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Matrix manifolds and the Jordan structure of the bialternate matrix product

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Abstract

The bialternate product of matrices was introduced at the end of the 19th century and recently revived as a computational tool in problems where real matrices with conjugate pairs of pure imaginary eigenvalues are important, i.e., in stability theory and Hopf bifurcation problems. We give a complete description of the Jordan structure of the bialternate product $2A \odot I_n$ of an $n \times n$ matrix A, thus extending several partial results in the literature. We use these results to obtain regular (local) defining systems for some manifolds of matrices which occur naturally in applications, in particular for manifolds with resonant conjugate pairs of pure imaginary eigenvalues. Such defining systems can be used analytically to obtain local parameterizations of the manifolds or numerically to set up Newton systems with local quadratic convergence. We give references to explicit numerical applications and implementations in software. We expect that the analysis provided in this paper can be used to further improve such implementations. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

The study of matrix manifolds presents interesting mathematical features [18,2] and also leads to numerical applications; we cite [8,9] among a long list of publications. Recently an interest emerged in manifolds of (real) matrices with one or more conjugate pairs of pure imaginary eigenvalues, and generalizations of this setting, see [6,7]. This leads to a combination of bialternate matrix product and bordered matrix methods. In [6] some analysis of the Jordan structure of the bialternate product matrix is given and the regularity of the obtained defining systems is discussed in a particular situation. We go further by describing the complete Jordan structure and proving the regularity in a more explicit way (in [6] the choice of two among four equations is essentially left unresolved so that the numerical code has to decide this choice by an optimization step). The present approach therefore has a potential for further applications in computational work.

The origins of the notion of a bialternate product go back to the paper [17] of Stéphanos (1900); Stéphanos' term is "composition bialternée". An extensive treatment is given in [3]; bialternate products were considered in [11–14] to compute Hopf bifurcations; their importance for multiple Hopf was established in [6].

Since we are interested in complex eigenvalues and since the complex Jordan normal form of a matrix is somewhat simpler than the real form, we prefer to formulate the results first for general complex matrices. In this way we avoid some awkward notational problems. However, when applied to real matrices (the case that is important in the applications) all matrix constructions that we consider will again yield real matrices. Also, the rank of a real matrix (over the real numbers) is the same as its rank as a complex matrix (over the complex numbers). Hence the results that we obtain on geometric multiplicities of eigenvalues, in particular, in Propositions 11 and 12 hold also for real matrices.

2. Basic properties of the bialternate product of matrices

We briefly recall the definition of the tensor product of two matrices (e.g., [15]). Let $\{e_i : 1 \leqslant i \leqslant n\}$ be the canonical base of unit vectors in $\mathbb{C}^n$. The tensor product $\mathbb{C}^n \otimes \mathbb{C}^n$ is the space $\mathbb{C}^{n^2}$ with formal base $\{e_i \otimes e_j : 1 \leqslant i, j \leqslant n\}$. If $x = \sum_{i=1}^n x_i e_i \in \mathbb{C}^n$, $y = \sum_{i=1}^n y_i e_i \in \mathbb{C}^n$ then we define $x \otimes y = \sum_{i=1}^n \sum_{j=1}^n x_i y_j e_i \otimes e_j \in \mathbb{C}^{n^2}$. The mapping $(x, y) \to x \otimes y$ is clearly bilinear.

Two matrices $A, B \in \mathbb{C}^{n \times n}$ can be identified with linear mappings $\mathbb{C}^n \to \mathbb{C}^n$. The tensor product $A \otimes B \in \mathbb{C}^{n^2 \times n^2}$ is then defined as the linear mapping from $\mathbb{C}^n \otimes \mathbb{C}^n$ into itself for which $(A \otimes B)(e_k \otimes e_l) = Ae_k \otimes Be_l = \sum_{i,j=1}^n a_{ik} b_{jl} (e_i \otimes e_j)$. It is easily seen that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{2.1}$$

for all $x, y \in \mathbb{C}^n$. Denoting by $(A \otimes B)_{(i,j),(k,l)}$ the $(e_i \otimes e_j)$-component of $(A \otimes B)(e_k \otimes e_l)$ we have

$$(A \otimes B)_{(i,j),(k,l)} = a_{ik} b_{jl}. \tag{2.2}$$

Proposition 1. *Let $A, A_1, A_2, B, B_1, B_2 \in \mathbb{C}^{n \times n}$. Then*
1. $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$.
2. $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$.
3. $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$.
4. $I_n \otimes I_n = I_{n^2}$.
5. *If A, B are nonsingular, then so is $A \otimes B$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.*

Proof. Obvious from (2.1). □

There exists a natural linear mapping $\sigma : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ defined by

$$\sigma(e_i \otimes e_j) = e_j \otimes e_i \quad (1 \leqslant i, j \leqslant n). \tag{2.3}$$

Clearly σ is an involution, i.e., $\sigma^2 = I_{\mathbb{C}^n \otimes \mathbb{C}^n}$ and one has $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in \mathbb{C}^n$ and $\sigma(A \otimes B) = (B \otimes A)\sigma$ for all $A, B \in \mathbb{C}^{n \times n}$.

The spectrum (set of eigenvalues) of σ is remarkable. For every pair of indices (i, j) with $1 \leqslant i, j \leqslant n$ we define the vectors

$$\zeta_{ij} = e_i \otimes e_j - e_j \otimes e_i, \qquad \eta_{ij} = e_i \otimes e_j + e_j \otimes e_i.$$

We denote by E_a and E_s, respectively, the subspaces of $\mathbb{C}^n \otimes \mathbb{C}^n$ spanned by all vectors of the form ζ_{ij}, respectively, η_{ij}. Then the following holds.

Proposition 2. *The operator σ has two eigenvalues, namely the eigenvalue -1 with algebraic and geometric multiplicity $\frac{1}{2}n(n-1)$ and the eigenvalue $+1$ with algebraic and geometric multiplicity $\frac{1}{2}n(n+1)$. The eigenspace corresponding to -1 is E_a and has a base consisting of all vectors ζ_{ij} $(n \geqslant i > j \geqslant 1)$; the eigenspace corresponding to $+1$ is E_s and has a base consisting of all vectors η_{ij} $(n \geqslant i \geqslant j \geqslant 1)$.*

Proof. Easy. □

Let $A, B \in \mathbb{C}^{n \times n}$. We define the bialternate product or biproduct matrix by

$$A \odot B = \frac{1}{2}(A \otimes B + B \otimes A). \tag{2.4}$$

Some properties of bialternate products are immediate.

Proposition 3. *Let* $A, B, B_1, B_2 \in \mathbb{C}^{n\times n}$. *Then*
1. $A \odot B = B \odot A$.
2. $A \odot (B_1 + B_2) = A \odot B_1 + A \odot B_2$.
3. $A \odot A = A \otimes A$.
4. *If* A *is nonsingular*, *then so is* $A \odot A$ *and* $(A \odot A)^{-1} = A^{-1} \odot A^{-1}$.
5. $\sigma(A \odot B) = (A \odot B)\sigma$.

Proof. The first four claims follow immediately from (2.4). The last one follows from $\sigma(A \odot B) = \frac{1}{2}\sigma(A \otimes B + B \otimes A) = \frac{1}{2}(B \otimes A + A \otimes B)\sigma = (A \odot B)\sigma$. □

Proposition 4. *For every* $A, B \in \mathbb{C}^{n\times n}$ *the spaces* E_a *and* E_s *are invariant subspaces of* $A \odot B$.

Proof. If $f \in E_a$ then $\sigma(A \odot B)f = (A \odot B)\sigma f = -(A \odot B)f$, i.e., $(A \odot B)f \in E_a$. The proof for E_s is similar. □

For our purposes only the restriction of $(A \odot B)$ to E_a is important. For the numerical applications we need a representation of this operator with respect to a suitable base of E_a. From Proposition 2 it follows that $(\zeta_{ij})_{i>j}$ is such a base. The representation of $A \odot B$ is then given by the following proposition (see [17]).

Proposition 5. *With respect to the base* $(\zeta_{ij})_{i>j}$ *the restriction of* $A \odot B$ *to* E_a *is represented by*

$$(A \odot B)_{(i,j),(k,l)} = \frac{1}{2}\left\{ \begin{vmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{vmatrix} + \begin{vmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \right\}. \tag{2.5}$$

Proof. By a straightforward computation the result follows. □

Convention. With an abuse of notation we will from now on identify the linear operator represented in the canonical base by $A \odot B$ with its restriction to the invariant subspace E_a and take the matrix representation on this invariant subspace in Proposition 5 as the standard one.

The special case of a bialternate product of the form $2A \odot I_n$ is so important that we simply call it the bialternate product of A. From Proposition 5 we obtain its explicit form. We find

$$(2A \odot I_n)_{(i,j),(k,l)} = \begin{cases} -a_{il} & \text{if } k = j, \\ a_{ik} & \text{if } k \neq i \text{ and } l = j, \\ a_{ii} + a_{jj} & \text{if } k = i \text{ and } l = j, \\ a_{jl} & \text{if } k = i \text{ and } l \neq j, \\ -a_{jk} & \text{if } l = i, \\ 0 & \text{else.} \end{cases} \tag{2.6}$$

One checks easily that $(2A \odot I_n)^{\mathrm{T}} = (2A^{\mathrm{T}} \odot I_n)$.

3. The Jordan structure of the bialternate product matrix

A remarkable property of the bialternate product matrix is that its eigenvalues and Jordan structure are completely determined by those of the original matrix. We shall indeed prove that if A and B are similar, then so are $2A \odot I_n$ and $2B \odot I_n$. To be precise, we have the following proposition.

Proposition 6. *Let $A, B \in \mathbb{C}^{n\times n}$ be two similar matrices and P a nonsingular matrix such that $B = PAP^{-1}$. Then $P \odot P$ is nonsingular and*

$$(P \odot P)(2A \odot I_n)(P \odot P)^{-1} = 2B \odot I_n.$$

Proof. By Proposition 3 $P \odot P$ is nonsingular. So it is sufficient to prove that

$$(P \odot P)(2A \odot I_n) = (2B \odot I_n)(P \odot P).$$

The left-hand side in this expression is the restriction to E_a of

$$(P \otimes P)(A \otimes I_n + I_n \otimes A) = PA \otimes P + P \otimes PA.$$

The right-hand side is the restriction to E_a of

$$(PAP^{-1} \otimes I_n + I_n \otimes PAP^{-1})(P \otimes P) = PA \otimes P + P \otimes PA.$$

Since both sides are equal the result follows. □

If $v, w \in \mathbb{C}^n$ then clearly $w \otimes v - v \otimes w \in E_a$. Furthermore, with respect to the basis $(\zeta_{ij})_{n \geqslant i > j \geqslant 1}$ defined before Proposition 2 we have

$$w \otimes v - v \otimes w = \sum_{n \geqslant i > j \geqslant 1} (v_j w_i - v_i w_j)\zeta_{ij}.$$

We set $N_b = n(n-1)/2$ and introduce the following definition.

Definition 1. Let $v, w \in \mathbb{C}^n$. The wedge product of v and w is the vector $v \wedge w \in \mathbb{C}^{N_b}$ with components $(v \wedge w)_{i,j} = v_j w_i - v_i w_j$ $(n \geqslant i > j \geqslant 1)$.

We can visualize the components of $v \wedge w$ as determinants of 2×2 blocks in the $n \times 2$ matrix with columns v, w. Formally, $v \wedge w$ is the representation of $w \otimes v - v \otimes w$ with respect to the canonical base of E_a.

Obviously, the wedge product $v \wedge w$ is linear with respect to both v and w and vanishes if and only if v, w are linearly dependent. It is anti-symmetric $(v \wedge w + w \wedge v = 0)$ and is determined up to a scalar multiple by the two-dimensional space that contains v, w. Conversely, if it is nonzero, then it defines this space completely.

Proposition 7. *If* $(v_i)_{i=1}^k$, $k \leqslant n$, *are linearly independent vectors in* $\mathbb{C}^n$, *then* $(v_i \wedge v_j)_{1 \leqslant j<i \leqslant k}$ *are linearly independent in* $\mathbb{C}^{N_b}$. *In particular, if* $k = n$ *then they form a base of* $\mathbb{C}^{N_b}$.

Proof. First assume that $k = n$ and let e_i denote the ith unit vector in $\mathbb{C}^n$ $(1 \leqslant i \leqslant n)$. Since every e_i is in the span of the $(v_i)_{1 \leqslant i \leqslant n}$ it follows that each $e_i \wedge e_j$ $(1 \leqslant j < i \leqslant n)$ is in the span of the $v_i \wedge v_j$. Since the $e_i \wedge e_j$ obviously form a base for $\mathbb{C}^{N_b}$ the result follows.

The case $k < n$ then follows from the fact that a linearly independent set of vectors can be extended to a base. □

We now prove the basic relation between bialternate product matrices and wedge products of vectors.

Proposition 8. *Let* $A, B \in \mathbb{C}^{n\times n}$ *and* $v, w \in \mathbb{C}^n$. *Then*

$$(A \odot B)(v \wedge w) = \frac{1}{2}(Av \wedge Bw - Aw \wedge Bv). \tag{3.1}$$

In particular,

$$(2A \odot I_n)(v \wedge w) = Av \wedge w + v \wedge Aw. \tag{3.2}$$

Proof. We provide a proof in the canonical base of E_a; a coordinate-free proof can also be given easily. Let i, j be integers with $n \geqslant i > j \geqslant 1$. Then

$$\begin{aligned}
[(2A \odot B)(v \wedge w)]_{i,j} &= \sum_{n \geqslant k>l \geqslant 1} (a_{ik}b_{jl} - a_{il}b_{jk} + a_{jl}b_{ik} - a_{jk}b_{il})(v_l w_k - v_k w_l) \\
&= \sum_{n \geqslant k,l \geqslant 1} (a_{ik}b_{jl} + a_{jl}b_{ik})(v_l w_k - v_k w_l) \\
&= (Bv)_j(Aw)_i - (Bw)_j(Av)_i + (Bw)_i(Av)_j - (Bv)_i(Aw)_j \\
&= (Av \wedge Bw)_{i,j} - (Aw \wedge Bv)_{i,j}.
\end{aligned}$$

This implies the proposition. □

We note that (3.1) completely defines $A \odot B$. The next result is well-known.

Proposition 9. *Let* $A \in \mathbb{C}^{n\times n}$ *and let* λ_1, λ_2 *be eigenvalues of* A *with corresponding eigenvectors* v_1, v_2. *If* v_1, v_2 *are linearly independent then* $v_1 \wedge v_2$ *is an eigenvector of* $2A \odot I_n$ *for the eigenvalue* $\lambda_1 + \lambda_2$.

Proof. By Proposition 8 we have

$$\begin{aligned}(2A \odot I_n)(v_1 \wedge v_2) &= Av_1 \wedge v_2 - Av_2 \wedge v_1 \\ &= \lambda_1 v_1 \wedge v_2 - \lambda_2 v_2 \wedge v_1 \\ &= (\lambda_1 + \lambda_2)(v_1 \wedge v_2). \quad \Box\end{aligned}$$

We now consider the Jordan structures in more detail. We need the following definition.

Definition 2. For $1 \leqslant l \leqslant n_1 \leqslant n_2$, we define $C(n_1, n_2, l)$ as the $l \times l$ matrix whose (i, j)th entry is the binomial coefficient

$$\binom{n_1 + n_2 - 2l}{n_1 - l + i - j}$$

with the understanding that this entry vanishes if $n_1 - l + i - j$ is not in the range $[0, n_1 + n_2 - 2l]$.

We note that with this definition the diagonal elements of $C(n_1, n_2, l)$ are never zero. We will need the following result.

Proposition 10. *Every matrix of the form* $C(n_1, n_2, l)$ $(1 \leqslant l \leqslant n_1)$ *as defined in Definition* 2 *is nonsingular.*

For a proof we refer to [16], Section I.3, Example 4 where a more general result is obtained. In [6] the special case $n_1 = n_2$ is proved by a direct argument.

Proposition 11. *Let* $A \in \mathbb{C}^{n \times n}$ *and let* λ_1, λ_2 *be two eigenvalues of* A *that belong to different Jordan blocks with dimensions* n_1 *and* n_2, *respectively, with* $1 \leqslant n_1 \leqslant n_2$. *Then the matrix* $2A \odot I_n$ *has* n_1 *Jordan blocks for the eigenvalue* $\lambda_1 + \lambda_2$, *one each of size* $1 + n_2 - n_1$, $3 + n_2 - n_1, \ldots, 2n_1 - 1 + n_2 - n_1$, *corresponding to this eigenvalue pair* (*total algebraic multiplicity of* $\lambda_1 + \lambda_2$ *for this eigenvalue pair*: $n_1 n_2$).

Proof. For simplicity of notation we prove this in the case $n_1 = 4$, $n_2 = 6$, the generalization being obvious. Let $v_1, \ldots, v_4$ be the generalized eigenvectors associated with λ_1, i.e., $(A - \lambda_1 I_n)v_i = v_{i-1}$ for $i = 2, 3, 4$ and $(A - \lambda_1 I_n)v_1 = 0$. Let $w_1, \ldots, w_6$ be the generalized eigenvectors associated with λ_2 in the same way. The generalized eigenspace of the eigenvalue $\lambda_1 + \lambda_2$ has dimension $n_1 n_2 = 24$ and is spanned by the vectors $V_{ij} = v_i \wedge w_j$, where $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 6$ (the linear independence of these vectors follows from Proposition 7). It is convenient to order them as follows:

$$\begin{array}{ccccccccc}
 & & & V_{46} & & & & & \\
 & & V_{36} & & V_{45} & & & & \\
 & V_{26} & & V_{35} & & V_{44} & & & \\
V_{16} & & V_{25} & & V_{34} & & V_{43} & & \\
 & V_{15} & & V_{24} & & V_{33} & & V_{42} & \\
 & & V_{14} & & V_{23} & & V_{32} & & V_{41} \\
 & & & V_{13} & & V_{22} & & V_{31} & \\
 & & & & V_{12} & & V_{21} & & \\
 & & & & & V_{11} & & &
\end{array} \tag{3.3}$$

The picture illustrates the action of $B = (2A \odot I_n) - (\lambda_1 + \lambda_2)I_{N_b}$ on the generalized eigenspace of $\lambda_1 + \lambda_2$ since V_{ij} is transformed into $V_{i-1,j} + V_{i,j-1}$ with the convention that such a vector vanishes if it is not in (3.3). If B acts p times on a vector in (3.3) then the resulting vector is a linear combination of vectors p rows down in (3.3); by induction we have

$$B^p V_{ij} = \sum_{r=0}^{p} \binom{p}{r} V_{i-p+r,j-r}. \tag{3.4}$$

Now, denote

$$E_k = \mathrm{Span}\{V_{ij} \mid i + j = k\}$$

for $k = 2, 3, \ldots, 10$.

Clearly B has four linearly independent singular vectors, namely V_{11}, $V_{12} - V_{21}$, $V_{13} - V_{22} + V_{31}$ and $V_{14} - V_{23} + V_{32} - V_{41}$ in E_2, E_3, E_4, E_5, respectively.

Now B^{12-2k} is a linear map from E_{12-k} to E_k for $k = 2, 3, 4, 5$. The result follows if these maps are all onto. From (3.4) we can compute an explicit representation of B^{12-2k} in terms of the basis $V_{6-k,6}, \ldots, V_{4,8-k}$ of E_{12-k} and the basis $V_{1,k-1}, \ldots, V_{k-1,1}$ of E_k. One finds

$$B^{12-2k} V_{5-k+j,7-j} = \sum_{r=0}^{12-2k} \binom{12-2k}{r} V_{-7+k+j+r,7-j-r}$$

for $j = 1, \ldots, k-1$. By requiring that $-7 + k + j + r = i$, i.e. $r = 7 - k + i - j$, we find that B^{12-2k} is represented by a matrix whose (i, j)th entry is

$$\binom{12-2k}{7-k+i-j} = \binom{10-2l}{4-l+j-i},$$

where we have set $l = k - 1$. So B^{12-2k} is represented by the square matrix $C(4, 6, l)^{\mathrm{T}}$. Since this matrix is nonsingular by Proposition 10, the proof is complete. $\square$

In Proposition 11 it does not matter if λ_1, λ_2 are equal or not. If another pair of Jordan blocks leads to the same eigenvalue sum $\lambda_1 + \lambda_2$ then this eigenvalue of $2A \odot I_n$ simply has the two collections of Jordan blocks; there is no interaction since the respective generalized eigenspaces have only the zero vector in common.

The case of eigenvalues within the same Jordan block must be considered separately.

Proposition 12. *Let $A \in \mathbb{C}^{n\times n}$ and let λ be an eigenvalue of A in a Jordan block with dimension k. Let $v_1, \ldots, v_k$ be a base of the generalized eigenspace that corresponds to λ, i.e., $(A-\lambda)v_k = v_{k-1}, \ldots, (A-\lambda)v_1 = 0$. Then*:

1. *If $k = 2l, l \geqslant 1$ then the matrix $2A \odot I_n$ has l Jordan blocks for the eigenvalue 2λ. The vectors*

$$v_1 \wedge v_2, v_1 \wedge v_4 - v_2 \wedge v_3, \ldots, v_1 \wedge v_{2l} - v_2 \wedge v_{2l-1} + \cdots + (-1)^{l+1} v_l \wedge v_{l+1}$$

are linearly independent eigenvectors of $2A \odot I_n$ and correspond to Jordan blocks with sizes $4l-3, 4l-7, \ldots, 1$, respectively.

2. *If $k = 2l+1, l \geqslant 1$ then the matrix $2A \odot I_n$ has l Jordan blocks for the eigenvalue of $2A \odot I_n$. The vectors*

$$v_1 \wedge v_2, v_1 \wedge v_4 - v_2 \wedge v_3, \ldots, v_1 \wedge v_{2l} - v_2 \wedge v_{2l-1} + \cdots + (-1)^{l+1} v_l \wedge v_{l+2}$$

are linearly independent eigenvectors of $2A \odot I_n$ and correspond to Jordan blocks with sizes $4l-1, 4l-5, \ldots, 3$, respectively.

Proof. We will prove the first statement in the case $l = 4$; the same argument applies for other values of l as well.

The generalized eigenspace corresponding to 2λ has dimension $\frac{1}{2}(7 \times 8) = 28$ and is spanned by the vectors $V_{ij} = v_i \wedge v_j$ with $1 \leqslant i < j \leqslant 8$ (these vectors are linearly independent by Proposition 7). It is natural to order them in the following scheme:

$$\begin{array}{ccccccc}
 & & & & & & V_{78} \\
 & & & & & V_{68} & \\
 & & & & V_{58} & & V_{67} \\
 & & & V_{48} & & V_{57} & \\
 & & V_{38} & & V_{47} & & V_{56} \\
 & V_{28} & & V_{37} & & V_{46} & \\
V_{18} & & V_{27} & & V_{36} & & V_{45} \\
 & V_{17} & & V_{26} & & V_{35} & \\
 & & V_{16} & & V_{25} & & V_{34} \\
 & & & V_{15} & & V_{24} & \\
 & & & & V_{14} & & V_{23} \\
 & & & & & V_{13} & \\
 & & & & & & V_{12}
\end{array} \tag{3.5}$$

The action of $B = 2A \odot I_n - 2\lambda I_{N_b}$ on the vectors V_{ij} can be read from (3.5) since this matrix transforms V_{ij} into $V_{i-1,j} + V_{i,j-1}$ with the convention that vectors not represented in (3.5) are zero.

Let us define

$$E_s = \mathrm{Span}\{V_{ij} \mid i + j = s\}$$

for $s = 3, \dots, 15$. It is clear that $V_{12} \in E_3$, $V_{14} - V_{23} \in E_5$, $V_{16} - V_{25} + V_{34} \in E_7$ and $V_{18} - V_{27} + V_{36} - V_{45} \in E_9$ are null vectors of B.

The result follows if we prove that B^{12} maps E_{15} onto E_3, B^8 maps E_{13} onto E_5, B^4 maps E_{11} onto E_7 and I_{N_b} maps E_9 onto E_9. We will prove the result for the case of B^8 acting on E_{13}, the other cases being similar.

For $i = 1, \dots, 4$, let c_i be the number of downward paths in (3.5) that connect V_{58} with $V_{i,9-i}$, or, obviously equivalently, $V_{i,9-i}$ with V_{14}. Similarly, let d_i be the number of downward paths in (3.5) that connect V_{67} with $V_{i,9-i}$, or, obviously equivalently, $V_{i,9-i}$ with V_{23}. With respect to the basis vectors in (3.5), B^4 as a map from E_{13} to E_9 is represented by the matrix

$$M_4 = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \\ c_4 & d_4 \end{pmatrix}.$$

We note that M_4 has full rank 2 because B is one-to-one on E_{13}, E_{12}, E_{11} and E_{10}. Furthermore, with respect to the basis vectors in (3.5) B^4 as a map from E_9 to E_5 is represented by the matrix M_4^{T}. Hence, with respect to the basis vectors in (3.5) B^8 as a map from E_{13} to E_5 is represented by the matrix $M_4^{\mathrm{T}} M_4$; since M_4 has full rank, so has $M_4^{\mathrm{T}} M_4$ by elementary linear algebra.

The second statement can be proved similarly. □

If there are two Jordan blocks with the same eigenvalue then Proposition 12 applies separately to each block. The eigenvalue 2λ of $2A \odot I_n$ simply has the two collections of Jordan blocks. There is also an interaction between the blocks which is described by Proposition 11, leading to still more Jordan blocks for the eigenvalue 2λ. This obviously generalizes easily to any number of pairs of Jordan blocks with the same eigenvalue sum.

Since $(2A \odot I_n)^{\mathrm{T}} = (2A^{\mathrm{T}} \odot I_n)$ all results obtained in Propositions 11 and 12 concerning the right eigenspaces of $2A \odot I_n$ carry over to the left eigenspaces, using left eigenvectors of A instead of right eigenvectors. A left eigenvector p of A for the eigenvalue λ is a nonzero vector such that $p^{\mathrm{H}} A = \lambda p^{\mathrm{H}}$ where p^{H} denotes the conjugate transposed vector, cf. [4], Section 7.1.1.

4. Hopf bifurcations and zero-sum eigenvalue pairs

In Hopf bifurcation problems, we are interested in the question whether a real matrix A has a conjugate pair of pure imaginary eigenvalues $\pm i\omega, \omega > 0$. From Proposition 11, we infer that in this case $2A \odot I_n$ has an eigenvalue $i\omega - i\omega = 0$, i.e., $2A \odot I_n$ is a singular matrix. In fact, we have the following complete result.

Proposition 13. *Let $A \in \mathbb{R}^{n\times n}$. Then $2A \odot I_n$ has rank defect 1 if and only if one of the following three conditions is satisfied:*
1. *A has a conjugate pair of algebraically simple eigenvalues $\pm i\omega$, $\omega > 0$ and no other pair of zero-sum eigenvalues.*
2. *A has a pair of eigenvalues $\pm\lambda$, $\lambda > 0$, both with geometric multiplicity one and at least one of them with algebraic multiplicity one; and no other pair of zero-sum eigenvalues.*
3. *A has eigenvalue 0 with geometric multiplicity 1 and algebraic multiplicity 2 or 3; and no other pair of zero-sum eigenvalues.*

Proof. By a careful inspection of Propositions 11 and 12 the result follows. □

To express that $2A \odot I_n$ is singular it is not necessary to compute the eigenvalues; e.g., the determinant of $2A \odot I_n$ is purely an algebraic function of the coefficients of A.

In fact, as in the case of zero eigenvalues of A itself (see [5]) the determinant function is usually not a good choice. As in [5] a rank defect can be detected by a bordering technique; however, we have to border $2A \odot I_n$ instead of A itself. For any choice of vectors $b, c \in \mathbb{R}^{N_b}$ and scalar $d \in \mathbb{R}$ such that the matrix

$$M(A) = \begin{pmatrix} 2A \odot I_n & b \\ c^{\mathrm{T}} & d \end{pmatrix} \tag{4.1}$$

is nonsingular, we define $q(A) \in \mathbb{R}^{N_b}, s(A) \in \mathbb{R}$ by

$$M(A)\begin{pmatrix} q(A) \\ s(A) \end{pmatrix} = \begin{pmatrix} 0_{N_b} \\ 1 \end{pmatrix}. \tag{4.2}$$

Now $s(A) = 0$ defines (locally) the matrices A for which $2A \odot I_n$ is singular (cf. [5], Propositions 4.1 and 4.2). It is useful to have also the derivatives of $s(A)$. Let z be any variable in A. By taking derivatives of (4.2) we find that

$$M\begin{pmatrix} q_z \\ s_z \end{pmatrix} + \begin{pmatrix} (2A_z \odot I_n)q \\ 0 \end{pmatrix} = 0. \tag{4.3}$$

This allows to compute s_z. If several derivatives are desired, then it is useful to solve the system

$$\begin{pmatrix} w^{\mathrm{T}}(A) & s(A) \end{pmatrix} M = \begin{pmatrix} 0_{N_b}^{\mathrm{T}} & 1 \end{pmatrix}, \tag{4.4}$$

which is a natural adjoint to (4.2). Multiplying (4.3) from the left with

$$\begin{pmatrix} w^{\mathrm{T}}(A) & s(A) \end{pmatrix},$$

we obtain

$$s_z + w^{\mathrm{T}}(2A_z \odot I_n)q = 0. \tag{4.5}$$

5. Defining functions for double Hopf points

Suppose that $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\pm \mathrm{i}\omega_1, \pm \mathrm{i}\omega_2$ where $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1 \neq \omega_2$. The last condition is usually expressed by saying that the two Hopf pairs are not 1:1 resonant. The distinction between the resonant and nonresonant cases is quite important in dynamical applications, see e.g., [10,13].

If there are no other zero-sum eigenvalue pairs, then by Proposition 11 the matrix $2A \odot I_n$ has rank defect 2. We choose $B, C \in \mathbb{R}^{N_b\times 2}$, $D \in \mathbb{R}^{2\times 2}$ such that

$$M(A) = \begin{pmatrix} 2A \odot I_n & B \\ C^{\mathrm{T}} & D \end{pmatrix} \tag{5.1}$$

is nonsingular at the double Hopf point. Then we define the $N_b \times 2$ matrix $Q(A)$ and the 2×2 matrix $S(A)$ by

$$M(A)\begin{pmatrix} Q(A) \\ S(A) \end{pmatrix} = \begin{pmatrix} 0_{N_b,2} \\ I_{2,2} \end{pmatrix}. \tag{5.2}$$

By [5], Proposition 4.1 the four entries of $S(A)$ vanish together if and only if $2A \odot I_n$ has rank defect 2.

The derivatives of S can be obtained in the now familiar way. Define the $N_b \times 2$ matrix $W(A)$ by

$$\begin{pmatrix} W^{\mathrm{T}}(A) & S(A) \end{pmatrix} M(A) = \begin{pmatrix} 0_{2,N_b} & I_{2,2} \end{pmatrix}. \tag{5.3}$$

Then S_z can be obtained from

$$S_z(A) + W^{\mathrm{T}}(2A_z(A) \odot I_n)Q = 0_{2,2}. \tag{5.4}$$

We note that $S(A)$ has four components while intuitively a double Hopf point is a codimension 2 phenomenon only (we will see that it is). So we can suspect that the four resulting equations are not independent. We will discuss this in a somewhat more general setting but first prove a lemma to elucidate the meaning of (5.4).

Lemma 1. *Let* $v_1, v_2, w_1, w_2 \in \mathbb{C}^n$ *and let* $1 \leqslant i, j \leqslant n$. *Then for any matrix* $A \in \mathbb{C}^{n\times n}$ *we have*

$$(w_1 \wedge w_2)^{\mathrm{T}}(2A \odot I_n)(v_1 \wedge v_2) = (w_1^{\mathrm{T}}Av_1)(w_2^{\mathrm{T}}v_2) + (w_2^{\mathrm{T}}Av_2)(w_1^{\mathrm{T}}v_1) - (w_1^{\mathrm{T}}Av_2)(w_2^{\mathrm{T}}v_1) - (w_2^{\mathrm{T}}Av_1)(w_1^{\mathrm{T}}v_2). \quad (5.5)$$

In particular,

$$(w_1 \wedge w_2)^{\mathrm{T}}(v_1 \wedge v_2) = (w_1^{\mathrm{T}}v_1)(w_2^{\mathrm{T}}v_2) - (w_2^{\mathrm{T}}v_1)(w_1^{\mathrm{T}}v_2). \quad (5.6)$$

If $1 \leqslant i, j \leqslant n$ *and* z *denotes the* (i, j)*th entry of* A, *then*

$$(w_1 \wedge w_2)^{\mathrm{T}}(2A_z \odot I_n)(v_1 \wedge v_2) = w_{1i}v_{1j}w_2^{\mathrm{T}}v_2 - w_{1i}v_{2j}w_2^{\mathrm{T}}v_1 - w_{2i}v_{1j}w_1^{\mathrm{T}}v_2 + w_{2i}v_{2j}w_1^{\mathrm{T}}v_1. \quad (5.7)$$

Proof. We prove (5.5); the other statements follow easily. We have

$$\begin{aligned}
&(w_1 \wedge w_2)^{\mathrm{T}}(2A \odot I_n)(v_1 \wedge v_2) \\
&\quad = \sum_{1\leqslant j<i\leqslant n} (w_1 \wedge w_2)_{i,j}(Av_1 \wedge v_2 + v_1 \wedge Av_2)_{i,j} \\
&\quad = \sum_{1\leqslant j<i\leqslant n} (w_{1j}w_{2i} - w_{2j}w_{2i})((Av_1)_j v_{2i} - (Av_1)_i v_{2j} + v_{1j}(Av_2)_i - v_{1i}(Av_2)_j) \\
&\qquad\qquad \text{(by Proposition 8)} \\
&\quad = \frac{1}{2}\sum_{1\leqslant i,j\leqslant n} (w_{1j}w_{2i} - w_{2j}w_{2i})((Av_1)_j v_{2i} - (Av_1)_i v_{2j} + v_{1j}(Av_2)_i - v_{1i}(Av_2)_j) \\
&\quad = (w_1^{\mathrm{T}}Av_1)(w_2^{\mathrm{T}}v_2) + (w_2^{\mathrm{T}}Av_2)(w_1^{\mathrm{T}}v_1) - (w_1^{\mathrm{T}}Av_2)(w_2^{\mathrm{T}}v_1) - (w_2^{\mathrm{T}}Av_1)(w_1^{\mathrm{T}}v_2).
\end{aligned}$$

This implies the proposition. □

Proposition 14. *In the nonresonant double Hopf situation the four gradient vectors contained in* (5.4) *span a two-dimensional space.*

Proof. Let $q_1^j + \mathrm{i}q_2^j$ denote the right eigenvector that corresponds to $\mathrm{i}\omega_j$ ($j = 1, 2$) and $p_1^j + \mathrm{i}p_2^j$ ($j = 1, 2$) be the left eigenvector. Since ω_1, ω_2 are positive and not equal we necessarily have

$$(p_i^j)^{\mathrm{T}} q_k^l = 0 \quad (j \neq l) \quad (5.8)$$

while by an appropriate choice of the vectors we may assume

$$(p_i^j)^{\mathrm{T}} q_k^j = \delta_{ik} \quad (j = 1, 2). \quad (5.9)$$

By Proposition 11, $2A \odot I_n$ has a two-dimensional right singular space spanned by the (complex) vectors $V_1 = (q_1^1 + \mathrm{i}q_2^1) \wedge (q_1^1 - \mathrm{i}q_2^1)$ and $V_2 = (q_1^2 + \mathrm{i}q_2^2) \wedge (q_1^2 - \mathrm{i}q_2^2)$; its left singular space is spanned by the (complex) vectors $W_1 = (p_1^1 + \mathrm{i}p_2^1) \wedge (p_1^1 - \mathrm{i}p_2^1)$ and $W_2 = (p_1^2 + \mathrm{i}p_2^2) \wedge (p_1^2 - \mathrm{i}p_2^2)$.

Now consider the four gradient vectors defined in the space of all $n \times n$ matrices by (5.4). Obviously their span is the same as that of the four vectors $W_i^{\mathrm{T}}(2A_z \odot I_n)V_j$ for $i, j = 1, 2$. But by Lemma 1 and (5.8) we have $W_i^{\mathrm{T}}(2A_z \odot I_n)V_j = 0$ if $i \neq j$. So we are left with the two gradient vectors $W_i^{\mathrm{T}}(2A_z \odot I_n)V_i$ for $i = 1, 2$.

From (5.9) it follows that $(p_1^1 + \mathrm{i}p_2^1)^{\mathrm{T}}(q_1^1 + \mathrm{i}q_2^1) = 0$, $(p_1^1 + \mathrm{i}p_2^1)^{\mathrm{T}}(q_1^1 - \mathrm{i}q_2^1) = 2$, $(p_1^1 - \mathrm{i}p_2^1)^{\mathrm{T}}(q_1^1 + \mathrm{i}q_2^1) = 2$, $(p_1^1 - \mathrm{i}p_2^1)^{\mathrm{T}}(q_1^1 - \mathrm{i}q_2^1) = 0$. By Lemma 1 we infer that

$$\begin{aligned}(W_1^{\mathrm{T}}(2A_{ij} \odot I_n)V_1) &= -2(p_1^1 + \mathrm{i}p_2^1)_i(q_1^1 - \mathrm{i}q_2^1)_j - 2(p_1^1 - \mathrm{i}p_2^1)_i(q_1^1 + \mathrm{i}q_2^1)_j \\ &= -4(p_1^1)_i(q_1^1)_j - 4(p_2^1)_i(q_2^1)_j.\end{aligned}$$

A similar formula holds of course for the other gradient vector. To prove that the two are linearly independent, suppose that there exist α_1, α_2 such that

$$\alpha_1((p_1^1)_i(q_1^1)_j + (p_2^1)_i(q_2^1)_j) + \alpha_2((p_1^2)_i(q_1^2)_j + (p_2^2)_i(q_2^2)_j) = 0 \tag{5.10}$$

for all i, j. Multiplying (5.10) with $(q_1^1)_i$ we find after summation over i that $\alpha_1(q_1^1)_j = 0$ for all j. This implies $\alpha_1 = 0$. Similarly $\alpha_2 = 0$. □

Proposition 15. *For each nonresonant double Hopf matrix $A_0 \in \mathbb{R}^{n\times n}$ there exists a neighborhood (in the space of all matrices $A \in \mathbb{R}^{n\times n}$) in which the double Hopf matrices form a manifold with codimension 2 and are characterized by the fact that $2A \odot I_n$ has rank defect 2.*

Proof. Let $A_0 = CJ_0C^{-1}$ be the Jordan decomposition of A_0. Obviously, J_0 has one-element Jordan blocks of the form $\mathrm{i}\omega_1, \mathrm{i}\omega_2, -\mathrm{i}\omega_1, -\mathrm{i}\omega_2$. By classical matrix perturbation theory (see e.g., [1]), there is a neighborhood of A_0 in which every matrix has corresponding Jordan blocks of the form $\mathrm{i}\omega_1 + \delta_1 + \mathrm{i}\delta_2$, $\mathrm{i}\omega_2 + \delta_3 + \mathrm{i}\delta_4$, $-\mathrm{i}\omega_1 + \delta_1 - \mathrm{i}\delta_2$, $-\mathrm{i}\omega_2 + \delta_3 - \mathrm{i}\delta_4$, where δ_1, δ_2, δ_3, δ_4 are smooth functions of the matrix entries and vanish at A_0. Clearly, $2A \odot I_n$ has rank defect 2 if and only if A is a double Hopf matrix if and only if $\delta_1 = \delta_3 = 0$. So it is sufficient to show that δ_1, δ_3 form a regular set of functions of the entries of A, i.e., that their $2 \times n^2$ Jacobian

$$\begin{pmatrix} (\delta_1)_A \\ (\delta_3)_A \end{pmatrix} \tag{5.11}$$

has full rank 2 at $A = A_0$. Now consider the real two-parameter unfolding $A(\alpha_1, \alpha_2) = CJ(\alpha_1, \alpha_2)C^{-1}$ where $J(\alpha_1, \alpha_2)$ is obtained by replacing in J_0 $\pm\mathrm{i}\omega_1$ by $\pm\mathrm{i}\omega_1 + \alpha_1$ and $\pm\mathrm{i}\omega_2$ by $\pm\mathrm{i}\omega_2 + \alpha_2$. Obviously, the mapping $(\alpha_1, \alpha_2)^{\mathrm{T}} \to$

$(\delta_1(A(\alpha_1,\alpha_2)),\delta_3(A(\alpha_1,\alpha_2)))^{\mathrm{T}}$ is the identity mapping; by the chain rule for derivatives this implies that (5.11) must have full rank. □

The above result naturally generalizes to three other cases.

Proposition 16. *Let $A_0 \in \mathbb{R}^{n\times n}$ be a matrix of one of the following four types.*
1. *A is nonresonant double Hopf (Type DH).*
2. *A has a pair of algebraically simple eigenvalues of the form $a + \mathrm{i}b$ and $-a - \mathrm{i}b$ with a, b both real and nonzero (Type RT).*
3. *A has four different nonzero real algebraically simple eigenvalues $-\lambda_1$, $-\lambda_2$, λ_1, λ_2 (Type DN).*
4. *A has algebraically simple eigenvalues $\pm\mathrm{i}\omega$, $\pm\lambda$ with $\omega, \lambda > 0$ (Type HN).*

Then there exists a neighborhood of A_0 (in the space of all matrices $A \in \mathbb{R}^{n\times n}$) in which the matrices of the same type form a manifold with codimension 2 and are characterized by the fact that $2A \odot I_n$ has rank defect 2.

Proof. The first case is Proposition 15; the other cases can be proved similarly. □

By Proposition 15, in the double Hopf case two of the four functions contained in the system

$$S(A) = 0, \tag{5.12}$$

where $S(A)$ is defined by (5.2) form a defining system for the two-dimensional manifold of double Hopf matrices near a given one. In fact, Proposition 14 provides sufficient information to make an a priori choice based on local information in the matrix under consideration. This remark applies equally in the three other cases described in Proposition 16.

6. Resonant double Hopf points

We now consider the 1:1 resonant case where A has double eigenvalues $\pm\mathrm{i}\omega$, $\omega > 0$ with geometric multiplicity 1. By Proposition 11, $2A \odot I_n$ has eigenvalue zero with two Jordan blocks with sizes 1 and 3, respectively. Hence $(2A \odot I_n)^2$ has rank defect 3. We define the $N_b \times 2$ matrix $Q_1(A)$ and the 2×2 matrix $S_1(A)$ by solving

$$M(A)\begin{pmatrix} Q_1(A) \\ S_1(A) \end{pmatrix} = \begin{pmatrix} Q(A) \\ S(A) \end{pmatrix}, \tag{6.1}$$

where $M(A)$ is defined as in (5.1) and $Q(A), S(A)$ obtained from (5.2). Obviously, we also have

$$\begin{pmatrix} (2A \odot I_n)^2 + BC^{\mathrm{T}} & B_1 \\ C_1^{\mathrm{T}} & D_1 \end{pmatrix} \begin{pmatrix} Q_1(A) \\ S_1(A) \end{pmatrix} = \begin{pmatrix} 0_{N_b,2} \\ I_{2,2} \end{pmatrix}, \tag{6.2}$$

where $B_1, C_1 \in \mathbb{R}^{N_b \times 2}$ and $D_1 \in \mathbb{R}^{2\times 2}$. Since BC^{T} has rank at most two, it follows that $(2A \odot I_n)^2 + BC^{\mathrm{T}}$ has rank defect at least 1, so by [5], Proposition 3.2, S_1 is singular. On the other hand, S_1 cannot be zero since then (6.1) would imply that $2A \odot I_n$ has Jordan blocks with sizes 2,2 instead of 1,3. So it is natural to add the condition

$$\det(S_1(A)) = 0 \tag{6.3}$$

to the conditions for double Hopf to obtain conditions for 1:1 resonant double Hopf. Before dealing with the regularity of this system we note that the derivatives of S_1 can be obtained in the now familiar way. Define the $N_b \times 2$ matrix $W_1(A)$ by

$$\begin{pmatrix} W_1^{\mathrm{T}}(A) & S_1(A) \end{pmatrix} M(A) = \begin{pmatrix} W^{\mathrm{T}}(A) & S(A) \end{pmatrix}. \tag{6.4}$$

Then S_{1z} can be obtained from

$$S_{1z}(A) + W_1^{\mathrm{T}}(2A_z \odot I_n)Q(A) + W^{\mathrm{T}}(2A_z \odot I_n)Q_1(A) = 0_{2,2} \tag{6.5}$$

(see [6], Proposition 3.5).

It is convenient to deal with the regularity issue in a more general situation.

Proposition 17. *Consider the following four types of matrices $A \in \mathbb{R}^{n\times n}$.*
1. *A is resonant double Hopf and the Hopf eigenvalues have geometric multiplicity* 1 (*Type RDH*).
2. *A has real eigenvalues* $\pm\lambda$, $\lambda > 0$, *each with algebraic multiplicity* 2 *and geometric multiplicity* 1 (*Type RDN*).
3. *A has algebraically simple eigenvalues* $\pm\lambda$, $\lambda > 0$, *and eigenvalue zero with algebraic multiplicity* 2 *and geometric multiplicity* 1 (*Type BTN*).
4. *A has algebraically simple eigenvalues* $\pm \mathrm{i}\omega$, $\omega > 0$, *and eigenvalue zero with algebraic multiplicity* 2 *and geometric multiplicity* 1 (*Type BTH*).

The matrices of each type form a manifold of codimension 3 *in the space of all* $\mathbb{R}^{n\times n}$ *matrices. If* A_0 *is in one of these classes then* $2A_0 \odot I_n$ *has rank defect* 2. *Furthermore, there exists a neighborhood of* A_0 (*in the space of all matrices* $A \in \mathbb{R}^{n\times n}$) *in which the matrices for which* $2A \odot I_n$ *has rank defect* 2 *form a manifold with codimension* 2. *In the case of RDH this neighborhood contains only matrices of the types RDH, DH and RT. For RDN it is RDN, RT and DN. For BTN it is BTN, DN and HN. For BTH it is BTH, HN and DH. In each case a regular set of defining functions for the manifold of codimension 2 is obtained by taking two of the functions contained in* (5.12) *for which the gradient system has full rank* 2. *In the cases RDN and RDH a regular set of defining functions for the* *manifold of codimension 3 is obtained by adding the condition* (6.3). *In the cases BTN and BTH one obtains a regular set of defining equations by adding the condition* $s_f(A) = 0$ *where* $s_f(A)$ *is obtained by solving*

$$\begin{pmatrix} A & b_f \\ c_f^{\mathrm{T}} & d_f \end{pmatrix} \begin{pmatrix} v_f \\ s_f \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \tag{6.6}$$

with $v_f \in \mathbb{R}^n$; $b_f, c_f \in \mathbb{R}^n$ *and* $d_f \in \mathbb{R}$ *are fixed and chosen in such a way that the square matrix in* (6.6) *is nonsingular in the codimension three point.*

Proof. We first consider the case RDN in some detail. There exist vectors v_1, v_2, v_3, v_4 and w_1, w_2, w_3, w_4, all in $\mathbb{R}^n$, such that $Av_1 = \lambda v_1$, $Av_2 = \lambda v_2 + v_1$, $Av_3 = -\lambda v_3$, $Av_4 = -\lambda v_4 + v_3$, $w_1^{\mathrm{T}} A = \lambda w_1^{\mathrm{T}}$, $w_2^{\mathrm{T}} A = \lambda w_2^{\mathrm{T}} + w_1^{\mathrm{T}}$, $w_3^{\mathrm{T}} A = -\lambda w_3^{\mathrm{T}}$, $w_4^{\mathrm{T}} A = -\lambda w_4 + w_3^{\mathrm{T}}$. We have $w_i^{\mathrm{T}} v_j = 0$ if $i \in \{1, 2\}$, $j \in \{3, 4\}$ or $i \in \{3, 4\}$, $j \in \{1, 2\}$ (different eigenvalues). Furthermore, we may assume that $w_i^{\mathrm{T}} v_j = 0$ if $i = j$ and $w_i^{\mathrm{T}} v_j = 1$ if $i \neq j$ and i, j correspond with the same eigenvalue.

By Proposition 11 $2A \odot I_n$ has the linearly independent right singular vectors $v_1 \wedge v_3$ and $v_1 \wedge v_4 - v_2 \wedge v_3$. Furthermore, $v_1 \wedge v_3$ is in the range of $2A \odot I_n$; in fact, $(2A \odot I_n)((v_1 \wedge v_4 + v_2 \wedge v_3)/2) = v_1 \wedge v_3$. Similar relations hold for $(2A \odot I_n)^{\mathrm{T}}$ if we replace every v_i by a w_i.

Let us first determine the dimensions of the relevant manifolds. In the Jordan form of A a diagonal block of the form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \tag{6.7}$$

appears with real universal unfolding

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ \delta_1 & \lambda + \delta_2 & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & \delta_3 & -\lambda + \delta_4 \end{pmatrix}. \tag{6.8}$$

For versality theory we refer to [1]; universality is understood with respect to the group of similarity transformations. By the same argument as in Proposition 15 one proves that the system $(\delta_i(A))_{1 \leqslant i \leqslant 4}$ has full rank 4. Furthermore, it is not hard to check that for matrices with a diagonal block (6.8) the matrix $2A \odot I_n$ has rank defect 2 if and only if $\delta_3 = -\delta_1$, $\delta_4 = -\delta_2$. This proves the assertion concerning the manifold with codimension 2.

Next, a matrix of the form (6.8) for which $2A \odot I_n$ has rank defect 2 is either of type RT (if $\delta_2^2 + 4\delta_1 < 0$) or type DN (if $\delta_2^2 + 4\delta_1 > 0$) or type RDN (if $\delta_2^2 + 4\delta_1 = 0$).

Since the system of three conditions for RDN ($\delta_1 + \delta_3 = 0,\ \delta_2 + \delta_4 = 0,\ \delta_2^2 + 4\delta_1 = 0$) has a Jacobian with full rank 3 at the origin, the assertion concerning the manifold with codimension 3 follows.

We now turn to the regularity of the systems obtained by the bordered matrix methods. First note that the columns of Q span the right singular space of $2A \odot I_n$, i.e., the same space as $v_1 \wedge v_3$ and $v_1 \wedge v_4 - v_2 \wedge v_3$. Similarly, the columns of W span the same space as $w_1 \wedge w_3$ and $w_1 \wedge w_4 - w_2 \wedge w_3$.

By Lemma 1 and the relations between the vectors v, w we have for $1 \leqslant i,\ j \leqslant n$ that

$$(w_1 \wedge w_3)^{\mathrm{T}}(2A_{i,j} \odot I_n)(v_1 \wedge v_3) = 0, \tag{6.9}$$

$$(w_1 \wedge w_3)^{\mathrm{T}}(2A_{i,j} \odot I_n)(v_1 \wedge v_4 - v_2 \wedge v_3) = w_{1i}v_{1j} - w_{3i}v_{3j}, \tag{6.10}$$

$$(w_1 \wedge w_4 - w_2 \wedge w_3)^{\mathrm{T}}(2A_{i,j} \odot I_n)(v_1 \wedge v_3) = w_{1i}v_{1j} - w_{3i}v_{3j}, \tag{6.11}$$

$$\begin{aligned}(w_1 \wedge w_4 - w_2 \wedge w_3)^{\mathrm{T}}&(2A_{i,j} \odot I_n)(v_1 \wedge v_4 - v_2 \wedge v_3) \\ &= -w_{1i}v_{2j} - w_{4i}v_{3j} - w_{2i}v_{1j} - w_{3i}v_{4j}.\end{aligned} \tag{6.12}$$

Obviously these four gradient vectors span at most a two-dimensional space. Multiplying (6.10) with v_{2i} and summing over i we obtain $-(w_1^{\mathrm{T}}v_2)v_{1j}$ for all j; this implies that the gradient vector in (6.10) is nonzero. On the other hand, multiplying it with v_{1i} and summing over i we find zero; multiplying (6.12) also with v_{1i} and summing over i we obtain $-w_2^{\mathrm{T}}v_1v_{1j}$ for all j; hence the two gradient vectors in (6.10) and (6.12) are linearly independent and span a two-dimensional space. So the equations corresponding to the second and fourth gradient vectors determine the two-dimensional manifold of matrices that we are considering.

Now let us look at the resonance condition $\det(S_1) = 0$. We know already that S_1 has rank 1 at the resonant point. Set

$$S_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

We first consider the special case that $s_{12} = s_{21} = s_{22} = 0$. Then necessarily $s_{11} \neq 0$ and $\det_z(S_1) = s_{11}(s_{22})_z$. Also,

$$(s_{22})_z = -W_1^{2\mathrm{T}}(2A_z \odot I_n)Q^2 - W^{2\mathrm{T}}(2A_z \odot I_n)Q_1^2, \tag{6.13}$$

where the upper index 2 in each case indicates that we take the second column out of a matrix with 2 columns. By the assumptions on S_1 we must necessarily have that

$$Q_1^2 = \alpha_r v_1 \wedge v_3 + \beta_r(v_1 \wedge v_4) + \gamma_r(v_2 \wedge v_3)$$

with $\beta_r + \gamma_r \neq 0$ and

$$Q^2 = (2A \odot I_n)Q_1^2 = (\beta_r + \gamma_r)(v_1 \wedge v_3).$$

$$W_1^2 = \alpha_l w_1 \wedge w_3 + \beta_l(w_1 \wedge w_4) + \gamma_l(w_2 \wedge w_3)$$

with $\beta_l + \gamma_l \neq 0$ and

$$W^2 = (2A \odot I_n)^{\mathrm{T}} W_1^2 = (\beta_l + \gamma_l)(w_1 \wedge w_3).$$

Comparing (6.13) with (6.9)–(6.12) we note that it is natural to rewrite Q_1^2 and W_1^2 as

$$Q_1^2 = \alpha_r v_1 \wedge v_3 + \beta_r(v_1 \wedge v_4 - v_2 \wedge v_3) + (\beta_r + \gamma_r)(v_2 \wedge v_3),$$

$$W_1^2 = \alpha_l w_1 \wedge w_3 + \beta_l(w_1 \wedge w_4 - w_2 \wedge w_3) + (\beta_l + \gamma_l)(w_2 \wedge w_3).$$

Inserting these expressions into (6.13) we obviously find a linear combination of the expressions in (6.10) and (6.11) plus an additional nonzero multiple of

$$\begin{aligned}(w_2 \wedge w_3)^{\mathrm{T}}(2A_{i,j} \odot I_n)(v_1 \wedge v_3) + (w_1 \wedge w_3)^{\mathrm{T}}(2A_{i,j} \odot I_n)(v_2 \wedge v_3) \\ = 2w_{3i}v_{3j}. \end{aligned} \tag{6.14}$$

By multiplying (6.10), (6.12) and (6.14) with v_{1i} and v_{2i} and summing over i it follows easily that these gradient vectors are linearly independent, so the result follows under the assumptions we made concerning S_1.

In the general case there exists by the Jordan decomposition theorem a 2×2 nonsingular matrix X such that $S_{1J} = X^{-1}S_1X$ has at resonance a nonzero element in the upper left entry and zeroes everywhere else.

Now define $B_J = BX, C_J = CX^{-\mathrm{T}}, D_J = X^{-1}DX$. Then the matrix

$$M_J = \begin{pmatrix} 2A \odot I_n & B_J \\ C_J^{\mathrm{T}} & D_J \end{pmatrix}$$

is obviously nonsingular and by trivial computations we find that

$$(M_J)^2 \begin{pmatrix} Q_1X \\ X^{-1}S_1X \end{pmatrix} = \begin{pmatrix} 0 \\ I_{2,2} \end{pmatrix}$$

in a neighborhood of the resonant matrix.

In this neighborhood we have $S_{1J}(A) = XS_J(A)X^{-1}$ and hence $\det(S_1(A)) = \det(S_{1J}(A))$. So the result follows from the special case that we considered first.

This proves the case RDN. The case RDH is similar. Now consider the case BTN. There exist a $\lambda > 0$ and vectors v_1, v_2, v_3, v_4 and w_1, w_2, w_3, w_4 in $\mathbb{R}^n$ such that $Av_1 = 0, Av_2 = v_1, Av_3 = \lambda v_3, Av_4 = -\lambda v_4$, $w_1^{\mathrm{T}}A = 0$, $w_2^{\mathrm{T}}A = w_1^{\mathrm{T}}$, $w_3^{\mathrm{T}}A = \lambda w_3^{\mathrm{T}}$,

$w_4^{\mathrm{T}} A = -\lambda w_4^{\mathrm{T}}$. We may assume that $w_i^{\mathrm{T}} v_j = 0$ for $i, j = 1,2,3,4$ except for the cases $w_1^{\mathrm{T}} v_2 = w_2^{\mathrm{T}} v_1 = w_3^{\mathrm{T}} v_3 = w_4^{\mathrm{T}} v_4 = 1$. The left singular vectors of $2A \odot I_n$ are the vectors $w_1 \wedge w_2$ and $w_3 \wedge w_4$. The right singular vectors are $v_1 \wedge v_2$ and $v_3 \wedge v_4$.

The universal unfolding of the Jordan form now has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \delta_1 & \delta_2 & 0 & 0 \\ 0 & 0 & \lambda + \delta_3 & 0 \\ 0 & 0 & 0 & -\lambda + \delta_4 \end{pmatrix}. \tag{6.15}$$

For a matrix with a diagonal block (6.15) the bialternate product matrix has rank defect 2 if and only if $\delta_2 = \delta_3 + \delta_4 = 0$; this proves the claim concerning the codimension two manifold. Furthermore, it is a BTN matrix if and only if $\delta_1 = \delta_2 = \delta_3 + \delta_4 = 0$. This proves the claim concerning the codimension three manifold.

Now we consider the regularity of the defining systems. By Lemma 1 and the relations between the vectors v, w we have for $1 \leqslant i, j \leqslant n$ that

$$(w_1 \wedge w_2)^{\mathrm{T}} (2A_{i,j} \odot I_n)(v_1 \wedge v_2) = -w_{2i} v_{1j} - w_{1i} v_{2j}, \tag{6.16}$$

$$(w_3 \wedge w_4)^{\mathrm{T}} (2A_{i,j} \odot I_n)(v_3 \wedge v_4) = w_{3i} v_{3j} + w_{4i} v_{4j}. \tag{6.17}$$

The two other candidates for a gradient vector are zero. Furthermore, the gradient that corresponds to the condition $s_f(A) = 0$ is given by

$$-w_1^{\mathrm{T}} A_{ij} v_1 = -w_{1i} v_{1j}. \tag{6.18}$$

It is easy to prove that the three gradient vectors with components given by (6.16)–(6.18) are linearly independent (multiply successively with w_{1j}, w_{2j} and w_{3j} and sum over j).

Finally, the case BTH is similar to BTN. □

Fig. 1 presents the eight types of matrices described in Propositions 16 and 17 and their possible interactions. As a typical application, one might compute a curve of points in a three parameter problem, expressing the requirement that $2A \odot I_n$ has rank defect 2. Then in the scheme of Fig. 1 we expect to move from each type to one of the two adjacent types, the types at the corners generically being isolated points on the computed curve. The numbers between brackets indicate the sizes of the Jordan blocks of $2A \odot I_n$ for the zero eigenvalue.

We note that there are other matrices A for which the bialternate product has rank defect 2. However, they have to lie on certain manifolds with codimension higher than 3 and the systems that we obtained may not be regular in such points. For example, consider the case where A has the eigenvalue 1 with algebraic multiplicity 2 and geometric multiplicity 1 and the eigenvalue -1 with

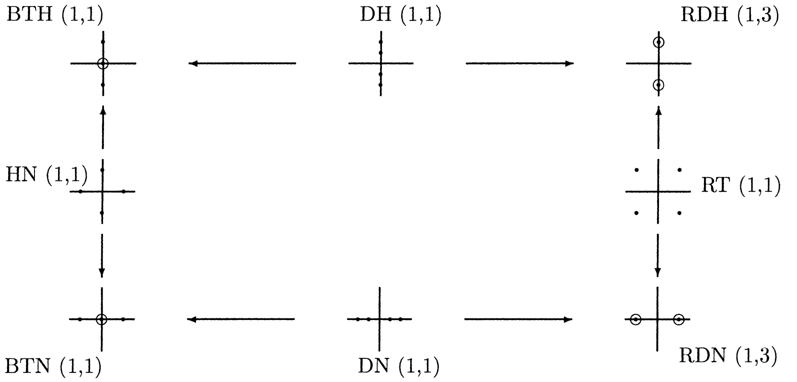

Fig. 1. Eight types of matrices A where $2A \odot I_n$ has rank defect 2.

algebraic multiplicity 3 and geometric multiplicity 1. Then by Proposition 11 $2A \odot I_n$ has rank defect 2 and its square has rank defect 4. In the setting of the preceding methods we have $S = S_1 = 0$. So the gradient vector of $\det(S_1)$ vanishes.

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