Available online at www.sciencedirect.com
SCIENCE DIRECT®

Physics Letters B 638 (2006) 531–537

PHYSICS LETTERS B

www.elsevier.com/locate/physletb

# Calculating four-loop tadpoles with one non-zero mass

Bernd A. Kniehl *, Anatoly V. Kotikov [1]

*II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*

Received 22 August 2005; received in revised form 5 April 2006; accepted 5 April 2006

Available online 12 June 2006

Editor: N. Glover

**Abstract**

An efficient method to calculate tadpole diagrams is proposed. Its capability is demonstrated by analytically evaluating two four-loop tadpole diagrams of current interest in the literature, including their $O(\epsilon)$ terms in $D = 4 - 2\epsilon$ space–time dimensions.



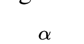
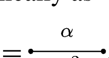
*PACS:* 11.15.-q; 12.38.Bx

*Keywords:* Quantum field theory; Multi-loop calculations; Tadpole diagrams

## 1. Introduction

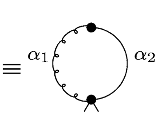
Calculations of higher-order corrections are very important for precision tests of the Standard Model in present and future high-energy-physics experiments. The complexity of such calculations and the final results strongly increases with the number of quantum loops considered, and one rapidly reaches the limits of the present realm of possibility when one attempts to exactly account for all mass scales of a given problem, already at the two-loop level. To simplify the calculations and also the final expressions, various types of expansions were proposed during the last couple of years (see Refs. [1–3] and references cited therein). These approaches usually provide a possibility to reduce the problem of evaluating complicated Feynman integrals to the calculation of tadpoles $T_{0,m}(\alpha_1, \alpha_2)$ and loops with massless propagators $L_q(\alpha_1, \alpha_2)$, defined in Eqs. (2) and (4), respectively, which have simple representations.

As a first step, let us to introduce some definitions to be used below. All the calculations are performed in Euclidean momentum space of dimension $D = 4 - 2\epsilon$ using dimensional regularisation with 't Hooft mass scale $\mu$. For the Euclidean integral measure, we use the short-hand notation $Dk \equiv \mu^{4-D} d^D k/\pi^{D/2}$. The dotted and solid lines of any diagram correspond to massless and massive Euclidean propagators, represented graphically as

$$\frac{1}{(k^2)^\alpha} =$$  $$, \qquad \frac{1}{(k^2+m^2)^\alpha} =$$  $$, \tag{1}$$

where $\alpha$ and $m$ denote the index and mass of the considered line, respectively. Unless stated otherwise, all solid lines have the same mass $m$. Lines with index 1 and mass $m$ are not marked.

Let us now discuss the rules used in our calculation. Firstly, massive tadpoles $T_{0,m}(\alpha_1, \alpha_2)$ are integrated according to the identity

$$T_{0,m}(\alpha_1, \alpha_2) \equiv \int \frac{Dk}{(k^2)^{\alpha_1}(k^2+m^2)^{\alpha_2}} \equiv$$  $$= \frac{R(\alpha_1, \alpha_2)}{(m^2)^{\alpha_1+\alpha_2-D/2}}, \tag{2}$$

* Corresponding author.
*E-mail addresses:* kniehl@desy.de (B.A. Kniehl), kotikov@theor.jinr.ru (A.V. Kotikov).

[1] Permanent address: Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna (Moscow region), Russia.

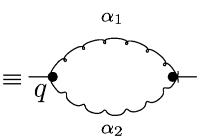
0370-2693 © 2006 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2006.04.057

where

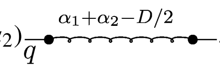
$$R(\alpha_1,\alpha_2)=\frac{\Gamma(D/2-\alpha_1)\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_2)\Gamma(D/2)} \tag{3}$$

and $\Gamma$ is Euler's gamma function. Secondly, massless loops $L_q(\alpha_1,\alpha_2)$ with the external momentum $q$ are integrated according to the identity

$$L_q(\alpha_1,\alpha_2)\equiv\int\frac{Dk}{(k^2)^{\alpha_1}(k-q)^{\alpha_2}}\equiv$$




$$= A(\alpha_1,\alpha_2)$$




(4)

where

$$A(\alpha_1,\alpha_2)=\frac{a(\alpha_1)a(\alpha_2)}{a(\alpha_1+\alpha_2-D/2)},\qquad a(\alpha)=\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}. \tag{5}$$

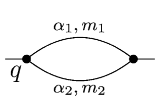
Recently, two examples of four-loop tadpoles were calculated in Ref. [4]. Certain $O(\epsilon^0)$ parts of these results, denoted as $N_{10}$ and $N_{20}$, were presented there only numerically.[2] Also the $O(\epsilon)$ terms beyond $N_{10}$ and $N_{20}$ are of current interest in the literature [6] and so far only known numerically. In conjunction with the results of Ref. [7], they allow us to derive analytic results for several four-loop master integrals which are indispensable for modern calculations at this level of accuracy [8].

The subject of this Letter is to advertise a powerful technique to evaluate such tadpoles analytically. The simplifications encountered in the examples at hand suggest that this technique might also be useful for more complicated diagrams. Therefore, we wish to introduce it to the interested reader.

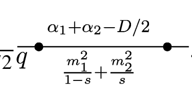
The content of this Letter is as follows. Section 2 explains the calculational technique in general. Sections 3 and 4 describe its application to the two integrals $N_{10}$ and $N_{20}$ of Ref. [4], respectively, also including their $O(\epsilon)$ terms. A summary is given in Section 5.

## 2. Technique

The core of the technique is a master formula to represent a loop with two massive propagators as an integral whose integrand contains a new propagator with a mass that depends on the variable of integration. This formula has the following form:




$$=\frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1\frac{ds}{(1-s)^{\alpha_1+1-D/2}s^{\alpha_2+1-D/2}}$$




(6)

It was introduced in Euclidean and Minkowski spaces in Refs. [9] and [10], respectively. Here, we are working in Euclidean space and will, thus, follow Ref. [9]. Some recent applications in Minkowski space may be found in Ref. [11]. Special cases of Eq. (6) were considered in Ref. [12], where a differential-equation method [12,13] was introduced. The latter only requires that quite simple diagrams are calculated directly. The results for more complicated diagrams may be reconstructed by integrating the results for simpler diagrams over external parameters. In many cases, the results for complicated diagrams may be obtained by integrating the one-loop tadpoles of Eq. (2). The method is now very popular for the calculation of complicated Feynman integrals; for recent articles, see Ref. [14] and references cited therein.

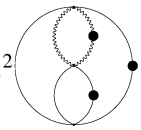
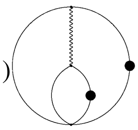
Here, we shall follow a similar strategy. Applying Eq. (6), we shall represent the results of Ref. [4] as integrals over one-loop tadpoles, which in turn contain propagators with masses that depend on the variable of integration. The case of $N_{10}$ will be considered in detail.

We would like to note that Eq. (6) can be applied successfully to diagrams containing one-loop self-energy subdiagrams. Such subdiagrams are frequently generated by the application of the integration-by-parts technique [15]. Such an application [12,13] leads to a differential equation for the original diagram (in exceptional cases, the equation degenerates to an algebraic relation) with an inhomogeneous term that depends on less complicated diagrams usually containing one-loop self-energy subdiagrams (see, for example, Refs. [9,10]).

[2] During the preparation of this article, we were informed by K.G. Chetyrkin about a paper [5] which also contains analytic results for $N_{10}$ and $N_{20}$. Our results are in full agreement with those of Ref. [5].

## 3. First example

As the first example, we consider the case of $N_{10}$ in Ref. [4], which may be represented graphically as[3]

$$m^2$$  $$+\, m^{2(1-\epsilon)}\Gamma(\epsilon)$$  $$= \frac{\Gamma^4(1+\epsilon)}{(m^2)^{4\epsilon}} N_{10}(\epsilon), \tag{7}$$

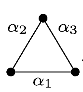
where $N_{10}(\epsilon) = [N_{10} + M_{10}\epsilon + O(\epsilon^2)]/(1-\epsilon)$. For $\epsilon = 0$, we recover $N_{10} = N_{10}(0)$. Applying Eq. (4) to the massless loop in the left diagram and Eq. (6) with $\alpha_1 = 1$ and $\alpha_2 = 2$ to the interior massive loops and to the loops with exterior propagators, we easily obtain the following representation for $N_{10}(\epsilon)$:

$$N_{10}(\epsilon) = \frac{m^2}{\epsilon}\left[\frac{1}{m^{2\epsilon}} J_1(0) - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} J_1(1)\right], \tag{8}$$

where

$$J_1(a) = \frac{(1-\epsilon)(m^2)^{4\epsilon}}{\Gamma(1+\epsilon)} \int_0^1 \frac{ds_1}{s_1\eta_1^\epsilon} \int_0^1 \frac{ds_2}{s_2\eta_2^\epsilon} T_{0,M_1,M_2}(1+a\epsilon, 1+\epsilon, 1+\epsilon), \tag{9}$$

$M_i^2 = m^2/\eta_i$, $\eta_i = s_i(1-s_i)$, and $T_{M_1,M_2,M_3}(\alpha_1,\alpha_2,\alpha_3)$, where the index $\alpha_i$ belongs to the mass $M_i$, is the one-loop tadpole involving three massive propagators,

$$T_{M_1,M_2,M_3}(\alpha_1,\alpha_2,\alpha_3) =$$ 

$$. \tag{10}$$

Expanding the propagator $(k^2 + M_2^2)^{-\alpha_2}$ in $T_{0,M_1,M_2}(\alpha_3,\alpha_1,\alpha_2)$ as

$$\frac{1}{(k^2+M_2^2)^{\alpha_2}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_2)}{n!\Gamma(\alpha_2)} \frac{\mu_{12}^n}{(k^2+M_1^2)^{n+1+\epsilon}}, \tag{11}$$

where $\mu_{ij} = M_i^2 - M_j^2$, and using Eq. (2) for the tadpole $T_{0,M_1}(\alpha_1,\alpha_2)$, we obtain

$$T_{0,M_1,M_2}(\alpha_3,\alpha_1,\alpha_2) = \frac{R(\alpha_3,\alpha_1+\alpha_2)}{(M_1^2)^{\bar{\alpha}-D/2}}\, {}_2F_1(\bar{\alpha}-D/2, \alpha_2; \alpha_1+\alpha_2; \bar{x}), \tag{12}$$

where $\bar{\alpha} = \alpha_1+\alpha_2+\alpha_3$, $\bar{x} = 1-x$, $x = M_2^2/M_1^2 = \eta_1/\eta_2$, and ${}_2F_1$ denotes a hyper-geometric function [16]. Setting $\alpha_i = 1 + a_i\epsilon$ and expanding the hyper-geometric function, we have

$$\begin{aligned}
&T_{0,M_1,M_2}(1+a_3\epsilon, 1+a_1\epsilon, 1+a_2\epsilon) \\
&\quad = \frac{1}{(M_1^2)^{1+(\bar{a}+1)\epsilon}} \frac{\Gamma(1+\epsilon)}{\bar{x}(1-\epsilon)} \biggl\{ -\ln x + \epsilon\left[\frac{a_2+a_3+1}{2}\ln^2 x - (a_1+a_2)\,\mathrm{Li}_2(\bar{x})\right] \\
&\qquad + \epsilon^2\biggl[\frac{(a_2+a_3+1)^2}{6}\ln^3 x - \bigl((a_3+1)(\bar{a}+1)-1\bigr)\zeta(2)\ln x + a_1(a_3+1)\ln x\,\mathrm{Li}_2(\bar{x}) + (a_1+a_2)^2\,\mathrm{Li}_3(\bar{x}) \\
&\qquad - \bigl(a_2(\bar{a}+1) - a_1(a_3+1)\bigr)\mathrm{S}_{1,2}(\bar{x})\biggr]\biggr\} + O\bigl(\epsilon^3\bigr),
\end{aligned} \tag{13}$$

where $\bar{a} = a_1+a_2+a_3$ and

$$\mathrm{S}_{n,m}(x) = \frac{(-1)^{n+m-1}}{(n-1)!m!} \int_0^1 \frac{dy}{y} \ln^{n-1}(y)\ln^m(1-xy), \qquad \mathrm{Li}_n(x) = \mathrm{S}_{n-1,1}(x) \tag{14}$$

denote the generalised and ordinary polylogarithms (see, for example, Ref. [17]), respectively. Thus, we obtain

[3] The graphic representations of $N_{10}$ and $N_{20}$ are adopted from Ref. [4]. A propagator with a point is equal to one with the index 2.

$$N_{10}(\epsilon) = \int_0^1 \frac{ds_1}{s_1\eta_1^\epsilon}\int_0^1 \frac{ds_2}{s_2\eta_2^\epsilon}\frac{\eta_1}{\bar{x}}\left\{-\frac{1}{2}\ln^2 x + \ln x \ln\eta_1 \right.$$
$$\left. + \epsilon\left[\frac{19}{6}\ln^3 x + 4\zeta(2)\ln x - \ln x \,\mathrm{Li}_2(\bar{x}) - \left(3\ln^2 x - 2\,\mathrm{Li}_2(\bar{x})\right)\ln\eta_1 + \frac{7}{2}\ln x \ln^2\eta_1\right]\right\} + O\left(\epsilon^2\right). \tag{15}$$

Exploiting the $s_1 \leftrightarrow s_2$ symmetry, we find

$$N_{10}(\epsilon) = \frac{1}{8}\int_0^1 ds_1 \int_0^1 \frac{ds_2}{\eta_2 - \eta_1}\left\{\ln x \ln(\eta_1\eta_2) + \epsilon\left[\frac{61}{12}\ln^3 x + 8\zeta(2)\ln x + \left(\mathrm{Li}_2(\bar{x}) - \mathrm{Li}_2\left(-\frac{\bar{x}}{x}\right)\right)\ln(\eta_1\eta_2)\right.\right.$$
$$\left.\left. + \frac{3}{4}\ln x \ln^2(\eta_1\eta_2)\right]\right\} + O\left(\epsilon^2\right). \tag{16}$$

We first concentrate on the $O(\epsilon^0)$ term $N_{10}$. Using the standard replacement $s_i = (1+\xi_i)/2$, we obtain

$$N_{10} = \frac{1}{2}\int_0^1 d\xi_1 \int_0^1 \frac{d\xi_2}{\xi_1^2 - \xi_2^2}\left(\ln^2\frac{1-\xi_1^2}{4} - \ln^2\frac{1-\xi_2^2}{4}\right)$$
$$= \frac{1}{2}\int_0^1 d\xi_1 \int_0^1 \frac{d\xi_2}{\xi_1^2 - \xi_2^2}\left\{\left[\ln^2\left(1-\xi_1^2\right) - 2\ln 4\ln\left(1-\xi_1^2\right)\right] - [\xi_1 \leftrightarrow \xi_2]\right\}. \tag{17}$$

Exploiting the $\xi_1 \leftrightarrow \xi_2$ symmetry, we find

$$N_{10} = \int_0^1 d\xi_1\left[\ln^2\left(1-\xi_1^2\right) - 2\ln 4\ln\left(1-\xi_1^2\right)\right]\int_0^1 \frac{d\xi_2}{\xi_1^2 - \xi_2^2} = -\frac{1}{2}\int_0^1 \frac{d\xi_1}{\xi_1}\ln\frac{1-\xi_1}{1+\xi_1}\left[\ln^2\left(1-\xi_1^2\right) - 2\ln 4\ln\left(1-\xi_1^2\right)\right]. \tag{18}$$

Evaluating the r.h.s. of Eq. (18), we obtain

$$N_{10} = \frac{17\pi^4}{720} + \frac{7}{2}\zeta(3)\ln 2 - 4\mathrm{S}_{1,3}(-1) = \frac{49\pi^4}{720} - \frac{1}{2}b_4, \tag{19}$$

where

$$b_4 = -\frac{1}{3}\left(\pi^2 - \ln^2 2\right)\ln^2 2 + 8\,\mathrm{Li}_4\left(\frac{1}{2}\right). \tag{20}$$

Eq. (19) coincides with the result obtained in Ref. [5].

The evaluation of the $O(\epsilon)$ term $M_{10}$ is rather tedious and cannot be described in this brief communication. Here, we merely list the result, which reads

$$M_{10} = -\frac{149\pi^4}{180}\ln 2 + \frac{279}{8}\zeta(5) - 7b_5, \tag{21}$$

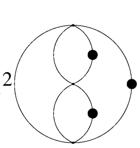
where

$$b_5 = \frac{1}{45}\left(5\pi^2 - 3\ln^2 2\right)\ln^3 2 + 8\,\mathrm{Li}_5\left(\frac{1}{2}\right). \tag{22}$$

We note in passing that also the other master integrals presented in Ref. [5] can be written in a more compact form if the combinations $b_4$ and $b_5$ of Eqs. (20) and (22), respectively, are introduced.

## 4. Second example

Let us consider the second example of Ref. [4], which may be graphically represented as

$m^2$  $$= \frac{\Gamma^4(1+\epsilon)}{(m^2)^{4\epsilon}}N_{20}(\epsilon), \tag{23}$$

where $N_{20}(\epsilon) = [N_{20} + M_{20}\epsilon + O(\epsilon^2)]/(1-\epsilon)$. For $\epsilon = 0$, we recover $N_{20} = N_{20}(0)$. Applying Eq. (6) with $\alpha_1 = 1$ and $\alpha_2 = 2$ to the two interior massive loops and to the loops involving exterior propagators, we easily obtain the following representation for

$N_{20}(\epsilon)$:

$$N_{20}(\epsilon) = \frac{(1-\epsilon)(m^2)^{4\epsilon}}{\Gamma(1+\epsilon)} \int_0^1 \frac{ds_1}{s_1\eta_1^\epsilon} \int_0^1 \frac{ds_2}{s_2\eta_2^\epsilon} \int_0^1 \frac{ds_3}{s_3\eta_3^\epsilon} T_{M_1,M_2,M_3}(1+\epsilon, 1+\epsilon, 1+\epsilon). \tag{24}$$

Expanding the propagators $(k^2+M_2^2)^{-\alpha_2}$ and $(k^2+M_3^2)^{-\alpha_3}$ in $T_{M_1,M_2,M_3}(\alpha_1,\alpha_2,\alpha_3)$ as in Eq. (11) and using Eq. (2) for the tadpole $T_{0,M_1}(0,\alpha_2)$, we obtain

$$\begin{aligned} T_{M_1,M_2,M_3}(\alpha_1,\alpha_2,\alpha_3) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_2)}{n!\Gamma(\alpha_2)} \sum_{l=0}^{\infty} \frac{\Gamma(l+\alpha_3)}{l!\Gamma(\alpha_3)} \frac{\Gamma(l+n+\bar{\alpha}-D/2)}{\Gamma(l+n+\bar{\alpha})} \frac{\mu_{12}^n \mu_{13}^l}{(M_1^2)^{n+l+\bar{\alpha}-D/2}} \\ &= \frac{R(0,\bar{\alpha})}{(M_1^2)^{\bar{\alpha}-D/2}} F_1\big(\bar{\alpha}-D/2, \alpha_2, \alpha_3; \bar{\alpha}; \bar{x}, \mu_{13}/M_1^2\big), \end{aligned}$$

where $F_1$ denotes an Appel hyper-geometric function [16]. Expanding the hyper-geometric function, we have

$$\begin{aligned} &T_{M_1,M_2,M_3}(1+a_1\epsilon, 1+a_2\epsilon, 1+a_3\epsilon) \\ &\quad = \frac{1}{(M_1^2)^{(\bar{a}+1)\epsilon}} \frac{\Gamma(1+\epsilon)}{\mu_{32}(1-\epsilon)} \left\{ \frac{M_2^2}{\mu_{12}} \ln \frac{M_2^2}{M_1^2} + \epsilon \left[ -\frac{a_2+1}{2} \frac{M_2^2}{\mu_{12}} \ln^2 \frac{M_2^2}{M_1^2} + \left( \bar{a}\frac{M_1^2}{\mu_{12}} - a_1 - a_3 \right) \mathrm{Li}_2\left(\frac{\mu_{12}}{M_1^2}\right) \right. \right. \\ &\qquad \left. \left. + a_3 \frac{M_1^2}{\mu_{12}} \mathrm{Li}_2\left(\frac{\mu_{13}}{M_1^2}\right) + a_3 \frac{M_2^2}{\mu_{12}} \left( \mathrm{Li}_2\left(\frac{\mu_{21}}{M_2^2}\right) - \mathrm{Li}_2\left(\frac{\mu_{23}}{M_2^2}\right)\right)\right] + O\big(\epsilon^2\big) \right\} + (2 \leftrightarrow 3). \end{aligned} \tag{25}$$

As in the previous section, we describe the evaluation of the $O(\epsilon^0)$ term in some detail, while we merely list the final result for the $O(\epsilon)$ term for reasons of space. At $O(\epsilon^0)$, the right-hand side of the Eq. (24) contains the tadpole $T_{M_1,M_2,M_3}(1,1,1)$, which can be calculated in an essentially simpler way. Indeed, using

$$\frac{1}{(k^2+M_1^2)(k^2+M_2^2)} = \frac{1}{\mu_{21}} \left( \frac{1}{k^2+M_1^2} - \frac{1}{k^2+M_2^2} \right) \tag{26}$$

and Eq. (2), we find

$$T_{M_1,M_2}(1,1) = \frac{\Gamma(\epsilon-1)}{\mu_{21}} \big[ \big(M_1^2\big)^{1-\epsilon} - \big(M_2^2\big)^{1-\epsilon} \big] = \frac{1}{\mu^{2\epsilon}} \frac{\Gamma(1+\epsilon)}{1-\epsilon} \left( \frac{1}{\epsilon} + \frac{M_1^2}{\mu_{21}} \ln \frac{M_1^2}{\mu^2} \right) + (M_1 \leftrightarrow M_2) + O(\epsilon), \tag{27}$$

$$T_{M_1,M_2,M_3}(1,1,1) = \frac{1}{\mu_{32}} \big[ T_{M_1,M_2}(1,1) - T_{M_1,M_3}(1,1) \big] = \frac{M_1^2}{\mu_{12}\mu_{13}} \ln \frac{M_1^2}{\mu^2} + (M_1 \leftrightarrow M_2) + (M_1 \leftrightarrow M_3) + O(\epsilon). \tag{28}$$

Notice that the right-hand side of Eq. (28) is $m^2$ and $\mu^2$ independent. Moreover, it coincides with Eqs. (24) and (25) at $O(\epsilon^0)$. Every term in Eqs. (27) and (28) contains only one logarithm depending on the variable $\eta_i$. Thus, the integrals $\int_0^1 ds_i$ can be evaluated at the end of the calculation (see Eq. (18), for a similar procedure). After some algebra, we obtain

$$N_{20} = \frac{9}{2}\zeta(3), \tag{29}$$

which agrees with Ref. [5].

Lack of space prevents us from going into details with our derivation of the $O(\epsilon)$ term $M_{20}$, and we merely list our result,

$$M_{20} = -\frac{4\pi^4}{15} - 27\zeta(3) + 3b_4, \tag{30}$$

where $b_4$ is defined in Eq. (20).

## 5. Conclusion

We demonstrated the usefulness of a simple technique for the analytic evaluation of four-loop tadpoles, which allows one to obtain the results by integrating one-loop tadpoles with masses that depend on integration variables. By means of this method, we calculated two Feynman diagrams which were presented in Ref. [4] in numerical form. Our results are in full agreement with the analytic results obtained just recently in Ref. [5].

We note that the integral $N_{10}$ was found in Ref. [5] with the aid of the PSLQ program [18], which is able to reconstruct the rational-number coefficients multiplying a given set of transcendental numbers from a high-precision numerical result, i.e. the algebraic structure of the result must be known a priori. Such structures are often more complicated, and they are in general not known from other considerations (see, for example, Ref. [19]), so that the PSLQ program cannot be applied. The method presented here does not suffer from such a limitation and should be applicable also for more complicated tadpoles depending on one [5] or two [20] non-zero masses. Moreover, it can be applied in combination with the PSLQ program. In fact, by evaluating some part of a complicated Feynman diagram by means of this method, the underlying set of transcendental numbers may emerge and can then be injected into the reconstruction of the full analytic result with the help of the PSLQ program.

## Acknowledgements

We are grateful to K.G. Chetyrkin for useful communications and for bringing Ref. [5] to our attention, and to O.L. Veretin for his interest in this work, for his helpful comments, for checking some of our formulae, and for technical advice related to the PSLQ program [18], which helped us in checking the $O(\epsilon)$ terms of the four-loop tadpoles $N_{10}$ and $N_{20}$. A.V.K. was supported in part by the RFBR Foundation under Grant No. 05-02-17645-a and the Heisenberg-Landau-Programm. This work was supported in part by BMBF Grant No. 05 HT4GUA/4 and HGF Grant No. NG-VH-008.

## References

[1] F.V. Tkachov, Report No. IYaI-P-0332, Moscow, 1983;
F.V. Tkachov, Report No. IYaI-P-0358, Moscow, 1984;
F.V. Tkachov, Int. J. Mod. Phys. A 8 (1993) 2041;
S.G. Gorishnii, S.A. Larin, Nucl. Phys. B 283 (1987) 452;
K.G. Chetyrkin, Teor. Mat. Fiz. 75 (1988) 26, Theor. Math. Phys. 75 (1988) 346;
K.G. Chetyrkin, Teor. Mat. Fiz. 76 (1988) 207, Theor. Math. Phys. 76 (1988) 809;
S.G. Gorishnii, Nucl. Phys. B 319 (1989) 633;
G.B. Pivovarov, F.V. Tkachov, Int. J. Mod. Phys. A 8 (1993) 2241;
S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Nucl. Phys. B 438 (1995) 278;
S.A. Larin, Phys. Lett. B 514 (2001) 366.
[2] V.A. Smirnov, Commun. Math. Phys. 134 (1990) 109;
V.A. Smirnov, Renormalization and Asymptotic Expansions, Birkhäuser, Basel, 1991;
V.A. Smirnov, Mod. Phys. Lett. A 10 (1995) 1485;
V.A. Smirnov, Phys. Lett. B 465 (1999) 226;
V.A. Smirnov, Applied Asymptotic Expansions in Momenta and Masses, Springer, Berlin, 2002.
[3] A.I. Davydychev, J.B. Tausk, Nucl. Phys. B 397 (1993) 123;
A.I. Davydychev, J.B. Tausk, Nucl. Phys. B 410 (1993) 325;
J. Fleischer, O.V. Tarasov, Z. Phys. C 64 (1994) 413;
J. Fleischer, M.Yu. Kalmykov, O.L. Veretin, Phys. Lett. B 427 (1998) 141.
[4] K.G. Chetyrkin, J.H. Kühn, P. Mastrolia, C. Sturm, Eur. Phys. J. C 40 (2005) 361.
[5] Y. Schröder, A. Vuorinen, JHEP 0506 (2005) 051.
[6] Y. Schröder, M. Steinhauser, JHEP 0601 (2006) 051;
K.G. Chetyrkin, M. Faisst, C. Sturm, M. Tentyukov, Nucl. Phys. B 742 (2006) 208;
K.G. Chetyrkin, J.H. Kühn, C. Sturm, Nucl. Phys. B 744 (2006) 121;
K.G. Chetyrkin, J.H. Kühn, C. Sturm, Report No. SFB/CPP-06-21, TTP06-15, hep-ph/0604234.
[7] B.A. Kniehl, A.V. Kotikov, A.I. Onishchenko, O.L. Veretin, Report No. DESY 06-074.
[8] B.A. Kniehl, A.V. Kotikov, Report No. DESY 06-073.
[9] J. Fleischer, A.V. Kotikov, O.L. Veretin, Phys. Lett. B 417 (1998) 163.
[10] J. Fleischer, A.V. Kotikov, O.L. Veretin, Nucl. Phys. B 547 (1999) 343.
[11] B.A. Kniehl, A.V. Kotikov, A.I. Onishchenko, O.L. Veretin, Nucl. Phys. B 738 (2006) 306.
[12] A.V. Kotikov, Phys. Lett. B 254 (1991) 158;
A.V. Kotikov, Phys. Lett. B 259 (1991) 314;
A.V. Kotikov, Phys. Lett. B 267 (1991) 123.
[13] E. Remiddi, Nuovo Cimento A 110 (1997) 1435.
[14] U. Aglietti, R. Bonciani, Nucl. Phys. B 698 (2004) 277;
U. Aglietti, R. Bonciani, G. Degrassi, A. Vicini, Phys. Lett. B 595 (2004) 432;
U. Aglietti, R. Bonciani, G. Degrassi, A. Vicini, Phys. Lett. B 600 (2004) 57;
W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia, E. Remiddi, Nucl. Phys. B 706 (2005) 245;
W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia, E. Remiddi, Nucl. Phys. B 712 (2005) 229;
R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, J.J. van der Bij, Nucl. Phys. B 716 (2005) 280;
S.P. Martin, D.G. Robertson, Comput. Phys. Commun. 174 (2006) 133;
M.Yu. Kalmykov, Nucl. Phys. B 718 (2005) 276;
R. Bonciani, A. Ferroglia, Phys. Rev. D 72 (2005) 056004;
M. Czakon, J. Gluza, T. Riemann, Phys. Rev. D 71 (2005) 073009.
[15] K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B 192 (1981) 159;
F.V. Tkachov, Phys. Lett. B 100 (1981) 65;
A.N. Vassiliev, Y.M. Pis'mak, Y.P. Khonkonen, Teor. Mat. Fiz. 47 (1981) 291, Theor. Math. Phys. 47 (1981) 465.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York, 1980.
[17] A. Devoto, D.W. Duke, Riv. Nuovo Cimento 7 (1984) 1.

[18] H.R.P. Ferguson, D.H. Bailey, RNR Technical Report No. RNR-91-032;
H.R.P. Ferguson, D.H. Bailey, S. Arno, NASA Technical Report No. NAS-96-005.
[19] D.J. Broadhurst, Eur. Phys. J. C 8 (1999) 311;
J. Fleischer, M.Yu. Kalmykov, A.V. Kotikov, Phys. Lett. B 462 (1999) 169;
J. Fleischer, M.Yu. Kalmykov, Phys. Lett. B 470 (1999) 168;
A.V. Kotikov, L.N. Lipatov, Nucl. Phys. B 582 (2000) 19;
A.V. Kotikov, L.N. Lipatov, Nucl. Phys. B 661 (2003) 19;
M.Yu. Kalmykov, O.L. Veretin, Phys. Lett. B 483 (2000) 315;
S. Laporta, Phys. Lett. B 549 (2002) 115.
[20] J.M. Chung, B.K. Chung, Phys. Rev. D 56 (1997) 6508;
J.M. Chung, B.K. Chung, Phys. Rev. D 59 (1999) 105014;
A.V. Kotikov, JHEP 9809 (1998) 001;
A.I. Davydychev, M.Yu. Kalmykov, Nucl. Phys. B 699 (2004) 3.