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Stabilization in elastic solids with voids ☆

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A B S T R A C T

In this paper we study the asymptotic behavior and the analyticity of the solutions of the one-dimensional porous-elasticity problem with thermal effect. Our main result is to prove the lack of exponential stability in case of the porous-elasticity with thermal effect when viscoelasticity is present. We prove the analyticity of the problem when a porous viscosity is present. We conclude by showing the impossibility of localization in time of the solutions in the isothermal case.

1. Introduction

Elasticity problems have attracted the attention of researchers from different fields interested in the temporal decay behavior of the solutions. In the one-dimensional case, for instance, it is known that combining the equations of elasticity with thermal effects provokes the exponential decay of the solution. If elastic solids with voids are considered, as in this paper, one should look into the theory of porous elastic materials. Here we deal with the theory established by Cowin and Nunziato [5,6,18]. As we are going to work with the theories where the thermal effects and viscosity effects are present we recall the contributions by Ieşan [8–11].

The analysis of the temporal decay in one-dimensional porous-elastic materials was started by Quintanilla [20]. The author showed that the dissipation given by the porous viscosity was not powerful enough to obtain exponential stability to the solutions, that is the decay of the solutions can be very slow. For this reason, several other dissipative mechanisms were considered in the recent contributions [3,4,14–16]. We recall the main conclusions with the help of a scheme:

$$
\boxed{\begin{array}{l}\text{Thermal effect}\\[1em]\text{Viscoelastic effect}\end{array}}
\;\longrightarrow\;
\boxed{\begin{array}{c}\text{Elasticity}\\ \updownarrow\\ \text{Porosity}\end{array}}
\;\longleftarrow\;
\boxed{\begin{array}{l}\text{Microthermal effect}\\[1em]\text{Viscoporous effect}\end{array}}
$$

If we take simultaneously one effect from the right square and another one from the left square, then we get exponential stability. However, if we consider two simultaneous effects from one square only, then we get slow decay. In fact, in this direction it is proved in [17], that some of the models studied decay polynomially with rates of decay that depends on the regularity of the initial data. Which means that the decay can be very slow provided the initial data is not regular. Recently, Z. Liu and B. Rao [12] and A. Batkai et al. [1] find sufficient conditions to get a polynomial decay of semigroup operators. These conditions depends essentially on the regularity of the initial data and also on some estimates of the

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resolvent operator. One interesting point about this result is that in the two references above there exists a lack of optimality concerning the polynomial rate of decay of the solutions. That is to say, the rate of decay is like $1/t^{1-\varepsilon}$ where such ε seems to appear for technical reasons. J. Muñoz Rivera and R. Quintanilla [17], find a polynomial decay for several porous-thermo-elastic models, which seems to be optimal in the sense that no additional parameter appears in the decay estimate, that is the parameter ε given in [1,12] is removed.

In the one-dimensional case the evolution equations for the theory of elastic solids with voids are given by

$$\rho u_{tt} = \mathbf{t}_x, \qquad \rho\kappa\varphi_{tt} = h_x + g, \qquad \rho T_0 \Xi_t = q_x. \tag{1.1}$$

Here, $\mathbf{t}$ is the stress, h is the equilibrated stress, g is the equilibrated body force, q is the heat flux and T_0 is the absolute temperature in the reference configuration which is assumed positive. The variables u, φ and Ξ are the displacement of the solid elastic material, the volume fraction and the entropy, respectively. We assume that ρ and κ are positive constants whose physical meaning is well known. In general, we can consider several dissipation mechanisms in this theory (see [11]). We here, restrict our attention to the case that the viscoelasticity is present and the viscosity at the microstructure is also present apart the temperature effect. That is in our case, we assume the following constitutive equations (see [11])

$$\mathbf{t} = \mu u_x + b\varphi - \beta\theta + \gamma u_{xt}, \qquad h = \delta\varphi_x + \eta\varphi_{xt} + k_1\theta_x, \qquad g = -bu_x - \xi\varphi + m\theta,$$

$$\rho\Xi = \beta u_x + c\theta + m\varphi, \qquad q = k\theta_x + k_2\varphi_{xt}.$$

It is assumed that the internal mechanical energy density is a positive definite form. Thus, the constitutive coefficients satisfy the conditions

$$\mu > 0, \qquad \delta > 0, \qquad \mu\xi > b^2. \tag{1.2}$$

The dissipation of the system is defined with the help of the function

$$\Pi = \gamma|u_{xt}|^2 + \eta|\varphi_{xt}|^2 + (k_1 + k_2)\varphi_{xt}\theta_x + k|\theta_x|^2.$$

Thus, when the dissipation is assumed we need to guarantee that this function is greater than zero (see condition (4.4)). In particular when we assume that η or k vanish then we also have $k_1 = k_2 = 0$. If we introduce the constitutive equations in the evolution equations, we obtain the field equations

$$\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x + \gamma u_{xxt}, \tag{1.3}$$

$$J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + m\theta + \eta\varphi_{xxt} + k_1\theta_{xx}, \tag{1.4}$$

$$c\theta_t = k^*\theta_{xx} - \beta u_{xt} - m\varphi_t + k_2^*\varphi_{xxt}. \tag{1.5}$$

Here $J = \rho\kappa$, $k^* = kT_0^{-1}$ and $k_2^* = k_2T_0^{-1}$, but in the sequel, we will omit the star.

As coupling is considered, b must be different from 0, but its sign does not matter in the analysis. As thermal effects is considered, we assume that the thermal capacity c and the thermal conductivity k are strictly positive. The sign of the coupling term β does not matter in the analysis neither. And as viscoelastic dissipation is assumed in the system, $\gamma > 0$. In the first part of the paper we assume that the porous dissipation is absent ($\eta = k_1 = k_2 = 0$).

Here we assume that the solutions satisfy the boundary conditions

$$u(0,t) = u(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0, \tag{1.6}$$

and the initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x), \qquad \theta(x,0) = \theta_0(x). \tag{1.7}$$

There are solutions (uniform in the variable x) that do not decay. To avoid these cases, we will also assume that

$$\int_0^\pi \varphi_0(x)\,dx = \int_0^\pi \varphi_1(x)\,dx = \int_0^\pi \theta_0(x)\,dx = 0. \tag{1.8}$$

Finally, any time we use the semigroup theory, we consider the complex phase space, that is the functions u,φ and θ will be of complex value. Instead, when we consider the evolution model, we consider the functions u,φ and θ as reals functions.

This paper is structured as follows. In Section 2 we state the equations for the one-dimensional porous-elasticity problem when the viscoelastic and thermal effect are present. We show that the problem is well posed and that there is not exponential decay of the solution. In Section 3 we use essentially the energy method to show the polynomial stability. Moreover using a result on [1] we are able to improve the polynomial rate of decay by taking more regular initial data. The difference of our work to [17] is that we consider also the viscoelastic effect in the porous-thermo-elastic problem. The point is that this extra thermal dissipation does not change the lack of exponential stability. In Section 4 we consider the model with an extra viscosity in the porous structure, and we show that the corresponding system is analytic, which in particular implies the exponential decay and the *spectrum determined growth property* (SDG-property). In the last section we prove the impossibility of localization of solutions in the isothermal case.

2. Well-posedness and the lack of exponential stability

In this section we prove the lack of exponential stability and that there exists only one solution to the problem

$$\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x + \gamma u_{xxt}, \tag{2.1}$$

$$J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + m\theta, \tag{2.2}$$

$$c\theta_t = k\theta_{xx} - \beta u_{xt} - m\varphi_t, \tag{2.3}$$

with the conditions (1.6)–(1.8). Here, the variables u, φ and θ are the displacement of the solid elastic material, the volume fraction and the temperature, respectively. The constitutive coefficients ρ, μ, γ, J, δ, ξ, c and k are positive constants and as coupling is considered b, β and m must be different from 0, but its sign does not matter in the analysis.

We consider the Hilbert space

$$\mathcal{H} = H_0^1(0,\pi) \times L^2(0,\pi) \times H_*^1(0,\pi) \times L_*^2(0,\pi) \times L_*^2(0,\pi),$$

where

$$H_*^m(0,\pi) = \left\{ w \in H^m(0,\pi);\ \int_0^\pi w\,dx = 0 \right\} \quad \text{and} \quad L_*^2(0,\pi) = \left\{ w \in L^2(0,\pi);\ \int_0^\pi w\,dx = 0 \right\}$$

with inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_0^\pi \left[\rho v\bar{v}^* + \mu u_x \bar{u}_x^* + J\phi\bar{\phi}^* + \delta\varphi_x\bar{\varphi}_x^* + \xi\varphi\bar{\varphi}^* + c\theta\bar{\theta}^* + b\big(u_x\bar{\varphi}^* + \bar{u}_x^*\varphi\big)\right] dx,$$

where $U = (u, v, \varphi, \phi, \theta)^T$ and $U^* = (u^*, v^*, \varphi^*, \phi^*, \theta^*)^T$. The corresponding norm in $\mathcal{H}$ is given by

$$\|U\|_{\mathcal{H}} = \int_0^\pi \left[\rho|v|^2 + \mu|u_x|^2 + J|\phi|^2 + \delta|\varphi_x|^2 + \xi|\varphi|^2 + c|\theta|^2 + 2b\,\mathrm{Re}\,u_x\bar{\varphi}\right] dx.$$

Let us introduce the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \rho^{-1}\mu D^2 & \rho^{-1}\gamma D^2 & \rho^{-1} bD & 0 & -\rho^{-1}\beta D \\ 0 & 0 & 0 & I & 0 \\ -J^{-1}bD & 0 & J^{-1}(\delta D^2 - \xi I) & 0 & J^{-1}mI \\ 0 & -c^{-1}\beta D & 0 & -c^{-1}mI & c^{-1}kD^2 \end{pmatrix}, \tag{2.4}$$

where I is the identity operator and $D^i = \frac{d^i}{dx^i}$. The initial–boundary value problem (2.2)–(1.6) is equivalent to problem

$$U_t = \mathcal{A}U, \qquad U(0) = U_0 \in \mathcal{D}(\mathcal{A}), \tag{2.5}$$

where $U_0 = (u_0, u_1, \varphi_0, \varphi_1, \theta_0)^T$ and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. The domain of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \big\{ U \in \mathcal{H};\ \mu u + \gamma v \in H^2 \cap H_0^1;\ \varphi, \theta \in H^2;\ \phi \in H^1;\ D\varphi = D\phi = D\theta = 0,\ x = 0, \pi \big\}.$$

Note that $\mathcal{A}$ is dissipative, that is

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\int_0^\pi \big(\gamma v_x^2 + k\theta_x^2\big)\,dx \leqslant 0. \tag{2.6}$$

Lemma 2.1. *Under the above notations we have that $0 \in \varrho(\mathcal{A})$, where $\varrho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.*

Proof. For any $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we want to find $U = (u, v, \varphi, \phi, \theta)^T \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}U = F, \tag{2.7}$$

in terms of the components we get

$$v = f_1, \tag{2.8}$$

$$\mu u_{xx} + b\varphi_x + \gamma v_{xx} - \beta\theta_x = \rho f_2, \tag{2.9}$$

$$\phi = f_3, \tag{2.10}$$

$$\delta\varphi_{xx} - bu_x - \xi\varphi + m\theta = Jf_4, \tag{2.11}$$

$$k\theta_{xx} - \beta v_x - m\phi = cf_5. \tag{2.12}$$

By (2.8) and (2.10) we have

$$v \in H_0^1(0,\pi) \quad \text{and} \quad \phi \in H_*^1(0,\pi). \tag{2.13}$$

By (2.8), (2.9) and (2.12) we can write

$$k\theta_{xx} = \beta(f_1)_x + mf_3 + cf_5 \in L_*^2(0,\pi). \tag{2.14}$$

We conclude that there exists a unique function $\theta \in H^2(0,\pi)$ satisfying (2.14). Then, the remanning point is to prove that there exist u and φ satisfying

$$\mu u_{xx} + b\varphi_x = F := -\gamma(f_1)_{xx} + \beta\theta_x + \rho f_2 \in H^{-1}(0,\pi), \tag{2.15}$$

$$\delta\varphi_{xx} - bu_x - \xi\varphi = G := -m\theta + Jf_4 \in L^2(0,\pi). \tag{2.16}$$

Introducing the space $W = H_0^1(0,\pi) \cap H_*^1(0,\pi)$, and denoting the bilinear

$$a(V,\widetilde{V}) = \mu\int_0^\pi u_x\overline{\widetilde{u}}_x\,dx - 2b\,\mathrm{Re}\int_0^\pi \varphi_x\overline{\widetilde{u}}\,dx + \delta\int_0^\pi \varphi_x\overline{\widetilde{\varphi}}_x\,dx + \xi\int_0^\pi \varphi\overline{\widetilde{\varphi}}\,dx$$

we conclude that $a(\cdot,\cdot)$ is a coercive, continuous bilinear operator over the Hilbert space W. Therefore there exists a solution to the variational equation

$$a(U,V) = \big\langle (F,G), V\big\rangle$$

that is equivalent do system (2.15)–(2.16). □

Under this conditions we have:

Theorem 2.2. *Under the above conditions we have that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$ of contractions over the space $\mathcal{H}$.*

Next we will prove that the semigroup $\mathcal{T}$ associated to systems (2.1)–(2.3) is not exponentially stable. This result was proved in [15]. Here, we propose an alternative proof. To do this we use Prüss result; see [19].

Theorem 2.3. *Let us consider $A : D(A) \subseteq H \to H$ a generator of a C_0-semigroup of contractions. Then e^{At} is exponentially stable if and only if*

(i) $i\mathbb{R} \subset \varrho(A)$;
(ii) $\|(i\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C$, $\forall\lambda \in \mathbb{R}$,

where I is the identity operator.

Under the above conditions we are able to show the main result of this section.

Theorem 2.4. *Let (u,φ,θ) be a solution of the problem determined by* (2.1)–(2.3) *with boundary conditions* (1.6) *and initial conditions* (1.7)*. If the initial data satisfy condition* (1.8)*, then the semigroup generated by operator $\mathcal{A}$ given in* (2.4) *is not exponentially stable.*

Proof. It suffices to show the existence of sequences $(\lambda_n)_n \subset i\mathbb{R}$ with $\lim_{n\to\infty}|\lambda_n| = \infty$ and $(U_n)_n \subset \mathcal{D}(\mathcal{A})$ to $(F_n)_n \subset \mathcal{H}$ such that $(\lambda_n I - \mathcal{A})U_n = F_n$ is bounded in $\mathcal{H}$ and

$$\lim_{n\to\infty}\|U_n\|_{\mathcal{H}} = \infty.$$

We choose $F \equiv F_n$ with $F = (0,0,0,g,0)^T$ where $g = J^{-1}\cos(nx)$. We have that F_n is bounded in $\mathcal{H}$ and the solution $U_n = U = (u,v,\varphi,\phi,\theta)^T$ to $(\lambda I - \mathcal{A})U = F$ has to satisfy

$$\lambda u = v,$$
$$\rho\lambda v - \mu u_{xx} - b\varphi_x - \gamma v_{xx} + \beta\theta_x = 0,$$
$$\lambda\varphi = \phi,$$
$$J\lambda\phi - \delta\varphi_{xx} + bu_x + \xi\varphi - m\theta = Jg,$$
$$c\lambda\theta - k\theta_{xx} + \beta v_x + m\phi = 0.$$

This will determine v, ϕ and we obtain for u, φ, θ:

$$\rho\lambda^2 u - \mu u_{xx} - b\varphi_x - \gamma\lambda u_{xx} + \beta\theta_x = 0, \tag{2.17}$$
$$J\lambda^2\varphi - \delta\varphi_{xx} + bu_x + \xi\varphi - m\theta = \cos(nx), \tag{2.18}$$
$$c\lambda\theta - k\theta_{xx} + \beta\lambda u_x + m\lambda\varphi = 0. \tag{2.19}$$

Because of the boundary conditions we can take solution of type

$$u = A\sin(nx), \qquad \varphi = B\cos(nx) \quad \text{and} \quad \theta = C\cos(nx), \tag{2.20}$$

for appropriate $A = A(\lambda)$, $B = B(\lambda)$ and $C = C(\lambda)$. Substituting (2.20) into (2.17)–(2.19), we find that A, B and C satisfy

$$\big(\rho\lambda^2 + \mu n^2 + \gamma\lambda n^2\big)A + bnB - \beta nC = 0, \tag{2.21}$$
$$bnA + \big(J\lambda^2 + \delta n^2 + \xi\big)B - mC = 1, \tag{2.22}$$
$$\beta\lambda nA + m\lambda B + \big(c\lambda + kn^2\big)C = 0. \tag{2.23}$$

Taking λ such that $J\lambda^2 + \delta n^2 + \xi = 0$. That is, $\lambda = \sqrt{\frac{1}{J}(\delta n^2 + \xi)}$, $i = \lambda_n$. Then system (2.21)–(2.23) is equivalent to

$$\big(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2\big)A + bnB - \beta nC = 0, \tag{2.24}$$
$$bnA - mC = 1, \tag{2.25}$$
$$\beta\lambda_n nA + m\lambda_n B + \big(c\lambda_n + kn^2\big)C = 0. \tag{2.26}$$

Solving the system (2.24)–(2.26), we have

$$A = \frac{b^2n^2(c\lambda_n + kn^2) + mbn^2\beta\lambda_n}{b^3n^3(c\lambda_n + kn^2) + 2mb^2n^3\beta\lambda_n - bnm^2\lambda_n(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2)},$$
$$B = \frac{-b^2n^2(c\lambda_n + kn^2)(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2) - b^2n^4\beta^2\lambda_n}{b^4n^4(c\lambda_n + kn^2) + 2mb^3n^4\beta\lambda_n - m^2b^2n^2\lambda_n(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2)}$$

and

$$C = \frac{m\lambda_n(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2) - bn^2\beta\lambda_n}{b^2n^2(c\lambda_n + kn^2) + 2mbn^2\beta\lambda_n - m^2\lambda_n(\rho\lambda_n^2 + \mu n^2 + \gamma\lambda_n n^2)}.$$

That is to say

$$nA \to \frac{kb^2}{kb^3 + bm^2\gamma\delta/J}, \qquad \frac{B}{n} \to \frac{kb^2\gamma}{kb^4 + b^2m^2\gamma\delta/J}, \qquad C \to \frac{m\gamma\delta/J}{kb^2 + m^2\gamma\delta/J},$$

as $n \to \infty$. From where we conclude that

$$B_n \to \infty, \quad \text{as } n \to \infty.$$

Using (2.1) and recalling the definition of φ_n, we get

$$\|U_n\|_{\mathcal{H}} \geqslant c\int_0^\pi |\varphi_n|^2\,dx = cB_n\frac{\pi}{2} \to \infty.$$

Which completes the proof. □

3. Polynomial decay

In this section we will prove that the time decay of the solutions of the problem determined by the system can be controlled by a polynomial. We prove the polynomial decay of solutions for the boundary conditions (1.6).

We recall a result due to A. Batkai et al. [1], for we can improve the polynomial rate of decay, by taking more regular initial data:

Theorem 3.1. *Assume that A is an operator invertible and the infinitesimal generator of a C_0-semigroup $T(t)$ over the Hilbert space H such that $\|T(t)\| \leqslant M$, $\forall t \geqslant 0$. Then the following statements are equivalent with a constant $\gamma > 0$:*

(i) $\|T(t)A^{-\gamma}\|_{\mathcal{L}(H)} \leqslant Ct^{-\beta}$, $t > 0$;
(ii) $\|T(t)A^{-\gamma\alpha}\|_{\mathcal{L}(H)} \leqslant C_\alpha t^{-\alpha\beta}$, $t > 0$, $\alpha > 0$.

We define the first order energy as

$$E_1(t,u,\varphi,\theta) = \frac{1}{2}\int_0^\pi \left[\rho|u_t|^2 + \mu u_x^2 + J|\varphi_t|^2 + \delta\varphi_x^2 + \xi\varphi^2 + c\theta^2 + 2b\varphi u_x\right] dx. \tag{3.1}$$

Then we introduce the second order energy as

$$E_2(t) = E_1(t, u_t, \varphi_t, \theta_t) \tag{3.2}$$

and the third order energy as

$$E_3(t) = E_1(t, u_x, \varphi_x, \theta_x). \tag{3.3}$$

After several integrations by parts, we can see that

$$\frac{dE_1}{dt} = -\int_0^\pi \left(\gamma|u_{xt}|^2 + k|\theta_x|^2\right) dx, \tag{3.4}$$

$$\frac{dE_2}{dt} = -\int_0^\pi \left(\gamma|u_{xtt}|^2\, dx + k|\theta_{xt}|^2\right) dx \tag{3.5}$$

and

$$\frac{dE_3}{dt} = -\int_0^\pi \left(\gamma|u_{xxt}|^2\, dx + k|\theta_{xx}|^2\right) dx. \tag{3.6}$$

Let us introduce the functional

$$S(t) = \int_0^\pi \left(\rho u u_t + J\varphi\varphi_t + \frac{\gamma}{2}|u_x|^2\right) dx.$$

Lemma 3.2. *Let us suppose that initial data $U_0 = (u_0, u_1, \varphi_0, \varphi_1, \theta_0)^T \in \mathcal{D}(\mathcal{A})$ then the following inequality*

$$\frac{dS}{dt} \leqslant \rho\int_0^\pi |u_t|^2\, dx + J\int_0^\pi |\varphi_t|^2\, dx - \gamma_1\int_0^\pi \left(u_x^2 + \varphi_x^2 + \varphi^2\right) dx + c_1\int_0^\pi \theta_x^2\, dx - 2b\int_0^\pi \varphi u_x\, dx \tag{3.7}$$

holds, where γ_1 and c_1 are positive and calculable constants.

Proof. Let us multiply Eq. (2.1) by u to get, we have

$$\frac{d}{dt}\int_0^\pi \rho u_t u\, dx = \rho\int_0^\pi |u_t|^2\, dx + \int_0^\pi \rho u_{tt} u\, dx = \int_0^\pi \rho|u_t|^2\, dx - \mu\int_0^\pi |u_x|^2\, dx - b\int_0^\pi \varphi u_x\, dx + \beta\int_0^\pi \theta u_x\, dx - \frac{d}{dt}\int_0^\pi \frac{\gamma}{2}|u_x|^2\, dx.$$

So, we have

$$\frac{d}{dt}\int_0^\pi \left(\rho u u_t + \frac{\gamma}{2}|u_x|^2\right) dx = \rho\int_0^\pi |u_t|^2\,dx - \mu\int_0^\pi |u_x|^2\,dx - b\int_0^\pi \varphi u_x\,dx + \beta\int_0^\pi \theta u_x\,dx. \tag{3.8}$$

Using Eq. (2.2), we have

$$\frac{d}{dt}\int_0^\pi J\varphi\varphi_t\,dx = J\int_0^\pi |\varphi_t|^2\,dx + \int_0^\pi J\varphi_{tt}\varphi\,dx = J\int_0^\pi |\varphi_t|^2\,dx - \delta\int_0^\pi |\varphi_x|^2\,dx - b\int_0^\pi \varphi u_x\,dx - \xi\int_0^\pi |\varphi|^2\,dx + m\int_0^\pi \theta\varphi\,dx. \tag{3.9}$$

Using Eqs. (3.8) and (3.9) and recalling the definition of S, we get

$$\frac{dS}{dt} = \rho\int_0^\pi |u_t|^2\,dx - \mu\int_0^\pi |u_x|^2\,dx - 2b\int_0^\pi \varphi u_x\,dx + \beta\int_0^\pi \theta u_x\,dx + J\int_0^\pi |\varphi_t|^2\,dx - \delta\int_0^\pi |\varphi_x|^2\,dx - \xi\int_0^\pi |\varphi|^2\,dx + m\int_0^\pi \theta\varphi\,dx.$$

Using the Young and Poincaré inequalities we obtain (3.7). □

Let us the functional

$$Q(t) = \frac{1}{m}\int_0^\pi c\theta\varphi_t\,dx.$$

Lemma 3.3. *Let us suppose that initial data $U_0 = (u_0, u_1, \varphi_0, \varphi_1, \theta_0)^T \in \mathcal{D}(\mathcal{A})$ then for any $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that*

$$\frac{dQ}{dt} \leqslant -\frac{1}{2}\int_0^\pi |\varphi_t|^2\,dx + c_\epsilon\int_0^\pi \left(|u_{xt}|^2 + |\theta_{xx}|^2 + |\theta_x|^2\right) dx + \epsilon\int_0^\pi \left(|u_x|^2 + |\varphi_x|^2\right) dx. \tag{3.10}$$

Proof. Using Eq. (2.3), we have

$$\frac{1}{m}\frac{d}{dt}\int_0^\pi c\theta\varphi_t\,dx = \frac{1}{m}\int_0^\pi c\theta\varphi_{tt}\,dx + \frac{1}{m}\int_0^\pi c\theta_t\varphi_t\,dx = \frac{c}{m}\int_0^\pi \theta\varphi_{tt}\,dx + \frac{k}{m}\int_0^\pi \theta_{xx}\varphi_t\,dx - \frac{\beta}{m}\int_0^\pi u_{xt}\varphi_t\,dx - \int_0^\pi |\varphi_t|^2\,dx. \tag{3.11}$$

From Eqs. (2.2) and (3.11) we have

$$\frac{1}{m}\frac{d}{dt}\int_0^\pi c\theta\varphi_t\,dx = -\frac{c}{mJ}\int_0^\pi (\delta\theta_x\varphi_x + b\theta u_x + \xi\theta\varphi)\,dx + \frac{c}{J}\int_0^\pi |\theta|^2\,dx + \frac{k}{m}\int_0^\pi \theta_{xx}\varphi_t\,dx - \frac{\beta}{m}\int_0^\pi u_{xt}\varphi_t\,dx - \int_0^\pi |\varphi_t|^2\,dx. \tag{3.12}$$

From the above inequality our conclusion follows. □

Now, we are in conditions to show the main result of this section.

Theorem 3.4. *Let (u, φ, θ) be a solution of the problem determined by* (2.1)–(2.3) *with boundary conditions* (1.6) *and initial conditions* (1.7)*. If the initial data satisfy condition* (1.8)*, then there exists a positive constant C such that*

$$E_1(t) \leqslant \frac{C\|(u_0, u_1, \varphi_0, \varphi_1, \theta_0)\|_{\mathcal{D}(\mathcal{A})}}{t}. \tag{3.13}$$

Moreover, if $(u_0, u_1, \varphi_0, \varphi_1, \theta_0) \in \mathcal{D}(\mathcal{A}^\alpha)$, then there exists a positive constant C_α such that

$$E_1(t) \leqslant \frac{C_\alpha\|(u_0, u_1, \varphi_0, \varphi_1, \theta_0)\|_{\mathcal{D}(\mathcal{A}^\alpha)}}{t^\alpha}. \tag{3.14}$$

Proof. We define the functional

$$\mathcal{L}(t) = S(t) + NQ(t) + N_1E_1(t) + N_3E_3(t),$$

where N, N_1 and N_3 are sufficiently greater to guarantee that $\mathcal{L}(t)$ is positive. From Lemmas 3.2 and 3.3 we have

$$\frac{d\mathcal{L}}{dt} \leqslant -\gamma_3 E_1(t), \tag{3.15}$$

where $\gamma_3 > 0$ can be calculated. Integration over $[0, t]$ implies

$$\mathcal{L}(t)+\gamma_3\int_0^t E_1(s)\,ds\leqslant \mathcal{L}(0). \tag{3.16}$$

Then

$$\frac{d}{dt}\big(tE_1(t)\big)=E_1(t)+t\frac{dE_1}{dt}\leqslant E_1(t). \tag{3.17}$$

Integration over $[0,t]$ and using (3.16), we have

$$tE_1(t)\leqslant\int_0^t E_1(s)\,ds\leqslant \gamma_3^{-1}\mathcal{L}(0).$$

which implies the polynomial decay. To improve the polynomial decay we use Theorem 3.1. □

4. Analyticity

In this section we prove the analyticity of the semigroup which defines the solutions of the problem (1.3)–(1.5) with the conditions (1.6)–(1.8). To guarantee that the system dissipates energy we also need to assume that the constitutive coefficients η, k_1 and k_2 satisfy the condition

$$(k_1+k_2)^2<4k\eta. \tag{4.1}$$

Here we consider $k_1,k_2\geqslant 0$. We note that the solutions of this problem can be generated by means of a semigroup of contractions. In fact, this semigroup is defined in the Hilbert space

$$\mathcal{H}=H_0^1(0,\pi)\times L^2(0,\pi)\times H_*^1(0,\pi)\times L_*^2(0,\pi)\times L_*^2(0,\pi)$$

by the operator

$$\mathcal{A}=\begin{pmatrix} 0 & I & 0 & 0 & 0\\ \rho^{-1}\mu D^2 & \rho^{-1}\gamma D^2 & \rho^{-1}bD & 0 & -\rho^{-1}\beta D\\ 0 & 0 & 0 & I & 0\\ -J^{-1}bD & 0 & J^{-1}(\delta D^2-\xi I) & J^{-1}\eta D^2 & J^{-1}(mI+k_1D^2)\\ 0 & -c^{-1}\beta D & 0 & c^{-1}(k_2D^2-mI) & c^{-1}kD^2\end{pmatrix} \tag{4.2}$$

where I is the identity operator. The initial–boundary value problem (1.3)–(1.5) with (1.6)–(1.7) is equivalent to solve the Cauchy problem

$$U_t=\mathcal{A}U,\qquad U(0)=U_0\in\mathcal{D}(\mathcal{A}), \tag{4.3}$$

where $U=(u,v,\varphi,\phi,\theta)^T$, $U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0)^T$ and $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$. The domain of $\mathcal{A}$ is

$$\begin{aligned}\mathcal{D}(\mathcal{A})=\big\{&U\in\mathcal{H};\ \mu u+\gamma v\in H^2(0,\pi)\cap H_0^1(0,\pi);\ \delta\varphi+\eta\phi+k_1\theta\in H_*^2(0,\pi);\\ &k\theta+k_2\phi\in H_*^2(0,\pi);\ D\varphi=D\phi=D\theta=0,\ x=0,\pi\big\}.\end{aligned}$$

Now, we recall the inner product in $\mathcal{H}$ defined at Section 2. We note that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ and

$$\operatorname{Re}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-\gamma\int_0^\pi v_x^2\,dx-k\int_0^\pi\theta_x^2\,dx-\eta\int_0^\pi\phi_x^2-(k_1+k_2)\operatorname{Re}\left(\int_0^\pi\theta_x\phi_x\,dx\right)\leqslant-\gamma\int_0^\pi v_x^2\,dx-M\int_0^\pi\big(k\theta_x^2+\eta\phi_x^2\big)\,dx\leqslant 0,$$

where $M=1-\frac{k_1+k_2}{2\sqrt{k\eta}}>0$ and $\gamma,k,\eta>0$. Then $\mathcal{A}$ is dissipative. As in Lemma 2.1 we have that 0 is in the resolvent of $\mathcal{A}$. Therefore, from Lummer–Phillips's theorem we conclude that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup.

To show the analyticity of the C_0-semigroup of contractions generated for operator A on a Hilbert space H, we have the following result due to Liu and Zheng (see [13]):

Theorem 4.1. *Let us consider $S(t)=e^{At}$ a C_0-semigroup of contractions generated for operator A in Hilbert space H. Suppose that*

$$\varrho(A)\supseteq\{i\beta;\ \beta\in\mathbb{R}\}\equiv i\mathbb{R}. \tag{4.4}$$

Then $S(t)$ is analytic if and only if

$$\overline{\lim_{|\beta|\to\infty}}\big\|\beta(i\beta I-A)^{-1}\big\|<\infty,\quad \beta\in\mathbb{R}, \tag{4.5}$$

holds.

The resolvent equation is given by

$$\lambda U - \mathcal{A}U = F \tag{4.6}$$

where

$$U = \begin{pmatrix} u \\ v \\ \varphi \\ \phi \\ \theta \end{pmatrix}, \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \quad \text{and} \quad \lambda \in \mathbb{C}.$$

To show the analyticity we shall take $\lambda = i\alpha, \alpha \in \mathbb{R}$. Written Eq. (4.6) with $\lambda = i\alpha$, $\alpha \in \mathbb{R}$, we have

$$i\alpha u - v = f_1, \tag{4.7}$$

$$i\alpha\rho v - \mu u_{xx} - b\varphi_x - \gamma v_{xx} + \beta\theta_x = \rho f_2, \tag{4.8}$$

$$i\alpha\varphi - \phi = f_3, \tag{4.9}$$

$$i\alpha J\phi - \delta\varphi_{xx} + bu_x + \xi\varphi - m\theta - \eta\phi_{xx} - k_1\theta_{xx} = Jf_4, \tag{4.10}$$

$$i\alpha c\theta - k\theta_{xx} + \beta v_x + m\phi - k_2\phi_{xx} = cf_5. \tag{4.11}$$

To show the main result of this section we need of the following lemmas.

Lemma 4.2. *For any $F \in \mathcal{H}$, there exists a constant $c_1 > 0$ such that*

$$\gamma\int_0^\pi |v_x|^2\,dx + M\int_0^\pi \big(k|\theta_x|^2 + \eta|\phi_x|^2\big)\,dx \leqslant c_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{4.12}$$

where $M = 1 - \frac{k_1+k_2}{2\sqrt{k\eta}} > 0$.

Proof. Multiplying Eqs. (4.7)–(4.11), respectively, for $-\mu\bar{u}_{xx}$, $\bar{v}$, $-\delta\bar{\varphi}_{xx}$ and $\xi\bar{\varphi}$, $\bar{\phi}$, $\bar{\theta}$, integrating from 0 to π and summing the equations, we find that

$$\begin{aligned} &i\alpha\int_0^\pi \big[\rho|v|^2 + \mu|u_x|^2 + \delta|\varphi_x|^2 + J|\phi|^2 + \xi|\varphi|^2 + c|\theta|^2\big]\,dx + \mu\int_0^\pi (u_x\bar{v}_x - \bar{u}_x v_x)\,dx + b\int_0^\pi (\varphi\bar{v}_x + u_x\bar{\phi})\,dx \\ &\quad + m\int_0^\pi (\phi\bar{\theta} - \theta\bar{\phi})\,dx + \delta\int_0^\pi (\varphi_x\bar{\phi}_x - \bar{\varphi}_x\phi_x)\,dx + \beta\int_0^\pi (\theta_x\bar{v} - \bar{\theta}_x v)\,dx + \xi\int_0^\pi (\varphi\bar{\phi} - \bar{\varphi}\phi)\,dx \\ &\quad + \int_0^\pi (k_1\theta_x\bar{\phi}_x + k_2\phi_x\bar{\theta}_x)\,dx + \int_0^\pi \big(\gamma|v_x|^2 + \eta|\phi_x|^2 + k|\theta_x|^2\big)\,dx = R \end{aligned} \tag{4.13}$$

where $|R| \leqslant \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$. Taking real part in Eq. (4.13) using the condition (4.1) and the definition of norm in $\mathcal{H}$, we have

$$\gamma\int_0^\pi |v_x|^2\,dx + \left(1 - \frac{k_1+k_2}{2\sqrt{k\eta}}\right)\int_0^\pi \big(k|\theta_x|^2 + \eta|\phi_x|^2\big)\,dx \leqslant c_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

where c_1 is a calculable positive constant. Our conclusion follows to $M = 1 - \frac{k_1+k_2}{2\sqrt{k\eta}} > 0$. $\square$

Lemma 4.3. *For any $F \in \mathcal{H}$, there exists $C > 0$ such that*

$$|\alpha|\|U\|_{\mathcal{H}} \leqslant C\||F\|_{\mathcal{H}}, \quad \forall\alpha \in \mathbb{R},$$

where U is the solution for (4.6) with $\lambda = i\alpha$.

Proof. Multiplying Eqs. (4.7)–(4.11), respectively, for $i\mu\bar{u}_{xx}$, $-i\bar{v}$, $i\delta\bar{\varphi}_{xx}$ and $-i\xi\bar{\varphi}$, $-i\bar{\phi}$, $-i\bar{\theta}$, integrating from 0 to π and summing the equations, we find that

$$\alpha \int_0^\pi \big[\rho|v|^2 + \mu|u_x|^2 + \delta|\varphi_x|^2 + J|\phi|^2 + \xi|\varphi|^2 + c|\theta|^2\big]\,dx - ib\int_0^\pi (\varphi\bar{v}_x + u_x\bar{\phi})\,dx + i\mu\int_0^\pi (v_x\bar{u}_x - \bar{v}_x u_x)\,dx$$
$$+ i\xi\int_0^\pi (\phi\bar{\varphi} - \varphi\bar{\phi})\,dx + im\int_0^\pi (\theta\bar{\phi} - \phi\bar{\theta})\,dx + i\beta\int_0^\pi (v\bar{\theta}_x - \bar{v}\theta_x)\,dx + i\delta\int_0^\pi (\phi_x\bar{\varphi}_x - \varphi_x\bar{\phi}_x)\,dx$$
$$- i\int_0^\pi (k_1\theta_x\bar{\phi}_x + k_2\phi_x\bar{\theta}_x)\,dx - i\int_0^\pi \big(\gamma|v_x|^2 + \eta|\phi_x|^2 + k|\theta_x|^2\big)\,dx = \widetilde{R} \tag{4.14}$$

where $|\widetilde{R}| \leqslant \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$. Multiplying Eqs. (4.7) and (4.9), respectively, for $ib\bar{\varphi}_x$ and $-ib\bar{u}_x$, integrating from 0 to π and summing the equations, we find that

$$\alpha\int_0^\pi 2b\,\mathrm{Re}(u_x\bar{\varphi})\,dx + ib\int_0^\pi (\phi\bar{u}_x - v\bar{\varphi}_x)\,dx = ib\int_0^\pi (f_1\bar{\varphi}_x - f_3\bar{u}_x)\,dx. \tag{4.15}$$

Summing (4.15) and (4.14) using (4.12) and the definition of the norm in $\mathcal{H}$, we have

$$\alpha\|U\|_{\mathcal{H}}^2 \leqslant \mathrm{Re}\Bigg\{ ib\int_0^\pi (\varphi\bar{v}_x - \bar{\varphi}v_x)\,dx + ib\int_0^\pi (u_x\bar{\phi} - \bar{u}_x\phi)\,dx - i\mu\int_0^\pi (v_x\bar{u}_x - \bar{v}_x u_x)\,dx$$
$$- i\delta\int_0^\pi (\phi_x\bar{\varphi}_x - \varphi_x\bar{\phi}_x)\,dx - i\beta\int_0^\pi (v\bar{\theta}_x - \bar{v}\theta_x)\,dx + i\xi\int_0^\pi (\phi\bar{\varphi} - \varphi\bar{\phi})\,dx$$
$$- i\int_0^\pi (k_1\theta_x\bar{\phi}_x + k_2\phi_x\bar{\theta}_x)\,dx - im\int_0^\pi (\theta\bar{\phi} - \phi\bar{\theta})\,dx \Bigg\} + c_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.16}$$

Using (4.12) we have that

$$\mathrm{Re}\Bigg\{ ib\int_0^\pi (\varphi\bar{v}_x - \bar{\varphi}v_x)\,dx \Bigg\} \leqslant c_2\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{3/2} \tag{4.17}$$

where c_2 is a calculable positive constant. Applying a similar idea as above we obtain an estimate analogous to the other term of (4.16). Therefore, of (4.16) we have that

$$\alpha\|U\|_{\mathcal{H}}^2 \leqslant c_3\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{3/2} + c_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Then

$$|\alpha|\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}, \tag{4.18}$$

where $C > 0$ and $\alpha > 0$ is sufficiently greater. From where our conclusion follows. □

Now, we are in conditions to show the main result of this section.

Theorem 4.4. *Let (u, φ, θ) be a solution of the problem determined by* (1.3)–(1.5) *with boundary conditions* (1.6) *and initial conditions* (1.7)*. If the initial data satisfy conditions* (1.8) *and* (4.1) *with $k_1, k_2 \geqslant 0$, then the semigroup generated by operator $\mathcal{A}$ given in* (4.2) *is analytic.*

Proof. We now use Theorem 4.1 to prove Theorem 4.4. We first prove (4.4). This consists of the following steps:

(i) It follows from the fact that 0 is in the resolvent of $\mathcal{A}$ and the contraction mapping theorem that for any real number λ with $|\lambda| < \|\mathcal{A}^{-1}\|^{-1}$, the operator $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible. Moreover, $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1})$.

(ii) If $\sup\{\|(i\lambda I - \mathcal{A})^{-1}\|,\ |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda I - \mathcal{A} = (i\lambda_0 I - \mathcal{A})\big(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}\big)$$

with $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$ is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that by choosing λ_0 as close to $\|\mathcal{A}^{-1}\|^{-1}$ as we can, the set $\{\lambda,\ |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $\|(i\lambda I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}^{-1}\|^{-1} + M^{-1})$.

(iii) Thus it follows from the argument in (ii) that if (4.4) is not true, then there is $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} \leqslant |\omega| < \infty$ such that the set $\{i\lambda,\ |\lambda| < |\omega|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{\|(i\lambda I - \mathcal{A})^{-1}\|,\ |\lambda| < |\omega|\} = \infty$. Therefore there exists a sequence of real numbers λ_n with $\lambda_n \to \omega$, $|\lambda| < |\omega|$ and a sequence of vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \theta_n)^T$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\left\|(i\lambda_n I - \mathcal{A})U_n\right\| \to 0, \quad \text{as } n \to \infty, \tag{4.19}$$

that is,

$$i\lambda_n u_n - v_n \to 0 \quad \text{in } H_0^1, \tag{4.20}$$

$$i\lambda_n \rho v_n - \mu D^2 u_n - bD\varphi_n - \gamma D^2 v_n + \beta D\theta_n \to 0 \quad \text{in } L^2, \tag{4.21}$$

$$i\lambda_n \varphi_n - \phi_n \to 0 \quad \text{in } H^1, \tag{4.22}$$

$$i\lambda_n J\phi_n - \delta D^2\varphi_n + bDu_n + \xi D\varphi_n - m\theta_n - \eta D^2\phi_n - k_1 D^2\theta_n \to 0 \quad \text{in } L^2, \tag{4.23}$$

$$i\lambda_n c\theta_n - kD^2\theta_n + \beta D v_n + m\phi_n - k_2 D^2\phi_n \to 0 \quad \text{in } L^2. \tag{4.24}$$

Taking the inner product of $(i\lambda I - \mathcal{A})U_n$ times U_n in $\mathcal{H}$ and then considering its real part yields

$$\gamma\|Dv_n\|^2 + M\big(k\|D\theta_n\|^2 + \eta\|D\phi_n\|^2\big) \to 0,$$

that is,

$$\|Dv_n\|, \|D\theta_n\|, \|D\phi_n\| \to 0. \tag{4.25}$$

From (4.20), (4.22) and (4.25) we have

$$\|Du_n\|, \|D\varphi_n\| \to 0. \tag{4.26}$$

Using the Poincaré inequality and the boundary conditions we find that

$$u_n, v_n \to 0 \quad \text{in } L^2.$$

Taking the inner product of (4.23) times ϕ_n and (4.24) times θ_n in L^2 and integrating by parts, we obtain

$$\phi_n, \theta_n \to 0 \quad \text{in } L^2. \tag{4.27}$$

Thus we have shown that $\|U_n\|_{\mathcal{H}}$ cannot be of unit norm and the proof of (4.4) is complete. We now prove (4.5). We write (4.6) with $\lambda = i\alpha$, $\alpha \in \mathbb{R}$. Then

$$U = (i\alpha I - \mathcal{A})^{-1}F.$$

From Lemma 4.3 we have

$$\left\|\alpha(i\alpha I - \mathcal{A})^{-1}F\right\|_{\mathcal{H}} = |\alpha|\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}.$$

Then

$$\overline{\lim_{|\alpha|\to\infty}}\left\|\alpha(i\alpha I - \mathcal{A})^{-1}\right\| < \infty.$$

Our conclusion follows from Theorem 4.1. $\square$

Remark 4.5. From Theorem 4.4 we conclude that:

(1) The analyticity also holds when $k_1 k_2 = 0$, provided (4.1) is valid.
(2) As consequence of the analyticity, the system (1.3)–(1.5) is *exponentially stable* and have the *spectrum determined growth property* (SDG-property). Moreover, the system has a regularity effect in the sense that the solution $U = (u, u_t, \varphi, \varphi_t, \theta)^T$ satisfies

$$U \in C^\infty\big(0, T; \mathcal{D}\big(\mathcal{A}^\infty\big)\big).$$

However, $\mathcal{D}(\mathcal{A})$ is not necessary a space regular, which in particular implies that the solution U is not in $C^\infty(]0, T[\times]0, L[)$ when the initial data is not necessary regular.

5. Impossibility of localization

The aim of this section is to show the impossibility of time localization of solutions for the isothermal version of the system (1.3)–(1.5). This is, we will consider the system

$$\rho u_{tt} = \mu u_{xx} + b\varphi_x + \gamma u_{xxt}, \tag{5.1}$$

$$J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + \eta\varphi_{xxt}, \tag{5.2}$$

with the conditions (1.6)–(1.8). It is possible to adapt the arguments used in Section 4 to prove that the solutions of this system decay in an exponential way. Thus, it is of interest to clarify if the solutions can vanish in a finite time. To prove the impossibility of localization of solutions of this system we will show the uniqueness of solutions of the backward in time problem. Thus, it will be suitable to recall that the system of equations which govern the backward in time problem is:

$$\rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma u_{xxt}, \tag{5.3}$$

$$J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - \eta\varphi_{xxt}. \tag{5.4}$$

Lemma 5.1. *Let (u, φ) be a solution of the problem determined by the system* (5.3)–(5.4)*, the null initial conditions, and the boundary conditions* (1.6)*. Then $u = \phi = 0$.*

Proof. Now, we state some basic relations. The first one we need follows from the Lagrange identity method and it could be found with the help of [2]. For a fixed $t \in (0, T)$, we use the identities

$$\frac{\partial}{\partial s}\big[\rho\dot{u}(s)\dot{u}(2t-s)\big] = \rho\ddot{u}(s)\dot{u}(2t-s) - \rho\dot{u}(s)\ddot{u}(2t-s), \tag{5.5}$$

$$\frac{\partial}{\partial s}\big[J\dot{\varphi}(s)\dot{\varphi}(2t-s)\big] = J\ddot{\varphi}(s)\dot{\varphi}(2t-s) - J\dot{\varphi}(s)\ddot{\varphi}(2t-s) \tag{5.6}$$

the basic equations (5.3), (5.4), the initial conditions and boundary conditions to obtain

$$\int_0^\pi \big[\rho|u_t|^2 + J|\varphi_t|^2\big]\,dx = \int_0^\pi \big[\mu|u_x|^2 + 2bu_x\varphi + \xi\varphi^2 + \delta|\varphi_x|^2\big]\,dx. \tag{5.7}$$

We can also obtain the relations

$$\frac{d}{dt}\left(\int_0^\pi \left(\frac{\gamma}{2}|u_x|^2 - \rho u u_t\right)dx\right) = \int_0^\pi \big(\mu|u_x|^2 + bu_x\varphi - \rho|u_t|^2\big)\,dx \tag{5.8}$$

and

$$\frac{d}{dt}\left(\int_0^\pi \left(\frac{\eta}{2}|\varphi_x|^2 - J\varphi\varphi_t\right)dx\right) = \int_0^\pi \big(\delta|\varphi_x|^2 + bu_x\varphi + \xi\varphi^2 - J|\varphi_t|^2\big)\,dx. \tag{5.9}$$

After addition we obtain

$$\frac{d}{dt}\left(\int_0^\pi \left(\frac{\gamma}{2}|u_x|^2 + \frac{\eta}{2}|\varphi_x|^2 - \rho u u_t - J\varphi\varphi_t\right)dx\right) = \int_0^\pi \big(\mu|u_x|^2 + \delta|\varphi_x|^2 + 2bu_x\varphi + \xi\varphi^2 - \rho|u_t|^2 - J|\varphi_t|^2\big)\,dx. \tag{5.10}$$

If we consider null initial conditions, in view of the relation (5.7), we obtain

$$\frac{1}{2}\int_0^\pi \big(\gamma|u_x|^2 + \eta|\varphi_x|^2\big)\,dx = \int_0^\pi (\rho u u_t + J\varphi\varphi_t)\,dx. \tag{5.11}$$

In view of (5.7) and (5.11), we obtain

$$H(t) = \int_0^\pi \big[\rho|u_t|^2 + J|\varphi_t|^2\big]\,dx \leqslant C\int_0^\pi (\rho u u_t + J\varphi\varphi_t)\,dx \tag{5.12}$$

where C is a calculable positive constant. We now use the Poincaré inequality which state that

$$\int_0^t |u|^2\,dx \leqslant \frac{4t^2}{\pi^2}\int_0^t |u_t|^2\,dx, \tag{5.13}$$

whenever $u(0)=0$ (see [7, p. 338]). If we consider

$$\mathcal{E}(t)=\int_0^t H(s)\,ds, \tag{5.14}$$

we have

$$\begin{aligned}\mathcal{E}(t) &\leqslant C\left(\int_0^t\int_0^\pi \rho u^2\,dx\,ds\right)^{1/2}\left(\int_0^t\int_0^\pi \rho|u_s|^2\,dx\,ds\right)^{1/2}+C\left(\int_0^t\int_0^\pi J\varphi^2\,dx\,ds\right)^{1/2}\left(\int_0^t\int_0^\pi J|\varphi_s|^2\,dx\,ds\right)^{1/2}\\ &\leqslant \frac{2tC}{\pi}\int_0^t\int_0^\pi\left[\rho|u_s|^2+J|\varphi_s|^2\right]dx\,ds.\end{aligned}$$

Thus

$$\mathcal{E}(t)\leqslant Dt\mathcal{E}(t), \tag{5.15}$$

where D is a calculable positive constant. If we take $t_0=(2D)^{-1}$, we obtain that $\mathcal{E}(t)=0$ for every $t\leqslant t_0$. It follows that $u=\dot{u}=\varphi=\dot{\varphi}=0$ for every $0\leqslant t\leqslant t_0$. Then, we can prove the same for $t\leqslant 2t_0$ and this process can be extended to $0\leqslant t<\infty$ and we obtain the uniqueness of solutions for the backward in time problem. □

To prove the impossibility of localization in time for the linear version of the forward in time problem is equivalent to show the uniqueness of solutions for the linear version of the backward in time problem. Thus, we can state the following:

Theorem 5.2. *Let (u,φ) be a solution of the problem determined by the system* (5.1)–(5.2)*, the initial conditions* (1.7) *and the boundary conditions* (1.6) *such that $u=\varphi\equiv 0$ after a finite time $t_0>0$. Then $u=\varphi\equiv 0$ for every $t\geqslant 0$.*

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