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Disjunctive and conjunctive normal forms of pseudo-Boolean functions

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Abstract

After showing that every pseudo-Boolean function (i.e. real-valued function with binary variables) can be represented by a disjunctive normal form (essentially the maximum of several weighted monomials), the concepts of implicants and of prime implicants are analyzed in the pseudo-Boolean context, and a consensus-type method is presented for finding all the prime implicants of a pseudo-Boolean function. In a similar way the concepts of conjunctive normal form, implicates and prime implicates, as well as the resolution method are examined in the case of pseudo-Boolean functions. © 2000 Elsevier Science B.V. All rights reserved.

1. Background

Let $B=\{0,1\}$. A function from the hypercube (n-cube) B^n to B is called a *Boolean function*. A function from B^n to the set $\mathscr{R}$ of real numbers is called a *pseudo-Boolean function*.

Since every $V=(v_1,\dots,v_n)$ in B^n is the characteristic vector of the subset $\{i : v_i=1\}$ of $\{1,\dots,n\}$, a pseudo-Boolean function is essentially a real-valued set function defined on the set of all subsets of an n-element set. This interpretation contributes to the role that pseudo-Boolean functions play in operations research. For example, in the theory of cooperative games with n players, the coalition values associated to sets of players can be described by pseudo-Boolean functions. Another way of looking at a pseudo-Boolean function is as a valuation on the Boolean lattice B^n, or as an assignment of numbers to the vertices of the n-cube.

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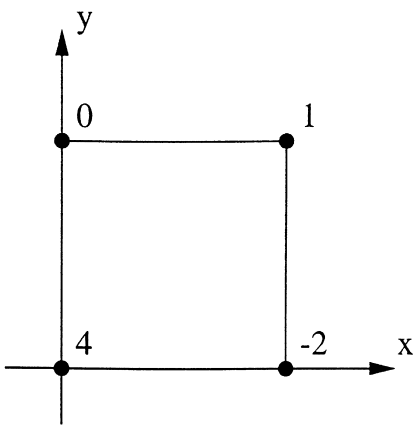

Fig. 1.

A pseudo-Boolean function can be defined in many different ways. For example, it can be specified by a value table:

x, y	$f(x, y)$
0,0	4
1,0	−2
0,1	0
1,1	1

(1)

This value table can also be displayed geometrically as in Fig. 1.

Pseudo-Boolean functions (pBf's) were introduced in [17], and extensively studied in [18]. These functions appear in numerous areas of discrete optimization, computer science, reliability theory, data analysis, graph theory, as well as in many interdisciplinary models of electronic circuit design, physics, telecommunications, etc.

It was seen in [18] that every pBf has a polynomial representation. For instance, the function defined in (1) has the polynomial expression

$$f(x, y) = 4 - 6x - 4y + 7xy. \tag{2}$$

It can be seen that the polynomial expression is multilinear (due to the idempotency law $x^2 = x$, which holds in B) and that this expression is unique (up to the order of terms, factors in the terms, etc.).

By using both the original variables x_j, and their *complements* $\bar{x}_j = 1 - x_j$, it can be shown that every pBf can be represented as an (*additive*) *posiform*, i.e. a polynomial depending on the original variables and their complements, and having only positive coefficients, with the possible exception of the constant term. The additive posiform representation however is not unique. The pBf (1) can for instance be represented as

$$f(x,y) = -4 + 8\bar{x}\bar{y} + 4\bar{x}y + 2x\bar{y} + 5xy \tag{3}$$

or as

$$f(x,y) = -6 + 6\bar{x} + 4\bar{y} + 7xy. \tag{4}$$

Recently [6], the cell flipping problem of VLSI design led to an interesting combinatorial optimization problem, consisting in the minimization of a pBf given as an expression using besides the arithmetic operators, also the *maximum operator* $\vee$. Such representations of the pBf (1) for instance are the following:

$$-2 + (6\bar{x}\bar{y} \vee 2\bar{x}y \vee 3xy) \tag{5}$$

or

$$-3 + (3y \vee 7\bar{x}\bar{y} \vee x\bar{y} \vee 4xy). \tag{6}$$

We shall call such expressions *disjunctive posiforms*, and shall study them in this paper. It should be mentioned that a computationally efficient solution of the VLSI problem studied in [6] became possible due to the fact that the formulation of the pBf describing it led directly to a disjunctive posiform, and that this disjunctive posiform had a very advantageous structure.

The disjunctive posiforms of a pBf do not possess the uniqueness characteristics of polynomial representations – thus simplification procedures are called for naturally. The consensus method for Boolean functions, introduced independently by Blake [5] and Quine [22] (see also Brown's text [7]), provides by analogy and generalization a basis for the simplification of the disjunctive posiforms of a pBf. We note that a consensus procedure for additive posiforms (such as (3) or (4)) was developed by Simeone [23]. Other extensions of the consensus method in the context of lattices appear in [9,13]. Also, an efficient consensus-based method was developed recently for finding all the maximal complete bipartite (not necessary induced) subgraphs of a graph [1].

In this paper we shall be concerned with disjunctive posiforms of pseudo-Boolean functions, their conjunctive analogues, as well as their simplifications, and the generalization of the consensus method to the pseudo-Boolean case. In fact a disjunctive posiform such as (5) can be rewritten as

$$(-2 + 6\bar{x}\bar{y}) \vee (-2 + 2\bar{x}y) \vee (-2 + 3xy). \tag{7}$$

We shall call this latter expression a *disjunctive normal form*, and we shall define *conjunctive normal forms* analogously. As the correspondence between (5) and (7) is obvious, we shall use either one of the two formulations according to which one of them is easier to handle in a particular situation.

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2. Basic pseudo-Boolean algebra

Any two pseudo-Boolean functions f and g can be added, multiplied and compared as follows:

$$(f+g)(V)=f(V)+g(V) \quad \text{for all } V\in B^n, \tag{8}$$

$$(fg)(V)=f(V)\times g(V) \quad \text{for all } V\in B^n, \tag{9}$$

$$f\leqslant g \Leftrightarrow \text{ for all } V\in B^n f(V)\leqslant g(V), \tag{10}$$

Eqs. (8) and (9) define a commutative ring structure on the set $\mathscr{R}^{(B^n)}$ of all pseudo-Boolean functions of n variables, with the constant 1 function as unit element. This ring is isomorphic to the Cartesian product ring $\mathscr{R}\times\cdots\times\mathscr{R}=\mathscr{R}^{2^n}$.

The order (10) is preserved by addition,

$$f\leqslant g \quad \text{and} \quad f'\leqslant g' \Rightarrow f+f'\leqslant g+g'.$$

The order relation (10) defines a distributive lattice structure on $\mathscr{R}^{B^n}$ with join and meet operations denoted by $\vee$ and $\wedge$ and usually called *disjunction and conjunction*, respectively. This lattice is isomorphic to the Cartesian product lattice $\mathscr{R}\times\cdots\times\mathscr{R}=\mathscr{R}^{2^n}$. (Distributivity follows from the fact that the ordered set $\mathscr{R}$, being totally ordered, is a distributive lattice.) Addition (8) is distributive over the lattice join and meet operations of $\mathscr{R}^{B^n}$,

$$f+(g\vee h)=(f+g)\vee(f+h),$$

$$f+(g\wedge h)=(f+g)\wedge(f+h).$$

We note that the $(\vee,+)$ structure of $\mathscr{R}^{(B^n)}$ is isomorphic, as a universal algebra of two binary operations, to the 2^nth power of the much studied $(\max,+)$ structure on $\mathscr{R}$. For the theory and applications of the latter, see e.g. Baccelli et al. [2] and Cuninghame-Green [8].

Join (disjunction), meet (conjunction) and addition operations are illustrated in the following examples, in $\mathscr{R}^{B^2}$, making use of the function f given by (1)–(6) and Fig. 1:

$$(4-4y)+(7xy-6x)=4-6x-4y+7xy=f,$$

$$(4-6x-4y)\vee(-2+2y+xy)=f,$$

$$f\wedge(5-3xy)=f.$$

Every real number can be identified with a constant pseudo-Boolean function of n variables, and thus the set $\mathscr{R}^{B^n}$ of all pseudo-Boolean functions of n variables can be viewed as a ring containing the real field $\mathscr{R}$ as a subring, i.e., as an *algebra over* $\mathscr{R}$. Thus $\mathscr{R}^{B^n}$ is also a 2^n-dimensional vector space over $\mathscr{R}$. For the same reason, $\mathscr{R}$ may also be viewed as a sublattice of $\mathscr{R}^{B^n}$.

In the theory of Boolean functions a *literal* is either the variable x, or its complement $1-x$, usually denoted by $\bar{x}$, and called a *complemented Boolean variable*.

A pBf having the form

$$a+bx, \tag{11}$$

where x is a variable and a and b are real constants, $b \neq 0$, will be called a *literal*. Thus a literal is a non-constant function depending only on a single variable. Examples of literals are

$$x,\ 1-x,\ 1+x,\ -2+3x,\ 3x,\ -x,$$

where x stands for any of the n variables.

Note that every literal has a unique expression as $a+b\tilde{x}$, where $b>0$ and $\tilde{x}$ is either a variable or a complemented variable, i.e. a Boolean literal. Obviously, every function $a+b\tilde{x},\ b>0$, is a literal.

Boolean functions, usually defined as functions $B^n \to B$, can be identified with those pseudo-Boolean functions whose range is contained in $\{0,1\}$. Thus Boolean functions constitute a finite sublattice of $\mathscr{R}^{B^n}$. Note that a literal (11) is a Boolean function if and only if one of the following conditions holds:

(i) $a=0,\ b=1$,

(ii) $a=1,\ b=-1$.

In case (i), $a+bx$ is just the variable x, while in case (ii), $a+bx$ is just the complemented variable $\bar{x}$. Note that, the sublattice of all Boolean functions is a *Boolean lattice* (i.e. a distributive and complemented lattice), and x and $\bar{x}$ are complements,

$$x \vee \bar{x} = 1,$$

$$x \wedge \bar{x} = 0,$$

here the constant functions 1 and 0 are the largest and smallest Boolean functions. In general, the complement of any Boolean function f is denoted by $\bar{f}$. Remark that the entire lattice $\mathscr{R}^{B^n}$ is not a complemented lattice.

The range $f[B^n]$ of any pseudo-Boolean function f, being a finite subset of $\mathscr{R}$, has a smallest as well as a greatest element. These are called the *minimum* and the *maximum* of f, and denoted $\min f$ and $\max f$, respectively, i.e.

$$\min f = \min\{f(V)\colon V \in B^n\},$$

$$\max f = \max\{f(V)\colon V \in B^n\}.$$

For example,

$$\min(x\bar{y}+2xy)=0,$$

$$\max(xy+2xy)=2.$$

The range of any literal $l=a+bx$ consists always of the two distinct real numbers a and $a+b$. If b is positive then

$$a=\min f, \quad a+b=\max f.$$

and if b is negative then

$$a+b=\min f, \quad a=\max f.$$

We define an *elementary conjunction* as a lattice meet (greatest lower bound) of one or more literals having the same minimum. Note that every constant function, say of value a, is of the form

$$(a+x)\wedge[(a+1)-x]$$

and it is therefore an elementary conjunction.

Similarly, an *elementary disjunction* is defined as a lattice join (least upper bound) of one or more literals having the same maximum. Every constant function is an elementary disjunction.

Examples of elementary conjunctions are

$$f_1=(3+x)\wedge(4-x),$$

$$f_2=(3+2x)\wedge(3+3\bar{x})\wedge(3+4y),$$

$$f_3=(3+2x)\wedge(3+4y)\wedge(3+2\bar{z}) \tag{12}$$

and elementary disjunctions are illustrated by

$$f_4=(3+x)\vee(4-x),$$

$$f_5=(3+2x)\vee(2+3\bar{x})\vee(1+4y),$$

$$f_6=(3+2x)\vee(1+4y)\vee(5-2z),$$
$$=(3+2x)\vee(1+4y)\vee(3+2\bar{z}). \tag{13}$$

Remark. Since the minimum of every Boolean literal is 0, the lattice meet of m such literals necessary constitutes an elementary conjunction

$$l_1\wedge\cdots\wedge l_m. \tag{14}$$

This lattice meet happens to coincide with the product

$$l_1\cdot\ldots\cdot l_m. \tag{15}$$

This coincidence fails in the case of pseudo-Boolean literals. For Boolean literals, clearly if $l_i=\bar{l}_j$ for two of the literals occurring in (14), then (14) is the constant 0 function. Otherwise (14) is precisely what is usually called an "*elementary conjunction*" *in Boolean function theory*, although in that case the notation (15) is more common. Such an elementary conjunction is never constant.

Similarly, the lattice join of m Boolean literals $l_1,\ldots,l_m$

$$l_1\vee\cdots\vee l_m \tag{16}$$

is necessarily an elementary disjunction and if $l_i=\bar{l}_j$ for two of the literals involved, then (16) is the constant 1 function. Otherwise (16) is precisely what is usually called in Boolean function theory an “*elementary disjunction*”. Such an elementary disjunction is never constant.

If a pseudo-Boolean function f is viewed as a function from B^n not to $\mathscr{R}$ but to some finite subset of $\mathscr{R}$ containing its range, then it corresponds to a “discrete function” in the sense of Davio et al. [9] and Störmer [24]. Theorems 1 and 2 will show that elementary conjunctions and disjunctions correspond essentially to the “cube functions” and “anticube functions” of [9] and [24]. Also, the following definitions of implicants, prime implicants, implicates and prime implicates coincide essentially with the corresponding notions in [9,24].

An *implicant* of a pseudo-Boolean function f is defined as an elementary conjunction g such that $g \leqslant f$. The implicant g is said to be a *prime implicant* if g is a maximal implicant of f, i.e. if there is no implicant g' such that

$$g < g' \leqslant f.$$

For example, all the following five elementary conjunctions are implicants of the function given by (1)–(2), but only the first and the last two are prime implicants:

$$(-2 + 2y), \tag{17}$$

$$(-2 + 3y) \wedge (-2 + 2x) = -2 + 2xy, \tag{18}$$

$$(-3 + 2y) \wedge (-3 + 2x) = -3 + 2xy, \tag{19}$$

$$(-2 + 3y) \wedge (-2 + 3x) = -2 + 3xy, \tag{20}$$

$$(-2 + \bar{x}). \tag{21}$$

An *implicate* of a pseudo-Boolean function f is defined as an elementary disjunction h such that $f \leqslant h$. The implicate h is said to be a *prime implicate* if h is a minimal implicate of f, i.e. if there is no implicate h' such that

$$f \leqslant h' < h.$$

For example

$$h = 4\bar{x} \vee (-3 + 7y) = 4(\bar{x} \vee y) \tag{22}$$

is an implicate of the function given by (1)–(2), and $h' = 1 + 3\bar{x}$ is a prime implicate such that $h' < h$.

Remark. (1) If f is a Boolean function that is not the constant 0, then another Boolean function g is an implicant or a prime implicant of f precisely when it is an implicant or a prime implicant in the usual sense of Boolean function theory. The constant 0 function does not have any implicants in traditional Boolean function theory.

(2) If f is a Boolean function that is not the constant 1, then another Boolean function h is an implicate or a prime implicate of f precisely when it is an implicate or a prime implicate in the usual sense of Boolean function theory. The constant 1 function does not have any implicates in Boolean function theory.

Theorem 1. *A pseudo-Boolean function f is an elementary conjunction if and only if it can be expressed as*

$$f = a + b(\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m), \tag{23}$$

where a and b are constants, $b > 0$ *and the* $\tilde{x}_i$ *are Boolean literals* $(m \geqslant 1)$.

Proof. If f is expressed as in (23) then, because $b > 0$,

$$f = a + (b\tilde{x}_1 \wedge \cdots \wedge b\tilde{x}_m) = \bigwedge_i (a + b\tilde{x}_i).$$

Note that if $\tilde{x}_i$ is a complemented Boolean variable $1 - x_i$, then

$$a + b\tilde{x}_i = a + b - bx = (a + b) + (-b)x.$$

This observation shows that f is an elementary conjunction.

Conversely, assume that f is an elementary conjunction,

$$f = \bigwedge_i (a_i + b_i x_i),$$

where the a_i and b_i are constants, $b_i \neq 0$, each x_i is a variable, and the functions $a_i + b_i x_i$ have a common minimum. This minimum, say a, is a_i if $b_i > 0$, and it is $a_i + b_i$ if $b_i < 0$. Define $\tilde{x}_i$ by

$$\tilde{x} = x_i \quad \text{if } b_i > 0,$$

$$\tilde{x}_i = 1 - x_i \quad \text{if } b_i < 0.$$

Then we have

$$\begin{aligned} f &= \left[\bigwedge_{b_i > 0} (a_i + b_i x_i) \right] \wedge \left[\bigwedge_{b_i < 0} (a_i + b_i x_i) \right] \\ &= \left[\bigwedge_{b_i > 0} (a_i + b_i \tilde{x}_i) \right] \wedge \left[\bigwedge_{b_i < 0} (a_i + b_i - b\tilde{x}_i) \right] \\ &= \left[\bigwedge_{b_i > 0} (a + b_i \tilde{x}_i) \right] \wedge \left[\bigwedge_{b_i < 0} (a - b_i \tilde{x}_i) \right] \\ &= a + \left[\bigwedge_{b_i > 0} b_i \tilde{x}_i \right] \wedge \left[\bigwedge_{b_i < 0} -b_i \tilde{x}_i \right] = a + \bigwedge_i |b_i| \tilde{x}_i. \end{aligned}$$

Letting $b = \min(\{b_i\colon b_i > 0\} \cup \{-b_i\colon b_i < 0\}) = \min_i |b_i|$ this latter expression of f equals

$$a + b \bigwedge_i \tilde{x}_i,$$

which is of the form prescribed by (23). $\square$

Applying Theorem 1 to the elementary conjunctions in (12) we find

$$f_1 = 3,$$
$$f_2 = 3,$$
$$f_3 = 3 + 2xy\bar{z}.$$

Note that a constant function a can always be written as $a + b(x \wedge \bar{x})$, for any $b > 0$.

Theorem 2. *A pseudo-Boolean function f is an elementary disjunction if and only if it can be expressed as*

$$f = a + b(\tilde{x}_1 \vee \cdots \vee \tilde{x}_m), \tag{24}$$

where a and b are constants, $b > 0$ and the $\tilde{x}_i$ are Boolean literals ($m \geqslant 1$).

Proof. The argument is similar to the proof of Theorem 1. If f is expressed as in (24) then, because of $b > 0$,

$$f = \bigvee_i (a + b\tilde{x}_i).$$

If $\tilde{x}_i$ is a Boolean complemented variable $1 - x$, then

$$a + b\tilde{x}_i = (a + b) + (-b)x.$$

Conversely, assume that f is an elementary disjunction

$$f = \bigvee_i (a_i + b_i x_i),$$

where the functions $a_i + b_i x_i$ have a common maximum. This maximum, say c, is $a_i + b_i$ if $b_i > 0$, a_i if $b_i < 0$. Define $\tilde{x}_i$ by

$$\tilde{x}_i = x_i \quad \text{if } b_i > 0,$$

$$\tilde{x}_i = 1 - x_i \quad \text{if } b_i < 0.$$

Let $a = \bigvee_i \min(a_i + b_i x_i)$, and let $b = c - a$. Then

$$f = \bigvee_i (a_i + b_i x_i) = \bigvee_i (a_i + b_i \tilde{x}_i) = a + b \bigvee_i \tilde{x}_i. \qquad \square$$

Applying Theorem 2 to the elementary disjunctions in (13) we find

$$f_4 = 4,$$
$$f_5 = 5,$$
$$f_6 = 3 + 2(x \vee y \vee \bar{z}).$$

Note that a constant function can always be written as $a + b(x \vee \bar{x})$.

Remark. (1) In general, a conjunction of arbitrary literals is not an elementary conjunction, e.g. $2x \wedge (1 + 3y)$. Also a disjunction of literals need not be an elementary disjunction, e.g. $2x \vee (1 + 3y)$.

(2) An elementary conjunction expressed as in (23) is a constant if and only if $\tilde{x}_i = 1 - \tilde{x}_j$ for some i, j. An elementary disjunction expressed as in (24) is a constant if and only if $\tilde{x}_i = 1 - \tilde{x}_j$ for some i, j.

(3) Every non-constant elementary conjunction, as well as every non-constant elementary disjunction, has a two-element range $\{a, a+b\}$, where a and b are as in the expressions (23) and (24).

(4) A non-constant elementary conjunction or disjunction expressed as in (23) or as in (24) is a Boolean function if and only if $a = 0,\ b = 1$.

(5) Expressions (23) and (24) are unique for non-constant elementary conjunctions and disjunctions, respectively, up to possible repetition and permutation of the Boolean literals $\tilde{x}$. This uniqueness does not extend to constant functions; in fact every constant has infinitely many expressions (23) and (24).

3. Disjunctive normal forms and other representations

An important fact in the theory of Boolean functions is the possibility of representing any Boolean function in a disjunctive normal form (i.e. as a disjunction of elementary conjunctions), as well as in a conjunctive normal form (i.e. as a conjunction of elementary disjunctions). Analogous facts will be seen to hold also for pseudo-Boolean functions.

Although some of the propositions of this section (Theorems 3 and 6) can be derived from the representation theorems of discrete functions appearing in [9,24], for the sake of completeness we shall present below direct proofs. It should be remarked that the concept of normal forms appears also in the treatment of real valued functions defined on the continuous unit cube $[0,1]^n$ (see [20,21]).

Property 1. *Every pseudo-Boolean function f can be expressed as a finite disjunction of elementary conjuctions having the same minimum a*:

$$f = \bigvee_i (a + b_i(\tilde{x}_{i_1} \wedge \cdots \wedge \tilde{x}_{im_i})), \quad b_i > 0,\ m_i \geqslant 1 \tag{25}$$

or as

$$f = a + \bigvee_i b_i(\tilde{x}_{i_1} \wedge \cdots \wedge \tilde{x}_{im_i}), \quad b_i > 0,\ m_i \geqslant 1. \tag{26}$$

Proof. The second representation is obviously equivalent to the first one, because of distributivity.

Given f, let a be any real number less than or equal to $\min f$. For each Boolean variable x, write x^0 for $1 - x$ and x^1 for x itself. For each $V \in B^n$, if $V = (v_1, \ldots v_n)$ let

$$t_V = x_1^{v_1} \wedge \cdots \wedge x_n^{v_n}. \tag{27}$$

Clearly, $t_V(V)=1$ but $t_V(W)=0$ for all $W \in B^n$ different from V. Let $b_V = f(V) - a$. Then

$$f = a + \bigvee_{V \in B^n} b_V t_V \tag{28}$$

and this expression is of the form (26). □

Any expression of a pseudo-Boolean function f in the form (25) is called a *disjunctive normal form* (*DNF*) representation of f. An expression of the form (26) is called a *disjunctive posiform* (*DPF*).

Elementary conjunctions of the form (27) are called *minterms*. From (28) it is clear that for each V, $a + b_V t_V$ is an implicant of f, which however is not necessarily a prime implicant of f.

For the function given by (1)–(2) a DNF using the four minterms is

$$(-3 + 7\bar{x}\bar{y}) \vee (-3 + x\bar{y}) \vee (-3 + 3\bar{x}y) \vee (-3 + 4xy)$$

and the corresponding DPF is

$$-3 + (7\bar{x}\bar{y} \vee x\bar{y} \vee 3\bar{x}y \vee 4xy).$$

Property 2. *For every implicant g of any pseudo-Boolean function f there is a prime implicant g' of f such that $g \leqslant g' \leqslant f$.*

Proof. If f is a constant, we can take $g' = f$. Assume therefore that f is not a constant.

Consider first the case where there is no constant k such that $g \leqslant k \leqslant f$. Let g be expressed as

$$g = a + b(\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m),$$

where $b > 0$ and $a + b = \max g > \min f$. Abbreviate $\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m$ as C. Thus

$$g = a + bC$$

and the Boolean function C is not constant. By assumption

$$g \nleqslant \min f.$$

Define

$$m_C = \min\{f(V)\colon C(V) = 1\}.$$

Since g is an implicant, $a + b \leqslant m_C$. Since we cannot have $a + b \leqslant f$ we must have $\min f < m_C$.

Consider the set S of those Boolean elementary conjunctions D for which $D \geqslant C$, and

$$\min\{f(V) : D(V) = 1\} = m_C.$$

As S is a finite set, containing in particular C, but not containing the constant 1 function, we can choose a maximal member P of S. Then

$$g' = \min f + (m_C - \min f)P$$

is a prime implicant of f greater than or equal to g. (Primality is due to the maximality of P in S, and $a+bC \leqslant g'$ follows from $C \leqslant P$ and $a+b \leqslant m_C$.)

Consider now the case when $g \leqslant k \leqslant f$ for some constant k. Since f is not constant, $k < \max f$. Choose any $V \in B^n$ such that

$$f(V) = \max f.$$

Let C be the minterm such that $C(V)=1$. Then for

$$g'' = k + (\max f - k)C,$$

we have $g \leqslant g'' \leqslant f$, g'' is an implicant of f, and there is no constant q with $g'' \leqslant q \leqslant f$. Applying the argument of the first case to g'' instead of g, we obtain a prime implicant g' of f such that $g \leqslant g'' \leqslant g' \leqslant f$. □

Properties 1 and 2 combined yield:

Theorem 3. *Every pseudo-Boolean function is the least upper bound of its prime implicants.*

Let $p = a + b(\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m)$ be a prime implicant of a non-constant pseudo-Boolean function f. Clearly $a = \min f$. Also, denoting by C the conjunction $\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m$, we must have

$$a + b = \min\{f(V) : C(V) = 1\}.$$

Thus p is fully determined by C. This leads to the conclusion that the number of prime implicants of any pseudo-Boolean function is finite (and obviously non-zero).

Consequently every pseudo-Boolean function f has a DNF (25) consisting of the join of all of its prime implicants. This particular DNF is called the *prime DNF*, or *canonical DNF* of the pseudo-Boolean function. (If f is constant, then it is understood that the only term of the join is the real constant in question.) The corresponding DPF (26) is also called the *prime* or *canonical* DPF.

For the function given by (1) and (2) the prime DNF is

$$(-2 + 6\bar{x}\bar{y}) \vee (-2 + 2\bar{x}y) \vee (-2 + 3xy)$$

and the prime DPF is

$$-2 + (6\bar{x}\bar{y} \vee 2\bar{x}y \vee 3xy).$$

Property 3. *Every pseudo-Boolean function f can be expressed as a finite conjunction of elementary disjunctions that have the same maximum t:*

$$f = \bigwedge_i (a_i + b_i(\tilde{x}_{i1} \vee \cdots \vee \tilde{x}_{im_i})), \quad b_i > 0, \ m_i \geqslant 1, \tag{29}$$

as well as

$$f = \bigwedge_i (t - b_i + b_i(\tilde{x}_{i1} \vee \cdots \vee \tilde{x}_{im_i})), \quad b_i > 0, \ m_i \geqslant 1 \tag{30}$$

or

$$f = t + \bigwedge_i (-b_i + b_i(\tilde{x}_{i1} \vee \cdots \vee \tilde{x}_{im_i})), \quad b_i > 0, \ m_i \geqslant 1. \tag{31}$$

Proof. The second and the third representations obviously follow from the first one.

Given f, let t be any real number greater than $\max f$. For each $V \in B^n$, $V = (v_1, \ldots, v_n)$ let

$$q_V = x_1^{1-v_1} \vee \cdots \vee x_n^{1-v_n}. \tag{32}$$

Clearly $q_V(V)=0$ but $q_V(W)=1$ for all $W \in B^n$ different from V. Let $b_V = t - f(V)$, $a_V = f(V)$. Then

$$f = \bigwedge_{V \in B^n} (a_V + b_V q_V) \tag{33}$$

and this expression is of form (29). □

Any expression of a pseudo-Boolean function f in form (29), with $a_i + b_i$ being the same for all i, is called a *conjunctive normal form* (*CNF*) representation of f. An expression of form (31) is called a *conjunctive posiform* (*CPF*). The lack of perfect symmetry between (25) and (29) is essentially due to the lack of duality between 0 and 1 in $\mathscr{R}$ (as opposed to perfect duality in a properly Boolean 0–1 environment).

Elementary disjunctions of form (32) are called *maxterms*. From (33) it is clear that for each V, $a_V + b_V q_V$ is an implicate of f, which, however, is not necessarily a prime implicate of f.

For the function given by (1)–(2), a CNF using the four maxterms is

$$[4 + (x \vee y)] \wedge [-2 + 7(\bar{x} \vee y)] \wedge [5(x \vee \bar{y})] \wedge [1 + 4(\bar{x} \vee \bar{y})].$$

Property 4. *For every implicate h of any pseudo-Boolean function f there is a prime implicate h' of f such that $f \leqslant h' \leqslant h$.*

Proof. The case analysis is analogous to that in the proof of Theorem 4. Therefore we give the full argument only for the case when there is no constant k such that $f \leqslant k \leqslant h$. Let h be expressed as

$$f = a + b(\tilde{x}_1 \vee \cdots \vee \tilde{x}_m).$$

Abbreviate $\tilde{x}_1 \vee \cdots \vee \tilde{x}_m$ as D. Thus

$$h = a + bD$$

and the Boolean function D is not constant. From the assumption that there is no constant k such that $f \leqslant k \leqslant h$, it follows that

$$\max f \nleqslant h.$$

Define

$$m_D = \max\{f(V) : D(V) = 0\}.$$

We must have $m_D < \max f$. Consider the set of those Boolean elementary disjunctions E that are less than or equal to D, $E \leqslant D$, and for which

$$\max\{f(V): E(V) = 0\} = m_D.$$

Let P be a minimal member of this finite set. Then

$$h' = m_D + (\max f - m_D)P$$

is a prime implicate of f, and $f \leqslant h' \leqslant h$. □

Properties 3 and 4 combined yield:

Theorem 4. *Every pseudo-Boolean function f is the greatest lower bound of its prime implicates.*

Let $p = a + b(\tilde{x}_1 \vee \cdots \vee \tilde{x}_m)$ be a prime implicate of a non-constant pseudo-Boolean function f. Clearly $a + b = \max f$. Also, denoting by D the disjunction $\tilde{x}_1 \vee \cdots \vee \tilde{x}_m$, we must have

$$a = \max\{f(V): D(V) = 0\}.$$

Thus p is fully determined by D. This leads to the conclusion that the number of prime implicates of any pseudo-Boolean function is finite (and obviously non-zero).

Consequently, every pseudo-Boolean function f has a CNF (29) consisting of the meet of all of its prime implicates. This particular CNF is called the *prime CNF*, or *canonical CNF* of the pseudo-Boolean function. (If f is constant, then it is understood that the only term of the meet is the real constant in question.) The corresponding CPF (31) is also called the *prime* or *canonical CPF*.

4. Special classes of pseudo-Boolean functions

Section 3 shows that the pseudo-Boolean DNF and CNF concepts generalize the corresponding well-known notions for Boolean functions. We shall outline in this section the main results presented in [11] concerning characterizations of monotone, Horn and quadratic pseudo-Boolean functions. Properties of disjunctive submodular and supermodular pseudo-Boolean functions are investigated in [12].

4.1. Monotone pseudo-Boolean functions

A pseudo-Boolean function f is called *monotone non-decreasing* if $V \leqslant W$ implies $f(V) \leqslant f(W)$, and it is called *monotone non-increasing* if $V \leqslant W$ implies $f(V) \geqslant f(W)$. The well-known characterizations of monotone Boolean functions were generalized in [3] to the case of discrete functions, while in [11] the following generalizations of them were given for the pseudo-Boolean case:

Proposition 1. *For any pseudo-Boolean function f the following conditions are equivalent*:
(i) *f is monotone non-decreasing*,
(ii) *some DNF of f contains no complemented variables*,
(iii) *the canonical DNF of f contains no complemented variables*,
(iv) *some CNF of f contains no complemented variables*,
(v) *the canonical CNF of f contains no complemented variables.*

Proposition 2. *For any pseudo-Boolean function f the following conditions are equivalent*:
(i) *f is monotone non-increasing*,
(ii) *in some DNF of f all variable occurrences are complemented*,
(iii) *in the canonical DNF of f all variable occurrences are complemented*,
(iv) *in some CNF of f all variable occurrences are complemented*,
(v) *in the canonical CNF of f all variable occurrences are complemented.*

In view of these characterizations, monotone non-decreasing and non-increasing functions are also called *positive* and *negative*, respectively.

4.2. Pseudo-Boolean Horn functions

A Boolean DNF is called *Horn* if it contains at most one complemented variable in each of its terms. A Boolean function is called *Horn* if it has a Horn DNF representation. It is known (see [10]) that a Boolean function is Horn if and only if for every $V, W \in B^n$ the following inequality holds:

$$f(VW) \leqslant f(V) \vee f(W). \tag{34}$$

Adopting the same definitions for pseudo-Boolean Horn DNFs and functions, the following result was proved in [11]:

Proposition 3. *A pseudo-Boolean function f is Horn if and only if it satisfies* (34) *for every V, $W \in B^n$.*

Sketch of proof. The "if" part is proved by showing that the canonical DNF is Horn, i.e. that no prime implicant of f is of the form $a+b\bar{x}\bar{y}P$. This is done by contradiction, using the non-implicants $a+b\bar{y}P$ and $a+b\bar{x}P$ to define appropriate V and W to violate (34). The "only if" part is proved by reduction to the Boolean case in [10]. □

4.3. Quadratic pseudo-Boolean functions

A Boolean DNF is called *quadratic* if every elementary conjunction of it contains at most two literals. A Boolean function is called *quadratic* if it has a quadratic DNF. It was proved in [10] that a necessary and sufficient condition for a Boolean function to be quadratic is that for every $U, V, W \in B^n$ the following inequality holds:

$$f(UV \vee UW \vee VW) \leqslant f(U) \vee f(V) \vee f(W). \tag{35}$$

Adopting the same definitions for quadratic pseudo-Boolean DNFs and functions, the following result was proved in [11]:

Proposition 4. *A pseudo-Boolean function is quadratic if and only if it satisfies* (35) *for every* $U, V, W \in B^n$.

Sketch of proof. The "if" part is shown, here too as in the Horn case, by contradiction, assuming the existence of a prime implicant of the form $a + b\tilde{x}\tilde{y}\tilde{z}P$ and then using the non-implicants $a + b\tilde{x}\tilde{y}P$, $a + b\tilde{x}\tilde{z}P$ and $a + b\tilde{y}\tilde{z}P$ to define U, V, W to violate (35). The "only if" part is established again by reduction to the Boolean case. □

5. Polynomial representation and normal forms

Besides DNF/DPF and CNF/CPF representations, pseudo-Boolean functions can be expressed in different other forms. Two such alternative forms will be considered in this section, and simple algorithmic procedures are given that transform one form of representation to the other one.

It is known (see [18]) that every pBf has a unique representation in the form

$$c_0 + c_1P_1 + \cdots + c_mP_m, \tag{36}$$

where $m \geqslant 0$, the m constants $c_1, \ldots, c_m$ are all non-zero, the P_i's are distinct and each P_i is a product of one or several (distinct) variables. (Uniqueness is of course meant up to the order of summands or variables within each summand.) As (36) is in fact a polynomial in several variables, it is referred to as the *polynomial representation* of f.

If in (36) we allow not only variables but also complemented variables as factors in each P_i, but require that each c_i be positive for $i \geqslant 1$ (c_0 remaining unconstrained), then (36) is called an *additive posiform representation* (see [23]). Every pBf has an additive posiform representation; indeed one can be obtained from the polynomial representation by repeatedly replacing some Boolean literal t by the expression $1 - \bar{t}$. Note that uniqueness no longer holds for posiform representations, e.g. $x_1 + \bar{x}_1 = 1$. The reverse passage from posiforms to polynomials is effected by replacing, in any additive posiform, each complemented variable $\bar{x}$ by $1 - x$.

The passage from an additive posiform to a DNF can be accomplished by the repeated application of the following two valid transformations:

$$cP + c'P' = cP \vee c'P' \vee (c + c')PP',$$

which holds if c and c' are positive, and the distributive law

$$(A \vee B) + C = (A + C) \vee (B + C),$$

that holds among all pBf's.

Example.

$$x + 2\bar{y} + z = (x \vee 2\bar{y} \vee 3x\bar{y}) + z$$
$$= x \vee 2\bar{y} \vee 3x\bar{y} \vee z \vee 2xz \vee 3\bar{y}z \vee 4x\bar{y}z.$$

Conversely, starting with any DNF (25), we obtain an additive posiform as follows: take some i with largest b_i and with $P_i = \tilde{x}_{i1} \wedge \cdots \wedge \tilde{x}_{im_i}$, rewrite (25) as

$$b_i P_i + \bigvee_{j \neq i, 1 \leqslant r \leqslant m_i} [a + b_j P_j (1 - \tilde{x}_{ir})], \tag{37}$$

then apply again the same procedure to the join expression in (37), and repeat.

Example.

$$(1 + 2x) \vee (1 + 3\bar{y}z) = 3\bar{y}z + [(1 + 2xy) \vee (1 + 2x\bar{z})]$$
$$= 3\bar{y}z + 2xy + [(1 + 2x\bar{z}\bar{x}) \vee (1 + 2x\bar{z}\bar{y})]$$
$$= 3\bar{y}z + 2xy + 1 + 2x\bar{z}\bar{y}$$
$$= 1 + 3\bar{y}z + 2xy + 2x\bar{z}\bar{y}.$$

Note that a short additive posiform may give rise to a large-size DNF, and conversely.

6. The consensus method for pseudo-Boolean functions

The pseudo-Boolean *consensus algorithm* that we shall now describe transforms an arbitrary DPF into another DPF of the same pBf, until it produces the canonical DPF of the function. The algorithm is a generalized version of the Boolean consensus algorithm (see [5,22]). It is also analogous to the consensus procedure for discrete functions described in [9]. We use DPF rather than DNF formulation as it allows a more concise notation.

The pseudo-Boolean consensus method has two basic steps that are applied to a DPF $c_0 + (c_1 P_1 \vee \cdots \vee c_m P_m)$:

(a) *absorption*: if $i \neq j$ and $c_i P_i \leqslant c_j P_j$ (i.e. $c_i \leqslant c_j$ and $P_i \leqslant P_j$) then delete $c_i P_i$,

(b) *adjunction of consensus*: if for some variable x, $P_i = xQ_i$, $P_j = \bar{x}Q_j$ for elementary conjunctions Q_i and Q_j such that $Q_i Q_j \neq 0$, and if for no k

$$\min(c_i, c_j) Q_i Q_j \leqslant c_k P_k,$$

then

- if $Q_i Q_j \neq 1$ adjoin $\min(c_i, c_j) Q_i Q_j$ to the disjunction to yield

$$c_0 + (c_1 P_1 \vee \cdots \vee c_m P_m \vee \min(c_i, c_j) Q_i Q_j),$$

- if $Q_i Q_j = 1$, then replace c_0 by $c_0 + \min(c_i, c_j)$, delete each $c_k P_k$ such that $c_k \leqslant \min(c_i, c_j)$, and replace every other c_k by $c_k - \min(c_i, c_j)$.

Observe that for a given input DPF, c_0 can never decrease during the execution of the algorithm, and it can only assume a finite number of values. Also, there are only a finite number of possibilities for the terms c_iP_i that may appear at various stages, and once a c_iP_i is absorbed it cannot re-appear before c_0 is incremented. Thus the algorithm cannot enter a loop and it always terminates after a finite number of basic steps.

If in the input DPF we have $c_0 = 0$, $c_1 = \cdots = c_m = 1$, which corresponds to a Boolean DNF, then absorption and adjunction of consensus have the same effect as in the Boolean case. Thus the Boolean case is embedded in the general pseudo-Boolean one. Since there is no polynomial time bound on Boolean consensus, pseudo-Boolean consensus is non-polynomial as well.

We shall prove below that if we apply the two steps of the pseudo-Boolean consensus method to a DPF in an arbitrary order, as long as they produce new expressions, we arrive after a finite number of iterations to the canonical DPF. In other words we shall prove the following

Theorem 5. *For any DPF* $c_0+(c_1P_1\vee\cdots\vee c_mP_m)$ *the following conditions are equivalent*:

(i) *the application of any of the two basic steps of the consensus algorithm does not result in a different DPF,*
(ii) *the DPF is canonical.*

Proof. We first show that (ii) implies (i), i.e. that none of the basic steps can be applied to a canonical DPF. Assume (ii).

Since the implicants $c_0 + c_iP_i$ are all distinct and prime, if $i \neq j$ we cannot have $c_iP_i \leqslant c_jP_j$, and hence absorption cannot apply.

If $P_i = xQ_i$, $P_j = \bar{x}Q_j$, $Q_iQ_j \neq 0$, then $c_0 + \min(c_i, c_j)Q_iQ_j$ is an implicant; clearly this implicant must be less than or equal to some prime implicant $c_0 + c_kP_k$, and therefore adjunction of consensus cannot apply.

Conversely, we shall establish that (i) implies (ii). Assume (i). Denote by f the pBf represented by the given DPF.

The fact that all the c_iP_i's are distinct follows immediately from the impossibility to perform absorptions.

Let us establish now the important auxiliary fact that for every $1 \leqslant i \leqslant m$

$$c_0 + c_i = \min\{f(V)\colon P_i(V) = 1\}. \tag{38}$$

If $c_0 + c_i$ were less than this minimum, we would have

$$P_i \leqslant \bigvee_{c_i < c_j} P_j.$$

Renaming the P_j appearing in this join expression as $R_1, \ldots, R_k$ we express P_i as a Boolean DNF

$$P_i = R_1P_i \vee \cdots \vee R_kP_i. \tag{39}$$

We claim that $P_i = R_h P_i$ for some $1 \leqslant h \leqslant k$. If not, then adjunction of consensus performed on the right-hand side of (39) would yield new Boolean elementary conjunctions. Without loss of generality, this would mean that there exists a variable x and elementary conjunctions Q_1 and Q_2 such that

$$R_1 = xQ_1,\ R_2 = \bar{x}Q_2, \qquad Q_1Q_2P_i \nleqslant R_hP_i \quad \text{for any } h.$$

This implies that $Q_1Q_2 \nleqslant R_h$ for all h, and consequently,

$$\min(c_{j_1}, c_{j_2})Q_1Q_2 \nleqslant c_jP_j \quad \text{for any } 1 \leqslant j \leqslant m,$$

where $R_1 = P_{j_1}$, $R_2 = P_{j_2}$. But adjunction of consensus cannot be performed on the original pseudo-Boolean DPF, and therefore for some j, $1 \leqslant j \leqslant m$, such that $c_i < c_j$

$$\min(c_{j_1}, c_{j_2})Q_1Q_2 \leqslant c_jP_j.$$

If $P_j = R_h$, this implies $Q_1Q_2 \leqslant R_h$ and contradicts $Q_1Q_2P_i \nleqslant R_hP_i$. Therefore $P_i = R_hP_i$ for some h as claimed, i.e. $P_i \leqslant P_j$ for some j with $c_i < c_j$. As this would allow absorption, we obtain a contradiction proving (38).

Let us show next that $c_0 = \min f$. If we had $c_0 < \min f$ then for every $V \in B^n$ we would have $P_i(V) = 1$ for some i such that $\min f \leqslant c_0 + c_i$, that is

$$\bigvee_{\min f \leqslant c_0 + c_i} P_i \equiv 1.$$

We could then adjoin consensus to the left-hand side disjunction. This means that for some variable x and some i, j with

$$\min f \leqslant c_0 + c_i, \quad \min f \leqslant c_0 + c_j,$$

there are elementary conjunctions Q_i, Q_j such that $P_i = xQ_i$, $P_j = \bar{x}Q_j$, $Q_iQ_j \neq 0$ and

$$Q_iQ_j \nleqslant P_k \quad \text{for any } k \text{ such that } \min f \leqslant c_0 + c_k.$$

This would imply a fortiori, for this particular pair i, j, that

$$\min(c_i, c_j)Q_iQ_j \nleqslant c_kP_k \quad \text{for all } 1 \leqslant k \leqslant m,$$

allowing adjunction of consensus in the original DPF, which is impossible. Thus $c_0 = \min f$.

Let us prove now that every non-constant implicant of the form $\min f + cP$ is less than or equal to some $c_0 + c_iP_i$. We have

$$\min f + cP \leqslant c_0 + (c_1P_1 \vee \cdots \vee c_mP_m),$$

$$cP \leqslant c_1P_1 \vee \cdots \vee c_mP_m,$$

$$P \leqslant \bigvee_{c \leqslant c_i} P_i,$$

$$P = \bigvee_{c \leqslant c_i} P_iP. \tag{40}$$

We claim that $P = P_iP$ for some i such that $c \leqslant c_i$. Were this not the case, the Boolean DNF on the right-hand side of (40) would allow adjunction of consensus, i.e. there would be indices i, j and a variable x such that

$$c \leqslant c_i, \quad c \leqslant c_j,$$
$$P_i = xQ_i, \quad P_j = \bar{x}Q_j, \quad Q_iQ_j \neq 0, \quad Q_iQ_jP \nleqslant P_kP \quad \text{for any } k.$$

This would imply $Q_iQ_j \nleqslant P_k$ for all k. But since there can be no adjunction of consensus to the original DPF, we must have

$$\min(c_i, c_j)Q_iQ_j \leqslant c_kP_k$$

for some $1 \leqslant k \leqslant m$ for which we would have $c \leqslant c_k$, and this contradicts $Q_iQ_j \nleqslant P_k$. Therefore $P = P_iP$ for some i with $c \leqslant c_i$ as claimed. This shows that the $c_0 + c_iP_i$ are precisely the prime implicants of f, completing the proof of (ii). □

Example of consensus algorithm. As input we take representation (6) of the function $f(x, y)$ with value table (1),

$$f(x, y) = -3 + (3y \vee 7\bar{x}\bar{y} \vee x\bar{y} \vee 4xy),$$

in order to obtain its canonical DPF. Applying now the steps of the pseudo-Boolean consensus method we obtain successively the following expressions of the pBf $f(x, y)$:

$$\begin{aligned} f(x, y) &= -3 + (3y \vee 7\bar{x}\bar{y} \vee 3\bar{x} \vee x\bar{y} \vee 4xy) \\ &= -3 + (3y \vee 7\bar{x}\bar{y} \vee 3\bar{x} \vee x\bar{y} \vee x \vee 4xy) \\ &= -3 + (3y \vee 7\bar{x}\bar{y} \vee 3\bar{x} \vee x \vee 4xy) \\ &= -2 + (2y \vee 6\bar{x}\bar{y} \vee 2\bar{x} \vee 3xy). \end{aligned} \tag{41}$$

7. Dual functions and the resolution method

In order to link the disjunctive and conjunctive expressions of pseudo-Boolean functions, we introduce the following notion of duality. We make use of complementation in the Boolean lattice B^n, denoting by $\bar{V}$ the complement of any $V \in B^n$. For a pseudo-Boolean function f, its *dual* f^{d} is then the pseudo-Boolean function defined on B^n by

$$f^{\mathrm{d}}(V) = 1 - f(\bar{V}).$$

For example, the dual of the function specified by (1) is given by the following value table

x, y	$f^{\mathrm{d}}(x, y)$
0, 0	0
1, 0	1
0, 1	3
1, 1	−3

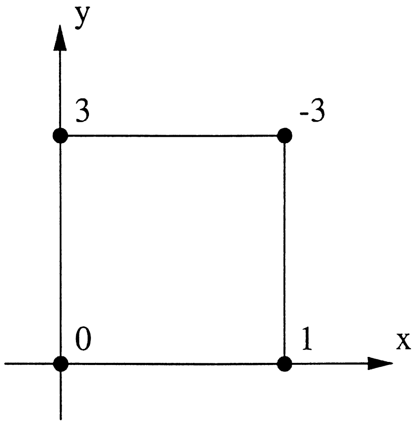

Fig. 2.

The polynomial representation is $x+3y-7xy$. The value table is displayed geometrically in Fig. 2.

Elementary properties of duals. (1) If f is a Boolean function, then f^{d} is its dual in the usual sense of Boolean function theory.

(2) Dualization is involutive, i.e. $(f^{\mathrm{d}})^{\mathrm{d}} = f$ for any pseudo-Boolean function f.

(3) Dualization is an affine transformation of the real vector space $\mathscr{R}^{B^n}$, i.e. if $a+b=1$ then

$$(af + bg)^{\mathrm{d}} = af^{\mathrm{d}} + bg^{\mathrm{d}}.$$

(4) Boolean literals are self-dual

$$x^{\mathrm{d}} = x, \quad (1-x)^{\mathrm{d}} = 1 - x.$$

(5) The dual of any constant is a constant, $0^{\mathrm{d}} = 1$, $1^{\mathrm{d}} = 0$, $(\frac{1}{2})^{\mathrm{d}} = \frac{1}{2}$.

(6) The dualization transformation is a dual order-automorphism of $\mathscr{R}^{B^n}$.

(7) We have $\min f = \min g$ if and only if $\max f^{\mathrm{d}} = \max g^{\mathrm{d}}$.

(8) The four transformations on $\mathscr{R}^{B^n}$ consisting of

(a) dualization, i.e. $f \mapsto f^{\mathrm{d}}$,

(b) reflection about $\frac{1}{2}$, i.e. $f \mapsto 1 - f$,

(c) argument complementation, $f \mapsto g$, given by $g(V) = f(\bar{V})$ for any $V \in B^n$

(d) identity, i.e. $f \mapsto f$

form a group isomorphic to $\mathscr{Z}_2^2$ under composition. This extends the analogous fact for Boolean functions, noted by Gottschalk [15] (discussed by Halmos [16] and Bioch and Ibaraki [4]). Observe that both dualization and reflection are dual-order automorphisms of $\mathscr{R}^{B^n}$, while argument complementation and the identity transformation are direct order automorphisms.

(9) The dual of any monotone non-decreasing function f is monotone non-decreasing, because

$$V \leqslant W \Rightarrow \bar{W} \leqslant \bar{V} \Rightarrow f(\bar{W}) \leqslant f(\bar{V}),$$

$$1 - f(\bar{V}) \leqslant 1 - f(\bar{W}),$$

$$f^{\mathrm{d}}(V) \leqslant f^{\mathrm{d}}(W).$$

Similarly, the dual of any monotone non-increasing function is monotone non-increasing.

(10) The dual of any literal $a + b\tilde{x}$ is the literal $1 - a - b + b\tilde{x}$.

(11) The duals of any literals having the same minimum have the same maximum. The duals of any literals having the same maximum have the same minimum.

(12) Dualization converts meets to joins and vice versa: $(f \wedge g)^{\mathrm{d}} = f^{\mathrm{d}} \vee g^{\mathrm{d}}$ and $(f \vee g)^{\mathrm{d}} = f^{\mathrm{d}} \wedge g^{\mathrm{d}}$. In particular if the $\tilde{x}_i$ are Boolean literals, then

$$(\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m)^{\mathrm{d}} = \tilde{x}_1 \vee \cdots \vee \tilde{x}_m,$$

$$(\tilde{x}_1 \vee \cdots \vee \tilde{x}_m)^{\mathrm{d}} = \tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m.$$

(13) Dualization converts elementary conjunctions to elementary disjunctions and vice versa. The dual of an elementary conjunction $c_0 + cP$, where $c > 0$ and $P = \tilde{x}_1 \wedge \cdots \wedge \tilde{x}_m$, is $1 - c_0 - c + cD$ where $D = P^{\mathrm{d}} = \tilde{x}_1 \vee \cdots \vee \tilde{x}_m$.

(14) The dual of a function f with a DPF

$$c_0 + (c_1 P_1 \vee \cdots \vee c_m P_m)$$

is

$$t + [(-c_1 + c_1 D_1) \wedge \cdots \wedge (-c_m + c_m D_m)],$$

where $t = 1 - c_0$ and $D_i = P_i^{\mathrm{d}}$.

(15) An elementary conjunction p is an implicant of a function f if and only if the elementary disjunction p^{d} is an implicate of f^{d}. Also, p is a prime implicant of f if and only if p^{d} is a prime implicate of f^{d}.

We shall present now an algorithm, called *resolution*, that when applied to any CPF of a pseudo-Boolean function g, yields the canonical CPF of g. Let g be given by the CPF

$$g = t + [(-c_1 + c_1 D_1) \wedge \cdots \wedge (-c_m + c_m D_m)], \tag{42}$$

where each c_i is positive and each D_i is a non-constant join of Boolean literals. Then the dual f of g is given by

$$f = c_0 + (c_1 P_1 \vee \cdots \vee c_m P_m),$$

where $c_0 = 1 - t$ and $P_i = D_i^{\mathrm{d}}$. The algorithm is a generalized version of the Boolean resolution algorithm. Its validity follows directly from the properties of dualization enumerated above.

The algorithm has two basic steps which have to be applied to a CPF given as in (42):

(a) *absorption*: if $(-c_j + c_jD_j) \leqslant (-c_i + c_iD_i)$, $i \neq j$ then delete $(-c_i + c_iD_i)$,

(b) *adjunction of resolution*: if for some variable x, $D_i = x \vee E_i$, $D_j = \bar{x} \vee E_j$, where the elementary disjunctions E_i and E_j are such that $E_i \vee E_j \neq 1$, and if the relations

$$\min(c_i, c_j) \leqslant c_k, \quad E_i \vee E_j \geqslant D_k$$

do not hold for any k then

- if $E_i \vee E_j \neq 0$, adjoin $-\min(c_i, c_j) + \min(c_i, c_j)(E_i \vee E_j)$ to the conjunction to yield,

$$t + [(-c_1 + c_1D_1) \wedge \cdots \wedge (-c_m + c_mD_m) \wedge (-\min(c_i, c_j) + \min(c_i, c_j)(E_i \vee E_j))],$$

- if $E \vee E_j = 0$, then replace t by $t - \min(c_i, c_j)$, delete each $(-c_k + c_kD_k)$ such that $c_k \leqslant \min(c_i, c_j)$ and replace each other c_k by $c_k - \min(c_i, c_j)$.

Note that the test $(-c_j + c_jD_j) \leqslant (-c_i + c_iD_i)$ for the applicability of absorption is equivalent to $c_i \leqslant c_j$ and $D_j \leqslant D_i$.

Analogous to Theorem 5 it can be proved that applying the pseudo-Boolean resolution method to any CPF, as long as it produces new expressions, results after a finite number of iterations in the canonical CPF of the given function.

8. Global and local minima and maxima

We conclude with some remarks concerning global and local extrema of a pseudo-Boolean function f.

(1) If f is given by any DNF (25) such that no term of the disjunction is constant, then

$$\max f = a + \max_i b_i.$$

We also have $a \leqslant \min f$, and if any term of the disjunction (25) is a prime implicant, then

$$\min f = a.$$

This is true in particular if the DNF is the canonical DNF. The consensus algorithm may therefore be viewed as a minimization algorithm. For instance, it appears from the last expression in (41) that -2 is the minimum of the function.

(2) If f is given by any CNF (29) such that no term of the conjunction is constant, then

$$\min f = \min_i a_i.$$

We also have $\max f \leqslant a_i + b_i$ (which is independent of i), and if any term of the conjunction (29) is a prime implicate, then

$$\max f = a_i + b_i.$$

This is true in particular if the CNF is the canonical CNF. The resolution algorithm may therefore be viewed as a maximization algorithm.

For quadratic Boolean DNF's it is obvious that the consensus method allows to decide in polynomial time whether a given DNF represents a constant 1 function. This decision problem is also solvable in polynomial time for Boolean Horn DNF's. It follows that for pseudo-Boolean functions given by quadratic or Horn DNF's, minimization requires only polynomial time. In [6], motivated by an electronic chip design problem, a polynomial time minimization algorithm was presented for quadratic pseudo-Boolean DNF's, using polynomial time reduction to Boolean 2-satisfiability.

Finally, we describe the *local extrema* of a pBf. In B^n, we say that V and W are *adjacent* if they differ in precisely one component. With respect to a given pseudo-Boolean function f, $V \in B^n$ is called a *strict local maximum* if $f(W) < f(V)$ for every adjacent W, and it is called a *strict local minimum* if $f(V) < f(W)$ for every adjacent W.

The following two theorems are easy to verify:

Theorem 6. *Let P be the minterm corresponding to some $V \in B^n$, i.e., $P(V)=1$ and $P(W)=0$ for all $W \neq V$. Then for any pBf f the following conditions are equivalent*:

(i) *V is a strict local maximum of f,*
(ii) *P appears in a term $a + b_i P$ of the canonical DNF of f,*
(ii) *P appears in a term $a + b_i P$ of every DNF,*
(iv) *there is a constant m such that in every DNF $\bigvee_i (a + b_i P_i)$ of f we have $P_i = P$ and $a + b_i = m$ for one of the terms $a + b_i P_i$.*

To illustrate the theorem above, consider again the function f given by (1)–(2). From (41), its canonical DNF is

$$(-2 + 2y) \vee (-2 + 6\bar{x}\bar{y}) \vee (-2 + 2\bar{x}) \vee (-2 + 3xy)$$

The two minterms appearing in the canonical DNF (xy and $\bar{x}\bar{y}$) define the two strict local maxima of the function ($(1,1)$ and $(0,0)$).

Theorem 7. *Let D be the maxterm corresponding to some $V \in B^n$, i.e., $D(V) = 0$ and $D(W) = 1$ for all $W \neq V$. Then for any pBf f the following conditions are equivalent*:

(i) *V is a strict local minimum of f,*
(ii) *D appears in a term $a_i + b_i D$ of the canonical CNF of f,*
(iii) *D appears in a term $a_i + b_i D$ of every CNF of f,*
(iv) *there is a constant m such that in every CNF $\bigwedge_i (a_i + b_i D_i)$ of f we have $D_i = D$ and $a_i = m$ for one of the terms $a_i + b_i D_i$.*

9. For further reading

The following references are also of interest to the reader: [14,19].

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