can be proved by similar arguments for BMW algebras, cyclotomic NW algebras and cyclotomic BMW algebras. Of course, we need Cox–De Visscher–Martin's results on the blocks of the Brauer algebras in characteristic 0 together with the results for the symmetric groups in [26]. We leave the detailed arguments to the reader. First, we recall some combinatorics.

Given a $\lambda \in \Lambda_1^+(n)$ and a node $(i, j) \in [\lambda]$, define h_{ij}^λ to be the (i, j)-hook length in $[\lambda]$ by

$$h_{ij}^\lambda = \lambda_i - j + \lambda'_j - i + 1,$$

where λ' is the dual partition of λ.

Theorem 6.4. *Suppose κ is a field with characteristic p. Let* $\det G_{f,\lambda}$ *be the Gram determinant associated to the cell module $\Delta(f, \lambda)$ of the Brauer algebra $\mathscr{W}_{1,n,\kappa}$. Then* $\det G_{f,\lambda} \neq 0$ *if and only if the following conditions hold*:

a) *λ is e-restricted in the sense that $\lambda_i - \lambda_{i+1} < e$ for all $i \geqslant 1$;*
b) *$\nu_p(h_{ac}^\lambda) = \nu_p(h_{ab}^\lambda)$ for any $(a, c), (a, b) \in [\lambda]$, where $\nu_p(z)$ is the exponent of p dividing z;*
c) *the parameter $u_1 \neq a 1_\kappa$ such that $a \in \mathbb{Z}$ is determined by the notion of "balanced pairs" in* [7] *over the field $\mathbb{C}$.*

At the end of the paper, we remark that we have proved the result for BMW algebras which is similar to Theorem 6.3 in [35].

References

[1] S. Ariki, On the semi-simplicity of the Hecke algebra of $(Z/rZ) \wr \mathfrak{S}_n$, J. Algebra 169 (1994) 216–225.
[2] S. Ariki, K. Koike, A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations, Adv. Math. 106 (1994) 216–243.
[3] S. Ariki, A. Mathas, The number of simple modules of the Hecke algebras of type $G(r,1,n)$, Math. Z. 233 (3) (2000) 601–623.
[4] S. Ariki, A. Mathas, H. Rui, Cyclotomic Nazarov–Wenzl algebras, Nagoya Math. J. 182 (2006) 47–134.
[5] J.S. Birman, H. Wenzl, Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313 (1989) 249–273.
[6] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann. of Math. 38 (1937) 857–872.
[7] A. Cox, M. De Visscher, P. Martin, The block of the Brauer algebra in characteristic zero, Represent. Theory 13 (2009) 272–308.
[8] R. Dipper, G. James, A. Mathas, Cyclotomic q-Schur algebras, Math. Z. 229 (1999) 385–416.
[9] J. Du, H. Rui, Based algebras and standard bases for quasi-hereditary algebras, Trans. Amer. Math. Soc. 350 (1998) 3207–3235.
[10] W.F. Doran IV, D.B. Wales, P.J. Hanlon, On the semisimplicity of the Brauer centralizer algebras, J. Algebra 211 (1999) 647–685.
[11] J. Enyang, Specht modules and semisimplicity criteria for Brauer and Birman–Murakami–Wenzl algebras, J. Algebraic Combin. 26 (3) (2007) 291–341.
[12] F.M. Goodman, Cellularity of cyclotomic Birman–Wenzl–Murakami algebras, J. Algebra 321 (2009) 3299–3320.
[13] F.M. Goodman, Comparison of admissibility conditions for cyclotomic Birman–Wenzl–Murakami algebras, J. Pure Appl. Algebra 214 (2010) 2009–2016.
[14] F.M. Goodman, Admissibility conditions for degenerate cyclotomic BMW algebras, Comm. Algebra 39 (2011) 452–461.
[15] F.M. Goodman, John Graber, Cellularity and the Jones basic construction, Adv. in Appl. Math. 46 (2011) 312–362.
[16] F.M. Goodman, John Graber, On cellular algebras with Jucys–Murphy elements, J. Algebra 330 (2011) 147–176.
[17] F.M. Goodman, H.M. Hauschild, Cyclotomic Birman–Wenzl–Murakami algebras, I: Freeness and realization as tangle algebras, J. Knot Theory Ramifications 18 (8) (2009) 1089–1127.
[18] F.M. Goodman, H.M. Hauschild, Cyclotomic Birman–Wenzl–Murakami algebras, II: Admissibility relations and freeness, Algebr. Represent. Theory 14 (1) (2011) 1–39.
[19] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
[20] J.J. Graham, G.I. Lehrer, The representation theory of affine Temperley–Lieb algebras, Enseign. Math. 44 (1998) 173–278.
[21] J.A. Green, Polynomial Representations of GL_n, Second corrected and augmented edition. With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker, Lecture Notes in Math., vol. 830, Springer, Berlin, 2007.
[22] R. Häring-Oldenburg, Cyclotomic Birman–Murakami–Wenzl algebras, J. Pure Appl. Algebra 161 (2001) 113–144.
[23] T. Hungerford, On the structure of principal ideal rings, Pacific J. Math. 25 (3) (1968) 543–547.
[24] G. James, A. Mathas, A q-analogue of the Jantzen–Schaper theorem, Proc. Lond. Math. Soc. 74 (1997) 241–274.
[25] G. James, A. Mathas, The Jantzen sum formula for cyclotomic q-Schur algebras, Trans. Amer. Math. Soc. 352 (2000) 5381–5404.
[26] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia Math. Appl., vol. 16, Addison–Wesley, Reading, MA, 1981.
[27] A. Kleshchev, Linear and Projective Representations of Symmetric Groups, Cambridge Tracts in Math., vol. 163, Cambridge Univ. Press, 2005.
[28] S. Lyle, A. Mathas, Blocks of cyclotomic Hecke algebras, Adv. Math. 216 (2007) 854–878.
[29] A. Mathas, Seminormal forms and Gram determinants for cellular algebras, J. Reine Angew. Math. 619 (2008) 141–173.
[30] J. Murakami, The Kauffman polynomial of links and representation theory, Osaka J. Math. 24 (1987) 745–758.
[31] M. Nazarov, Young's orthogonal form for Brauer's centralizer algebra, J. Algebra 182 (1996) 664–693.
[32] H. Rui, M. Si, On the structure of cyclotomic Nazarov–Wenzl algebras, J. Pure Appl. Algebra 212 (10) (2008) 2209–2235.
[33] H. Rui, M. Si, The representation theory of cyclotomic BMW algebras, II, Algebr. Represent. Theory, doi:10.1007/s10468-010-9249-z, in press.
[34] H. Rui, M. Si, Gram determinants and semisimplicity criteria for Birman–Wenzl algebras, J. Reine Angew. Math. 631 (2009) 153–180.
[35] H. Rui, M. Si, Blocks of Birman–Murakami–Wenzl algebras, Int. Math. Res. Not. IMRN 2 (2011) 452–486.
[36] H. Rui, J. Xu, The representation theory of cyclotomic BMW algebras, J. Pure Appl. Algebra 213 (2009) 2262–2288.
[37] M. Si, On the structure and representations of cyclotomic Nazarov–Wenzl algebras, PhD thesis, ECNU, 2008, in Chinese.
[38] S. Wilcox, S. Yu, On the cellularity of the cyclotomic Birman–Murakami–Wenzl algebras, J. Lond. Math. Soc., in press.
[39] S. Wilcox, S. Yu, On the freeness of the cyclotomic BMW algebras: admissibility and an isomorphism with the cyclotomic Kauffman tangle algebras, preprint, 2009.
[40] S. Wilcox, S. Yu, The cyclotomic BMW algebra associated with the two string type B braid group, Comm. Algebra, in press.
[41] J. Xu, Cyclotomic Birman–Murakami–Wenzl algebras $\mathscr{W}_{r,2}$, thesis for Master degree, East China Normal University, June, 2006, in Chinese.
[42] Shona Yu, The cyclotomic Birman–Murakami–Wenzl algebras, PhD thesis, Sydney University, 2007.