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Remarks on cyclotomic and degenerate cyclotomic BMW algebras ☆

Frederick M. Goodman

Department of Mathematics, University of Iowa, Iowa City, IA, United States

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ABSTRACT

We relate the structure of cyclotomic and degenerate cyclotomic BMW algebras, for arbitrary parameter values, to that for admissible parameter values. In particular, we show that these algebras are cellular. We characterize those parameter sets for affine BMW algebras over an algebraically closed field that permit the algebras to have non-trivial cyclotomic quotients.

1. Introduction

This paper is a contribution to the study of affine and degenerate affine Birman–Wenzl–Murakami (BMW) algebras. In order to study the finite dimensional representation theory of these infinite dimensional algebras, one introduces cyclotomic quotients, which are BMW analogues of cyclotomic and degenerate cyclotomic Hecke algebras (see [2,1,15]).

A peculiar feature of the cyclotomic algebras is that the parameters cannot be chosen arbitrarily; that is, unless the parameters satisfy certain relations, the algebras (defined over a field) collapse to cyclotomic or degenerate cyclotomic Hecke algebras. These "obligatory" conditions did not seem adequate at first to develop the representation theory. Consequently, several authors, notably Ariki,

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E-mail address: frederick-goodman@uiowa.edu.

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Mathas and Rui [3], Wilcox and Yu [23], and Rui and Xu [21] introduced stronger "admissibility" conditions under which the algebras could be shown to have a well-behaved representation theory.

Up until now, the cyclotomic algebras have been studied only under the assumption of admissibility of the parameters. Despite the successes achieved, this was not satisfactory, since *a priori* the admissibility requirement might be too restrictive to capture the entire finite dimensional representation theory of the affine algebras.

In this paper, we extend the analysis of cyclotomic and degenerate cyclotomic BMW algebras to include the case of non-admissible parameters. We show that the structure and representation theory of the cyclotomic algebras with non-admissible parameters can be derived from that of the algebras with admissible parameters.

1.1. Background

Affine and cyclotomic BMW algebras and their degenerate versions arise naturally by several different "affinization" processes. One such process amounts to making the Jucys–Murphy elements in the ordinary BMW or Brauer algebras into variables, retaining the relations between these elements and the standard generators of the BMW or Brauer algebras. This point of view was stressed by Nazarov, in defining degenerate affine BMW algebras [17]. For the BMW algebras, there is a geometric affinization process: The ordinary BMW algebras can be realized as algebras of tangles in the disc cross the interval, modulo Kauffman skein relations [16]. To affinize these algebras, one should replace the disc by the annulus; alternatively, one replaces the ordinary braid group by the affine or type B braid group. This is the motivation cited by Häring-Oldenburg [14] for introducing affine and cyclotomic BMW algebras. Finally, Orellana and Ram provide an affinization of Schur–Weyl duality [18] which produces representations of the affine braid group by $\check{R}$-matrices of a quantum group; for symplectic or orthogonal quantum groups, this process yields representations of cyclotomic BMW algebras (over the complex numbers, with special parameters).

As mentioned above, degenerate affine BMW algebras were introduced by Nazarov [17] under the name *affine Wenzl algebras*. The cyclotomic quotients of these algebras were introduced by Ariki, Mathas, and Rui in [3] and studied further by Rui and Si in [19], under the name *cyclotomic Nazarov–Wenzl algebras*. Affine and cyclotomic BMW algebras were introduced by Häring-Oldenburg in [14] and studied by Goodman and Hauschild Mosley [10–12,5], Rui, Xu, and Si [21,20], Wilcox and Yu [23,24,22,25], and Ram, Orellana, Daugherty and Virk [18,4].

The papers cited above study the algebras under the assumption of admissibility. It has been shown that the algebras with admissible parameters are cellular [3,24,25,21,20,5,8,9]; simple modules over a field have been classified [19,20]; and the non-degenerate cyclotomic BMW algebras have been shown to be isomorphic to algebras of tangles [11,12,24,22,25].

1.2. Summary of results

In this note, we show that the structure of the cyclotomic and degenerate cyclotomic BMW algebras for general parameters can be derived from the admissible case. An affine (resp. degenerate affine) BMW algebra A_n contains a copy of the finite dimensional BMW algebra (resp. Brauer algebra) B_n and an additional "affine" generator y_1, satisfying several relations with the generators of B_n. A cyclotomic quotient is determined by a polynomial relation

$$(y_1 - u_1) \cdots (y_1 - u_r) = 0. \tag{1.1}$$

Denote the cyclotomic quotient determined by (1.1) by $A_{n,r}(u_1, \ldots, u_r)$. Let $J_{n,r}(u_1, \ldots, u_r)$ denote the ideal generated by the "contraction" e_1 in $A_{n,r}(u_1, \ldots, u_r)$. Then we have a short exact sequence

$$0 \to J_{n,r}(u_1, \ldots, u_r) \to A_{n,r}(u_1, \ldots, u_r) \to H_n(u_1, \ldots, u_r) \to 0, \tag{1.2}$$

where $H_n(u_1, \ldots, u_r)$ is the cyclotomic Hecke algebra (resp. degenerate cyclotomic Hecke algebra). Admissibility of the parameters means that $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$ is linearly independent in

$A_{2,r}(u_1,\dots,u_r)$; this condition translates into specific conditions on the parameters of the algebra which are discussed in Sections 3 and 4. Suppose now that we are working over a field and that admissibility fails, but $e_1 \neq 0$; then there exists a d with $0 < d < r$ such that $\{e_1, y_1e_1, \dots, y_1^{d-1}e_1\}$ is linearly independent in $A_{2,r}(u_1,\dots,u_r)$ but $\{e_1, y_1e_1, \dots, y_1^{d}e_1\}$ is linearly dependent. We say that the parameters are d-semi-admissible. We show that there exists a subset $\{v_1,\dots,v_d\} \subset \{u_1,\dots,u_r\}$ such that

(1) $A_{n,d}(v_1,\dots,v_d)$ has admissible parameters, and
(2) $J_{n,d}(v_1,\dots,v_d) \cong J_{n,r}(u_1,\dots,u_r)$.

Thus we have

$$0 \to J_{n,d}(v_1,\dots,v_d) \to A_{n,r}(u_1,\dots,u_r) \to H_n(u_1,\dots,u_r) \to 0. \tag{1.3}$$

Two consequences of this analysis are the following:

(1) The cyclotomic algebras are cellular, under very mild hypotheses; in particular, when the ground ring is a field, the algebras are always cellular.
(2) Every finite dimensional simple module of an affine (resp. degenerate affine) BMW algebra over an algebraically closed field factors through a cyclotomic (resp. degenerate cyclotomic) BMW algebra with admissible parameters, or through a cyclotomic (resp. degenerate cyclotomic) Hecke algebra.

The latter result is a step towards classifying the simple modules of the affine and degenerate affine BMW algebras over an algebraically closed field.

The main results of Ariki, Mathas and Rui [3] regarding degenerate cyclotomic BMW algebras depend on the hypothesis that 2 is invertible in the ground ring. We point out in that this hypothesis can be eliminated; see Section 3.

Finally, we characterize those parameter sets for affine BMW algebras over an algebraically closed field that permit the algebras to have non-trivial cyclotomic quotients, or equivalently, finite dimensional modules M with $e_1M \neq 0$; see Theorem 7.9. The analogous result for degenerate affine BMW algebras was proved in [3]; we have made a minor improvement by removing the restriction that the characteristic of the field should be different from 2; see Theorem 7.1.

2. Preliminaries

2.1. Definition of degenerate affine and cyclotomic BMW algebras

Fix a positive integer n and a commutative ring S with multiplicative identity. Let $\Omega = \{\omega_a\colon\ a \geqslant 0\}$ be a sequence of elements of S.

Definition 2.1. (See [17].) The *degenerate affine BMW algebra* $\widehat{\mathcal{N}}_{n,S} = \widehat{\mathcal{N}}_{n,S}(\Omega)$ is the unital associative S-algebra with generators $\{s_i, e_i, y_j\colon\ 1 \leqslant i < n \text{ and } 1 \leqslant j \leqslant n\}$ and relations:

(1) (Involutions) $s_i^2 = 1$, for $1 \leqslant i < n$.
(2) (Affine braid relations)
  (a) $s_is_j = s_js_i$ if $|i-j| > 1$.
  (b) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $1 \leqslant i < n-1$.
  (c) $s_iy_j = y_js_i$ if $j \neq i, i+1$.
(3) (Idempotent relations) $e_i^2 = \omega_0e_i$, for $1 \leqslant i < n$.
(4) (Compression relations) $e_1y_1^ae_1 = \omega_ae_1$, for $a > 0$.
(5) (Commutation relations)
  (a) $s_ie_j = e_js_i$, and $e_ie_j = e_je_i$ if $|i-j| > 1$.

(b) $e_i y_j = y_j e_i$, if $j \neq i, i+1$.
(c) $y_i y_j = y_j y_i$, for $1 \leqslant i, j \leqslant n$.
(6) (Tangle relations)
(a) $e_i s_i = e_i = s_i e_i$, for $1 \leqslant i \leqslant n-1$.
(b) $s_i e_{i+1} e_i = s_{i+1} e_i$, and $e_i e_{i+1} s_i = e_i s_{i+1}$, for $1 \leqslant i \leqslant n-2$.
(c) $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, and $s_{i+1} e_i e_{i+1} = s_i e_{i+1}$, for $1 \leqslant i \leqslant n-2$.
(d) $e_{i+1} e_i e_{i+1} = e_{i+1}$, and $e_i e_{i+1} e_i = e_i$, for $1 \leqslant i \leqslant n-2$.
(7) (Skein relations) $s_i y_i - y_{i+1} s_i = e_i - 1$, and $y_i s_i - s_i y_{i+1} = e_i - 1$, for $1 \leqslant i < n$.
(8) (Anti-symmetry relations) $e_i(y_i + y_{i+1}) = 0$, and $(y_i + y_{i+1}) e_i = 0$, for $1 \leqslant i < n$.

Definition 2.2. (See [3].) Fix an integer $r \geqslant 1$ and elements $u_1, \ldots, u_r$ in S. The *degenerate cyclotomic BMW algebra* $\mathcal{N}_{n,S,r} = \mathcal{N}_{n,S,r}(\Omega; u_1, \ldots, u_r)$ is the quotient of the degenerate affine BMW algebra $\widehat{\mathcal{N}}_{n,S}(\Omega)$ by the cyclotomic relation $(y_1 - u_1) \cdots (y_1 - u_r) = 0$.

Note that, due to the symmetry of the relations, $\widehat{\mathcal{N}}_{n,S}$ has a unique S-linear algebra involution $*$ (that is, an algebra anti-automorphism of order 2) such that $e_i^* = e_i$, $s_i^* = s_i$, and $y_i^* = y_i$ for all i. The involution passes to cyclotomic quotients.

2.2. Definition of affine and cyclotomic BMW algebras

Fix an integer $n \geqslant 0$, and a commutative ring S with invertible elements ρ and q, and a sequence of elements $\Omega = (\omega_a)_{a \geqslant 0}$, satisfying

$$\rho^{-1} - \rho = \big(q^{-1} - q\big)(\omega_0 - 1). \tag{2.1}$$

Definition 2.3. (See [14].) The *affine BMW algebra* $\widehat{\mathcal{W}}_{n,S} = \widehat{\mathcal{W}}_{n,S}(\rho, q, \Omega)$ is the unital associative S-algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i $(1 \leqslant i \leqslant n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
(2) (Affine braid relations)
(a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| \geqslant 2$.
(b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geqslant 2$.
(3) (Idempotent relation) $e_i^2 = \omega_0 e_i$.
(4) (Compression relations) For $j \geqslant 1$, $e_1 y_1^j e_1 = \omega_j e_1$.
(5) (Commutation relations)
(a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geqslant 2$.
(b) $y_1 e_j = e_j y_1$ if $j \geqslant 2$.
(6) (Tangle relations)
(a) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i \pm 1} e_i = \rho e_i$.
(b) $e_i e_{i \pm 1} e_i = e_i$.
(c) $g_i g_{i \pm 1} e_i = e_{i \pm 1} e_i$ and $e_i g_{i \pm 1} g_i = e_i e_{i \pm 1}$.
(7) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 g_1 = e_1 = g_1 y_1 g_1 y_1 e_1$.

Definition 2.4. (See [14].) Fix an integer $r \geqslant 1$ and invertible elements $u_1, \ldots, u_r$ in S. The *cyclotomic BMW algebra* $\mathcal{W}_{n,S,r} = \mathcal{W}_{n,S,r}(\rho, q, \Omega; u_1, \ldots, u_r)$ is the quotient of the affine BMW algebra $\widehat{\mathcal{W}}_{n,S}(\rho, q, \Omega)$ by the cyclotomic relation $(y_1 - u_1) \cdots (y_1 - u_r) = 0$.

As in the degenerate case, $\widehat{\mathcal{W}}_{n,S}$ has a unique S-linear algebra involution $*$ such that $e_i^* = e_i$ and $g_i^* = g_i$, for all i, and $y_1^* = y_1$. The involution passes to cyclotomic quotients.

2.3. Admissibility

Notation 2.5. Let $\mathcal{A}_{n,S,r}$ denote either the cyclotomic BMW algebra $\mathcal{W}_{n,S,r}$ (with parameters ρ, q, $\Omega = (\omega_a)_{a \geqslant 0}$, and $u_1, \ldots, u_r$) or the degenerate cyclotomic BMW algebra $\mathcal{N}_{n,S,r}$ (with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \ldots, u_r$) over a commutative ring S. Let

$$p(u) = (u - u_1) \cdots (u - u_r) = \sum_{j=0}^{r} a_j u^j. \tag{2.2}$$

The coefficients a_j for $j < r$ are signed elementary symmetric functions in $u_1, \ldots, u_r$, namely $a_j = (-1)^{r-j} \varepsilon_{r-j}(u_1, \ldots, u_r)$, and $a_r = 1$.

Lemma 2.6. *The left ideal $\mathcal{A}_{2,S,r} e_1$ in $\mathcal{A}_{2,S,r}$ is equal to the S-span of $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$.*

Proof. For both the cyclotomic and degenerate cyclotomic BMW algebras, it is easy to check using the relations that the S-span of $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$ is invariant under multiplication by the generators on the left. $\square$

Lemma 2.7. *Assume that e_1 is not a torsion element over S in $\mathcal{A}_{2,S,r}$. Then the elements ω_j, $j \geqslant 0$, satisfy the following recursion relation*:

$$\sum_{j=0}^{r} a_j \omega_{j+\ell} = 0, \quad \textit{for all } \ell \geqslant 0. \tag{2.3}$$

Proof. Multiply the cyclotomic condition: $\sum_{j=0}^{r} a_j y_1^j = 0$ by y_1^ℓ, and then multiply from both sides by e_1. Use the compression and idempotent relations to obtain $\sum_{j=0}^{r} a_j \omega_{j+\ell} e_1 = 0$. Since e_1 is not a torsion element over S, the result follows. $\square$

Definition 2.8. Consider the cyclotomic or degenerate cyclotomic BMW algebras over a commutative ring S with suitable parameters. We say that the parameters are *admissible* if $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$ is linearly independent over S in $\mathcal{A}_{2,S,r}$.

For both the cyclotomic and degenerate cyclotomic BMW algebras, admissibility as defined above translates into explicit conditions on the parameters. We review this for the two classes of algebras separately in the following two sections.

3. Admissibility for degenerate cyclotomic BMW algebras

Consider the degenerate cyclotomic BMW algebras $\mathcal{N}_{n,S,r}$ with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \ldots, u_r$ over a commutative ring S. Define $a_0, \ldots, a_{r-1}$ by (2.2).

Lemma 3.1. *(See [7, Lemma 4.1].) Suppose that $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$ is linearly independent over S in $\mathcal{N}_{2,S,r}$. Then the parameters satisfy the following relations*:

$$\sum_{\mu=0}^{r-j-1} \omega_\mu a_{\mu+j+1} = -2\delta_{(r\,-\,j \text{ is odd})} a_j + \delta_{(j \text{ is even})} a_{j+1}, \tag{3.1}$$

for $0 \leqslant j \leqslant r - 1$.

We are going to show that admissibility (i.e. linear independence of $\{e_1, y_1e_1, \dots, y_1^{r-1}e_1\}$) is equivalent to the parameters satisfying conditions (2.3) and (3.1).

Lemma 3.2. *(See [7, Lemma 4.4].) There exist universal polynomials $H_a(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ for $a \geqslant 0$, symmetric in $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, with integer coefficients, such that whenever S is a commutative ring with parameters $\Omega = (\omega_a)_{a\geqslant 0}$ and $u_1, \dots, u_r$ satisfying* (2.3) *and* (3.1)*, one has*

$$\omega_a = H_a(u_1, \dots, u_r) \quad \text{for } a \geqslant 0. \tag{3.2}$$

Conversely, if $\omega_a = H_a(u_1, \dots, u_r)$ for $a \geqslant 0$, then the parameters satisfy (2.3) *and* (3.1)*.*

Proof. The system of relations (3.1) is a unitriangular linear system of equation for the variables $\omega_0, \dots, \omega_{r-1}$. In fact, if we list the equations in reverse order then the matrix of coefficients is

$$\begin{bmatrix} 1 & & & & \\ a_{r-1} & 1 & & & \\ a_{r-2} & a_{r-1} & 1 & & \\ \vdots & & \ddots & \ddots & \\ a_1 & a_2 & \cdots & a_{r-1} & 1 \end{bmatrix}.$$

Solving the system for $\omega_0, \dots, \omega_{r-1}$ gives these quantities as polynomial functions of $a_0, \dots, a_{r-1}$, thus symmetric polynomials in $u_1, \dots, u_r$. The recursion relations $\sum_{j=0}^{r} a_j\omega_{j+m} = 0$, for all $m \geqslant 0$ yield (3.2) for $a \geqslant r$. The converse is obvious, since the ω_a given by (3.2) are the solutions of Eqs. (2.3) and (3.1). $\square$

3.1. The admissibility condition of Ariki, Mathas, and Rui

Ariki, Mathas and Rui used a different approach to admissibility for degenerate cyclotomic BMW algebras in their fundamental work [3]. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ and t be algebraically independent indeterminants over $\mathbb{Z}$. Define symmetric polynomials $q_a(\boldsymbol{u})$ in $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ by

$$\prod_{i=1}^{r} \frac{1 + \boldsymbol{u}_i t}{1 - \boldsymbol{u}_i t} = \sum_{a\geqslant 0} q_a(\boldsymbol{u})t^a.$$

The polynomials q_a are known as *Schur q-functions*. Define

$$\eta_a^{\pm}(\boldsymbol{u}) = q_{a+1}(\boldsymbol{u}) \pm \frac{(-1)^{r-1}}{2} q_a(\boldsymbol{u}) + \frac{1}{2}\delta_{a,0}, \tag{3.3}$$

for $a \geqslant 0$. Then (cf. [3, Lemma 3.8])

$$\sum_{a\geqslant 0} \eta_a^{\pm}(\boldsymbol{u})t^{-a} = \left(\frac{1}{2} - t\right) + \left(t \pm \frac{(-1)^{r-1}}{2}\right) \prod_{i=1}^{r} \frac{t + u_i}{t - u_i}, \tag{3.4}$$

as one sees by expanding the series, using the definition of the Schur q-functions. Ostensibly, $\eta_a^{\pm}(\boldsymbol{u}) \in \mathbb{Z}[1/2, \boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$, but actually:

Lemma 3.3.

(1) $q_0(\boldsymbol{u}) = 1$.
(2) *For $a \geqslant 1$, $q_a \equiv 2p_a(\boldsymbol{u}) \bmod 4\mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$, where p_a denotes the a-th power sum symmetric function.*
(3) $\eta_a^{\pm}(\boldsymbol{u}) \in \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$.

Proof. Part (1) is obvious. Using the identity:

$$\frac{1+vt}{1-vt} = 1 + 2vt(1-vt)^{-1} = 1 + 2vt + 2v^2t^2 + 2v^3t^3 + \cdots,$$

one sees that the coefficient of t^a in $\prod_{i=1}^r \frac{1+\boldsymbol{u}_i t}{1-\boldsymbol{u}_i t}$ is $2\sum_i u_i^a$ plus a sum of terms divisible by 4. This gives (2), and (3) follows as well. □

Example 3.4. Consider a ring S of characteristic 2 and $u_1, \dots, u_r \in S$. Then $q_a(u_1, \dots, u_r) = 0$ for $a \geqslant 1$, but $\frac{1}{2}q_a(u_1, \dots, u_r) = \sum_i u_i^a$; that is, we consider $\frac{1}{2}q_a$ in $\mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$, and then evaluate at $(u_1, \dots, u_r) \in S^r$. Furthermore,

$$\eta_0^+(u_1, \dots, u_r) = \delta_{(r \text{ is odd})} \quad \text{and} \quad \eta_a^+(u_1, \dots, u_r) = p_a(u_1, \dots, u_r),$$

for $a \geqslant 1$.

Definition 3.5. (See [3].) Let S be a commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. Say that the parameters are $(u_1, \dots, u_r)$-admissible, or that Ω is $(u_1, \dots, u_r)$-admissible, if for all $a \geqslant 0$,

$$\omega_a = \eta_a^+(u_1, \dots, u_r). \tag{3.5}$$

Lemma 3.6. *(Cf. [7, Lemma 5.1].)*

(1) $\eta_a^+(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r) = H_a(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$, *where H_a are the polynomials as in Lemma* 3.2.
(2) *Let S be a commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. The parameters are $(u_1, \dots, u_r)$-admissible if and only if they satisfy* (2.3) *and* (3.1)*.*

Proof. Part (1) is proved in [7, Section 5] by showing that the sequence $(\eta_a^+(\boldsymbol{u}))_{a \geqslant 0}$ satisfies (2.3) and (3.1); that is,

$$\sum_{j=0}^{r} \boldsymbol{a}_j \eta_{j+\ell}^+(\boldsymbol{u}) = 0, \quad \text{for all } \ell \geqslant 0, \tag{3.6}$$

and

$$\sum_{\mu=0}^{r-j-1} \eta_\mu^+(\boldsymbol{u}) \boldsymbol{a}_{\mu+j+1} = -2\delta_{(r-j \text{ is odd})} \boldsymbol{a}_j + \delta_{(j \text{ is even})} \boldsymbol{a}_{j+1}, \tag{3.7}$$

for $0 \leqslant j \leqslant r-1$, where $\boldsymbol{a}_j = (-1)^{r-j} \varepsilon_{r-j}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$. Part (2) follows from part (1) together with Definition 3.5 and Lemma 3.2. □

3.2. *Recovering the results of Ariki, Mathas, and Rui*

The main results of [3] regarding degenerate cyclotomic BMW algebras are stated for ground rings S in which 2 is invertible. The primary reason for this restriction on the ground ring was that it seemed to be needed in order to use the quantities $\eta_a^+(u_1, \dots, u_r)$, which play a central role in [3], via Definition 3.5. Using Lemma 3.3, the restriction on the ground ring can be eliminated. We proceed to outline how the proofs have to be adjusted.

Let us define a universal ring with $(u_1, \dots, u_r)$-admissible parameters. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ be indeterminants over $\mathbb{Z}$. Let $\mathcal{Z} = \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$; define $\boldsymbol{a}_j = (-1)^j \varepsilon_{r-j}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ for $0 \leqslant j \leqslant r$, where ε_k is the k-th elementary symmetric function, and define $\boldsymbol{\omega}_a$ for $a \geqslant 0$ by

$$\boldsymbol{\omega}_a = H_a(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r) = \eta_a^+(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r) \quad \text{for } a \geqslant 0. \tag{3.8}$$

The parameters $\boldsymbol{\Omega} = (\boldsymbol{\omega}_a)_{a \geqslant 0}$ and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ are $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$-admissible by definition. (This is the same construction as in [3, page 105] except that we don't need to adjoin 1/2 to the ring.) If S is any commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$, such that Ω is $(u_1, \dots, u_r)$-admissible then there is a unique algebra homomorphism from $\mathcal{Z}$ to S taking $\boldsymbol{u}_j \mapsto u_j$. Since Ω is $(u_1, \dots, u_r)$-admissible, it follows that $\boldsymbol{\omega}_a \mapsto \omega_a$ for all $a \geqslant 0$. For all $n \geqslant 0$, we have

$$\mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r) \cong \mathcal{N}_{n,\mathcal{Z},r}(\boldsymbol{\Omega}; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r) \otimes_{\mathcal{Z}} S. \tag{3.9}$$

See [11, Remark 3.4] for a justification.

Let S be any commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$ (with no conditions imposed on the parameters). We recall a construction of a spanning set in $\mathcal{N}_n = \mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r)$ from [3]. Remark that there is a homomorphism from the Brauer algebra $\mathcal{B}_n(\omega_0)$ with parameter ω_0 to $\mathcal{N}_{n,S,r}$ taking $s_i \mapsto s_i$ and $e_i \mapsto e_i$; this follows from the presentation of the Brauer algebra cited in [3, Proposition 2.7]. For a Brauer diagram γ, we will also write γ for the image of γ in $\mathcal{N}_{n,S,r}$. The "r-regular monomials" in $\mathcal{N}_{n,S,r}$ are defined to be the elements

$$y^{\boldsymbol{p}} \gamma y^{\boldsymbol{q}}, \tag{3.10}$$

where γ is a Brauer diagram, $y^{\boldsymbol{p}} = {y_1}^{p_1} \cdots {y_n}^{p_n}$, and $y^{\boldsymbol{q}} = {y_1}^{q_1} \cdots {y_n}^{q_n}$; moreover, p_i and q_i are non-negative integers, in the interval $0, 1, \dots, r-1$, and $p_i = 0$ unless the i-th vertex at the bottom of γ is the left endpoint of a horizontal strand, and $q_i = 0$ unless the i-th vertex at the top of γ is either the left endpoint of a horizontal strand, or the top endpoint of a vertical strand. Note that there are at most n strictly positive exponents p_i or q_i, and the number of r-regular monomials is $r^n(2n-1)!!$.

Proposition 3.7. *(See [3, Proposition 2.15].) Let S be any commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. For all $n \geqslant 0$, $\mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r)$ is spanned over S by the set of r-regular monomials. Furthermore, the ideal $\mathcal{N}_n e_{n-1} \mathcal{N}_n$ is spanned by those r-regular monomials $y^{\boldsymbol{p}} \gamma y^{\boldsymbol{q}}$ such that γ has at least two horizontal strands.*

Remark 3.8. It may appear from the presentation in [3] that this result depends on the invertibility of 2 in the ground ring and on an additional condition on the parameters (called "admissibility" in [3], see Definition 2.10 in that paper). However, in fact, the result does not depend on any additional assumptions. From Theorem 2.12 in [3], one only needs the statement that the degenerate affine BMW algebra is spanned by regular monomials, and the argument for this part of Theorem 2.12 is valid over an arbitrary ring. The argument given for Proposition 2.15 itself in [3] is also valid over an arbitrary ring.

Theorem 3.9. *(See [3].) Let $F = \mathbb{Q}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ denote the field of fractions of $\mathcal{Z}$. Then the algebra $\mathcal{N}_{n,F,r}(\boldsymbol{\Omega}; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ is split-semisimple of dimension $r^n(2n-1)!!$.*

This theorem is proved by explicit construction of sufficiently many irreducible representations.

Corollary 3.10. *(Cf. [3, Theorem 5.5].) Let S be a commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. Assume that Ω is $(u_1, \dots, u_r)$-admissible. Then for all $n \geqslant 0$, $\mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r)$ is a free S-module with basis the set of r-regular monomials.*

Proof. Because of (3.9), it suffices to show that $\mathcal{N}_{n,\mathcal{Z},r}(\boldsymbol{\Omega}; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ is a free $\mathcal{Z}$-module with basis the set $\mathcal{M}$ of r-regular monomials. By Proposition 3.7, $\mathcal{M}$ is a spanning set, and $\mathcal{M} \otimes 1 := \{m \otimes 1: m \in \mathcal{M}\}$ is a spanning set in $\mathcal{N}_{n,\mathcal{Z},r} \otimes_{\mathcal{Z}} F = \mathcal{N}_{n,F,r}$. But by Theorem 3.9, the latter algebra over F has dimension $r^n(2n-1)!!$, and hence $\mathcal{M} \otimes 1$ is linearly independent over F. It follows that $\mathcal{M}$ is linearly independent over $\mathcal{Z}$. □

The following theorem concerns cellularity of degenerate cyclotomic BMW algebras. The definition of cellularity is given in Section 6.

Theorem 3.11. *(Cf. [3, Theorem 7.17].) Let S be a commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. Assume that Ω is $(u_1, \dots, u_r)$-admissible. Then $\mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r)$ is a cellular algebra of rank $r^n(2n-1)!!$.*

Proof. Because of (3.9), it suffices to prove this when $S = \mathcal{Z}$. For this special case, one can follow the proof in [3, Theorem 7.17] substituting Corollary 3.10 for [3, Theorem 5.5]. For an alternative treatment of cellularity, see [9, Section 6.5]. □

3.3. *Equivalence of admissibility conditions*

The following theorem establishes the equivalence of the various admissibility criteria for degenerate cyclotomic BMW algebras.

Theorem 3.12. *(Cf. [7, Theorem 5.2].) Let S be a commutative ring with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$. Consider the degenerate cyclotomic BMW algebra $\mathcal{N}_2 = \mathcal{N}_{2,S,r}(\Omega; u_1, \dots, u_r)$. The following are equivalent:*

(1) *The parameters are admissible, i.e. $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\}$ is linearly independent over S in $\mathcal{N}_2$.*
(2) *$\{y_1^a e_1 y_1^b, s_1 y_1^a y_2^b, y_1^a y_2^b: 0 \leqslant a, b \leqslant r-1\}$ is an S-basis of $\mathcal{N}_2$.*
(3) *The parameters satisfy (2.3) and (3.1).*
(4) *The parameters are $(u_1, \dots, u_r)$-admissible.*

Proof. (1) ⇒ (3) results from Lemmas 2.7 and 3.1. We have (3) ⇔ (4) by Lemma 3.6. The implication (4) ⇒ (2) is a special case of Corollary 3.10. Finally, the implication (2) ⇒ (1) is trivial. □

4. Admissibility for cyclotomic BMW algebras

Fix an integral domain S with parameters ρ, q, $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$; assume that ρ and q are invertible and that Eq. (2.1) holds. Consider the cyclotomic BMW algebras $\mathcal{W}_{n,S,r} = \mathcal{W}_{n,S,r}(\rho, q, \Omega; u_1, \dots, u_r)$.

4.1. *Admissibility conditions of Wilcox and Yu*

Explicit relations on the parameters that are equivalent to admissibility (i.e. linear independence of $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\}$) have been found by Wilcox and Yu [23,22,25]. The form of these relations depends on whether $q^2 \neq 1$ is satisfied in S. Note that the conditions $q^2 \neq 1$ (in the non-degenerate case) and $\mathrm{char}(S) \neq 2$ (in the degenerate case) should be regarded as analogous.

Theorem 4.1. *(See Wilcox and Yu [23].) Let S be an integral domain with parameters ρ, q, $\Omega = (\omega_a)_{a \geqslant 0}$, and $u_1, \dots, u_r$ satisfying Eq. (2.1) and $(q - q^{-1}) \neq 0$. The following conditions are equivalent:*

(1) *$\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\} \subseteq \mathcal{W}_{2,S,r}$ is linearly independent over S.*

(2) *The parameters satisfy the recursion relation* (2.3) *and the following relations*:

$$\big(q-q^{-1}\big)\left[\sum_{j=1}^{r-\ell} a_{j+\ell}\omega_j\right] = -\rho(a_\ell - a_{r-\ell}/a_0) + \big(q-q^{-1}\big)\left[\sum_{j=\max(\ell+1,\lceil r/2\rceil)}^{\lfloor(\ell+r)/2\rfloor} a_{2j-\ell} - \sum_{j=\lceil \ell/2\rceil}^{\min(\ell,\lceil r/2\rceil-1)} a_{2j-\ell}\right] \tag{4.1}$$

for $1 \leqslant \ell \leqslant r-1$, *and*

$$\rho = \pm a_0 \quad \textit{if } r \textit{ is odd}, \quad \textit{and} \quad \rho \in \big\{q^{-1}a_0, -qa_0\big\} \quad \textit{if } r \textit{ is even}. \tag{4.2}$$

Note that Eqs. (2.1), (4.1), and (4.2) determine $\omega_0, \dots, \omega_{r-1}$ and ρ in terms of q, $u_1, \dots, u_r$ while the recursion relation (2.3) determines ω_a for $a \geqslant r$.

In [22] and [25] Wilcox and Yu derive explicit relations on the parameters that are equivalent to linear independence of $\{e_1, y_1e_1, \dots, y_1^{r-1}e_1\}$ also in the case that $q-q^{-1}=0$; their new conditions reduce to those of Theorem 4.1 in the case $q-q^{-1}\neq 0$.

4.2. The admissibility criterion of Rui and Xu

Rui and Xu [21], following [3], take a different approach to admissibility for cyclotomic BMW algebras when $q-q^{-1}\neq 0$. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, $\boldsymbol{\rho}$, $\boldsymbol{q}$, and t be algebraically independent indeterminants over $\mathbb{Z}$. Define

$$G(t) = G(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r; t) = \prod_{\ell=1}^{r} \frac{t-\boldsymbol{u}_\ell}{t\boldsymbol{u}_\ell - 1}. \tag{4.3}$$

Let

$$Z(t) = Z(t; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{\rho}, \boldsymbol{q}) = -\boldsymbol{\rho}^{-1} + \big(\boldsymbol{q}-\boldsymbol{q}^{-1}\big)\frac{t^2}{t^2-1} + A(t)G\big(t^{-1}\big), \tag{4.4}$$

where

$$A(t) = \begin{cases} \rho^{-1}(\prod_j \boldsymbol{u}_j) + (\boldsymbol{q}-\boldsymbol{q}^{-1})t/(t^2-1) & \text{if } r \text{ is odd, and} \\ \rho^{-1}(\prod_j \boldsymbol{u}_j) - (\boldsymbol{q}-\boldsymbol{q}^{-1})t^2/(t^2-1) & \text{if } r \text{ is even.} \end{cases} \tag{4.5}$$

Definition 4.2. (See Rui and Xu [21].) Let S be an integral domain with parameters ρ, q, $\Omega = (\omega_a)_{a\geqslant 0}$ and $u_1, \dots, u_r$ satisfying (2.1) and $q-q^{-1}\neq 0$. One says that the parameters are $(u_1, \dots, u_r)$-*admissible*, or that Ω is $(u_1, \dots, u_r)$-*admissible*, if

$$\big(q-q^{-1}\big)\sum_{a\geqslant 0}\omega_a t^{-a} = Z(t; u_1, \dots, u_r, \rho, q), \tag{4.6}$$

where Z is defined in Eqs. (4.4) and (4.5).

Remark 4.3. Let S be an integral domain with $(u_1, \dots, u_r)$-admissible parameters, as in Definition 4.2. With $p = \prod_j u_j$, we have

$$\rho = \pm p \quad \text{if } r \text{ is odd}, \quad \text{and} \quad \rho \in \{q^{-1}p, -qp\} \quad \text{if } r \text{ is even}. \tag{4.7}$$

The condition (4.7) on ρ was included in the definition of u-admissibility in [21], but it actually follows from (2.1) and (4.6), as explained in [6, Remark 3.10].

4.3. Equivalence of admissibility conditions

The following theorem establishes the equivalence of the various admissibility conditions for cyclotomic BMW algebras, in case the ground ring is integral and $q - q^{-1} \neq 0$.

Theorem 4.4. *(See [6, Theorem 4.4].) Let S be an integral domain with parameters ρ, q, $\Omega = (\omega_a)_{a \geqslant 0}$, and $u_1, \dots, u_r$ satisfying Eq.* (2.1) *and* $(q - q^{-1}) \neq 0$. *The following are equivalent*:

(1) $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\} \subseteq \mathcal{W}_{2,S,r}$ *is linearly independent over S.*
(2) *The parameters satisfy the recursion relation* (2.3) *and the conditions* (4.1) *and* (4.2) *of Wilcox and Yu.*
(3) *Ω is $(u_1, \dots, u_r)$-admissible.*

5. Semi-admissibility

Let $\mathcal{A}_{n,S,r} = \mathcal{A}_{n,S,r}(u_1, \dots, u_r)$ denote either the cyclotomic BMW algebra $\mathcal{W}_{n,S,r}$, with parameters ρ, q, $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$, or the degenerate cyclotomic BMW algebra $\mathcal{N}_{n,S,r}$, with parameters $\Omega = (\omega_a)_{a \geqslant 0}$ and $u_1, \dots, u_r$, over an integral domain S.

From here on, we impose the following standing assumption:

Assumption 5.1. The ground ring S is an integral domain, and the left ideal $\mathcal{A}_{2,S,r} e_1 \subseteq \mathcal{A}_{2,S,r}$ is torsion free as an S-module.

This assumption holds, in particular, whenever S is a field.

Under Assumption 5.1, exactly one of the following three possibilities must hold:

(1) $e_1 = 0$ in $\mathcal{A}_{2,S,r}$. In this case, $e_{n-1} = 0$ in $\mathcal{A}_{n,S,r}$ for all $n \geqslant 2$. The (degenerate) cyclotomic BMW algebras reduce to (degenerate) cyclotomic Hecke algebras.
(2) The parameters are admissible, i.e. $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\}$ is linearly independent over S in $\mathcal{A}_{2,S,r}$. This case has been studied in the literature and is well understood.
(3) There is a d with $0 < d < r$ such that $\{e_1, y_1 e_1, \dots, y_1^{d-1} e_1\}$ is linearly independent over S in $\mathcal{A}_{2,S,r}$, but $\{e_1, y_1 e_1, \dots, y_1^{d} e_1\}$ is linearly dependent. This case remains to be investigated.

Definition 5.2. Consider the cyclotomic or degenerate cyclotomic BMW algebras $\mathcal{A}_{n,S,r}$ over an integral domain S with suitable parameters. Let $0 < d < r$. We say that the parameters are *d-semi-admissible* if $\{e_1, y_1 e_1, \dots, y_1^{d-1} e_1\}$ is linearly independent over S in $\mathcal{A}_{2,S,r}$, but $\{e_1, y_1 e_1, \dots, y_1^{d} e_1\}$ is linearly dependent.

Suppose d-semi-admissibility of the parameters. Then there is a polynomial of $p_0(u) \in S[u]$ of degree d such that $p_0(y_1) e_1 = 0$ but $r(y_1) e_1 \neq 0$ for any non-zero polynomial $r(u) \in S[u]$ of degree less than d. Let F denote the field of fractions of S, and write $p(u) = (u - u_1) \cdots (u - u_r) \in S[u]$. Since $p(y_1) = 0$, it follows that $p_0(u)$ divides $p(u)$ in $F[u]$. Because of unique factorization in $F[u]$, we have (after renumbering the roots u_i of $p(u)$) $p_0(u) = \alpha (u - u_1) \cdots (u - u_d)$ for some non-zero α in F. In fact $\alpha \in S$, since it is the leading coefficient of $p_0(u)$. Then we have $\alpha (y_1 - u_1) \cdots (y_1 - u_d) e_1 = 0$. Because we assumed $\mathcal{A}_{2,S,r} e_1$ is torsion-free over S, we can conclude that $(y_1 - u_1) \cdots (y_1 - u_d) e_1 = 0$. Thus without loss of generality, we can take $p_0(u) = (u - u_1) \cdots (u - u_d)$.

Assumption 5.3. For the remainder of this section, we assume the parameters of $\mathcal{A}_{n,S,r}$ are d-semi-admissible for some d with $0 < d < r$. Assume without loss of generality that $p_0(y_1)e_1 = 0$, where $p_0(u) = (u-u_1)\cdots(u-u_d) = \sum_{j=0}^{d} b_j u^j$.

Lemma 5.4. *There is a surjective homomorphism $\theta : \mathcal{A}_{n,S,r}(u_1,\ldots,u_r) \to \mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$ taking generators to generators. Moreover, θ maps the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,r}(u_1,\ldots,u_r)$ onto the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$.*

Proof. The existence of the surjective homomorphism θ is evident because the generators of $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$ satisfy the defining relations of $\mathcal{A}_{n,S,r}(u_1,\ldots,u_r)$.

In general, if A and B are algebras and $\varphi : A \to B$ is a surjective algebra homomorphism, then for any $e \in A$, we have $\varphi(AeA) = B\varphi(e)B$. In particular, θ maps the ideal generated by e_{n-1} in $\mathcal{A}_{n,Sr}(u_1,\ldots,u_r)$ onto the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,d}(u_1,\ldots,u_e)$. □

Lemma 5.5.

(1) *The sequence $\Omega = (\omega_a)_{a\geqslant 0}$ satisfies the recurrence relation $\sum_{j=0}^{d} b_j \omega_{j+\ell} = 0$ for all $\ell \geqslant 0$.*
(2) *The parameters $\Omega = (\omega_a)_{a\geqslant 0}$ and $u_1,\ldots,u_d$ in the degenerate case (respectively, $\rho, q, \Omega = (\omega_a)_{a\geqslant 0}$, and $u_1,\ldots,u_d$ in the non-degenerate case) are admissible. That is, the set $\{e_1, y_1e_1,\ldots,y_1^{d-1}e_1\}$ is linearly independent over S in $\mathcal{A}_{2,S,d}(u_1,\ldots,u_d)$.*

Proof. For part (1), multiply the equation $p_0(y_1)e_1 = 0$ by $e_1 y_1^{\ell}$ on the left, and employ the idempotent and compression relations to get $\sum_{j=0}^{d} b_j \omega_{j+\ell} e_1 = 0$. The conclusion follows since e_1 is not a torsion element over S.

We should pause to see why something needs to be proved for part (2). We have assumed that $\{e_1, y_1e_1,\ldots,y_1^{d-1}e_1\} \subseteq \mathcal{A}_{2,S,r}(u_1,\ldots,u_r)$ is linearly independent, and we have to prove that $\{e_1, y_1e_1,\ldots,y_1^{d-1}e_1\} \subseteq \mathcal{A}_{2,S,d}(u_1,\ldots,u_d)$ is linearly independent. The latter set is the image of the former under the algebra homomorphism $\theta : \mathcal{A}_{2,S,r}(u_1,\ldots,u_r) \to \mathcal{A}_{2,S,d}(u_1,\ldots,u_d)$.

Consider the degenerate case. Apply the proof of (1) ⇒ (3) in Theorem 3.12 to the linearly independent set $\{e_1, y_1e_1,\ldots,y_1^{d-1}e_1\} \subseteq \mathcal{N}_{2,S,r}(u_1,\ldots,u_r)$. This yields the analogue of condition (3.1) with r replaced by d and a_j by b_j, namely

$$\sum_{\mu=0}^{d-j-1} \omega_\mu b_{\mu+j+1} = -2\delta_{(d-j \text{ is odd})} b_j + \delta_{(j \text{ is even})} b_{j+1}, \tag{5.1}$$

for $0 \leqslant j \leqslant d-1$. Part (1) of this lemma together with the implication (3) ⇒ (1) in Theorem 3.12, applied now to $\mathcal{N}_{2,S,d}(\Omega; u_1,\ldots,u_d)$, gives the conclusion (2).

For the non-degenerate case, one uses the theorem of Wilcox and Yu (Theorem 4.1 in the case $q - q^{-1} \neq 0$, or [22] in general) in the same manner. □

5.1. A spanning set for $\mathcal{W}_{n,S,r}$

In this section, write $\mathcal{W}_n$ for $\mathcal{W}_{n,S,r}(\rho, q, \Omega; u_1,\ldots,u_r)$.

Define elements y_j' and y_j'' for $j \geqslant 1$ in the affine or cyclotomic BMW algebras by

$$y_1' = y_1'' = y_1,$$
$$y_j' = g_{j-1} y_{j-1}' g_{j-1}^{-1} \quad \text{and} \quad y_j'' = g_{j-1}^{-1} y_{j-1}'' g_{j-1} \quad \text{for } j \geqslant 2.$$

Since the elements y_j' and y_j'' are all conjugate, we have $p(y_j') = (y_j' - u_1)\cdots(y_j' - u_r) = 0$, for all j, and similarly for the elements y_j''.

Lemma 5.6. *In any affine or cyclotomic BMW algebra, e_i and g_i commute with y'_j and y''_j if $j \notin \{i, i+1\}$.*

Proof. We will prove the commutation relations for the element y'_j; the proof for the elements y''_j is essentially the same.

For $i \geqslant 2$, e_i and g_i commute with y_1 and with $g_1^{\pm 1}, \dots, g_{i-2}^{\pm 1}$, hence with y'_j for $j < i$. One sees from the defining relations that

$$g_i g_{i+1} e_i g_{i+1}^{-1} g_i^{-1} = e_{i+1} \quad \text{and} \quad {g_i}^{-1} {g_{i+1}}^{-1} e_i g_{i+1} g_i = e_{i+1}, \tag{5.2}$$

for all i. Using this, and the already established commutation relation $[e_{i+1}, y'_i] = 0$, we have

$$\begin{aligned} e_i y'_{i+2} &= e_i (g_{i+1} g_i) y'_i \big(g_i^{-1} g_{i+1}^{-1}\big) \\ &= (g_{i+1} g_i) e_{i+1} y'_i \big(g_i^{-1} g_{i+1}^{-1}\big) \\ &= (g_{i+1} g_i) y'_i e_{i+1} \big(g_i^{-1} g_{i+1}^{-1}\big) \\ &= (g_{i+1} g_i) y'_i \big(g_i^{-1} g_{i+1}^{-1}\big) e_i = y'_{i+2} e_i. \end{aligned} \tag{5.3}$$

Similarly, using the braid relations and the commutation relation $[g_{i+1}, y'_i] = 0$, we obtain that $[g_i, y'_{i+2}] = 0$. If $j \geqslant i+3$, we have

$$y'_j = (g_{j-1} \cdots g_{i+2}) y'_{i+2} \big(g_{i+2}^{-1} \cdots g_{j-1}^{-1}\big), \tag{5.4}$$

and we see that g_i and e_i commute with y'_j because they commute with all the factors on the right hand side of (5.4). □

Lemma 5.7. *In $\mathcal{W}_n$, we have $p_0(y'_j) e_i = e_i p_0(y'_j) = 0$ for all $j \neq i+1$. The same statement holds with y'_j replaced by y''_j.*

Proof. We will verify explicitly that $p_0(y'_j) e_i = 0$ for $j \neq i+1$. An identical argument shows the same with y'_j replaced by y''_j, and the statement $e_i p_0(y'_j) = e_i p_0(y''_j) = 0$ for $j \neq i+1$ follows as well by applying the involution $*$.

We first show that $p_0(y'_j) e_j = 0$ for all j, by induction. This is given for $j = 1$. If $p_0(y'_j) e_j = 0$ holds for some particular value of j, then

$$0 = g_j g_{j+1} p_0\big(y'_j\big) e_j g_{j+1}^{-1} g_j^{-1} = p_0\big(y'_{j+1}\big) e_{j+1},$$

and our assertion follows.

Next we check that $p_0(y'_j) e_i = 0$ for all $j \leqslant i$, by induction on $i - j$. We have already checked the case $j = i$. If this holds for some particular $j \leqslant i$, then

$$p_0\big(y'_j\big) e_{i+1} = p_0\big(y'_j\big) e_{i+1} e_i e_{i+1} = e_{i+1} p_0\big(y'_j\big) e_i e_{i+1} = 0.$$

It remains to check that $p_0(y'_j) e_i = 0$ for $i \leqslant j-2$. We have

$$\begin{aligned} p_0\big(y'_j\big) e_i &= g_{j-1} \cdots (g_{i+1} g_i) p_0\big(y'_i\big) \big({g_i}^{-1} {g_{i+1}}^{-1}\big) \cdots {g_{j-1}}^{-1} e_i \\ &= g_{j-1} \cdots (g_{i+1} g_i) \big[p_0\big(y'_i\big) e_{i+1}\big] \big({g_i}^{-1} {g_{i+1}}^{-1}\big) \cdots {g_{j-1}}^{-1} = 0, \end{aligned}$$

since $p_0(y'_i) e_{i+1} = 0$ by the previous part of the proof. □

We now describe a certain basis of the affine BMW algebra $\widehat{\mathcal{W}}_n = \widehat{\mathcal{W}}_{n,S}(\rho, q, \Omega)$ that was introduced in [5, Section 3.2]. Given a permutation $\pi \in \mathfrak{S}_n$, with reduced expression $\pi = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, let $g_\pi = g_{i_1} g_{i_2} \cdots g_{i_\ell}$ in $\widehat{\mathcal{W}}_n$; in fact, g_π is independent of the choice of the reduced expression of π, see [5, Section 2.4]. Fix an integer f with $0 \leqslant 2f \leqslant n$, and let γ be a Brauer diagram with $2f$ horizontal strands and $s = n - 2f$ vertical strands. Then γ has a unique factorization in the Brauer algebra of the form

$$\gamma = \alpha(e_1 e_3 \cdots e_{2f-1})\pi \beta^{-1}, \tag{5.5}$$

where π is a permutation of $\{2f+1, \ldots, n-1, n\}$ and α and β are in a certain subset $\mathcal{D}_{f,n}$ of $\mathfrak{S}_n$ described in [5, Section 3.2]. Consider a sequence of n integers

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (a_1, a_3, \ldots, a_{2f-1}, b_1, b_3, \ldots, b_{2f-1}, c_{2f+1}, \ldots, c_n).$$

Corresponding to γ and the sequence $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, we let $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ be the following element of $\widehat{\mathcal{W}}_n$,

$$T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = g_\alpha y''^{\boldsymbol{a}} (e_1 e_3 \cdots e_{2f-1}) g_\pi y''^{\boldsymbol{c}} y'^{\boldsymbol{b}} (g_\beta)^*,$$

where

$$y''^{\boldsymbol{a}} = y_1''^{a_1} y_3''^{a_3} \cdots y_{2f-1}''^{a_{2f-1}},$$
$$y'^{\boldsymbol{b}} = y_{2f-1}'^{b_{2f-1}} \cdots y_3'^{b_3} y_1'^{b_1},$$

and

$$y''^{\boldsymbol{c}} = y_n''^{c_n} \cdots y_{2f+2}''^{c_{2f+2}} y_{2f+1}''^{c_{2f+1}}.$$

If γ has no horizontal strands (i.e. γ is a permutation diagram), the elements $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ still make sense, but then $f = 0$, α and β are trivial, $\gamma = \pi$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are empty sequences. We have

$$T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = T_{\gamma,\boldsymbol{c}} = g_\gamma y''^{\boldsymbol{c}}.$$

It is shown in [5, Section 3.2] that the set of $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$, as γ ranges over Brauer diagrams and $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ ranges over n-tuples of integers forms an S-basis of $\widehat{\mathcal{W}}_n$, and, moreover, the subset corresponding to Brauer diagrams with $2f > 0$ horizontal strands, forms a basis of the ideal $\widehat{\mathcal{W}}_n e_{n-1} \widehat{\mathcal{W}}_n$.

Let $b'(n)$ denote the number of Brauer diagrams on n strands with at least one horizontal strand, $b'(n) = (2n-1)!! - n!$.

Lemma 5.8. *The ideal $\mathcal{W}_n e_{n-1} \mathcal{W}_n$ is spanned by a set of $d^n b'(n)$ elements. The algebra $\mathcal{W}_n$ is spanned by a set of $d^n b'(n) + r^n n!$ elements.*

Proof. We also write $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ for the image of that element in the cyclotomic BMW algebra $\mathcal{W}_n$. The set of all $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ spans $\mathcal{W}_n$, while those with γ a Brauer diagram with $2f > 0$ horizontal strands span the ideal $\mathcal{W}_n e_{n-1} \mathcal{W}_n$.

If γ is a permutation diagram, then we can write any element $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = T_{\gamma,\boldsymbol{c}}$ as a linear combination of elements $T_{\gamma,\boldsymbol{c}'}$, with $0 \leqslant c_i' \leqslant r$, using the relations $p(y_j'') = (y_j'' - u_1) \cdots (y_j'' - u_r) = 0$.

In the following, take $f > 0$ and let γ be a Brauer diagram with $2f$ horizontal strands. We claim that any element $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ can be written as a linear combination of elements $T_{\gamma,\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}'}$ where a_i', b_i', and c_i' lie in the interval $0, 1, \ldots, d-1$. Using the commutation relations of Lemma 5.6, we can write

$$T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = g_\alpha \big(y_1''^{a_1} e_1\big)\big(y_3''^{a_3} e_3\big) \cdots \big(y_{2f-1}''^{a_{2f-1}} e_{2f-1}\big) g_\pi y''^{\boldsymbol{c}} y'^{\boldsymbol{b}} (g_\beta)^*.$$

Now, using Lemma 5.7, we can write any such element as a linear combination elements $T_{\gamma,\boldsymbol{a}',\boldsymbol{b},\boldsymbol{c}}$ with the a_i' in the desired interval. Using the commutation relations again, we can also write

$$T_{\gamma,\boldsymbol{a}',\boldsymbol{b},\boldsymbol{c}} = g_\alpha y''^{\boldsymbol{a}'} g_\pi y''^{\boldsymbol{c}} \big(e_{2f-1} y_{2f-1}'^{b_{2f-1}}\big) \cdots \big(e_3 y_3'^{b_3}\big)\big(e_1 y_1'^{b_1}\big)(g_\beta)^*,$$

and using Lemma 5.7, we can write any such element as a linear combination of elements $T_{\gamma,\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}}$ with the b_i' in the desired interval. Finally, e_{2f-1} commutes with g_π and with all y_{2f+j}''. Using $e_{2f-1} p_0(y_{2f+j}'') = 0$, we can reduce any $T_{\gamma,\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}}$ to a linear combination elements $T_{\gamma,\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}'}$ with the c_i' in the desired interval.

It follows that $\mathcal{W}_n$ is spanned by elements $T_{\gamma,\boldsymbol{c}}$, where γ is a permutation diagram and $0 \leqslant c_i \leqslant r$, and by elements $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ where γ is a Brauer diagram with at least 2 horizontal strands and $0 \leqslant a_i, b_i, c_i \leqslant d$. Moreover, the latter set spans $\mathcal{W}_n e_{n-1} \mathcal{W}_n$. $\square$

5.2. A spanning set for $\mathcal{N}_{n,S,r}$

In this section, write $\mathcal{N}_n$ for $\mathcal{N}_{n,S,r}(\Omega; u_1, \dots, u_r)$.

Consider first the free non-commutative polynomial algebra in the generators $\{s_i, e_i, y_j\colon 1 \leqslant i < n$ and $1 \leqslant j \leqslant n\}$. Assign degrees to the generators, $\deg(e_i) = \deg(s_i) = 0$, $\deg(y_j) = 1$. This makes the non-commutative polynomial algebra into a graded algebra. As the homomorphic image of a graded algebra is a filtered algebra, the degenerate cyclotomic BMW algebra $\mathcal{N}_n$ is filtered by degree, as is the ideal $\mathcal{N}_n e_{n-1} \mathcal{N}_n$. Let $\mathcal{G} = \mathrm{gr}(\mathcal{N}_n)$ denote the associated graded algebra. We will write e_i, s_i, y_j also for the images of these elements in $\mathcal{G}$.

Note that $(\mathcal{N}_n)_0$, the degree zero part of $\mathcal{N}_n$, is the unital subalgebra generated by $\{s_i, e_i\colon 1 \leqslant i < n\}$. The canonical map from $\mathcal{N}_n$ to $\mathcal{G}$ restricts to an algebra isomorphism from $(\mathcal{N}_n)_0$ to $\mathcal{G}_0$.

To produce a spanning set in the ideal $\mathcal{N}_{n,S,r} e_{n-1} \mathcal{N}_{n,S,r}$, it suffices to produce a spanning set in $\mathcal{G} e_{n-1} \mathcal{G}$.

Lemma 5.9. *In $\mathcal{G}$, we have $p_0(y_j) e_i = e_i p_0(y_j) = 0$ for all $j \neq i+1$.*

Proof. In $\mathcal{G}$, the elements y_i become conjugate, $s_i y_i s_i = y_{i+1}$. It follows that the proof of Lemma 5.7 carries over unchanged (replacing y_j' with y_j and g_i with s_i everywhere). $\square$

We have already discussed the surjective homomorphism from the Brauer algebra $\mathcal{B}_n(\omega_0)$ with parameter ω_0 to $(\mathcal{N}_n)_0$ taking $s_i \mapsto s_i$ and $e_i \mapsto e_i$; see the discussion just before Proposition 3.7. For a Brauer diagram γ, we will also write γ for the image of γ in $(\mathcal{N}_n)_0$ and in $\mathcal{G}_0$. According to Proposition 3.7, $\mathcal{N}_n$ is spanned by the set of r-regular monomials

$$y^{\boldsymbol{p}} \gamma y^{\boldsymbol{q}}. \tag{5.6}$$

Furthermore, the ideal $\mathcal{N}_n e_{n-1} \mathcal{N}_n$ is spanned by those elements $y^{\boldsymbol{p}} \gamma y^{\boldsymbol{q}}$ such that γ has $2f > 0$ horizontal strands.

If γ is a permutation diagram, then $p_i = 0$ for all i and

$$y^{\boldsymbol{p}} \gamma y^{\boldsymbol{q}} = \gamma y^{\boldsymbol{q}} := T_{\gamma,\boldsymbol{q}}.$$

If γ is not a permutation diagram, then using the factorization of γ in Eq. (5.5), and using $s_i y_i = y_{i+1} s_i$ in $\mathcal{G}$, the image of the element (3.10) in $\mathcal{G}$ can be written as

$$
\begin{aligned}
y^{\boldsymbol{p}}\gamma y^{\boldsymbol{q}} &= y^{\boldsymbol{p}}\alpha(e_1e_3\cdots e_{2f-1})\pi\beta^{-1}y^{\boldsymbol{q}} \\
&= \alpha y^{\alpha^{-1}(\boldsymbol{p})}(e_1e_3\cdots e_{2f-1})\pi y^{\beta^{-1}(\boldsymbol{q})}\beta^{-1},
\end{aligned} \tag{5.7}
$$

where $y^{\alpha^{-1}(\boldsymbol{p})} = {y_1}^{p_{\alpha(1)}}\cdots {y_n}^{p_{\alpha(n)}}$ and $y^{\beta^{-1}(\boldsymbol{p})} = {y_1}^{p_{\beta(1)}}\cdots {y_n}^{p_{\beta(n)}}$. Taking into account the restrictions on $\boldsymbol{p}$ and $\boldsymbol{q}$, this can be written in the form

$$T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} = \alpha y^{\boldsymbol{a}}(e_1e_3\cdots e_{2f-1})\pi y^{\boldsymbol{c}}y^{\boldsymbol{b}}\beta^{-1} \tag{5.8}$$

where

$$
\begin{gathered}
y^{\boldsymbol{a}} = {y_1}^{a_1}{y_3}^{a_3}\cdots {y_{2f-1}}^{a_{2f-1}}, \\
y^{\boldsymbol{c}} = \prod_{2f+1\leqslant j\leqslant n} y_j^{c_j},
\end{gathered}
$$

and

$$y^{\boldsymbol{b}} = {y_1}^{b_1}{y_3}^{b_3}\cdots {y_{2f-1}}^{b_{2f-1}}.$$

Lemma 5.10. *The ideal $\mathcal{N}_n e_{n-1}\mathcal{N}_n$ is spanned by a set of $d^n b'(n)$ elements. The algebra $\mathcal{N}_n$ is spanned by a set of $d^n b'(n) + r^n n!$ elements.*

Proof. It is enough to work instead in the associated graded algebra $\mathcal{G}$. We have that $\mathcal{G}$ is spanned by the elements $T_{\gamma,\boldsymbol{c}}$, where γ is a permutation diagram and $0 \leqslant c_i \leqslant r-1$ for all i, and by the elements $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ where γ is a Brauer diagram with at least 2 horizontal strands. The argument of Lemma 5.8, with Lemma 5.7 replaced by Lemma 5.9, shows that any $T_{\gamma,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$, where γ has horizontal strands, can be written as a linear combination of elements $T_{\gamma,\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}'}$, with $0 \leqslant a_i, b_i, c_i \leqslant d-1$. Moreover, the latter set of elements spans $\mathcal{G}e_{n-1}\mathcal{G}$. $\square$

5.3. Freeness of $\mathcal{A}_{n,S,r}$

Let us recall from Lemma 5.4 that there is a surjective algebra homomorphism

$$\theta : \mathcal{A}_{n,S,r}(u_1,\ldots,u_r) \to \mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$$

and that θ maps the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,r}(u_1,\ldots,u_r)$ onto the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$.

Proposition 5.11. *θ induces an isomorphism from the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,r}(u_1,\ldots,u_r)$ onto the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$.*

Proof. Write $\langle e_{n-1}\rangle_r$ for the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,r}(u_1,\ldots,u_r)$ and $\langle e_{n-1}\rangle_d$ for the ideal generated by e_{n-1} in $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$.

The parameters of $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$ are admissible, by Lemma 5.5. Hence, we know that $\mathcal{A}_{n,S,d}(u_1,\ldots,u_d)$ is a free S module of rank $d^n(2n-1)!!$, and $\langle e_{n-1}\rangle_d$ is free of rank

$$d^n\big((2n-1)!! - n!\big) = d^n b'(n),$$

where $b'(n)$ denotes the number of Brauer diagrams on n strands with at least one horizontal strand. We know that $\langle e_{n-1}\rangle_r$ has a spanning set of the same cardinality by Lemmas 5.8 and 5.10. Therefore, $\theta : \langle e_{n-1}\rangle_r \to \langle e_{n-1}\rangle_d$ is an isomorphism. (In fact, if $\mathbb{B}$ is spanning set of $\langle e_{n-1}\rangle_r$ of cardinality $d^n b'(n)$, then $\theta(\mathbb{B})$ spans $\langle e_{n-1}\rangle_d$. Since S is an integral domain and $\langle e_{n-1}\rangle_d$ is free over S with a basis of the same cardinality, it follows that $\theta(\mathbb{B})$ is a basis of $\langle e_{n-1}\rangle_d$. Therefore, $\mathbb{B}$ is a basis of $\langle e_{n-1}\rangle_r$, and θ is an isomorphism.) $\square$

Theorem 5.12. *For all $n \geqslant 0$, $\mathcal{A}_{n,S,r}$ is a free S-module of rank $d^n b'(n) + r^n n!$, and $\mathcal{A}_{n,S,r}$ imbeds in $\mathcal{A}_{n+1,S,r}$.*

Proof. The ideal $\langle e_{n-1}\rangle_r$ is free of rank $d^n b'(n)$, by Proposition 5.11, and the quotient $\mathcal{A}_{n,S,r}/\langle e_{n-1}\rangle_r$ is isomorphic to the cyclotomic Hecke algebra or degenerate cyclotomic Hecke algebra, which is free of rank $r^n n!$. Therefore, $\mathcal{A}_{n,S,r}$ is free of rank $d^n b'(n) + r^n n!$.

We have given spanning sets of the same cardinality in Lemmas 5.8 and 5.10, and hence those spanning sets are actually S-bases. It is straightforward to check that the homomorphism from $\mathcal{A}_{n,S,r}$ to $\mathcal{A}_{n+1,S,r}$ taking generators to generators maps the given basis of $\mathcal{A}_{n,S,r}$ injectively into the basis of $\mathcal{A}_{n+1,S,r}$. Therefore the map is injective. $\square$

6. Cellularity

The following is a slight weakening of the original definition of cellularity from Graham and Lehrer [13].

Definition 6.1. (See [13].) Let R be an integral domain and A a unital R-algebra. A *cell datum* for A consists of an algebra involution $*$ of A; a partially ordered set $(\Lambda, \geqslant)$ and for each $\lambda \in \Lambda$ a set $\mathcal{T}(\lambda)$; and a subset $\mathcal{C} = \{c^\lambda_{s,t}\colon\ \lambda \in \Lambda \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq A$; with the following properties:

(1) $\mathcal{C}$ is an R-basis of A.
(2) For each $\lambda \in \Lambda$, let $\breve{A}^\lambda$ be the span of the $c^\mu_{s,t}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $s \in \mathcal{T}(\lambda)$, and $a \in A$, there exist coefficients $r^s_v(a) \in R$ such that for all $t \in \mathcal{T}(\lambda)$:

$$ac^\lambda_{s,t} \equiv \sum_v r^s_v(a) c^\lambda_{v,t} \mod \breve{A}^\lambda.$$

(3) $(c^\lambda_{s,t})^* \equiv c^\lambda_{t,s} \mod \breve{A}^\lambda$ for all $\lambda \in \Lambda$ and, $s, t \in \mathcal{T}(\lambda)$.

A is said to be a *cellular algebra* if it has a cell datum.

For brevity, we will write that $(\mathcal{C}, \Lambda)$ is a cellular basis of A. In the original definition in [13] it is required that $(c^\lambda_{s,t})^* = c^\lambda_{t,s}$. All the conclusions of [13] remain valid with the weaker definition, and, in fact, the two definitions are equivalent if 2 is invertible in R. The main advantage of the weaker definition is that it allows a graceful treatment of extensions.

Definition 6.2. Let A be an algebra with involution and let J be a $*$-invariant ideal. Say that J is a *cellular ideal* if it satisfies the axioms for a cellular algebra (except for being unital) with cellular basis

$$\{c^\lambda_{s,t}\colon\ \lambda \in \Lambda_J \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq J$$

and we have, as in point (2) of the definition of cellularity,

$$ac^\lambda_{s,t} \equiv \sum_v r^s_v(a) c^\lambda_{v,t} \mod \breve{J}^\lambda$$

not only for $a \in J$ but also for $a \in A$.

Lemma 6.3 *(On extensions of cellular algebras). If J is a cellular ideal in A, and $H = A/J$ is cellular (with respect to the involution induced from the involution on A), then A is cellular.*

Proof. Let $(\Lambda_J, \geqslant)$ be the partially ordered set in the cell datum for J and $\mathcal{C}_J$ the cellular basis. Let $(\Lambda_H, \geqslant)$ be the partially ordered set in the cell datum for H and $\{\bar{h}^{\mu}_{u,v}\}$ the cellular basis. Let $\Lambda = \Lambda_J \cup \Lambda_H$, with partial order agreeing with the original partial orders on Λ_J and on Λ_H and with $\lambda > \mu$ if $\lambda \in \Lambda_J$ and $\mu \in \Lambda_H$. A cellular basis of A is $\mathcal{C}_J \cup \{h^{\mu}_{s,t}\}$, where $h^{\mu}_{s,t}$ is any lift of $\bar{h}^{\mu}_{s,t}$. $\square$

Theorem 6.4. *Consider the sequence $\mathcal{A}_{n,S,r}$ of cyclotomic or degenerate cyclotomic BMW algebras over an integral domain S. Suppose that Assumption* 5.1 *holds. Then*

(1) *$\mathcal{A}_{n,S,r}$ imbeds in $\mathcal{A}_{n+1,S,r}$ for all $n \geqslant 0$.*
(2) *$\mathcal{A}_{n,S,r}$ is a cellular algebra.*

Proof. In the case $e_1 = 0$ in $\mathcal{A}_{2,S,r}$, the cyclotomic or degenerate cyclotomic BMW algebras reduce to cyclotomic or degenerate cyclotomic Hecke algebras; in this case the results are known. If the parameters are admissible, these results are obtained in the papers cited in the introduction.

It remains to verify the results in the semi-admissible case. We already have shown in the semi-admissible case that $\mathcal{A}_{n,S,r}$ is a free S module, and that $\mathcal{A}_{n,S,r}$ imbeds in $\mathcal{A}_{n+1,S,r}$. Adopt the notation and conventions of Section 5. We know that $\mathcal{A}_{n,S,d}(u_1, \dots, u_d)$ has admissible parameters by Lemma 5.5, and therefore is a cellular algebra by the papers cited in the introduction. Moreover, $\langle e_{n-1}\rangle_d$ is a cellular ideal in $\mathcal{A}_{n,S,d}(u_1, \dots, u_d)$. It follows that $\langle e_{n-1}\rangle_r$ is a cellular ideal in $\mathcal{A}_{n,S,r}$, with cellular basis $\{\theta^{-1}(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})\}$, where $\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$ is a cellular basis of $\langle e_{n-1}\rangle_d$. The crucial point regarding the expansion of $a\theta^{-1}(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})$ in terms of basis elements, for $a \in \mathcal{A}_{n,S,r}$ follows because $a\theta^{-1}(c^{\lambda}_{\mathfrak{s},\mathfrak{t}}) = \theta^{-1}(\theta(a)c^{\lambda}_{\mathfrak{s},\mathfrak{t}})$.

Since $\mathcal{A}_{n,S,r}/\langle e_{n-1}\rangle_r$ is isomorphic to the cyclotomic Hecke algebra, or degenerate cyclotomic Hecke algebra, which is cellular, it follows from Lemma 6.3 that $\mathcal{A}_{n,S,r}$ is cellular. $\square$

Corollary 6.5. *Any cyclotomic or degenerate cyclotomic BMW algebra over a field is cellular.*

Proof. In case the ground ring is a field, Assumption 5.1 holds automatically. $\square$

Corollary 6.6. *Let F be an algebraically closed field and consider an affine* (*resp. degenerate affine*) *BMW algebra $\widehat{\mathcal{A}}_{n,F}$ over F. Let M be a simple finite dimensional $\widehat{\mathcal{A}}_{n,F}$-module. If $e_1 M = 0$, then M factors through a cyclotomic* (*resp. degenerate cyclotomic*) *Hecke algebra. If $e_1 M \neq 0$, then M factors through cyclotomic* (*resp. degenerate cyclotomic*) *BMW algebra with admissible parameters.*

Proof. In the degenerate case, this result is contained in [3, Theorem 7.19 and Proposition 3.11] (but with the hypothesis that the characteristic of the field is $\neq 2$.)

Because the field is algebraically closed, the minimal polynomial of y_1 on M factors over F. Hence M factors through some cyclotomic (resp. degenerate cyclotomic) BMW algebra. If $e_1 M = 0$, then M factors through the corresponding cyclotomic (resp. degenerate cyclotomic) Hecke algebra. If $e_1 M \neq 0$, then the parameters of the cyclotomic (resp. degenerate cyclotomic) BMW algebra must be either admissible or semi-admissible.

Let us assume a cyclotomic (resp. degenerate cyclotomic) BMW algebra $\mathcal{A}_{n,F,r} = \mathcal{A}_{nF,r}(u_1, \dots, u_r)$ with d-semi-admissible parameters $(d < r)$. Then M is the simple head of a cell module Δ^{λ}, and since $e_1 M \neq 0$, the cell module belongs to the ideal $\langle e_{n-1}\rangle_r$. But the cell modules belonging to $\langle e_{n-1}\rangle_r$ factor through $\theta : \mathcal{A}_{nF,r}(u_1, \dots, u_r) \to \mathcal{A}_{nF,d}(u_1, \dots, u_d)$, and the latter algebra has admissible parameters. $\square$

The following proposition depends only on the material in this paper up through Lemma 5.5.

Proposition 6.7. *Let F be an algebraically closed field and consider an affine* (*resp. degenerate affine*) *BMW algebra $\widehat{\mathcal{A}}_{n,F}$ over F. The following are equivalent*:

(1) *There exist* $r > 0$ *and* $u_1, \ldots, u_r \in F$ *such that the parameters of* $\widehat{\mathcal{A}}_{n,F}$ *together with* $u_1, \ldots, u_r$ *are admissible.*
(2) $\widehat{\mathcal{A}}_{n,F}$ *admits a finite dimensional module on which* e_1 *is non-zero.*

Proof. If (1) holds, then $\mathcal{A}_{n,F,r}(u_1, \ldots, u_r)$ is a finite dimensional $\widehat{\mathcal{A}}_{n,F}$ module on which $e_1 \neq 0$. If (2) holds, let $u_1, \ldots, u_r$ be the roots of the minimal polynomial of y_1 acting on M. The module M factors through the cyclotomic algebra $\mathcal{A}_{n,F,r}(u_1, \ldots, u_r)$. Since $e_1 M \neq 0$, it follows that $e_1 \neq 0$ in $\mathcal{A}_{n,F,r}(u_1, \ldots, u_r)$ and hence also in $\mathcal{A}_{2,F,r}(u_1, \ldots, u_r)$. Since F is a field, Assumption 5.1 holds for $\mathcal{A}_{2,F,r}(u_1, \ldots, u_r)$. Therefore, the parameters of $\mathcal{A}_{2,F,r}(u_1, \ldots, u_r)$ are either admissible or d-semi-admissible for some d with $0 < d < r$. In the latter case, after renumbering the roots u_i, $\mathcal{A}_{2,F,d}(u_1, \ldots, u_d)$ has admissible parameters, by Lemma 5.5. Thus (1) holds. □

7. Rationality of parameters for affine algebras

7.1. Rationality of parameters for degenerate affine BMW algebras

Ariki, Mathas, and Rui call the parameter set Ω of a degenerate affine (or cyclotomic) BMW algebra *rational* if the generating function $\sum_{a \geqslant 0} \omega_a t^{-a}$ is a rational function. They prove the following theorem, under the additional hypothesis that the characteristic of the field is different from 2.

Theorem 7.1. *Consider the degenerate affine BMW algebra* $\widehat{\mathcal{N}}_n$, $n \geqslant 2$, *over an algebraically closed field* F, *with parameters* $\Omega = (\omega_a)_{a \geqslant 0}$. *Suppose that* $e_1 \neq 0$ *in* $\widehat{\mathcal{N}}_n$. *The following are equivalent.*

(1) *The generating function* $\sum_{a \geqslant 0} \omega_a t^{-a}$ *is a rational function in* $F(t)$.
(2) Ω *satisfies a linear homogeneous recursion; i.e. there exist* $r > 0$, $N \geqslant 0$ *and* $a_0, a_1, \ldots, a_{r-1} \in F$ *such that* $\omega_{r+\ell} + \sum_{j=0}^{r-1} a_j \omega_{j+\ell} = 0$, *for all* $\ell \geqslant N$.
(3) *There exist* $r > 0$ *and* $a_0, a_1, \ldots, a_{r-1} \in F$ *such that* $\omega_{r+\ell} + \sum_{j=0}^{r-1} a_j \omega_{j+\ell} = 0$, *for all* $\ell \geqslant 0$.
(4) *There exist* $r > 0$ *and* $u_1, \ldots, u_r \in F$ *such that the parameters* Ω *and* $u_1, \ldots, u_r$ *are admissible.*
(5) $\widehat{\mathcal{N}}_n$ *admits a finite dimensional module on which* e_1 *is non-zero.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) is easy, and (3) $\Leftarrow$ (4) holds by Lemma 2.7. Proposition 6.7 gives (4) $\Leftrightarrow$ (5). The implication (1) $\Rightarrow$ (4) is proved in [3, Proposition 3.11] under the assumption that the characteristic of the field is not equal to 2. So it remains only to prove this implication for a field of characteristic 2. This will be done with the aid of two lemmas. □

Lemma 7.2. *Consider the degenerate affine BMW algebra* $\widehat{\mathcal{N}}_{2,S}$ *over a ring* S, *with parameters* $\Omega = (\omega_a)_{a \geqslant 0}$. *Suppose that* e_1 *is not a torsion element over* S. *Then*:

(1) $2\omega_{2a+1} = -\omega_{2a} + \sum_{b=1}^{2a+1} (-1)^{b-1} \omega_{b-1} \omega_{2a+1-b}$ *for* $a \geqslant 0$.
(2) *If the characteristic of* S *is* 2, *then* $\omega_{2a} = \omega_a^2$ *for* $a \geqslant 0$.

Proof. Part (1) is [3, Corollary 2.4]. If the characteristic is 2, then the equation in part (1) simplifies to $\omega_{2a} = \omega_a^2$. □

The proof of the following lemma was suggested by Kevin Buzzard, via `mathoverflow.net`.

Lemma 7.3. *Let* F *be an algebraically closed field of characteristic* 2. *Suppose that* $\Omega = (\omega_a)_{a \geqslant 0}$ *satisfies a linear homogeneous recursion, as in Theorem* 7.1(2) *and* $\omega_{2a} = \omega_a^2$ *for* $a \geqslant 0$. *Then there exist distinct* $u_1, \ldots, u_d \in F$ *such that* $\omega_a = \sum_{i=1}^{d} u_i^a$ *for all* $a \geqslant 1$, *and* $\omega_0 \in \{0, 1\}$.

Proof. Our assumptions include $\omega_0 = \omega_0^2$. Thus $\omega_0 \in \{0, 1\}$. Let $v_1, \ldots, v_m$ be the distinct roots of the characteristic polynomial of the linear recursion relation satisfied by Ω. Then there exist polynomials

$h_1, \ldots, h_m$ such that $\omega_a = \sum_{i=1}^m h_i(a) v_i^a$ for $a \geqslant N$. Let α_i be the constant term of h_i for each i. Since $\operatorname{char}(F) = 2$, we have $h_i(2a) = \alpha_i$ for all a. For $a \geqslant N$,

$$\sum_i \alpha_i v_i^{4a} = \omega_{4a} = \omega_{2a}^2 = \sum_i \alpha_i^2 v_i^{4a}. \tag{7.1}$$

Because the characteristic of F is 2, each element has a unique 2^k-th root for all $k \geqslant 1$; in particular all the v_i^4 are distinct, so Eq. (7.1) implies that $\alpha_i^2 = \alpha_i$ for all i, i.e. $\alpha_i \in \{0, 1\}$. Let $u_1, \ldots, u_d$ be the list of those v_j such that $\alpha_j = 1$. Then we have $\omega_{2a} = \sum_i u_i^{2a}$ for $a \geqslant N$. For an arbitrary $a \geqslant 1$, chose k such that $2^{k-1} a \geqslant N$. Then ω_a is the unique 2^k-th root of $\omega_{2^k a} = \sum_i u_i^{2^k a}$, namely $\omega_a = \sum_i u_i^a$. □

Conclusion of the proof of Theorem 7.1. Let us prove (1) ⇒ (4) when the characteristic of the field is 2. Since the ground ring is a field and $e_1 \neq 0$, we have $\omega_{2a} = \omega_a^2$ for $a \geqslant 0$, by Lemma 7.2. Hence, by Lemma 7.3, there exist $u_1, \ldots, u_d \in F$ such that

$$\omega_a = p_a(u_1, \ldots, u_d) = p_a(u_1, \ldots, u_d, 0)$$

for $a \geqslant 1$ and $\omega_0 \in \{0, 1\}$. Using Example 3.4 and Definition 3.5, Ω is either $(u_1, \ldots, u_d)$-admissible or $(u_1, \ldots, u_d, 0)$-admissible, so by Theorem 3.12, on equivalent conditions for admissibility, condition (4) holds. □

Corollary 7.4. *(See Rui and Si [19].) Assume* $\operatorname{char}(F) \neq 2$. *The conditions of Theorem* 7.1 *are equivalent to the existence of a simple finite dimensional module on which e_1 is non-zero, as long as Ω is not the zero sequence or $n \neq 2$.*

Proof. By the results of [19], a degenerate cyclotomic BMW algebra $\mathcal{N}_{n,F,r}(\Omega; u_1, \ldots, u_r)$ with admissible parameters has a simple module on which e_1 is non-zero, as long as Ω is not the zero sequence or $n \neq 2$. (Rui and Si assumed $\operatorname{char}(F) \neq 2$, and I have not checked whether their results remain valid in characteristic 2.) □

7.2. Rationality of parameters for affine BMW algebras

We are going to obtain a result analogous to Theorem 7.1 for the affine BMW algebras.

Lemma 7.5. *Consider an affine BMW algebra $\widehat{\mathcal{W}}_{n,S}$ with parameters ρ, q, and $\Omega = (\omega_a)_{a \geqslant 0}$.*

(1) *There exist elements $\omega_{-a} \in S$ such that $e_1 y_1^{-a} e_1 = \omega_{-a} e_1$ for $a \geqslant 1$.*
(2) *Suppose that e_1 is not a torsion element over S. Then*:

$$-\omega_a + \omega_{-a} + \rho\big(q - q^{-1}\big) \sum_{i=1}^{a} (\omega_{a-i}\omega_{-i} - \omega_{a-2i}) = 0 \quad \textit{for } a \geqslant 1. \tag{7.2}$$

(3) *Suppose that S is an integral domain, that $q - q^{-1} \neq 0$, and that e_1 is not a torsion element over S. Then*:

$$\left[\sum_{a \geqslant 0} \omega_a t^{-a} - \frac{t^2}{t^2 - 1} + \frac{\rho^{-1}}{q - q^{-1}}\right]\left[\sum_{b \geqslant 1} \omega_{-b} t^{-b} - \frac{1}{t^2 - 1} - \frac{\rho^{-1}}{q - q^{-1}}\right]$$
$$= \frac{t^2}{(t^2 - 1)^2} - \frac{1}{(q - q^{-1})^2}. \tag{7.3}$$

Proof. Statement (1) is from [10, Corollary 3.13]. Statement (2) is proved in [21, Lemma 2.17] and (in a different but equivalent form) in [10, Corollary 3.13] and [11, Lemma 2.6]. Eq. (7.3) appears as (2.30) in [21]. If S is integral and $q - q^{-1} \neq 0$, then (7.2) is equivalent to (7.3). To see this, expand the left side of (7.3) and isolate the coefficient of t^{-n} for each $n \geqslant 0$. $\square$

Remark 7.6. The equivalence of (7.2) and (7.3) seems to be implicit in [21]. The left side of (7.3) can also be written as:

$$\left[\sum_{a\geqslant 0}\omega_a t^{-a} - \frac{t^2}{t^2-1} + \frac{\rho^{-1}}{q-q^{-1}}\right]\left[\sum_{b\geqslant 0}\omega_{-b} t^{-b} - \frac{t^2}{t^2-1} - \frac{\rho}{q-q^{-1}}\right].$$

Ram et al. [4] have given an interesting non-inductive direct proof of (7.3).

Lemma 7.7. *Consider a cyclotomic BMW algebra $\mathcal{W}_{n,S,r}$ with parameters ρ, q, and $\Omega = (\omega_a)_{a\geqslant 0}$ and $u_1, \dots, u_r$. Let a_i be given by Eq. (2.2). If e_1 is not a torsion element over S, then $\sum_{j=0}^{r} a_j \omega_{j+\ell} = 0$ for all $\ell \in \mathbb{Z}$.*

Proof. Same as the proof of Lemma 2.7. $\square$

Lemma 7.8. *Consider an affine BMW algebra $\widehat{\mathcal{W}}_{n,F}$ with parameters ρ, q, and $\Omega = (\omega_a)_{a\geqslant 0}$ over a field F. Suppose that there exist $r > 0$ and $a_0, a_1, \dots, a_{r-1} \in S$ such that $\omega_{a+r} + \sum_{j=0}^{r-1} a_j \omega_{j+a} = 0$ for all $a \in \mathbb{Z}$. Then $w^+(t) = \sum_{a\geqslant 0} \omega_a t^{-a}$ and $w^-(t) = \sum_{b\geqslant 1} \omega_{-b} t^{-b}$ are rational functions in $F(t)$ and $w^-(t) = -w^+(t^{-1})$. Moreover, $w^+(0) = 0$ and $w^+(\infty) = \omega_0$.*

Proof. Let $p(t) = t^r + \sum_{j=0}^{r-1} a_j t^j$. Then one computes, using the recursion on $(\omega_a)_{a\in\mathbb{Z}}$, that $p(t)w^+(t) = q_1(t)$, where q_1 is an explicit polynomial of degree $\leqslant r$. Similarly, $p(t)w^-(t^{-1}) = q_2(t)$. Using the recursion again, one sees that $q_1 = -q_2$. The coefficient of t^r in $q_1(t)$ is ω_0 and the constant term is zero; this gives $w^+(0) = 0$ and $w^+(\infty) = \omega_0$. $\square$

Theorem 7.9. *Consider an affine BMW algebra $\widehat{\mathcal{W}}_n$ over an algebraically closed field F, with parameters ρ, q, and $\Omega = (\omega_a)_{a\geqslant 0}$. Suppose that $e_1 \neq 0$ in $\widehat{\mathcal{W}}_n$. Consider the following statements:*

(1) *$w^+(t) = \sum_{a\geqslant 0} \omega_a t^{-a}$ and $w^-(t) = \sum_{b\geqslant 1} \omega_{-b} t^{-b}$ are rational functions in $F(t)$ and $w^-(t) = -w^+(t^{-1})$. Moreover, $w^+(t)$ does not have a pole at 0 or at ∞.*
(2) *There exist $r > 0$ and $a_0, a_1, \dots, a_{r-1} \in F$ such that $\omega_{r+\ell} + \sum_{j=0}^{r-1} a_j \omega_{j+\ell} = 0$, for all $\ell \in \mathbb{Z}$.*
(3) *There exist $r > 0$ and $u_1, \dots, u_r \in F$ such that the parameters ρ, q, Ω, and $u_1, \dots, u_r$ are admissible.*
(4) *$\widehat{\mathcal{W}}_n$ admits a finite dimensional module on which e_1 is non-zero.*

The following implications hold:

$$(1) \Leftarrow (2) \Leftarrow (3) \Leftrightarrow (4).$$

If $q - q^{-1} \neq 0$, then all the conditions are equivalent.

Proof. The implication (1) $\Leftarrow$ (2) is from Lemma 7.8, and (2) $\Leftarrow$ (3) from Lemma 7.7. The equivalence (3) $\Leftrightarrow$ (4) comes from Proposition 6.7.

It remains to prove (1) $\Rightarrow$ (3) if $q - q^{-1} \neq 0$. Assume (1). Since the ground ring is a field and e_1 is assumed to be non-zero, (7.3) holds. But by assumption, we have that $w^+(t) = \sum_{a\geqslant 0} \omega_a t^{-a}$ and $w^-(t) = \sum_{b\geqslant 1} \omega_{-b} t^{-b}$ are rational functions, and $w^-(t) = -w^+(t^{-1})$. Substituting in (7.3), and writing

$$h(t) = -\frac{t^2}{t^2-1} + \frac{\rho^{-1}}{q-q^{-1}},$$

we have

$$-\big[w^+(t)+h(t)\big]\big[w^+\big(t^{-1}\big)+h\big(t^{-1}\big)\big] = \frac{t^2}{(t^2-1)^2} - \big(q-q^{-1}\big)^{-2}. \tag{7.4}$$

Define

$$B(t) = \big(q-q^{-1}\big)^{-1} + \frac{t}{t^2-1} = \frac{(t+q)(t-q^{-1})}{(q-q^{-1})(t^2-1)}.$$

Note that

$$-B(t)B\big(t^{-1}\big) = \frac{t^2}{(t^2-1)^2} - \big(q-q^{-1}\big)^{-2}. \tag{7.5}$$

We can write $w^+(t)$ in the form

$$w^+(t) = -h(t) + B(t)A_0 t^m \frac{\prod_{\ell=1}^s (tu_\ell - 1)}{\prod_{j=1}^r (t-v_j)},$$

where $m \in \mathbb{Z}$, $A_0 \in F$, no u_ℓ or v_j is zero, and $u_\ell \neq v_j^{-1}$ for all j, ℓ. Then, taking into account Eqs. (7.4) and (7.5) we have

$$1 = A_0^2 \frac{\prod_{\ell=1}^s (tu_\ell - 1)(t^{-1}u_\ell - 1)}{\prod_{j=1}^r (t-v_j)(t^{-1}-v_j)} = A_0^2 (-t)^{r-s} \frac{\prod_\ell u_\ell}{\prod_j v_j} \frac{\prod_{\ell=1}^s (t-u_\ell^{-1})(t-u_\ell)}{\prod_{j=1}^r (t-v_j^{-1})(t-v_j)}. \tag{7.6}$$

Considering the restrictions placed on the u_ℓ and v_j, we must have $r = s$, $A_0^2 = 1$, and the multisets $\{u_1, \dots, u_s\}$ and $\{v_1, \dots, v_s\}$ coincide. Thus

$$w^+(t) = -h(t) + (-1)^\alpha B(t) t^m \prod_{j=1}^s \frac{tu_j - 1}{t-u_j}, \tag{7.7}$$

with $\alpha \in \{0,1\}$ and $m \in \mathbb{Z}$. Because w^+ does not have a pole at 0 or ∞, we have $m = 0$. Using the definition of $h(t)$, we have finally

$$w^+(t) = \frac{t^2}{t^2-1} - \rho^{-1}\big(q-q^{-1}\big)^{-1} + (-1)^\alpha B(t) \prod_{j=1}^s \frac{tu_j - 1}{t-u_j}. \tag{7.8}$$

Moreover, using $w^+(\infty) = \omega_0$, we obtain that

$$(\omega_0 - 1)\big(q-q^{-1}\big) = -\rho^{-1} + (-1)^\alpha \prod_j u_j,$$

and (2.1) implies that $\rho = (-1)^\alpha \prod_j u_j$. Now there are four cases to consider, according to the parity of α and of s.

Case 1, $\alpha = 0$ *and* s *is odd.* Then $\rho = \prod_j u_j$. Comparing the expression (7.8) for $w^+(t)$ with the formulas (4.4) and (4.5) and Definition 4.2, we see that the parameters ρ, q, Ω are $(u_1, \ldots, u_s)$-admissible.

Case 2, $\alpha = 1$ *and* s *is odd.* Then $\rho = -\prod_j u_j$. Let $v = (u_1, \ldots, u_s, -1, 1)$. Then

$$w^+(t) = \frac{t^2}{t^2 - 1} - \rho^{-1}\big(q - q^{-1}\big)^{-1} - B(t)\left(\prod_{j=1}^{s} u_j\right) \prod_{j=1}^{s} \frac{t - u_j^{-1}}{t - u_j}$$

$$= \frac{t^2}{t^2 - 1} - \rho^{-1}\big(q - q^{-1}\big)^{-1} + B(t)\left(\prod_{j=1}^{s+2} v_j\right) \prod_{j=1}^{s+2} \frac{t - v_j^{-1}}{t - v_j},$$

and $\rho = -\prod_{j=1}^{s} u_j = \prod_{j=1}^{s+2} v_j$. Again, comparing with the formulas of Section 4.2, we see that the parameters ρ, q, Ω are $(u_1, \ldots, u_s, -1, 1)$-admissible.

Case 3, $\alpha = 0$ *and* s *is even.* Then $\rho = \prod_j u_j$. By a similar calculation as in Case 2, one checks that the parameters ρ, q, Ω are $(u_1, \ldots, u_s, 1)$-admissible.

Case 4, $\alpha = 1$ *and* r *is even.* Then $\rho = -\prod_j u_j$. By a similar calculation again, one checks that the parameters ρ, q, Ω are $(u_1, \ldots, u_s, -1)$-admissible.

In each of the four cases, there exists $r > 0$ and $v_1, \ldots, v_r$ such that ρ, q, Ω and $v_1, \ldots, v_r$ satisfy the Rui–Xu criterion for admissibility. Thus we have shown (1) $\Rightarrow$ (3) when $q - q^{-1} \neq 0$. $\square$

Corollary 7.10. *(See Rui and Si [20].) Assume $q - q^{-1} \neq 0$. The conditions of Proposition* 7.9 *are equivalent to the existence of a simple finite dimensional module on which e_1 is non-zero, as long as Ω is not the zero sequence or $n \neq 2$.*

Proof. By the results of [20], a cyclotomic BMW algebra $\mathcal{W}_{n,S,r}(\rho, q, \Omega; u_1, \ldots, u_r)$ with admissible parameters and $q - q^{-1} \neq 0$ has a simple module on which e_1 is non-zero, as long as Ω is not the zero sequence or $n \neq 2$. $\square$

Conjecture 7.11. *Theorem* 7.9 *remains valid when $q - q^{-1} = 0$.*

8. Construction of examples of semi-admissible parameters

Examples of cyclotomic (resp. degenerate cyclotomic) BMW algebras with semi-admissible parameters can easily be constructed. For the sake of clarity, we carry this out for the degenerate cyclotomic BMW algebras only; non-degenerate cyclotomic BMW algebras with $q^2 \neq 1$ can be treated in a similar way, using the admissibility criterion of Rui and Xu [21].

Let S be an integral domain with $1/2 \in S$. Take $0 < d < r$ and $u_1, \ldots, u_r \in S$. Assume that $u_i \neq \pm u_j$ for any i, j and that $u_i \neq \pm 1/2$ for any i. Let $p(u) = \prod_{1 \leqslant j \leqslant r}(u - u_j)$ and $p_0(u) = \prod_{1 \leqslant j \leqslant d}(u - u_j)$. Define ω_a for $a \geqslant 0$ via the $(u_1, \ldots, u_d)$-admissibility criterion of [3],

$$\omega_a = q_{a+1}(u_1, \ldots, u_d) + \frac{1}{2}(-1)^{d-1} q_a(u_1, \ldots, u_d) + \frac{1}{2}\delta_{a,0}. \tag{8.1}$$

By [3], this is equivalent to

$$\sum_{a \geqslant 0} \omega_a u^{-a} = 1/2 - u + \big(u - (-1)^d/2\big) \prod_{j=1}^{d} \frac{u + u_j}{u - u_j}. \tag{8.2}$$

By the implication (4) $\Rightarrow$ (1) in Theorem 3.12 (which is from [3]), the parameters $\Omega = (\omega_a)_{a\geqslant 0}$ and $u_1, \ldots, u_d$ are admissible; i.e. the set $\{e_1, y_1e_1, \ldots, y_1^{d-1}e_1\}$ is linearly independent over S in $\mathcal{N}_{2,S,d}(\Omega, u_1, \ldots, u_d)$.

Now consider $\mathcal{N}_{2,S,r}(\Omega; u_1, \ldots, u_r)$. Since we have an algebra map

$$\theta : \mathcal{N}_{2,S,r}(\Omega; u_1, \ldots, u_r) \to \mathcal{N}_{2,S,d}(\Omega; u_1, \ldots, u_d),$$

we have $\{e_1, y_1e_1, \ldots, y_1^{d-1}e_1\}$ is linearly independent over S in $\mathcal{N}_{2,S,r}(\Omega; u_1, \ldots, u_r)$. Let r' be maximal such that $\{e_1, y_1e_1, \ldots, y_1^{r'-1}e_1\}$ is linearly independent in $\mathcal{N}_{2,S,r}(\Omega; u_1, \ldots, u_r)$. Then by the argument following Definition 5.2, there is a subset $\{v_1, \ldots, v_{r'}\}$ of $\{u_1, \ldots, u_r\}$ such that $p_1(y_1)e_1 := \prod_{j=1}^{r'}(y_1 - v_j)e_1 = 0$, and $h(y_1)e_1 \neq 0$ for any polynomial h of degree less than r'. Now by Lemma 5.5 and Theorem 3.12, the set of parameters $\Omega, v_1, \ldots, v_{r'}$ satisfies the $(v_1, \ldots, v_{r'})$-admissibility conditions. Hence we also have

$$\sum_{a\geqslant 0} \omega_a u^{-a} = 1/2 - u + \bigl(u - (-1)^{r'}/2\bigr) \prod_{j=1}^{r'} \frac{u + v_j}{u - v_j}. \tag{8.3}$$

Comparing Eqs. (8.2) and (8.3), and taking into account the assumptions on $\{u_1, \ldots, u_r\}$, we conclude that $d = r'$ and $\{v_1, \ldots, v_d\} = \{u_1, \ldots, u_d\}$. Thus the parameters $\Omega, u_1, \ldots, u_r$ are d-semi-admissible.

References

[1] Susumu Ariki, Representations of Quantum Algebras and Combinatorics of Young Tableaux, Univ. Lecture Ser., vol. 26, American Mathematical Society, Providence, RI, 2002, translated from the 2000 Japanese edition and revised by the author. MR MR1911030 (2004b:17022).

[2] Susumu Ariki, Kazuhiko Koike, A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations, Adv. Math. 106 (2) (1994) 216–243. MR MR1279219 (95h:20006).

[3] Susumu Ariki, Andrew Mathas, Hebing Rui, Cyclotomic Nazarov–Wenzl algebras, Nagoya Math. J. 182 (2006) 47–134. MR MR2235339.

[4] Z. Daugherty, A. Ram, R. Virk, Affine and degenerate affine BMW algebras: The center, preprint, arXiv:1105.4207, 2011.

[5] Frederick M. Goodman, Cellularity of cyclotomic Birman–Wenzl–Murakami algebras, in: Special Issue in Honor of Gus Lehrer, J. Algebra 321 (11) (2009) 3299–3320.

[6] Frederick M. Goodman, Comparison of admissibility conditions for cyclotomic Birman–Wenzl–Murakami algebras, J. Pure Appl. Algebra 214 (11) (2010) 2009–2016.

[7] Frederick M. Goodman, Admissibility conditions for degenerate cyclotomic BMW algebras, Comm. Algebra 39 (2) (2011) 452–461.

[8] Frederick M. Goodman, John Graber, Cellularity and the Jones basic construction, in: Special Issue in Honor of Dennis Stanton, Adv. in Appl. Math. 46 (1–4) (2011) 312–362.

[9] Frederick M. Goodman, John Graber, On cellular algebras with Jucys Murphy elements, J. Algebra 330 (1) (2011) 147–176.

[10] Frederick M. Goodman, Holly Hauschild, Affine Birman–Wenzl–Murakami algebras and tangles in the solid torus, Fund. Math. 190 (2006) 77–137. MR MR2232856.

[11] Frederick M. Goodman, Holly Hauschild Mosley, Cyclotomic Birman–Wenzl–Murakami algebras I: freeness and realization as tangle algebras, J. Knot Theory Ramifications 18 (2009) 1089–1127.

[12] Frederick M. Goodman, Holly Hauschild Mosley, Cyclotomic Birman–Wenzl–Murakami algebras, II: admissibility relations and freeness, Algebr. Represent. Theory 14 (1) (2011) 1–39. MR 2763289.

[13] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1) (1996) 1–34. MR MR1376244 (97h:20016).

[14] Reinhard Häring-Oldenburg, Cyclotomic Birman–Murakami–Wenzl algebras, J. Pure Appl. Algebra 161 (1–2) (2001) 113–144. MR MR1834081 (2002c:20055).

[15] Alexander Kleshchev, Linear and Projective Representations of Symmetric Groups, Cambridge Tracts in Math., vol. 163, Cambridge University Press, Cambridge, 2005. MR MR2165457 (2007b:20022).

[16] Hugh Morton, Antony Wassermann, A basis for the Birman–Wenzl algebra, unpublished manuscript, 1989, pp. 1–29; revised, 2000; placed on the arXiv by H.R. Morton in 2010, arXiv:1012.3116.

[17] Maxim Nazarov, Young's orthogonal form for Brauer's centralizer algebra, J. Algebra 182 (3) (1996) 664–693. MR MR1398116 (97m:20057).

[18] Rosa Orellana, Arun Ram, Affine braids, Markov traces and the category $\mathcal{O}$, in: Algebraic Groups and Homogeneous Spaces, in: Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 423–473. MR 2348913 (2008m:17034).

[19] Hebing Rui, Mei Si, On the structure of cyclotomic Nazarov–Wenzl algebras, J. Pure Appl. Algebra 212 (10) (2008) 2209–2235. MR MR2418167.

[20] Hebing Rui, Mei Si, The representations of cyclotomic BMW algebras, II, Algebr. Represent. Theory (2010) 1–29, http://dx.doi.org/10.1007/s10468-010-9249-z.
[21] Hebing Rui, Jie Xu, The representations of cyclotomic BMW algebras, J. Pure Appl. Algebra 213 (12) (2009) 2262–2288.
[22] Stewart Wilcox, Shona Yu, On the freeness of the cyclotomic BMW algebras: admissibility and an isomorphism with the cyclotomic Kauffman tangle algebras, preprint, arXiv:0911.5284, 2009.
[23] Stewart Wilcox, Shona Yu, The cyclotomic BMW algebra associated with the two string type B braid group, Comm. Algebra 39 (11) (2011) 4428–4446.
[24] Stewart Wilcox, Shona Yu, On the cellularity of the cyclotomic Birman–Murakami–Wenzl algebras, J. Lond. Math. Soc., in press.
[25] Shona Yu, The cyclotomic Birman–Murakami–Wenzl algebras, PhD thesis, University of Sydney, 2007, arXiv:0810.0069.