Global Properties of Tensor Rank

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ABSTRACT

The dependence of tensor rank on the underlying ring of scalars is considered. It is shown that the integers are, in a certain sense, the worst scalars. A ring of scalars can be improved by adjoining algebraic elements but not by adjoining indeterminates. The real closed fields are the best scalars among ordered rings, and the algebraically closed fields are best among all rings. Let $B(R^{m\times n\times p})$ be the maximum tensor rank of any $m\times n\times p$ array of elements from the ring R. A generalization of Gaussian elimination shows that $B(R^{n\times n\times n})\leqslant\frac{3}{4}n^2$ for most useful rings R. For every R, $B(R^{m\times n\times p})\geqslant mnp/(m+n+p)$, and slightly stronger lower bounds are proven for R a field.

1. INTRODUCTION

The number of multiplications needed to evaluate a set of bilinear forms in noncommuting indeterminates depends on the coefficients of the bilinear forms and on the ring R from which the constants used in the computation are drawn. Tensor rank is an algebraic function expressing this dependence. Let the set of bilinear forms be $D=\{f_k(x,y)|f_k(x,y)=\sum_{i,j}t_{ijk}x_iy_j,\ 1\leqslant k\leqslant p\}$, where the t_{ijk}'s are constants from R. Throughout this paper the ring R is assumed to be a commutative ring with identity. The multiplications to be counted are those between functions of the x_i's and y_j's, sometimes called active multiplications. It is assumed that R-linear combinations of the x_i's and y_j's can be evaluated at a comparatively low cost. This problem has been studied quite extensively [1–3, 5, 7–10, 12]. The reasons for the above assumptions are explained in many of these references and will not be repeated here.

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Let $T=(t_{ijk})$ be an $m\times n\times p$ array of elements of R. Let $Rk_R(T)$ be the least integer r such that the following decomposition can be made:

$$t_{ijk}=\sum_{l=1}^{r} u_{li}v_{lj}w_{lk}, \qquad \begin{cases} 1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n,\\ 1\leqslant k\leqslant p,\end{cases}$$

where the u's, v's, and w's are elements of R. The number $Rk_R(T)$ is called the tensor rank of T over R.

A basic result appearing in several of the above references states that the tensor rank of T over R is exactly equal to the number of active multiplications necessary and sufficient to evaluate a set of bilinear forms in noncommuting indeterminates using constants from R. This number, in turn, determines the asymptotic running time of algorithms for a variety of computations, most notably matrix multiplication. Thus, the computational complexity of an important class of problems is determined by tensor rank, a purely algebraic concept.

Note that when T is a matrix ($p=1$), the tensor rank of T coincides with the usual linear algebraic rank of T when R is a field. In this sense tensor rank extends the notion of rank used in linear algebra. The word "tensor" is suggested by the fact that tensor rank is invariant under invertible changes of coordinates.

LEMMA 1. *Let $A=(\alpha_{ij})\in R^{m\times m}$, $B=(\beta_{ij})\in R^{n\times n}$, and $\Gamma=(\gamma_{ij})\in R^{p\times p}$ be invertible matrices. Let T and $T'\in R^{m\times n\times p}$ be related by*

$$t'_{i'j'k'}=\sum_{i,j,k} t_{ijk}\alpha_{ii'}\beta_{jj'}\gamma_{kk'}.$$

Then $Rk_R(T)=Rk_R(T')$.

Proof. If $t_{ijk}=\sum_{l=1}^{r}u_{li}v_{lj}w_{lk}$, then $t'_{ijk}=\sum_{l=1}^{r}(uA)_{li}(vB)_{lj}(w\Gamma)_{lk}$. If $t'_{ijk}=\sum_{l=1}^{r}u_{li}v_{lj}w_{lk}$, then $t_{ijk}=\sum_{l=1}^{r}(uA^{-1})_{li}(vB^{-1})_{lj}(w\Gamma^{-1})_{lk}$. ∎

Tensor rank is a function whose domain consists of pairs (R,T) where R is a ring and T is an $m\times n\times p$ array of elements from R. This array can have any number of dimensions, but the three-dimensional case is most interesting because of the connection with bilinear forms. Previous work on tensor rank has developed techniques for obtaining upper and lower bounds on the value of $Rk_R(T)$ for specific R and T. This can be viewed as trying to evaluate the tensor rank function at a particular point (R,T) in its domain.

A new approach to the study of tensor rank consists of investigating properties of the tensor rank function other than its values at individual points. The results reported here are of two types. The first type involves the dependence of tensor rank on the ring R. The second type concerns the dependence of tensor rank on the dimensions of T. In particular, bounds on the function $B(R^{m\times n\times p}) = \max_{T\in R^{m\times n\times p}}\{Rk_R(T)\}$ are determined.

2. DEPENDENCE ON RING OF CONSTANTS

Two simple observations can be made concerning the dependence of $Rk_R(T)$ on R.

LEMMA 2. *If R is a subring of S, then $Rk_R(T) \geqslant Rk_S(T)$.*

LEMMA 3. *If S is the image of R under a homomorphism ϕ, then $Rk_R(T) \geqslant Rk_S(\phi(T))$, $\phi(T)$ performed elementwise.*

The proofs of Lemmas 2 and 3 follow directly from the definition of tensor rank.

Lemma 3 has several applications. Let Z_q be the ring of integers modulo q. One can often take advantage of finiteness to compute $Rk_{Z_q}(\phi(T))$ for an integer array T, where $\phi(n) = n \bmod q$. By Lemma 3, this yields a lower bound for $Rk_Z(T)$, the tensor rank of T over the integers. Hopcroft and Kerr used this technique to obtain lower bounds on the complexity of matrix multiplication using integer constants [6].

Another application of Lemma 3 is in proving a theorem which sets the integers apart from other rings of constants used to evaluate sets of bilinear forms. Every commutative ring with identity R contains a homomorphic image of the integers. The homomorphism ϕ_R maps $n \in Z$ to $\sum_{i=1}^{n} 1_R \in R$, where 1_R is the multiplicative identity element in R. Having made this observation, the next theorem follows from Lemmas 2 and 3.

THEOREM 1. *Let $T \in Z^{m\times n\times p}$. The tensor rank of $\phi_R(T)$ over R is maximized when R is the ring of integers, Z—i.e., $Rk_Z(T) \geqslant Rk_R(\phi_R(T))$.*

Almost every set of bilinear forms which is important for computing has integer coefficients, in fact $\{-1,0,1\}$ coefficients. If r active multiplications are sufficient to evaluate such a set using integer scalars, then Theorem 1 says that r active multiplications are sufficient using scalars from any commutative ring with identity. In a sense, the ring of integers is the worst possible ring of scalars.

If the integers are the worst possible scalars, which rings are better? The relation $Rk_R(T) < Rk_Z(T)$ means that a certain set of polynomials with integer coefficients has a solution in R but not in Z. It is easy to see how this can happen if R is a ring obtained by adjoining elements which satisfy polynomials over Z. For example, let $R = Z[\frac{1}{2}] \cong Z[x]/(2x-1)$ be the ring of polynomials in $\frac{1}{2}$ with integer coefficients. Let

$$T = \begin{array}{cc} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \end{array} \in Z^{2\times2\times2}$$

It is not difficult to show that $Rk_Z(T) = 3$, while $Rk_R(T) = 2$. The corresponding decompositions use

$$U = (u_{li}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad V = (v_{lj}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad W = (w_{lk}) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{in } Z,$$

and

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{in } R.$$

Irrational algebraic numbers can also be used to decrease tensor rank. Consider the following example.

$$T' = \begin{array}{cc} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & \end{array} \in Z^{2\times2\times2}.$$

Let $R = Z[\frac{1}{2}]$ and $S = R[\sqrt{2}\,]$. Then $Rk_R(T') = 3$ and $Rk_S(T') = 2$. The corresponding decompositions are

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{in } R \text{ (or } Z\text{)},$$

and

$$U=\begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix}, \quad V=\begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix}, \quad W=\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2}/2 \\ 1 & -\sqrt{2}/2 \end{bmatrix} \quad \text{in } S.$$

More detailed information on the effect of algebraic numbers on tensor rank is contained in [4, 8, 14].

Suppose an indeterminate x is adjoined to R. Then x satisfies no nontrivial polynomial over R, and therefore it never affects tensor rank.

THEOREM 2. *If $S=R[x]$ is the ring of polynomials in x with coefficients from R, then $Rk_S(T)=Rk_R(T)$ for all $T\in R^{m\times n\times p}$.*

Proof. By Lemma 2, $Rk_S(T)\leqslant Rk_R(T)$. Let $\phi:S\to R$ be the homomorphism whose kernel is generated by x. By Lemma 3, $Rk_R(T)\leqslant Rk_S(T)$. ∎

Starting from the integers and adjoining rational numbers, one obtains Q, the field of rational numbers. The first example above shows that this action decreases the tensor ranks of certain arrays. Extending Q by algebraic numbers yields $\bar{Q}$, the field of algebraic numbers. Again, tensor ranks decrease, as evidenced by the second example. Can this process go further? There are no algebraic extensions of $\bar{Q}$, and Theorem 2 shows that simple transcendental extensions do not decrease tensor ranks. This suggests a negative answer to the question. The next theorem proves that no array with elements from $\bar{Q}$ has lower tensor rank in an extension R of $\bar{Q}$ than it has in $\bar{Q}$.

THEOREM 3. *Let K be an algebraically closed field. Let T be an element of $K^{m\times n\times p}$, and let R be a ring containing K. Then $Rk_R(T)=Rk_K(T)$.*

Proof. By Lemma 2, $Rk_R(T)\leqslant Rk_K(T)$. Consider the system of mnp polynomials in the u_{li}, v_{lj}, w_{lk}.

$$p_{ijk}(u_{li},v_{lj},w_{lk})=t_{ijk}-\sum_{l=1}^{r}u_{li}v_{lj}w_{lk}. \tag{1}$$

Let $x_1,\dots,x_\xi$ be the u's, v's, and w's, where $\xi=r(m+n+p)$. If $Rk_K(T)>r$, the p_{ijk} have no common zero in K. By Hilbert's *Nullstellensatz*, the p_{ijk} have no common zero in K if and only if the ideal generated by them in $K[x_1,\dots,x_\xi]$ contains 1 [13]. But then the ideal generated by the p_{ijk} in

$R[x_1,\ldots,x_\xi]$ also contains 1, so the p_{ijk} have no common zero in R, either. Therefore $Rk_K(T) \leqslant Rk_R(T)$, and the proof is complete. ■

One implication of Theorem 3 is that if a set of bilinear forms with algebraic coefficients can be evaluated with r active multiplications using complex scalars, then the same job can be done using algebraic scalars. One need not worry about possible improvements using nonalgebraic complex numbers. This fact leads one to conjecture that a similar theorem might be true for the real numbers. Such a theorem is true, and its proof rests on the following real version of the *Nullstellensatz.*

THEOREM 4 (Real *Nullstellensatz*). *Let Q be the rational numbers, and let $\hat{Q}$ be its real algebraic closure. Let $I \subseteq Q[x_1,\ldots,x_n]$ be an ideal. Then the elements of I have a common zero in $\hat{Q}$ if and only if I does not contain an element of the form*

$$1+\sum a_i f_i^2, \qquad 0<a_i\in Q, \quad f_i\in Q[x_1,\ldots,x_n].$$

Proof. See [11]. ■

The real version of Theorem 3 follows from the real *Nullstellensatz* in the same way that Theorem 3 follows from Hilbert's *Nullstellensatz.*

THEOREM 5. *Let $\hat{Q}$ be the real algebraic closure of Q. Let T be an element of $\hat{Q}^{m\times n\times p}$, and let $\mathbf{R}$ be the real numbers. Then $Rk_{\mathbf{R}}(T) = Rk_{\hat{Q}}(T)$.*

Proof. By Lemma 2, $Rk_{\mathbf{R}}(T) \leqslant Rk_{\hat{Q}}(T)$. Let $x_1,\ldots,x_\xi$ be the u's, v's, and w's, and let $I \subseteq \hat{Q}[x_1,\ldots,x_\xi]$ be the ideal generated by the p_{ijk}'s defined in (1). If $Rk_{\hat{Q}}(T) > r$, I must contain an element of the form $g = 1+\sum a_i f_i^2$ by Theorem 4. Any common zero of the p_{ijk}'s in $\mathbf{R}$ would also be a zero of g. Since $g>0$ for $x_1,\ldots,x_\xi \in \mathbf{R}$, $Rk_{\mathbf{R}}(T) > r$, and the proof is complete. ■

The real algebraic closure of Q is $\hat{Q} = \overline{Q} \cap \mathbf{R}$, the real algebraic numbers. The interpretation of Theorem 5 is that any set of bilinear forms which can be evaluated with r active multiplications using real scalars can also be evaluated using real algebraic scalars and the same number of active multiplications. Transcendental numbers such as π and e do not help. Theorems 4 and 5 can be generalized by making the following substitutions: $Q \leftarrow F =$ any ordered field, $\hat{Q} \leftarrow \hat{F} =$ the real algebraic closure of F, $\mathbf{R} \leftarrow R =$ any ordered ring containing $\hat{F}$.

This completes our discussion of the dependence of tensor rank on the ring of scalars. The main conclusions of this section are summarized as follows. The tensor rank of an integer array is maximized in the integers. It is minimized over ordered rings in the real algebraic numbers and over all rings in the algebraic numbers.

3. MAXIMUM VALUES OF TENSOR RANK

Let $B(R^{m\times n\times p})$ be the maximum tensor rank of any array in $R^{m\times n\times p}$. This function is useful in studying the dependence of tensor rank on the array dimensions, m, n, and p. Trivial bounds on $B(R^{m\times n\times p})$ are given by the following lemmas. Since tensor rank is unchanged by permutations of subscripts, we will always assume $m\leqslant n\leqslant p$, without loss of generality.

LEMMA 4. $B(R^{m\times n\times p})\leqslant mn$.

Proof. Let $T\in R^{m\times n\times p}$. Then $t_{ijk}=\sum_{l=1}^{mn}u_{li}v_{lj}w_{lk}$, where $u_{li}=\delta_{\alpha i}$, $v_{lj}=\delta_{\beta j}$, $w_{lk}=t_{\alpha\beta k}$ whenever $l=(\alpha-1)n+\beta$, and where $1\leqslant\alpha\leqslant m$, $1\leqslant\beta\leqslant n$, and δ is the Kronecker delta. ■

LEMMA 5.

(i) $B(R^{m\times n\times p})\geqslant p$ *if* $p<mn$.
(ii) $B(R^{m\times n\times p})=mn$ *if* $p\geqslant mn$.

Proof. Let the k-planes (k-plane = matrix of t_{ijk}'s with a fixed value of k) of T be chosen from the set $E_{ij}\subseteq R^{m\times n}$ of matrices with 1 in the (i,j) position and zeros elsewhere. The tensor rank of T is then the number of different k-planes it contains. This number can be made to be $\min\{mn,p\}$. The other half of (ii) comes from Lemma 4. ■

A better upper bound on $B(R^{m\times n\times p})$ can be obtained by a generalization of Gaussian elimination. Gaussian elimination on an $m\times n$ matrix A can be described as follows. Let $r^t=$(row 1 of A) and $c=$(column 1 of A)$/a_{11}$. Then cr^t is a rank-one matrix which "matches" A in the first row and column:

$$A=cr^t+\left[\begin{array}{c|c}0&0\\\hline 0&A^{(1)}\end{array}\right],$$

where $A^{(1)}$ is an $(m-1)\times(n-1)$ matrix. Repeating the process on $A^{(1)}$ eventually leads to a decomposition of the form

$$A=(a_{ij})=\sum_{l=1}^{m} c_{li}r_{lj}.$$

This process required division of some elements of A by others. Care must be taken to ensure that this division can be done in R.

THEOREM 6. *Let R be a principal ideal domain (PID).*

(1) $B(R^{m\times n\times p})\leqslant p+B(R^{(m-1)\times(n-1)\times p})$.
(2) $B(R^{m\times n\times p})\leqslant n+B(R^{(m-1)\times n\times(p-1)})$.
(3) $B(R^{m\times n\times p})\leqslant m+B(R^{m\times(n-1)\times(p-1)})$.

Proof. We prove only (1). Proofs of (2) and (3) are similar. Let M be an array in $R^{m\times n\times p}$. The first step is to find N such that $Rk_R(N)=Rk_R(M)$ and that n_{11k} divides every entry of N for $1\leqslant k\leqslant p$. Let g be the greatest common divisor of the entries of M. We begin by showing that there is an invertible change of coordinates (see Lemma 1) relating M to some M' whose $(1,1,1)$ entry is g. If the ideal (m_{111}) contains every entry of M, there is nothing to prove. If $m_{ijk}\notin(m_{111})$, one can construct an invertible change of coordinates (A,B,Γ) relating M to $M^{(1)}$, where $(m^{(1)}_{111})$ properly contains (m_{111}).

The matrices A,B,Γ are constructed by composing several simple matrices. An identity matrix with one nonzero off-diagonal element added is called an "elementary plane operator" because it adds a multiple of one plane of M to another. An identity matrix in which a principal submatrix is modified to be $\begin{bmatrix} a & c \\ b & d\end{bmatrix}$ is called a "gcd producer for x and y" if a and b are chosen so that $xa+yb=\gcd(x,y)$, and c and d are chosen so that $ad-bc=1=\gcd(a,b)$. Note that $\begin{bmatrix} a & c \\ b & d\end{bmatrix}$ is invertible and that

$$[x\quad y]\begin{bmatrix} a & c \\ b & d\end{bmatrix}=[\gcd(x,y)\quad xc+yd].$$

If $m_{ijk}\notin(m_{111})$ can be chosen on a "line" containing m_{111} (i.e., 2 of the 3 coordinates i,j,k are 1), then A, B, and Γ may be chosen to be two identity matrices and a gcd producer for m_{111} and m_{ijk}. If m_{ijk} can be chosen in a plane containing m_{111} (i.e., one of i,j,k is 1), then two elementary plane operations can be used to reduce this case to the first case. If none of i,j,k can be chosen to be 1, two more elementary plane operations reduce this case to the second case.

If $(m^{(1)}_{111})$ contains every entry of $M^{(1)}$, the process terminates. Otherwise it repeats on $M^{(1)}$. It must eventually terminate because an ideal in a PID can be properly extended only finitely many times. Let this final array be M'.

Another invertible change of coordinates adds multiples of the front plane to every other plane, causing the $(1,1,k)$-elements to become g for $1 \leqslant k \leqslant p$. Let this array be N; $Rk(N) = Rk(M)$ by Lemma 1, since they are related by an invertible change of coordinates. The procedure for obtaining M' is constructive as long as for every pair x, y of elements of R, another pair a, b can be constructed such that $xa + yb$ is the greatest common divisor of x and y. This condition holds in the integers, for example.

Perform one step of Gaussian elimination on each plane of N. The required divisions can be done, because g divides every element of N. The result is

$$N = (n_{ijk}) = \sum_{l=1}^{p} c_{li} r_{lj} e_{lk} + \bar{n}^{(1)}_{ijk},$$

where $E = (e_{lk})$ is a $p \times p$ identity matrix, and

$$\bar{N}^{(1)} = \left(\bar{n}^{(1)}_{ijk}\right) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & N_1^{(1)} \end{array}\right] \cdots \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & N_p^{(1)} \end{array}\right].$$

Let $N^{(1)} \in R^{(m-1)\times(n-1)\times p}$ be the array formed by dropping the first row and column of each plane of $\bar{N}^{(1)}$. Obviously, $Rk_R(N^{(1)}) = Rk_R(\bar{N}^{(1)})$. Therefore, $Rk_R(M) = Rk_R(N) \leqslant p + Rk_R(\bar{N}^{(1)}) = p + Rk_R(N^{(1)}) \leqslant p + B(R^{(m-1)\times(n-1)\times p})$. This holds for all $M \in R^{m\times n\times p}$, so $B(R^{m\times n\times p}) \leqslant p + B(R^{(m-1)\times(n-1)\times p})$. ■

THEOREM 7. *Let R be a PID. Then*[1]

$$B(R^{n\times n\times n}) \leqslant \left\lceil \tfrac{3}{4} n^2 \right\rceil.$$

[1]The symbols $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the least integer greater than or equal to x and the greatest integer less than or equal to x, respectively.

Proof. Apply part (1) of Theorem 6 $\lceil n/2\rceil$ times. $B(R^{n\times n\times n})\leqslant n\lceil n/2\rceil + B(R^{\lfloor n/2\rfloor\times\lfloor n/2\rfloor\times n})$. By Lemma 4, $B(R^{\lfloor n/2\rfloor\times\lfloor n/2\rfloor\times n})\leqslant\lfloor n/2\rfloor^2$. Therefore, $B(R^{n\times n\times n})\leqslant n\lceil n/2\rceil+\lfloor n/2\rfloor^2=\lceil\frac{3}{4}n^2\rceil$. ∎

Theorem 7 gives an upper bound on the tensor rank of $n\times n\times n$ arrays over a PID. A theorem of Dobkin [2] gives the same bound when R is the field of complex numbers and n is even.

Theorem 6 has the following generalization for d-dimensional arrays.

THEOREM 8.

$$B\left(R^{n_1\times\cdots\times n_d}\right)\leqslant\frac{\prod_{i=1}^{d} n_i}{n_j n_k}+B\left(R^{n_1\times\cdots\times(n_j-1)\times\cdots\times(n_k-1)\times\cdots\times n_d}\right).$$

Proof. Similar to Theorem 6. ∎

Theorem 8 can be used to prove the following d-dimensional analog of Theorem 7.

THEOREM 9. *Let R be a PID, and let $R^{n\times\cdots\times n}$ denote he set of all d-dimensional arrays of size $n\times\cdots\times n$. Then*

$$\lim_{n\to\infty}\frac{B\left(R^{n\times\cdots\times n}\right)}{n^{d-1}}\leqslant\frac{1}{2}+\frac{1}{2(d-1)}.$$

Proof. Apply Theorem 8 d times with j and k taking the values $j_i=2i-1 \bmod d$, and $k_i=2i \bmod d$, for $1\leqslant i\leqslant d$. The result is

$$B\left(R^{n\times\cdots\times n}\right)\leqslant dn^{d-2}+B\left(R^{(n-2)\times\cdots\times(n-2)}\right).$$

The solution to this recurrence with appropriate initial values is of the form

$$B\left(R^{n\times\cdots\times n}\right)\leqslant\frac{d}{2(d-1)}n^{d-1}+(\text{lower-order terms}).$$

The theorem follows after dividing this equation by n^{d-1} and taking the limit as $n\to\infty$. ∎

Lower bounds for $B(R^{m\times n\times p})$ can be obtained by counting arguments when R is finite.

THEOREM 10. *Let R be a finite commutative ring with identity. Then $B(R^{m\times n\times p}) \geqslant mnp/(m+n+p)$. If, in addition, R is a field with q elements, then $B(R^{m\times n\times p}) \geqslant mnp/[m+n+p-2\log_q(q-1)]$.*

Proof. There are q^{mnp} elements in $R^{m\times n\times p}$. The number of these whose tensor rank is one is at most q^{m+n+p}, because any such element has a decomposition $T=(t_{ijk})=u_i v_j w_k$ for some $u\in R^m$, $v\in R^n$, $w\in R^p$. The number of elements of tensor rank not exceeding k is at most $q^{k(m+n+p)}$, which implies $B(R^{m\times n\times p})(m+n+p) \geqslant mnp$.

Let R be a field. If the tensor rank of T is one, $T=(t_{ijk})=u_i v_j w_k = (\alpha\beta u_i)(\alpha^{-1}v_j)(\beta^{-1}w_k)$. As α and β range over the $q-1$ nonzero elements of R, $(q-1)^2$ different decompositions of T are produced. The number of different arrays of tensor rank one is at most $q^{m+n+p}/(q-1)^2$. The number of different arrays of tensor rank not exceeding k is at most $q^{k(m+n+p)}/(q-1)^{2k}$. Therefore,

$$\left[q^{m+n+p}(q-1)^{-2}\right]^{B(R^{m\times n\times p})} \geqslant q^{mnp},$$

and

$$B(R^{m\times n\times p}) \geqslant \frac{mnp}{m+n+p-2\log_q(q-1)}. \qquad \blacksquare$$

It can easily be shown using the mean-value theorem that $\log_q(q-1)$ behaves roughly as $1-(q\log q)^{-1}$ as q gets large. In fact, $\log_q(q-1)=1-(\theta\log q)^{-1}$ for some θ between $q-1$ and q. Thus, the second lower bound for $B(R^{m\times n\times p})$ quickly approximates $mnp/(m+n+p-2)$ as q increases. The advantage of this bound over the first lower bound is illustrated by the following example. Let R be a finite field with at least four elements. The first bound gives $B(R^{4\times 4\times 4}) \geqslant 6$, while the second gives $B(R^{4\times 4\times 4}) \geqslant 7$.

Lower bounds for $B(R^{m\times n\times p})$ similar to those of Theorem 10 can be obtained when R is infinite.

THEOREM 11. *Let R be an infinite commutative ring with identity. Then $B(R^{m\times n\times p}) \geqslant mnp/(m+n+p)$.*

Proof. Let $r = B(R^{m\times n\times p})$. Let $s = m+n+p$. Consider the following system of mnp polynomials in rs variables:

$$f_{ijk}(u_{li}, v_{lj}, w_{lk}) = \sum_{l=1}^{r} u_{li} v_{lj} w_{lk}, \tag{2}$$

$1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant p$. For convenience, let the variables also be called $x_1, \dots, x_{rs}$.

If $rs < mnp$, there are more polynomials than variables, so the polynomials are algebraically dependent. Hence, one can construct a nontrivial polynomial $Q(y_1, \dots, y_{mnp})$ such that

$$Q(f_{111}(x_1, \dots, x_{rs}), \dots, f_{mnp}(x_1, \dots, x_{rs})) = 0$$

as a polynomial in $x_1, \dots, x_{rs}$. Since R is infinite and Q is nontrivial, some mnp-tuple of values $(t_{111}, \dots, t_{mnp})$ from R satisfies $Q(t_{111}, \dots, t_{mnp}) \neq 0$. Consider the array $T = (t_{ijk}) \in R^{m\times n\times p}$. By the definition of r, $Rk_R(T) \leqslant r$. Therefore $t_{ijk} = f_{ijk}(x_1, \dots, x_{rs})$ for some values of $x_1, \dots, x_{rs}$, by the definition of tensor rank and f_{ijk}. Hence $Q(t_{111}, \dots, t_{mnp}) = 0$ contradicting the choice of T. The conclusion is that $r \geqslant mnp/s$. ∎

A special case of Theorem 11 says that $B(R^{n\times n\times n}) \geqslant \frac{1}{3}n^2$. The theorem can be strengthened if R is a field.

THEOREM 12. *Let R be an infinite field. Then $B(R^{m\times n\times p}) \geqslant mnp/(m+n+p-2)$.*

Proof. Let $r = B(R^{m\times n\times p})$, and $s = m+n+p$. Let f_{ijk} and $x_1, \dots, x_{rs}$ be as in the proof of Theorem 11. Let $\eta = (\eta_1, \dots, \eta_r)$ and $\kappa = (\kappa_1, \dots, \kappa_r)$ be sequences of r integers, $1 \leqslant \eta_l \leqslant n$, $1 \leqslant \kappa_l \leqslant p$. Let $\xi = r(s-2)$. For each pair (η, κ), form a new set of polynomials $f_{ijk}^{(\eta,\kappa)}(\bar{x}_1, \dots, \bar{x}_\xi)$ from $f_{ijk}(x_1, \dots, x_{rs})$ by replacing u_{li} with $\bar{u}_{li}$, v_{lj} with $\bar{v}_{lj}$ for $j \neq \eta_l$, $v_{l\eta_l}$ with 1, w_{lk} with $\bar{w}_{lk}$ for $k \neq \kappa_l$, and $w_{l\kappa_l}$ with 1 in (2). As before, let the $\bar{x}$'s be new names for the $\bar{u}$'s, $\bar{v}$'s, and $\bar{w}$'s.

Suppose $\xi < mnp$. Then for each pair (η, κ) there are more polynomials than variables in the set $f_{ijk}^{(\eta,\kappa)}(\bar{x}_1, \dots, \bar{x}_\xi)$. As in the proof of Theorem 11, construct for each (η, κ) a nontrivial polynomial $Q^{(\eta,\kappa)}(y_1, \dots, y_{mnp})$ such that $Q^{(\eta,\kappa)}(f_{111}^{(\eta,\kappa)}(\bar{x}_1, \dots, \bar{x}_\xi), \dots, f_{mnp}^{(\eta,\kappa)}(\bar{x}_1, \dots, \bar{x}_\xi)) = 0$ as a polynomial in the $\bar{x}$'s. Let $Q(y_1, \dots, y_{mnp})$ be the product of all the $Q^{(\eta,\kappa)}$'s, and choose $T = (t_{ijk})$ such that $Q(t_{111}, \dots, t_{mnp}) \neq 0$.

Since $Rk_R(T) \leqslant r$, $t_{ijk} = f_{ijk}(x_1, \dots, x_{rs})$ for all i, j, k and some choice of u_{li}, v_{lj}, and w_{lk} (also called $x_1, \dots, x_{rs}$) from R. We may assume each set $\{v_{lj} | 1 \leqslant j \leqslant n\}$ and $\{w_{lk} | 1 \leqslant k \leqslant p\}$ contains a nonzero element (otherwise, set $u_{li} = 0$ for $1 \leqslant i \leqslant m$, and $v_{l1} = w_{l1} = 1$). Choose one nonzero element from each set:

$v_{l\eta_l} = \nu_l \neq 0$, $w_{l\kappa_l} = \omega_l \neq 0$. This defines η_l, κ_l, ν_l, and ω_l. Set $\bar{u}_{li} = \nu_l \omega_l u_{li}$, $\bar{v}_{lj} = v_{lj}/\nu_l$, and $\bar{w}_{lk} = w_{lk}/\omega_l$. For these values of η, κ, and the $\bar{x}_i$,

$$f_{ijk}^{(\eta,\kappa)}(\bar{x}_1,\dots,\bar{x}_\xi) = f_{ijk}(x_1,\dots,x_{rs}) = t_{ijk}.$$

Hence $Q(t_{111},\dots,t_{mnp}) = Q\big(f_{111}^{(\eta,\kappa)}(\bar{x}_1,\dots,\bar{x}_\xi),\dots,f_{mnp}^{(\eta,\kappa)}(\bar{x}_1,\dots,\bar{x}_\xi)\big) = 0$, contradicting the choice of T. The conclusion is that $\xi \geqslant mnp$, so $r \geqslant mnp/(s-2)$. ∎

Generalizations of Theorems 10, 11 and 12 to d dimensions can be proven with no additional difficulties.

THEOREM 13. *Let $B(R^{n_1\times\cdots\times n_d})$ be the maximum tensor rank of any array in $R^{n_1\times\cdots\times n_d}$. Let $P = \prod_{i=1}^d n_i$ and $S = \sum_{i=1}^d n_i$. Then*

(i) $B(R^{n_1\times\cdots\times n_d}) \geqslant P/S$ *if R is a commutative ring with identity;*

(ii) $B(R^{n_1\times\cdots\times n_d}) \geqslant P/[S-(d-1)\log_q(q-1)]$ *if R is a finite field with q elements;*

(iii) $B(R^{n_1\times\cdots\times n_d}) \geqslant P/(S-d+1)$ *if R is an infinite field.*

The value of $B(R^{n\times n\times n})$ is determined by Theorems 7, 10, and 12 to within a constant factor: $\frac{1}{3}n^2 \leqslant B(R^{n\times n\times n}) \leqslant \lceil \frac{3}{4}n^2 \rceil$ when R is a PID, and $\frac{1}{3}n^2 \leqslant B(R^{n\times n\times n}) \leqslant n^2$ in any case. For d dimensions,

$$\frac{1}{d}n^{d-1} \leqslant B(R^{n\times\cdots\times n}) \leqslant \frac{1}{2}\left(\frac{d}{d-1}\right)n^{d-1} + (\text{lower-order terms})$$

when R is a PID, and

$$\frac{1}{d}n^{d-1} \leqslant B(R^{n\times\cdots\times n}) \leqslant n^{d-1}$$

in any case. The asymptotic rate of growth of $B(R^{n\times\cdots\times n})$ is determined, but the constant is still unknown.

4. CONCLUSION

The main results of this paper are summarized by the following relations.

(1) $Rk_{\mathbf{Z}}(T) \geqslant Rk_R(T)$.
(2) $Rk_R(T) = Rk_{R[x]}(T)$.
(3) $Rk_{\mathbf{Z}}(T) \geqslant Rk_{\mathbf{Z}[\alpha]}(T) \geqslant Rk_{\mathbf{Z}[\alpha,\beta]}(T) \geqslant \cdots \geqslant Rk_{\bar{Q}}(T)$ for $\alpha, \beta \in \bar{Q}$.
(4) $Rk_{\bar{Q}}(T) = Rk_R(T)$ if $\bar{Q} \subseteq R$.
(5) $Rk_{\bar{Q}\cap\mathbf{R}}(T) = Rk_{\mathbf{R}}(T)$, $\mathbf{R}$ = real numbers.

(6) $B(R^{m\times n\times p}) \leqslant \min\{m+B(R^{m\times(n-1)\times(p-1)}),\ n+B(R^{(m-1)\times n\times(p-1)}),\ p+B(R^{(m-1)\times(n-1)\times p})\}$ for R a PID.

(7) $B(R^{m\times n\times p}) \geqslant mnp/(m+n+p)$.

(8) $B(F^{m\times n\times p}) \geqslant mnp/(m+n+p-2)$ for F an infinite field.

(9) $B(F^{m\times n\times p}) \geqslant mnp/[m+n+p-2\log_q(q-1)]$ for F a finite field with q elements.

(10)

$$\frac{1}{d}n^{d-1} \leqslant B(R^{n\times\cdots\times n}) \leqslant \frac{1}{2}\left(\frac{d}{d-1}\right)n^{d-1}+o(n^{d-1})$$

for R a PID and d = number of dimensions.

The following are some open problems related to the properties of tensor rank discussed here.

(1) It has been shown that only algebraic numbers are needed to produce a minimum-rank decomposition of an integer array. Each algebraic number used in such a decomposition can be described by its irreducible polynomial over the integers. Give bounds on the degrees of these polynomials and on the sizes of their coefficients.

(2) How much can the tensor rank decrease when complex scalars are used instead of reals? How about real scalars instead of rationals?

(3) It has been shown that $B(R^{n\times n\times n}) = O(n^2)$. What is the constant? Conjecture: At least for algebraically closed fields R, $\lim_{n\to\infty} B(R^{n\times n\times n})/n^2 = \frac{1}{3}$, i.e., the lower bound is essentially correct.

(4) Theorems 10–12 imply the existence of sets of n bilinear forms in $x_1,\dots,x_n$ and $y_1,\dots,y_n$ which require $O(n^2)$ active multiplications to evaluate. Current lower-bound techniques are limited to proving lower bounds of only $O(n)$ active multiplications. Find families of examples requiring $O(n^2)$ multiplications and/or techniques for proving nonlinear lower bounds on the complexities of sets of bilinear forms.

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