Multivalued Quasi-Variational Inclusions and Multivalued Accretive Equations

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Abstract—The purpose of this paper is to introduce and study a class of more general multivalued quasi-variational inclusions in Banach spaces. By using the accretive operator technique, we establish the equivalence between these kinds of multivalued quasi-variational inclusions and a kind of multivalued accretive equations in Banach space. Then, invoking Michael's selection theorem and Nadler's theorem, some existence theorems and iterative algorithms for solving this kind of multivalued quasi-variational inclusions in Banach spaces are established and suggested. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. Useful and important generalizations of variational inequalities are variational inclusions.

Recently, in [1-4], Noor introduced and studied the following class of important variational inclusions.

Let $H$ be a real Hilbert space, $C(H)$ be a family of all nonempty compact subsets of $H$. Let $T, V : H \rightarrow C(H)$ be the multivalued mappings and $g : H \rightarrow H$ be a single-valued mapping.
Let $A(\cdot, \cdot) : H \times H \rightarrow H$ be a maximal monotone mapping with respect to the first argument. For a given nonlinear mapping $N(\cdot, \cdot) : H \times H \rightarrow H$ consider the problem of finding $u \in H$, $w \in T(u)$, $y \in V(u)$, such that

$$\theta \in N(u, y) + A(g(u), u),$$

which is called the multivalued quasi-variational inclusion problem. By using the resolvent operator technique, some existence theorem of solution and iterative approximation theorems for solving these kinds of variational inclusions are established.

The purpose of this paper is to introduce and study a class of more general multivalued quasi-variational inclusions in Banach spaces without compactness conditions. By using the accretive operator technique, we establish the equivalence between these kinds of multivalued quasi-variational inclusions and certain multivalued accretive equations in Banach space. Then, invoking Michael’s selection theorem [5] and Nadler’s theorem [6], some existence theorems and iterative algorithms for solving these kinds of multivalued quasi-variational inclusions in Banach spaces are established and suggested. The results presented in this paper generalize, improve, and unify the corresponding results of Noor [1–4], Ding [7], Jung and Morales [8], Zeng [9], and Chang, Kim and Cho [10–12].

2. PRELIMINARIES

Throughout this paper, we assume that $E$ is a real Banach space, $E^*$ is the topological dual space of $E$, $CB(E)$ is the family of all nonempty closed and bounded subset of $E$, $\langle \cdot, \cdot \rangle$ is the dual pair between $E$ and $E^*$, $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \quad A, B \in CB(E).$$

$D(T)$ and $R(T)$ denote the domain and range of $T$, respectively, $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \| \cdot \| f \| = \| x \| \}, \quad x \in E.$$

Remark 2.1. If $E$ is a uniformly smooth Banach space, then, the normalized duality mapping $J$ is single-valued and uniformly continuous on any bounded subset of $E$. (See [13].)

Definition 2.1. Let $A : D(A) \subset E \rightarrow 2^E$ be a set-valued mapping and $\phi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing function with $\phi(0) = 0$.

1. Mapping $A$ is said to be accretive if, for any $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$, such that

$$\langle u - v, j(x - y) \rangle \geq 0,$$

for all $u \in Ax$, $v \in Ay$.

2. Mapping $A$ is said to be $\phi$-strongly accretive if, for any $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$, such that, for any $u \in Ax$, $v \in Ay$,

$$\langle u - v, j(x - y) \rangle \geq \phi(\| x - y \|) \| x - y \|.$$

3. Mapping $A$ is said to be $\phi$-strongly pseudo-contractive if, for any given $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$, such that, for any $u \in Ax$, $v \in Ay$,

$$\langle u - v, j(x - y) \rangle \leq (\| x - y \| - \phi(\| x - y \|)) \| x - y \|.$$

4. Mapping $A$ is said to be $m$-accretive, if $A$ is accretive and $(I + \rho A)(D(A)) = E$, for all $\rho > 0$, where $I$ is the identity mapping.

5. Mapping $A$ is said to be $\phi$-expansive if, for any $x, y \in D(A)$ and any $u \in Ax$, $v \in Ay$,

$$\| u - v \| \geq \phi(\| x - y \|).$$
REMARK 2.2.

(1) It is easy to see that, if \( A \) is \( \phi \)-strongly accretive, then, \( A \) is \( \phi \)-expansive.

(2) If \( A : D(A) \subset E \to E \) is \( \phi \)-strongly accretive, then, \( I - A : D(A) \to E \) is \( \phi \)-strongly pseudo-contractive.

DEFINITION 2.2. Let \( T, V, G, Z : E \to CB(E) \) be four set-valued mappings, \( A(\cdot, \cdot) \) and \( N(\cdot, \cdot) : E \times E \to E \) be two nonlinear mappings. For any given \( f \in E \) and \( \lambda > 0 \), we consider the following problem.

Find \( q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q) \), such that

\[
f \in \left\{ \begin{array}{lr} N(w, v) + \lambda A(g, z) \end{array} \right. \tag{2.1}
\]

This problem is called the general multivalued quasi-variational inclusion problem in Banach spaces.

A number of problems arising in pure and applied sciences can be reduced to the study of this kind of variational inclusions (see, for example, [14,15]).

Next, we consider some special cases of problem (2.1).

(1) If \( G = g : E \to E \) is a single-valued mapping and \( A(g(u), y) = A(g(u)) \), for all \( y \in E \), then, problem (2.1) is equivalent to finding \( q \in E, w \in T(q), v \in V(q) \), such that

\[
f \in \left\{ \begin{array}{lr} N(w, v) + \lambda A(g(q)) \end{array} \right. \tag{2.2}
\]

This problem is called the set-valued variational inclusion in Banach space which was introduced and studied in Chang, Kim and Cho [10–12].

(2) If \( E = H \) is a Hilbert space, \( A(\cdot, \cdot) : H \times H \to H \) is a maximal monotone mapping with respect to the first argument, \( G = g : E \to E \) a single-valued mapping and \( Z = I \) (the identity mapping), then, the problem (2.1) is equivalent to finding \( q \in H, w \in T(q), v \in V(q) \), such that

\[
\theta \in \left\{ \begin{array}{lr} N(w, v) + A(g(q), q) \end{array} \right. \tag{2.3}
\]

This problem is called multivalued quasi-variational inclusion problem in Hilbert space which was introduced and studied in Noor [1–4], by using the resolvent equation technique and under some additional conditions.

(3) If \( E = H \), a Hilbert space and \( A(\cdot, u) = \partial \phi(\cdot, u) \), the subdifferential of \( \phi(\cdot, u) \), where \( \phi(\cdot, u) : H \times H \to R \cup \{+\infty\} \) is a proper convex lower semicontinuous functional with respect to the first argument, then, the problem (2.3) is equivalent to finding \( u \in H, w \in T(u), v \in V(u) \), such that

\[
\langle N(w, v), y - g(u) \rangle + \phi(y, u) - \phi(g(u), v) \geq 0, \quad \text{for all } y \in H. \tag{2.4}
\]

This problem is called the set-valued mixed quasi-variational inequality, which was introduced and studied by Noor [16].

(4) If \( G = I, V = 0, T = I, S : E \to E \) is a single-valued mapping and \( N(x, y) = S(x) \), for all \( (x, y) \in E \times E \), then, problem (2.1) is equivalent to finding \( q \in E \), such that

\[
f \in S(q) + \lambda Aq. \tag{2.5}
\]

This problem was introduced and studied in [8,17].

(5) If the function \( \phi(\cdot, \cdot) \) is the indicator function of a closed convex-valued set \( K(u) \) in \( H \), that is,

\[
\phi(u, u) = \begin{cases} 0, & \text{if } u \in K(u), \\ +\infty, & \text{otherwise}, \end{cases}
\]

then,

\[
f \in S(q) + \lambda Aq. \tag{2.5}
\]

This problem was introduced and studied in [8,17].
then, problem (2.4) is equivalent to finding \( u \in H, w \in T(u), v \in V(u), g(u) \in K(u), \)

\[ \langle N(w,v), x - g(u) \rangle \geq 0. \]  

(2.6)

This problem has been considered by Noor [16], using the projection method and the implicit Wiener-Hopf equation technique.

(6) If \( K^*(u) = \{ v \in H, \langle u, v \rangle \geq 0, \forall v \in K(u) \} \) is a polar cone of the convex-valued cone \( K(u) \) in \( H \), then, problem (2.6) is equivalent to finding \( u \in H, w \in T(u), v \in V(u), \) such that

\[ g(u) \in K(u), \quad N(w,v) \in K^*(u), \quad \text{and} \quad \langle N(w,v), g(u) \rangle = 0, \]  

(2.7)

which is called the multi-valued implicit complementarity problem. (See [3,4].)

The above observations show that for a suitable choice of the mappings \( T, V, G, Z, A, N, \) and space \( E, \) one can obtain a number of known and new classes of variational inequalities, variational inclusions and the corresponding optimization problems from the multi-valued quasi-variational inclusion problem (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in mathematical, physical, and engineering sciences in a general and unified framework.

Related to the multivalued quasi-variational inclusion (2.1), we now consider the following multivalued operator equation,

\[ f \in H(x) + \lambda K(x), \quad x \in E, \]  

(2.8)

where \( H, K : E \to 2^E \) are two multivalued mappings defined by

\[ H(x) = N(T(x), V(x)), \quad K(x) = A(G(x), Z(x)), \quad x \in E. \]  

(2.9)

We have the following result.

**Lemma 2.1.** For given \( f \in E, \lambda > 0, (q, w, v, g, z) \) (where \( q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q) \)) is a solution of (2.1), if and only if \( q \in E \) is a solution of (2.8).

**Proof.** If \( (q, w, v, g, z) \) is a solution of (2.1), then, we have

\[ f \in N(w,v) + \lambda A(g,z) \subset N(T(q), V(q)) + \lambda A(G(q), Z(q)) = H(q) + \lambda K(q). \]

i.e., \( q \) is a solution of (2.8).

Conversely, if \( q \in E \) is a solution of (2.8), then, we have

\[ f \in H(q) + \lambda K(q) = N(T(q), V(q)) + \lambda A(G(q), Z(q)). \]

Therefore, there exist \( w \in T(q), v \in V(q), g \in G(q), z \in Z(q) \), such that

\[ f \in N(w,v) + \lambda A(g,z), \]

i.e., \( (q, w, v, g, z) \) is a solution of (2.1).

This completes the proof of Lemma 2.1.

**Definition 2.3.** Let \( T, V : E \to 2^E \) be two set-valued mappings, \( N(\cdot, \cdot) : E \times E \to E \) be a nonlinear mapping and \( \phi : [0, \infty) \to [0, \infty) \) be a strictly increasing function with \( \phi(0) = 0 \).

(1) Mapping \( x \mapsto N(x,y) \) is said to be \( \phi \)-strongly accretive with respect to the mapping \( T \) if, for any \( x_1, x_2 \in E \), there exists \( j(x_1 - x_2) \in J(x_1 - x_2) \), such that, for any \( u_1 \in T x_1, u_2 \in T x_2 \) and, for any \( y \in E, \)

\[ \langle N(u_1, y) - N(u_2, y), j(x_1 - x_2) \rangle \geq \phi(\|x_1 - x_2\|) \|x_1 - x_2\|. \]

(2) Mapping \( y \mapsto N(x, y) \) is said to be accretive with respect to the mapping \( V \) if, for any \( y_1, y_2 \in E \), there exists \( j(y_1 - y_2) \in J(y_1 - y_2) \), such that, for any \( v_1 \in V y_1, v_2 \in V y_2 \) and, for any \( x \in E, \)

\[ \langle N(x, v_1) - N(x, v_2), j(y_1 - y_2) \rangle \geq 0. \]
DEFINITION 2.4. Let $T : E \rightarrow CB(E)$ be a set-valued mapping and $D(\cdot, \cdot)$ be the Hausdorff metric on $CB(E)$. $T$ is said to be $\xi$-Lipschitzian continuous, if, for any $x, y \in E$,

$$D(Tx, Ty) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant.

In order to prove our main results, the following lemmas will be needed.

LEMMA 2.2. (See [18].) Let $E$ be a real Banach space and $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$ the following inequality holds,

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle,$$

for all $j(x + y) \in J(x + y)$.

LEMMA 2.3. (See [12].) Let $E$ be a real smooth Banach space, $T, V : E \rightarrow 2^E$ be two set-valued mappings and $N(\cdot, \cdot) : E \times E \rightarrow E$ be a nonlinear mapping satisfying the following conditions.

1. Mapping $x \mapsto N(x, y)$ is $\phi$-strongly accretive with respect to the mapping $T$.
2. Mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping $V$.

Then, the mapping $H : E \rightarrow 2^E$ defined by,

$$Hx = N(Tx, Vx),$$

is $\phi$-strongly accretive.

LEMMA 2.4. (See [5].) Let $X$ and $Y$ be two Banach spaces, $T : X \rightarrow 2^Y$ be a lower semicontinuous mapping with nonempty closed convex values. Then, $T$ admits a continuous selection, i.e., there exists a continuous mapping $h : X \rightarrow Y$, such that $h(x) \in T(x)$, for each $x \in X$.

LEMMA 2.5. (See [12].) Let $E$ be a real uniformly smooth Banach space and $T : E \rightarrow 2^E$ be a lower semicontinuous $m$-accretive mapping. Then, the following conclusions hold.

1. $T$ admits a continuous and $m$-accretive selection.
2. In addition, if $T$ is also $\phi$-strongly accretive, then, $T$ admits a continuous, $m$-accretive and $\phi$-strongly accretive selection.

LEMMA 2.6. (See [6].) Let $E$ be a complete metric space, $T : E \rightarrow CB(E)$ be a set-valued mapping. Then, for any given $\varepsilon > 0$ and, for any given $x, y \in E$, $u \in Tx$, there exists $v \in Ty$, such that

$$d(u, v) \leq (1 + \varepsilon) D(Tx, Ty),$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$.

LEMMA 2.7. (See [19].) Let $E$ be a uniformly smooth Banach space and $A : D(A) \subseteq E \rightarrow 2^E$ be a $m$-accretive and $\phi$-expansive mapping, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. Then, $A$ is surjective.

3. EXISTENCE THEOREM OF SOLUTIONS FOR MULTIVALUED QUASI-VARIATIONAL INCLUSIONS

In this section, we shall establish an existence theorem of solutions for multivalued quasi-variational inclusion (2.1). We have the following result.
THEOREM 3.1. Let $E$ be a real uniformly smooth Banach space, $T, V, G, Z : E \rightarrow CB(E)$ be four continuous set-valued mappings, $\phi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing function with $\phi(0) = 0$, $A(\cdot, \cdot)$ and $N(\cdot, \cdot) : E \times E \rightarrow E$ be two continuous nonlinear mappings satisfying the following conditions.

(i) Mapping $x \mapsto N(x, y)$ is $\phi$-strongly accretive with respect to mapping $T$ and mapping $x \mapsto A(x, y)$ is $\phi$-strongly accretive with respect to mapping $G$.

(ii) Mapping $y \mapsto N(x, y)$ is accretive with respect to mapping $V$ and mapping $y \mapsto A(x, y)$ is accretive with respect to mapping $Z$.

Then, for any given $f \in E$, $\lambda > 0$, there exist $q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q)$, such that $(q, w, v, g, z)$ is a solution of the multivalued quasi-variational inclusion (2.1).

PROOF. It follows from Conditions (i), (ii), and Lemma 2.2, that mappings $H, K : E \rightarrow 2^E$ defined by (2.9) are $\phi$-strongly accretive. Since $N, A, T, V, G, Z$ all are continuous, and so, $H$ and $K$ are two continuous and $\phi$-strongly accretive mappings. By [20], $H$ and $K$ are $m$-accretive and $\phi$-strongly accretive mappings. By Lemma 2.4 (2), $H$ admits a continuous $\phi$-strongly accretive and $m$-accretive selection $h : E \rightarrow E$ and $K$ admits a continuous $\phi$-strongly accretive and $m$-accretive selection $k : E \rightarrow E$, such that

$$h(x) \in H(x) = N(Tx, Vx), \quad \forall x \in E;$$
$$k(x) \in K(x) = A(Gx, Zx), \quad \forall x \in E.$$

Next, we consider the following single-valued, $\phi$-strongly accretive and $m$-accretive equation,

$$f = h(x) + \lambda k(x), \quad \lambda > 0. \quad (3.1)$$

By Remark 2.2 and Lemma 2.6, for any given $f \in E$ and $\lambda > 0$, there exists a unique solution $q \in E$ of (3.1), such that

$$f = h(q) + \lambda k(q).$$

The uniqueness of $q \in E$ is a direct consequence of $\phi$-strongly accretivity of the mapping $h + \lambda k$. Since

$$h(q) + \lambda k(q) \in N(T(q), V(q)) + \lambda A(G(q), Z(q)),$$

there exist $w \in T(q), v \in V(q), g \in G(q), z \in Z(q)$, such that

$$f = N(w, v) + \lambda A(g, z).$$

This completes the proof of Theorem 3.1.

4. APPROXIMATE PROBLEM OF SOLUTIONS FOR MULTIVALUED QUASI-VARIATIONAL INCLUSION (2.1)

In Theorem 3.1, we have proved that under suitable conditions, there exists $(q, w, v, g, z)$ (where $q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q)$), which is a solution of multivalued quasi-variational inclusion (2.1). In this section, we shall study the approximate problem of solution $(q, w, v, g, z)$ of variational inclusion (2.1).

For the purpose, we first invoke Michael’s selection theorem and Nadler’s theorem to suggest the following algorithms for solving this kind of variational inclusions.

ALGORITHM. Let $\{\alpha_n\}$ be a sequence in $[0, 1], f \in E$ be any given point and $\lambda > 0$ be any given positive number. Let us define a mapping $S : E \rightarrow E$ by

$$Sx = f + x - h(x) - \lambda k(x), \quad x \in E.$$
Let \( \{u_n\} \) be any given sequence in \( E \). For any given \( x_0 \in E, w_0 \in T(x_0), v_0 \in V(x_0), g_0 \in G(x_0), z_0 \in Z(x_0) \), compute the sequence \( \{x_n\}, \{w_n\}, \{v_n\}, \{g_n\}, \{z_n\} \), by the iterative schemes, such that

\[
\begin{align*}
(i) & \quad x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Sx_n + u_n, \\
(ii) & \quad h(x_n) = N(w_n, v_n), \quad k(x_n) = A(g_n, z_n), \\
(iii) & \quad w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1})), \\
(iv) & \quad v_n \in V(x_n), \quad \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(V(x_n), V(x_{n+1})), \\
(v) & \quad g_n \in G(x_n), \quad \|g_n - g_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(G(x_n), G(x_{n+1})), \\
(vi) & \quad z_n \in Z(x_n), \quad \|z_n - z_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(Z(x_n), Z(x_{n+1})), \\
& \quad n = 0, 1, 2, \ldots.
\end{align*}
\]

The sequence \( \{x_n\} \) defined by (4.1) is called the Mann type iterative sequence with errors.

We have the following results.

**Theorem 4.1.** Let \( E \) be a real uniformly smooth Banach space, \( T, V, G, Z : E \to CB(E) \) be four set-valued mappings, satisfying the following conditions.

(iii) \( T, V, G, Z \) are \( \mu, \xi, \eta, \beta \)-Lipschitzian continuous mappings, respectively.

Let \( N(\cdot, \cdot) \) and \( A(\cdot, \cdot) : E \times E \to E \) be two continuous nonlinear mappings satisfying Conditions (i) and (ii) in Theorem 3.1. Let \( H \) and \( K \) be the \( m \)-accretive and \( \phi \)-strongly accretive mappings defined by (2.9). Let \( h(x) \) and \( k(x) : E \to E \) be the \( \phi \)-strongly accretive and \( m \)-accretive selection of \( H(x) \) and \( K(x) \), respectively. For any given \( f \in E \) and \( \lambda > 0 \), define a mapping \( S : E \to E \) by

\[
Sx = f + x - h(x) - \lambda k(x), \quad x \in E.
\]

Let \( \{\alpha_n\} \) be a sequence in \( [0, 1] \) and \( \{u_n\} \) be a given sequence in \( E \), satisfying the following conditions,

(a) \( \alpha_n \to 0 \),

(b) \( \sum_{n=0}^{\infty} \alpha_n = \infty \),

(c) \( u_n = u'_n + u''_n, \forall n \geq 0 \),

and

\[
\sum_{n=0}^{\infty} \|u'_n\| < \infty \quad \text{and} \quad \|u''_n\| = 0(\alpha_n).
\]

Then, for any given \( x_0 \in E, w_0 \in T(x_0), v_0 \in V(x_0), g_0 \in G(x_0), z_0 \in Z(x_0) \), the iterative sequence \( \{x_n\}, \{w_n\}, \{v_n\}, \{g_n\}, \{z_n\} \) defined by (4.1) converge strongly to the solution \( q, w, v, g, z \) where \( q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q) \), of multivalued quasi-variational inclusion (2.1), respectively, if and only if the sequence \( \{Sx_n\} \) is bounded.

**The Proof of Necessity of Theorem 4.1.** If the sequence \( \{x_n\} \) converges strongly to \( q \in E \), then, \( \{x_n\} \) is bounded. Since \( S \) is continuous, \( \{Sx_n\} \) is bounded.

**The Proof of Sufficiency of Theorem 4.1.** Suppose that \( \{Sx_n\} \) is bounded in \( E \). By Condition (c), \( u_n = u'_n + u''_n \) and \( \|u''_n\| = 0(\alpha_n) \forall n \geq 0 \). Hence, there exists a sequence \( \{\epsilon_n\} \) with \( \epsilon_n \geq 0 \) and \( \epsilon_n \to 0 \) (\( n \to \infty \)), such that \( \|u'_n\| = \alpha_n \epsilon_n \). Therefore, we have

\[
\|u_n\| \leq \|u'_n\| + \|u''_n\| = \|u'_n\| + \alpha_n \epsilon_n, \quad n \geq 0.
\]
Let
\[ M = \sup_{n \geq 0} \{ \|Sx_n - q\| + \epsilon_n \} + \|x_0 - q\|. \]

By induction, we can prove that
\[ \|x_n - q\| \leq M + \sum_{i=0}^{n-1} \|u'_n\|, \quad n \geq 1. \quad (4.3) \]

In fact, when \( n = 1 \), we have
\[
\|x_1 - q\| = \|(1 - \alpha_0)x_0 + \alpha_0Sx_0 + u_0 - q\|
\leq (1 - \alpha_0)\|x_0 - q\| + \alpha_0\|Sx_0 - q\| + \|u_0\|
\leq (1 - \alpha_0)\|x_0 - q\| + \alpha_0\{\|Sx_0 - q\| + \epsilon_0\} + \|u_0\|
\leq M + \|u'_1\|.
\]

Suppose that (4.3) is true for \( n = k \), \( k \geq 1 \), we prove that (4.3) is also true for \( n = k + 1 \).
In fact, we have
\[
\|x_{k+1} - q\| = \|(1 - \alpha_k)x_k + \alpha_kSx_k + u_k - q\|
\leq (1 - \alpha_k)\|x_k - q\| + \alpha_k\|Sx_k - q\| + \|u_k\|
\leq (1 - \alpha_k)\left\{ M + \sum_{i=0}^{k-1} \|u'_i\| \right\} + \alpha_k\{\|Sx_k - q\| + \epsilon_k\} + \|u_k\|
\leq (1 - \alpha_k)\left\{ M + \sum_{i=0}^{k-1} \|u'_i\| \right\} + \alpha_kM + \|u'_k\|
\leq M + \sum_{i=0}^{k} \|u'_n\|.
\]

Equation (4.3) is proved. Therefore, \( \{x_n\} \) is bounded. By the assumptions \( \{Sx_n\} \) and \( \{u_n\} \) are bounded, therefore, there exists a constant \( K > 0 \), such that
\[
\max_{n \geq 0} \{ \|x_n\| + \|q\| + \|u_n\| + \|Sx_n\| \} \leq K. \quad (4.4)
\]

By using inequalities (4.1), (4.2), (4.3), and Lemma 2.1, we have
\[
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(x_n - q) + \alpha_n(Sx_n - q) + u_n\|^2
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Sx_n - q, J(x_{n+1} - q) \rangle
+ 2 \langle u_n, J(x_{n+1} - q) \rangle
= (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Sx_n - q, J(x_n - q) \rangle
+ 2\alpha_n \langle Sx_n - q, J(x_{n+1} - q) \rangle - J(x_n - q) \rangle + 2 \langle u_n, J(x_{n+1} - q) \rangle.
\quad (4.5)
\]

Now, we consider the third term on the right side of (4.5). From (4.4) and Conditions (a) and (c), we have
\[
\|x_{n+1} - q - (x_n - q)\| = \|x_{n+1} - x_n\|
\leq \alpha_n \|Sx_n - x_n\| + \|u_n\|
\leq \alpha_n \{\|Sx_n - x_n\| + \epsilon_n\} + \|u'_n\| \to 0 \quad (n \to \infty).
\quad (4.6)
\]
In Remark 2.1, we have pointed out that, if $E$ is a uniformly smooth Banach space, then, $J$ is a single-valued and uniformly continuous on any bounded subset of $E$. Hence, $\|J(x_{n+1} - q) - J(x_n - q)\| \to 0$, as $n \to \infty$, and so,

$$\gamma_n := |\langle Sx_n - q, J(x_{n+1} - q) - J(x_n - q) \rangle|$$
$$\leq \|Sx_n - q\| \|J(x_{n+1} - q) - J(x_n - q)\|$$
$$\leq K \|J(x_{n+1} - q) - J(x_n - q)\| \to 0, \quad \text{as } n \to \infty. \quad (4.7)$$

Next, we consider the second term on the right side of (4.5).

Since $h + \lambda k$ is $\phi$-strongly accretive, by Remark 2.2, $S$ is $\phi$-strongly pseudo-contractive. Again, since $q$ is the unique solution of equation

$$f = h(x) + \lambda k(x), \quad x \in E,$$

hence, $q$ is a fixed point of $S$. Therefore, we have

$$2 \alpha_n \langle Sx_n - q, J(x_n - q) \rangle \leq 2 \alpha_n \left\{ \|x_n - q\|^2 - \phi(\|x_n - q\|) \|x_n - q\| \right\} \quad (4.8)$$

Substituting (4.7) and (4.8) into (4.5), we have

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2 \|x_n - q\|^2$$
$$+ 2 \alpha_n \left\{ \|x_n - q\|^2 - \phi(\|x_n - q\|) \|x_n - q\| \right\}$$
$$+ 2 \alpha_n \gamma_n + 2 \|u_n\| \cdot K$$
$$\leq (1 - \alpha_n)^2 \|x_n - q\|^2$$
$$+ 2 \alpha_n \left\{ \|x_n - q\|^2 - \phi(\|x_n - q\|) \|x_n - q\| \right\}$$
$$+ 2 \alpha_n \left( \gamma_n + \epsilon_n K \right) + 2 \|u_n\| \cdot K$$
$$\leq \|x_n - q\|^2 - \alpha_n \phi(\|x_n - q\|) \|x_n - q\|$$
$$- \alpha_n \left\{ \phi(\|x_n - q\|) \|x_n - q\| - \alpha_n \|x_n - q\|^2 - 2 (\gamma_n + \epsilon_n) K \right\}$$
$$+ 2 \|u_n\| \cdot K. \quad (4.9)$$

Let

$$\sigma = \inf_{n \geq 0} \|x_n - q\|. \quad (4.10)$$

Next, we prove that $\sigma = 0$.

Suppose the contract $\sigma > 0$. Hence, $\|x_n - q\| \geq \sigma > 0, \forall n \geq 0$. By the strictly increasing property of $\phi$ with $\phi(0) = 0$, we have $\phi(\sigma) > 0$. Hence, from (4.9) we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n \phi(\sigma) \|x_n - q\| + 2 \|u_n\| \cdot K. \quad (4.11)$$

Since $\alpha_n \to 0, \gamma_n \to 0, \epsilon_n \to 0$, there exists $n_0$, such that, for all $n \geq n_0$, we have

$$\phi(\sigma) \|x_n - q\|^2 - 2 (\gamma_n + \epsilon_n) K > 0.$$

Therefore, for $n \geq n_0$, we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n \phi(\|x_n - q\|) + 2 \|u_n\| \cdot K,$$

i.e.,

$$\alpha_n \phi(\|x_n - q\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2 \|u_n\| \cdot K, \quad \forall n \geq n_0.
Hence, for any \( m \geq n_0 \), we have
\[
\phi(\sigma) \alpha \sum_{n=n_0}^m \alpha_n \leq \|x_{n_0} - q\|^2 - \|x_{m+1} - q\|^2 + 2K \sum_{n=n_0}^m \|u'_n\| \\
\leq \|x_{n_0} - q\|^2 + 2K \sum_{n=n_0}^m \|u'_n\|.
\]
Letting \( m \to \infty \), we have
\[
\infty \leq \|x_{n_0} - q\|^2 + 2K \sum_{n=n_0}^\infty \|u'_n\| < \infty.
\]
This is a contradiction. Hence, we know that \( \sigma = 0 \). Therefore, there exists a subsequence \( \{x_{n_j}\} \subset \{x_n\} \), such that
\[
x_{n_j} \to q \quad (n_j \to \infty).
\]
This implies that
\[
x_{n_{j+1}} = (1 - \alpha_{n_j}) x_{n_j} + \alpha_{n_j} Sx_{n_j} + u_{n_j} \to q \quad (n_j \to \infty),
\]
and so,
\[
x_{n_{j+2}} = (1 - \alpha_{n_{j+1}}) x_{n_{j+1}} + \alpha_{n_{j+1}} Sx_{n_{j+1}} + u_{n_{j+1}} \to q \quad (n_j \to \infty).
\]
By induction, we can prove that, for all \( i \geq 0 \),
\[
x_{n_{j+i}} \to q \quad (n_j \to \infty).
\]
Therefore, we have \( x_n \to q \).

Besides, since \( T \) is \( \mu \)-Lipschitzian, \( V \) is \( \xi \)-Lipschitzian, \( G \) is \( \eta \)-Lipschitzian, and \( Z \) is \( \beta \)-Lipschitzian, it follows from (iii), (iv), (v), and (vi) in (4.1) that
\[
\|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1})) \\
\leq \mu \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\| ,
\]
\[
\|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(V(x_n), V(x_{n+1})) \\
\leq \xi \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\| ,
\]
\[
\|g_n - g_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(G(x_n), G(x_{n+1})) \\
\leq \eta \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\| ,
\]
\[
\|z_n - z_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(Z(x_n), Z(x_{n+1})) \\
\leq \beta \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\|.
\]
This implies that \( \{w_n\}, \{v_n\}, \{g_n\}, \{z_n\} \) all are Cauchy sequence. Let \( w_n \to w^*, v_n \to v^*, g_n \to g^*, z_n \to z^* \). Next, we prove that
\[
w^* = w, \quad v^* = v, \quad g^* = g, \quad \text{and} \quad z^* = z.
\]
where \((q, w, v, g, z), q \in E, w \in T(q), v \in V(q), g \in G(q), z \in Z(q)\) is the solution of multivalued quasi-variational inclusion (2.1). In fact, since
\[
d(w^*, Tq) \leq d(w^*, w_n) + d(w_n, Tq) \\
\leq d(w^*, w_n) + D(Tx_n, Tq) \\
\leq d(w^*, w_n) + \mu \|x_n - q\| \to 0, \quad \text{as } n \to \infty,
\]
this implies that \(w^* \in Tq\). Again, since
\[
d(w^*, w) \leq d(w^*, w_n) + d(w_n, w) \\
\leq d(w^*, w_n) + D(Tx_n, Tq) \\
\leq d(w^*, w_n) + \mu \|x_n - q\| \to 0, \quad \text{as } n \to \infty.
\]
This implies that \(w^* = w\).

By the same way, we can prove that \(v^* \in V(q), v^* = v, g^* = g\) and \(z^* \in Z(q), z^* = z\).

Summing up the above arguments, we have prove that the sequences \(\{x_n\}, \{w_n\}, \{v_n\}, \{g_n\}, \{z_n\}\) defined by (4.1) converge strongly to the solution \(q, w, v, g, z\) of multivalued variational inclusion (2.1), respectively.

This completes the proof of Theorem 4.1.

REMARK 4.1. Theorem 4.1 generalizes, improves and unifies the corresponding recent results of Ding [7], Chang, Kim and Cho [10-12], Jung and Morales [8], and Zeng [9]. Especially, Theorem 4.1 generalizes and improves [1-4] in the following aspects.

1. The compactness condition Noor puts on the mappings \(T, V\) in [1-4] all are removed.
2. The setting of Hilbert space in [1-4] is generalized to Banach space.
3. The multivalued quasi-variational inclusions discussed in [1-4] are a special case of (2.1) with \(G\) being a single-valued mapping and \(Z = I\) (identity mapping on \(E\)).
4. The resulted consequence in our paper is a necessary and sufficient condition for sequence \(\{x_n\}\) defined by (4.1) converging to a solution of (2.1).
5. The method used in [1-4] is resolvent operator technique, but the method used in our paper is the accretive operator technique and Michael’s selection theorem.

REFERENCES


