Spline Interpolation and Wavelet Construction

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The method of Dubuc and Deslauriers on symmetric interpolatory subdivision is extended to study the relationship between interpolation processes and wavelet construction. Refinable and interpolatory functions are constructed in stages from B-splines. Their method constructs the filter sequence (its Laurent polynomial) of the interpolatory function as a product of Laurent polynomials. This provides a natural way of splitting the filter for the construction of orthonormal and biorthogonal scaling functions leading to orthonormal and biorthogonal wavelets. Their method also leads to a class of filters which includes the minimal length Daubechies compactly supported orthonormal wavelet coefficients. Examples of “good” filters are given together with results of numerical experiments conducted to test the performance of these filters in data compression.

Key Words: refinable function; wavelets; interpolatory function; uniform B-spline; Euler–Frobenius polynomial; Riesz basis; biorthogonal basis; orthonormal basis; transition operator; subdivision algorithm; cascade algorithm; Condition E.

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1. INTRODUCTION

Let \((h(k))_{k=0}^{N}\) be a finitely supported real sequence satisfying

\[ \sum_{k=0}^{N} h(k) = 1. \]  \hspace{1cm} (1.1)

A function (or distribution) \(\phi\) which satisfies the equation

\[ \phi(x) = \sum_{k=0}^{N} 2h(k)\phi(2x - k), \quad x \in \mathbb{R} \]  \hspace{1cm} (1.2)

is called a \textit{refinable function} (or distribution). Equation (1.2) is called a \textit{refinement equation}. The sequence \(h\) is called a \textit{filter sequence} or \textit{refinement mask}. If \(h\) satisfies the fundamental condition

\[ \sum_{k \in \mathbb{Z}} h(2k) = \sum_{k \in \mathbb{Z}} h(2k + 1) = \frac{1}{2}, \]  \hspace{1cm} (1.3)

then it is called a \textit{fundamental filter sequence}. Clearly, if \(h\) is fundamental then it satisfies (1.1).

It is well known (see [3, 8]) that if \(h\) satisfies (1.1), there exists a compactly supported refinable distribution \(\phi\), unique up to a constant multiple, such that \(\hat{\phi}(0) \neq 0\). Here \(\hat{\phi}\) denotes the Fourier transform of \(\phi\). Further, \(\hat{\phi}\) admits the infinite product representation

\[ \hat{\phi}(u) = \prod_{j=1}^{\infty} \hat{\phi}(u/2^j), \quad u \in \mathbb{R}, \]  \hspace{1cm} (1.4)

where

\[ \hat{\phi}(u) := \sum_{k=0}^{N} h(k)e^{-iku} \]

is the Fourier transform of the sequence \(h\). The complex polynomial

\[ H(z) := \sum_{k=0}^{N} h(k)z^k, \quad z \in \mathbb{C} \setminus \{0\}, \]  \hspace{1cm} (1.5)

will be called the \textit{Z-transform} of the sequence \(h\). Clearly, \(H(e^{-iu}) = \hat{h}(u)\).

In order for \(\phi\) to be a sufficiently smooth function, it is necessary that the Fourier transform \(\hat{h}(u)\) of its filter sequence \((h(k))_{k=0}^{N}\) satisfies

\[ \hat{h}(u) = \left(\frac{1 + e^{-iu}}{2}\right)^m \hat{g}(u), \]  \hspace{1cm} (1.6)
for some positive integer $m$ (see [3, 8]). Condition (1.6) is satisfied for $m = 1$ if $h$ is fundamental. The exponent $m$ also gives the number of vanishing moments for the corresponding wavelets.

The construction of filter sequences satisfying (1.6) and the corresponding refinable functions is an important problem in both the theory and the applications of wavelets and in subdivision schemes. Once the refinable functions are known, the wavelets can be constructed by a standard technique (see [3, 8, 18]). The most popular filter sequences are those associated with Daubechies’ compactly supported orthonormal wavelets [9], Chui and Wang’s semiorthogonal cardinal spline wavelets [4], and the biorthogonal wavelets of Cohen et al. [6]. Multidimensional filters for smooth refinable functions and the corresponding subdivision schemes have been constructed in a recent paper of Riemenschneider and Shen [26].

If a refinable function $\phi$ satisfying (1.2) is orthonormal, i.e., its integer shifts form an orthonormal set, then its filter sequence satisfies

$$
\sum_{k=0}^{N} 2h(k - 2j)h(k) = \delta_0(j), \quad (1.7)
$$

where

$$
\delta_k(j) = \begin{cases} 
1 & \text{for } j = k \\
0 & \text{otherwise}.
\end{cases}
$$

A filter sequence which satisfies (1.7) is called a conjugate quadrature filter (CQF). It follows from (1.7) that a necessary condition for $h$ to be a CQF is that its autocorrelation $a$, defined by

$$
a(l) := \sum_{k=0}^{N} h(k - l)h(k), \quad l = -N, -N + 1, \ldots, N,
$$

satisfies $a(2j) = \delta_0(j)/2$.

Recently, Strang [27] and Micchelli [20] have pointed out an interesting connection between Lagrange interpolation and Daubechies’ compactly supported orthonormal wavelet coefficients. The connection can be described succinctly as follows. Let $l_j, j = -N + 1, \ldots, N$, be the Lagrange fundamental polynomial functions of degree $2N - 1$ satisfying

$$
l_j(k) = \delta_0(k), \quad j, k = -N + 1, \ldots, N,
$$

and define a sequence $a$ by

$$
a(-2j) = l_j(0)/2, \quad a(1 - 2j) = l_j(1/2)/2, \quad j = -N + 1, \ldots, N,
$$

and $a(j) = 0$, otherwise. Then the Fourier transform of $a$,

$$
\tilde{a}(u) = \sum_{k=-2N+1}^{2N-1} a(k)e^{-iku},
$$
is nonnegative and so can be factored as $\tilde{a}(u) = |\tilde{b}(u)|^2$, where

$$
\tilde{b}(u) = \sum_{k=0}^{2N-1} b(k)e^{-au},
$$

and $b$ is the filter sequence for the compactly supported orthonormal refinable function of Daubechies [9]. The connection in this form appeared in the work of Deslauriers and Dubuc [10] on interpolatory subdivision.

In [20], Micchelli has also constructed filter sequences using the Lagrange fundamental functions for interpolation by exponentials. The resulting sequence is a CQF satisfying the fundamental condition (1.3). In this case a longer sequence is required in order that the corresponding wavelet has the same smoothness and the same number of vanishing moments as Daubechies’ wavelets. The success of this approach depends on the property that the Lagrange fundamental functions form a partition of unity and in the case of polynomial interpolation, they reproduce polynomials up to a certain degree. Therefore, any interpolation and approximation process which reproduces polynomials can be used in the same way to generate filter sequences satisfying the fundamental assumption (1.3) and perhaps also condition (1.6).

A continuous function $f$ is called a fundamental function if its restriction to the integers $f | \mathbb{Z} = \delta_0$. A refinable function which is fundamental is said to be interpolatory.

The object of this paper is to construct refinable functions and interpolatory functions from the approximation and interpolation processes using uniform $B$-splines, and to use them in the construction of filter sequences satisfying (1.7), and the corresponding orthonormal and biorthogonal wavelet bases. The method gives the filter (its Laurent polynomial) as a product of Laurent polynomials. Using a result of Schoenberg on the Euler–Frobenius polynomials [23] and one of Goodman and Micchelli [12] on the solution of a generalized Bezout equation, we show that the Laurent polynomial of the filter sequence does not vanish on the unit circle, except at $z = -1$. This allows us to split the factors in three natural ways to obtain orthonormal refinable functions and nonorthonormal refinable functions with biorthogonal shifts. This is done in Section 3. Section 2 gives a general formulation of the problem and some results which will be needed in the later part of the paper. Although our focus is on uniform $B$-splines, these results involve general refinable functions. The algorithms for computing the filter coefficients are given in Section 4, which also demonstrates some examples of ‘‘good’’ filter sequences constructed in Section 3. Results of numerical experiments conducted to test the performance of the filters in discrete wavelet transforms and data compression are given in Section 5.

## 2. NEW FILTERS FROM REFINABLE FUNCTIONS

Suppose that $\phi$ is a continuous refinable function satisfying (1.2), where $h = (h(k))_{k=-N}^N$ is a finitely supported fundamental sequence. Then $\phi$ is compactly supported with $\text{supp}(\phi) \subset [0, N]$, and $\phi(0) = \phi(N) = 0$. Since $h$ is fundamental, the shifts of $\phi$ reproduce constant functions. Hence, $\phi$ can be normalized so that
\[ \sum_{j \in \mathbb{Z}} \phi(x - j) = 1, \quad x \in \mathbb{R}. \quad (2.1) \]

In particular,
\[ \sum_{j \in \mathbb{Z}} \phi(j) = 1. \quad (2.2) \]

If \( \phi|_\mathbb{Z} \) is the restriction of \( \phi \) to the integers, then \( \phi|_\mathbb{Z} \neq 0 \). Let \( H(z) \) be the \( Z \)-transform of \( h \), and
\[ \Phi(z) := \sum_{k=1}^{N-1} \phi(k) z^k, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.3) \]

With the sequence \( h \) we define a bounded linear operator \( W_h : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) by
\[ (W_h b)(k) := \sum_{j \in \mathbb{Z}} 2^k (2k - j) b(j), \quad k \in \mathbb{Z}, \quad b \in l^2(\mathbb{Z}). \quad (2.4) \]

The operator \( W_h \) is the \textit{transition operator} for the filter sequence \( h \). It is also called the \textit{wavelet-Galerkin operator} (see [15, 16]). Clearly, if \( \text{supp}(b) \subset \{0, 1, \ldots, N\} \) then \( \text{supp}(W_h b) \subset \{0, 1, \ldots, N\} \). Therefore, the set \( l([0, N]) \) which comprises all sequences with support in \( \{0, 1, \ldots, N\} \) is an invariant subspace of \( W_h \) of dimension \( N + 1 \). The restriction of \( W_h \) to \( l([0, N]) \) will be denoted by \( \hat{W}_h \).

\textbf{Proposition 2.1.} Suppose that \( \phi \) is the compactly supported distribution satisfying (1.2). If \( \hat{h}(u) \neq 0 \) for \(-\pi < u < \pi\), then \( \hat{\phi}(u) \neq 0 \) for \( u \in [0, 2\pi) \).

\textit{Proof.} Suppose that \( \hat{\phi}(u_0) = 0 \) for some \( u_0 \in [0, 2\pi) \). Then \( u_0 \neq 0 \), since \( \hat{\phi}(0) \neq 0 \). By (1.4),
\[ \hat{\phi}(u_0/2) \hat{h}(u_0/2) = \hat{\phi}(u_0) = 0. \]

Since \( u_0/2 \neq \pi \), \( \hat{h}(u_0) \neq 0 \). It follows that \( \hat{\phi}(u_0/2) = 0 \). Repeating the process inductively gives \( \hat{\phi}(u_0/2^n) = 0 \) for all \( n = 0, 1, \ldots \). By continuity, \( \hat{\phi}(0) = 0 \), which is impossible. \( \blacksquare \)

Suppose \( \phi \) is a continuous refinable function satisfying (1.2). Using \( \phi \), we define a filter sequence \( a \) by
\[ a(2j) := \frac{1}{2} \phi(j), \quad j = 1, 2, \ldots, N - 1, \]
\[ a(2j - 1) := \frac{1}{2} \phi \left( j - \frac{1}{2} \right), \quad j = 1, 2, \ldots, N, \quad (2.5) \]
and \( a(k) = 0, k \neq 1, 2, \ldots, 2N - 1 \), and let
\[ A(z) := \sum_{k=1}^{2N-1} a(k) z^k, \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.6) \]
**Proposition 2.2.** Let \( H(z) \) be the \( Z \)-transform of the filter sequence \( h \) and suppose that \( \Phi(z) \) and \( A(z) \) are defined in (2.3) and (2.6), respectively. Then

\[
A(z) = H(z)\Phi(z), \quad z \in \mathbb{C} \setminus \{0\}, \quad (2.7)
\]
and

\[
\Phi(z^2) = A(z) + A(-z), \quad z \in \mathbb{C} \setminus \{0\}. \quad (2.8)
\]

Further, \( a \) is fundamental.

**Proof.** By (2.5), (2.6), and (1.2),

\[
A(z) = \sum_{j} \frac{1}{2} \phi(j)z^{2j} + \sum_{j} \frac{1}{2} \phi\left(j - \frac{1}{2}\right)z^{2j-1}
\]

\[
= \sum_{j} h(k)\phi(2j - k)z^{2j} + \sum_{j} h(k)\phi(2j - 1 - k)z^{2j-1}
\]

\[
= \sum_{j} h(k)\phi(j - k)z^j,
\]

which gives (2.7).

By (2.5),

\[
A(z) + A(-z) = 2 \sum_{j=1}^{N-1} a(2j)z^{2j} = \sum_{j=1}^{N-1} \phi(j)z^{2j} = \Phi(z^2),
\]

and (2.8) is established.

Evaluating (2.7) at \( z = 1 \) and using (1.1) and (2.2) gives \( \sum a(j) = 1 \). Similarly, (2.2) and (2.8) imply that \( \sum a(2j) = 1/2 \). Hence \( a \) is fundamental. \( \blacksquare \)

Suppose that \( \varphi \in L^2(\mathbb{R}) \) is the solution of the refinement equation

\[
\varphi(x) = \sum_{k=1}^{2N-1} 2a(k)\varphi(2x - k), \quad x \in \mathbb{R},
\]

and let \( V(\varphi) \) be the closed linear span of integer shifts of \( \varphi \). Recall that \( V(\varphi) \) has approximation order \( m \) if it reproduces algebraic polynomials of degree \( <m \).

**Corollary 2.1.** If \( H(z) = ((1 + z)/2)^mG(z) \), for some positive integer \( m \), with \( G(-1) \neq 0 \), then \( V(\varphi) \) has approximation order \( m \).

### 3. Cardinal B-Splines Revisited

This section studies the connection between the approximation and interpolation processes by uniform B-splines and the nonorthonormal refinable functions whose refinement masks are related to the Euler–Frobenius polynomials. The forward B-spline \( \varphi(x) \) of degree \( n-1 \) is refinable and is the solution of the refinement equation
\[ Q_n(x) = \sum_{k=0}^{n} \frac{1}{2^{n-1}} \binom{n}{k} Q_n(2x - k), \quad x \in \mathbb{R}. \] (3.1)

Its Fourier transform is given by
\[ \hat{Q}_n(u) = \left(\frac{1 - e^{-iu}}{iu}\right)^n, \quad u \in \mathbb{R}. \] (3.2)

The polynomial
\[ \Pi_{n-1}(z) := (n-1)! \sum_{j=0}^{n-2} Q_n(j+1)z^j \] (3.3)

is the Euler–Frobenius polynomial of degree \( n - 2 \). The coefficients of \( \Pi_{n-1}(z) \) are known as the Eulerian numbers (see [1]). The polynomial \( \Pi_{n-1}(z) \) is a reciprocal polynomial having simple negative zeros (see [24]).

Let \( n \) be a fixed positive integer. Then \( Q_n \) is refinable with refinement mask
\[ H_n(z) = \left(\frac{1 + z}{2}\right)^n. \] (3.4)

Further,
\[ \Phi_n(z) = \frac{z \Pi_{n-1}(z)}{(n-1)!}, \] (3.5)

where
\[ \Phi_n(z) := \sum_{j=1}^{n-1} Q_n(j)z^j. \]

Using the forward B-spline \( Q_n(x) \), we define the sequence \( a_n \) by
\[ a_n(2j) := \frac{1}{2} Q_n(j), \quad j = 1, \ldots, n-1, \]
\[ a_n(2j - 1) := \frac{1}{2} Q_n\left(j - \frac{1}{2}\right), \quad j = 1, \ldots, n, \] (3.6)

and \( a_n(k) = 0 \) otherwise. By (2.7), (3.4), and (3.5), the Z-transform \( A_n(z) \) of \( a_n \) is related to the Euler–Frobenius polynomials by
\[ A_n(z) = H_n(z) \Phi_n(z) = z \left( \frac{1 + z}{2} \right)^n \prod_{m=1}^{n-1} \left( \frac{z}{n-1} \right)! . \quad (3.7) \]

For our subsequent analysis, we require some auxiliary results. The first, on the bounds of \( \Phi_n(e^{-iu}) \), is essentially due to Schoenberg [23]. For even \( n = 2m \), \( \Phi_n(e^{-iu}) \) is nonvanishing for \( u \in \mathbb{R} \), and for odd \( n = 2m + 1 \), it has a simple zero at \(-1\). To complete Schoenberg’s result for our analysis, we define the polynomial \( C_{2m+1} \) by

\[ \Phi_{2m+1}(z) = \left( \frac{1 + z}{2} \right) \Psi_{2m+1}(z), \quad z \in \mathbb{C} \setminus \{0\} . \quad (3.8) \]

Then \( \Psi_{2m+1}(e^{-iu}) \) is nonvanishing on the unit circle.

**Lemma 3.1.** If \( m \geq 1 \),

\[ 0 < \gamma_{2m+2} \leq |\Phi_{2m+2}(e^{-iu})| \leq 1, \quad \text{for all } u \in \mathbb{R} , \quad (3.9) \]

and

\[ 0 < (2m + 1) \gamma_{2m+2} \leq |\Psi_{2m+1}(e^{-iu})| \leq 1, \quad \text{for all } u \in \mathbb{R} , \quad (3.10) \]

where

\[ \gamma_{2m+2} := 2 \left( \frac{2}{\pi} \right)^{2m+2} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^{2m+2}} . \quad (3.11) \]

Further, the bounds are exact.

In particular, for any integer \( n \geq 3 \),

\[ |\Phi_n(e^{-iu})| \leq 1, \quad u \in \mathbb{R} . \quad (3.12) \]

**Proof.** A proof of the inequalities (3.9) can be found in [23]. We shall adapt Schoenberg’s proof to establish (3.10).

For any integer \( n \geq 3 \),

\[ \Phi_n(e^{-iu}) = \sum_{j=1}^{n-1} Q_n(j)e^{-iju} . \]

By the Poisson summation formula,

\[ \Phi_n(e^{-iu}) = \sum_{k \in \mathbb{Z}} \hat{Q}_n(u + 2\pi k), \quad u \in \mathbb{R} . \]

Using the formula (3.2) for the Fourier transform of \( Q_n \) in the last sum leads to
\[
\Phi_n(e^{-iu}) = \left( \frac{1 - e^{-iu}}{iu} \right)^n \sum_{k \in \mathbb{Z}} \frac{1}{(u + 2\pi k)^n}, \quad u \in \mathbb{R}. \quad (3.13)
\]

If \( n = 2m + 1 \) is odd, splitting the sum on the right of (3.13) into two sums, one over the negative integers and the other over the nonnegative integers, and making a change of index in one of the summations, we have

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(u + 2\pi k)^{2m+1}} = \sum_{k=1}^{\infty} \left( \frac{1}{(u + 2\pi(k - 1))^{2m+1}} - \frac{1}{(2\pi k - u)^{2m+1}} \right).
\]

Define

\[
a_k(u) := u + 2\pi(k - 1) \quad \text{and} \quad b_k(u) := 2\pi k - u.
\]

For \( u \in [0, 2\pi) \), \( a_k(u) \) and \( b_k(u) \) are nonnegative for \( k = 1, 2, \ldots \), and positive for \( k > 1 \). A straightforward calculation leads to

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(u + 2\pi k)^{2m+1}} = 2(\pi - u)S_{2m+1}(u), \quad (3.14)
\]

where

\[
S_{2m+1}(u) := \sum_{k=1}^{\infty} (b_k(u)^{2m} + b_k(u)^{2m-1}a_k(u) + \cdots + a_k(u)^{2m})/(a_k(u)b_k(u))^{2m+1}.\]

Combining (3.13) and (3.14) gives

\[
\Phi_{2m+1}(e^{-iu}) = 2(\pi - u)\left( \frac{1 - e^{-iu}}{iu} \right)^{2m+1}S_{2m+1}(u). \quad (3.15)
\]

For \( u \in (0, 2\pi) \), \( S_{2m+1}(u) > 0 \). Hence,

\[
|\Phi_{2m+1}(e^{-iu})| = 2|\pi - u|(2 \sin u/2)^{2m+1}S_{2m+1}(u), \quad u \in [0, 2\pi). \quad (3.16)
\]

On the other hand, \( \Phi_{2m+1}(z) = z \Pi_{2m} (z)/(2m)! \), and \( \Pi_{2m}(z) \) is a reciprocal polynomial of degree \( 2m - 1 \) with simple negative zeros. Hence \(-1\) is a zero and the other zeros are \(-\lambda_j\), \(-\lambda_j^{-1}\), \( j = 1, 2, \ldots, m - 1 \), such that

\[
\lambda_1 < \cdots < \lambda_{m-1} < 1 < \lambda_{m-1}^{-1} < \cdots < \lambda_1^{-1}.
\]

Therefore, we can write

\[
\Psi_{2m+1}(z) = Bz \prod_{j=1}^{m-1} (z + \lambda_j)(z + \lambda_j^{-1}),
\]
where $B$ is a nonzero constant. Taking the modulus of the polynomial restricted to the unit circle gives
\[
|\Psi_{2m+1}(e^{-iu})| = K \prod_{j=1}^{m-1} (1 + \lambda_j e^{-iu})(1 + \lambda_j e^{iu}),
\]
where $K$ denotes a positive constant. Now for each $j$,
\[
(1 + \lambda_j e^{-iu})(1 + \lambda_j e^{iu}) = 4\lambda_j \cos u/2 + (1 - \lambda_j)^2.
\]
Hence,
\[
|\Psi_{2m+1}(e^{-iu})| = K \prod_{j=1}^{m-1} (\cos^2 u/2 + \kappa_j), \quad u \in \mathbb{R},
\]
where
\[
\kappa_j := (1 - \lambda_j)^2/4\lambda_j > 0.
\]
It follows that $|\Psi_{2m+1}(e^{-iu})|$ is an even function, decreasing on the interval $[0, \pi]$. Hence
\[
\gamma_{2m+1} := |\Psi_{2m+1}(-1)| = |\Psi_{2m+1}(e^{-iu})| \leq |\Psi_{2m+1}(1)| = |\Phi_{2m+1}(1)| = 1.
\]
By (3.16) and (3.8),
\[
\gamma_{2m+1} := |\Psi_{2m+1}(-1)| = \lim_{u \to \pi} \frac{2(\pi - u)}{\cos u/2} (2 \sin u/2)^{2m+1} S_{2m+1}(u)
= 2(2m + 1) \left( \frac{2}{\pi} \right)^{2m+2} \sum_{j=1}^{m} \frac{1}{(2k - 1)^{2m+2}} = (2m + 1) \gamma_{2m+2},
\]
which establishes the left inequality in (3.10). ☐

**Lemma 3.2.** Let $n \equiv 3$ be an integer. For $k = -n + 2, -n + 3, \ldots, n - 2$, define
\[
q_n(k) := \sum_{j=1}^{n-1} Q_n(j) Q_n(j - k), \quad (3.17)
\]
and let $\tilde{W}_n$ be the corresponding transition operator restricted to its invariant subspace $l([-n + 2, n - 2])$. Then the spectral radius $r(\tilde{W}_n) \leq 2$.

**Proof.** Let $\lambda$ be an eigenvalue of $\tilde{W}_n$, and $b \in l([-n + 2, n - 2])$ be the corresponding eigenvector. Then
\[ \lambda \hat{b}(u) = |\Phi_a(e^{-iu/2})|^2 \hat{b}(u/2) + |\Phi_a(-e^{-iu/2})|^2 \hat{b}(u/2 + \pi), \] (3.18)

where

\[ \hat{b}(u) := \sum_{k=-n+2}^{n-2} b(k) e^{-iku}. \]

Let \( M = \max \{ \hat{b}(u) : u \in \mathbb{R} \} = \hat{b}(u_0) > 0 \). Then (3.18) gives

\[ |\lambda| M \leq |\Phi_a(e^{-iu_0/2})|^2 |\hat{b}(u_0/2)| + |\Phi_a(-e^{-iu_0/2})|^2 |\hat{b}(u_0/2 + \pi)| \]
\[ \leq (|\Phi_a(e^{-iu_0/2})|^2 + |\Phi_a(-e^{-iu_0/2})|^2) M. \]

By Lemma 3.1, the above inequalities give \( \lambda M \leq 2M \). Hence \( \lambda \leq 2. \]

**Theorem 3.1.** Suppose \( a_n \) is defined by (3.6). If \( n \geq 3 \), there exists a unique compactly supported function \( \varphi = \varphi_n \), with \( \text{supp}(\varphi) \subset [1, 2n - 1] \), satisfying the refinement equation

\[ \varphi(x) = \sum_{j=1}^{2n-1} 2a_n(j) \varphi(2x - j). \] (3.19)

Further, \( \varphi_{2m} \in C^{2m-1-\epsilon}(\mathbb{R}) \) and \( \varphi_{2m+1} \in C^{2m+1-\epsilon}(\mathbb{R}) \) for every \( \epsilon > 0 \).

The refinable function \( \varphi \) is stable; i.e., its integer shifts form a Riesz basis of their closed linear span \( V(\varphi) \) in \( L^2(\mathbb{R}) \). Further, \( V(\varphi) \) contains polynomials of degree \( n - 1 \) if \( n \) is even and of degree \( n \) if \( n \) is odd.

**Proof.** Since \( a_n \) is fundamental, there exists a compactly supported distribution \( \varphi = \varphi_n \) satisfying (3.19), and its Fourier transform is given by

\[ \hat{\varphi}(u) = \prod_{j=1}^{\infty} \hat{a}_n(u/2^j), \quad u \in \mathbb{R}, \] (3.20)

where \( \hat{a}_n(u) := A_n(e^{-iu}) \). If we restrict \( z = e^{-iu} \) to the unit circle, (3.7) gives

\[ \hat{a}_n(u) = \left( \frac{1 + e^{-iu}}{2} \right)^n \Phi_n(e^{-iu}). \] (3.21)

Further,

\[ \hat{a}_{2m}(u) = \left( \frac{1 + e^{-iu}}{2} \right)^{2m} \Phi_{2m}(e^{-iu}), \quad \Phi_{2m}(-1) \neq 0, \]
\[ \hat{a}_{2n+1}(u) = \left( \frac{1 + e^{-i\alpha}}{2} \right)^{2m+2} \Psi_{2n+1}(e^{-iu}), \quad \Psi_{2n+1}(-1) \neq 0. \]

By Lemma 3.1,
\[
|\Phi_{2m}(e^{-iu})| \leq 1 \quad \text{and} \quad |\Psi_{2n+1}(e^{-iu})| \leq 1, \quad \text{for all } u \in \mathbb{R}.
\]

It follows from Proposition 4.8 of Cohen et al. [6] that
\[
|\varphi_{2m}(u)| \leq C(1 + |u|)^{-2m}
\]
and
\[
|\varphi_{2n+1}(u)| \leq C(1 + |u|)^{-2m-2}.
\]

Hence \( \varphi_{2m} \in C^{2m-1-\epsilon} \) and \( \varphi_{2n+1} \in C^{2m+1-\epsilon} \), for every \( \epsilon > 0 \).

By (3.21), the eigenvalues of \( \tilde{W}_{r_{\alpha}} \), the transition operator for the autocorrelation \( r_{\alpha} \) of \( a_{\alpha} \), are
\[
{1, 1/2, \ldots, 1/2^{2m-1}} \cup \{ \gamma/2^{2n}: \gamma \text{ an eigenvalue of } \tilde{W}_{r_{\alpha}} \}.
\]

Since \( |\gamma| \leq 2 \), by Lemma 3.2, \( \tilde{W}_{r_{\alpha}} \) has a simple eigenvalue 1 and all its other eigenvalues lie inside the unit circle. Hence the cascade algorithm
\[
\varphi_k(x) = \sum_{j=1}^{2n-1} 2a_{\alpha}(j)\varphi_{k-1}(2x - j), \quad k = 1, 2, \ldots,
\]
converges in \( L^2(\mathbb{R}) \) to \( \varphi \) (see [13, 14, 28, 29]) for any compactly supported initial function \( \varphi_0 \) for which the shifts form a partition of unity. Hence, \( \text{supp}(\varphi) \subseteq [1, 2n - 1] \).

Since \( \varphi \) is compactly supported, it is stable if and only if
\[
\sum_{j \in \mathbb{Z}} |\hat{\varphi}(u + 2\pi j)|^2 \neq 0, \quad \text{for all } u \in [0, 2\pi).
\]

Equivalently, \( \varphi \) is stable if and only if for each \( u \in [0, 2\pi) \), the sequence \( (\hat{\varphi}(u + 2\pi j))_{j \in \mathbb{Z}} \neq 0 \). Since \( \hat{a}_{\alpha}(u) \neq 0 \) except at \( u = \pi \), it follows from Proposition 2.1 that \( \hat{\varphi}(u) \neq 0 \), for \( u \in [0, 2\pi) \). Hence for \( u \in [0, 2\pi) \), the sequence \( (\hat{\varphi}(u + 2\pi j))_{j \in \mathbb{Z}} \neq 0 \).

If \( n \) is even, \( \Pi_{n-1}(-1) \neq 0 \), and if \( n \) is odd, \( \Pi_{n-1}(z) \) has a simple zero at \( z = -1 \). It follows from the second equation of (3.7) that \( V(\varphi_{n}) \) contains polynomials of degree \( n - 1 \) if \( n \) is even and it contains polynomials of degree \( n \) if \( n \) is odd.
For any integer \( M \geq 0 \), let
\[
A_{n,M}(z) := \left( \frac{1 + z}{2} \right)^{2M} A_n(z) = \left( \frac{1 + z}{2} \right)^{2M+\lambda} \phi_n(z),
\]
(3.22)

and
\[
\hat{W}_{n,M} := 2(a_{n,M}(2k-j))^{2M+2n-1},
\]
(3.23)

where
\[
A_{n,M}(z) := \sum_{j=1}^{2M+2n-1} a_{n,M}(j) z^j.
\]

The eigenvalues of \( W_{n,M} \) comprise \( 1/2^j, j = 0, 1, \ldots, 2M + n - 1 \), and \( \lambda/2^{2M+\lambda} \), where the \( \lambda \)'s are the eigenvalues of the matrix \( (Q_n(2i-j)3_{j=1}^{n-1} \) which is nonsingular by the Schoenberg–Whitney theorem. Therefore, \( W_{n,M} \) is nonsingular and the linear system
\[
\hat{W}_{n,M} \beta_{n,M} = \delta_{n+M}
\]
(3.24)

has a unique \( \beta_{n,M} \in \mathbb{R}^{2n+2M-1} \).

For a positive integer \( k \), let
\[
b_k(j) := \begin{cases} 
\frac{1}{2^j} \binom{k}{j} & j = 0, 1, \ldots, k \\
0 & \text{otherwise}
\end{cases}
\]

**Theorem 3.2.** For any integer \( M \geq 0 \), define
\[
\alpha_{n,M} = b_{2M} * \beta_{n,M},
\]
(3.25)

and
\[
L_{n,M}(x) = \sum_{j=1}^{2n+4M-1} \alpha_{n,M}(j) Q_n(x + n + M - j/2), \quad x \in \mathbb{R}.
\]
(3.26)

Then \( L_{n,M} \) is a compactly supported fundamental spline function. Further,
\[
\alpha_{n,M}(1) = \alpha_{n,M}(2n + 4M - 1) = 0,
\]

and \( \text{supp}(L_{n,M}) \subseteq [-n - M + 1, n + M - 1] \).
Proof. Evaluating (3.26) at an integer \( k \), and using the refinement equation (3.1), gives

\[
L_n,M(k) = \sum_{j=1}^{2n+4M-1} \alpha_n,M(j) \sum_{l=0}^{n} 2b_n(l)Q_n(2k + 2n + 2M - j - l) \\
= \sum_{j=1}^{2n+4M-1} 2\alpha_n,M(j)(b_n,M)(2k + 2n + 2M - j).
\]

The relation (3.7) then gives

\[
L_n,M(k) = \sum_{j=1}^{2n+4M-1} 2\alpha_n,M(j)a_n(2k + 2n + 2M - j), \quad k \in \mathbb{Z}.
\]

By (3.25),

\[
L_n,M(k) = \sum_{j=1}^{2n+4M-1} 2\beta_n,M(j)a_n(2k + 2n + 2M - j) \\
= \sum_{j=1}^{2n+2M-1} 2\beta_n,M(j)(b_n,M)(2k + 2n + 2M - j) \\
= \sum_{j=1}^{2n+2M-1} 2\beta_n,M(j)a_n,M(2k + 2n + 2M - j) \\
= (\bar{W}_n,M\beta_n,M)(k + n + M) \\
= \delta_n,M(k + n + M) = \delta_n(k).
\]

Observe that the first and the last row of the matrix \( \bar{W}_n,M \) are

\[(a_n,M(1), 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, a_n,M(2n + 2M - 1)),\]

respectively. Since \( \bar{W}_n,M \) is invertible, it follows that \( a_n,M(1) \) and \( a_n,M(2n + 2M - 1) \) are nonzero. Hence (3.24) implies that \( \beta_n,M(1) = \beta_n,M(2n + 2M - 1) = 0 \). It follows from (3.25) that \( \alpha_n,M(1) = \alpha_n,M(2n + 4M - 1) = 0 \). Therefore,

\[
L_n,M(x) := \sum_{j=2}^{2n+4M-2} \alpha_n,M(j)Q_n(x + n + M - j/2), \quad x \in \mathbb{R},
\]

and it is clear that \( \text{supp}(L_n,M) \subset [-n - M + 1, n + M - 1] \). \( \square \)

Remark 1. The fundamental spline function \( L_n,0 \), for the case \( M = 0 \), has been constructed by Qi [21]. Qi’s result has been extended by Dahmen et al. [7] to splines with nonuniform knots. A multidimensional analogue of Qi’s results can be found in [25].
Using the fundamental function \( L_{n,M} \), we now define another filter sequence \( c_{n,M} \) by (3.27)

\[
c_{n,M}(2j) := \frac{1}{2} L_{n,M}(j), \quad j = -n - M + 2, \ldots, n + M - 2,
\]

\[
c_{n,M}(2j - 1) := \frac{1}{2} L_{n,M}(j - 1/2), \quad j = -n - M + 2, \ldots, n + M - 1,
\]

and \( c_{n,M}(k) = 0, \ k \neq -2n - 2M + 3, -2n - 2M + 4, \ldots, 2n + 2M - 3 \), and let

\[
C_{n,M}(z) := \sum_{k=-2n-2M+3}^{2n+2M-3} c_{n,M}(k)z^k, \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.28)
\]

**Theorem 3.3.** Let \( \bar{B}_{n,M}(z) \) be the \( \mathbb{Z} \)-transform of \( \beta_{n,M} \). Then

\[
C_{n,M}(z) = z^{-2(n+M)} \left( \frac{1 + z}{2} \right)^{2M+n} \bar{B}_{n,M}(z) \Phi_{n}(z), \quad z \in \mathbb{C} \setminus \{0\}, \quad (3.29)
\]

and

\[
C_{n,M}(z) + C_{n,M}(-z) = 1, \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.30)
\]

**Proof.** By (3.1) and (3.26),

\[
L_{n,M}(x) = \sum_{j=1}^{3n+4M-1} 2(b_n * \alpha_{n,M})(j) Q_n(2x + 2n + 2M - j), \quad x \in \mathbb{R}.
\]

Evaluating the equation at the integers leads to

\[
L_{n,M}(k) = 2(\alpha_{n,M} b_n * Q_n | \mathbb{Z})(2k + 2n + 2M),
\]

and

\[
L(k - 1/2) = 2(\alpha_{n,M} b_n * Q_n | \mathbb{Z})(2k - 1 + 2n + 2M).
\]

It follows from (3.28) that

\[
C_{n,M}(z) = \frac{1}{2} \left( \sum_{k=-n-M+2}^{n+M-2} L(k)z^{2k} + \frac{1}{2} \sum_{k=-n-M+2}^{n+M-1} L(k - 1/2)z^{2k-1} \right)
\]

\[
= \sum_{k=-n-M+2}^{n+M-2} (\alpha_{n,M} b_n * Q_n | \mathbb{Z})(2k + 2n + 2M)z^{2k} + \sum_{k=-n-M+2}^{n+M-1} (\alpha_{n,M} b_n * Q_n | \mathbb{Z})(2k - 1 + 2n + 2M)z^{2k-1}
\]
\[
\sum_{j=-2n-2M+3}^{2n+2M-3} (\alpha_{n,M} \ast b_n \ast Q_n | z) (j + 2n + 2M) z^j
\]

\[
= z^{-2M} \sum_{k=3}^{4n+4M-3} (\beta_{n,M} \ast b_{2M} \ast b_n \ast Q_n | z) (k) z^k
\]

\[
= z^{-2n-M} \left( \frac{1 + z^2}{2} \right)^{n+2M} B_{n,M}(z) \Phi_n(z).
\]

By (3.27),

\[
C_{n,M}(z) + C_{n,M}(-z) = \sum_j L(j) z^j, \quad z \in \mathbb{C} \setminus \{0\},
\]

which gives (3.30), since \( L \) is fundamental. ■

The relation (3.30) is equivalent to

\[
\left( \frac{1 + z^2}{2} \right)^{2M+n} \Phi_n(z) \tilde{B}_{n,M}(z) + \left( \frac{1 - z^2}{2} \right)^{2M+n} \Phi_n(-z) \tilde{B}_{n,M}(-z) = z^{2(M+n)}. \quad (3.31)
\]

**Lemma 3.3.** Let \( \tilde{B}_{n,M}(z) \) be the unique solution of (3.31). Then

\[
|\tilde{B}_{n,M}(e^{-iu})| > 0, \quad \text{for all } -\pi < u < \pi. \quad (3.32)
\]

**Proof.** We shall first consider the case where \( n = 2m \) is even. Then (3.31) can be written as

\[
z^{-(M+n)} \left( \frac{1 + z^2}{2} \right)^{2(M+n)} (z^{-m} \Phi_{2m}(z)) (z^{-(M+2m)} \tilde{B}_{2m,M}(z))
\]

\[
+ z^{-(M+n)} \left( \frac{1 - z^2}{2} \right)^{2(M+n)} (z^{-m} \Phi_{2m}(-z)) (z^{-(M+2m)} \tilde{B}_{2m,M}(-z)) = 1. \quad (3.33)
\]

Since \( (Q_{2m}(j))_{j=1}^{2m-1} \) and \( (\beta_{2m,M}(j))_{j=2}^{2M+2m-2} \) are symmetric, by restricting \( z \) to the unit circle, i.e., \( z = e^{-iu} \), standard trigonometric identities transform (3.33) into

\[
(1 - y)^{M+n} \tilde{\Phi}_{M+2m-2}(y) + y^{M+n} \tilde{\Phi}_{M+2m-2}(1 - y) = 1, \quad (3.34)
\]

where

\[
\tilde{\Phi}_{M+2m-2}(y) := e^{i(M+2m)u} \tilde{B}_{2m,M}(e^{-iu}),
\]

and

\[
y := \sin^2 u/2, \quad 0 \leq u \leq \pi.
\]
Since \((1 + z)^{-1} 2M + n \Phi_n(z)\) has only negative zeros, by Lemmas 2.3 and 2.4 of Goodman and Micchelli [12],

\[
|\mathcal{B}_{M+2m-2}(y)| > 0, \quad \text{for all } 0 < y < 1.
\]

Equivalently,

\[
|\mathcal{B}_{2mM}(e^{-iu})| > 0, \quad \text{for all } 0 < u < \pi. \tag{3.35}
\]

Further, \(\mathcal{B}_{2mM}(1) = 1\). Hence (3.35) holds for \(0 \leq u < \pi\), and since \((\beta_{2mM}(j))_{j=2}^{2M+4m-2}\) is symmetric, it holds for all \(-\pi < u < \pi\).

If \(n = 2m + 1\) is odd, \(\Phi_{2m+1}\) has a simple zero at \(z = -1\). Therefore \(\Phi_{2m+1}(z) = ((1 + z)/2) \Psi_{2m}(z)\), where \(\Psi_{2m}(-1) \neq 0\), and (3.31) can be written as

\[
z^{-(M+m+1)} \left(\frac{1 + z}{2}\right)^{2(M+m+1)} (\psi_m^+(z)) (z^{-(M+2m+1)} \mathcal{B}_{2m+1M}(z))
\]

\[
+ z^{-(M+m+1)} \left(\frac{1 - z}{2}\right)^{2(M+m+1)} (\psi_m^-(z)) (z^{-(M+2m+1)} \mathcal{B}_{2m+1M}(-z)) = 1.
\]

Similarly, this equation transforms to

\[(1 - y)^{M+m+1} \tilde{\psi}_{m-1}(y) \mathcal{B}_{M+2m-1}^-(y)
\]

\[+ y^{M+m+1} \tilde{\psi}_{m-1}(1 - y) \mathcal{B}_{M+2m-1}^+(1 - y) = 1, \tag{3.36}\]

where

\[
\tilde{\psi}_{m-1}(y) := e^{im \psi_{2m}(e^{-iu})},
\]

\[
\mathcal{B}_{M+2m-1}^-(y) := e^{i(M+2m+1)u} \mathcal{B}_{2m+1M}(e^{-iu}),
\]

and

\[
y := \sin^2 u/2, \quad 0 \leq u \leq \pi.
\]

The result follows by the same argument as before. \(\blacksquare\)

**Theorem 3.4.** Suppose that \(f\) is a continuous function satisfying

\[
f(x) = \sum_{j=-2n-2M+3}^{2n+2M-3} 2c_{nM}(j) f(2x - j), \quad x \in \mathbb{R}, \tag{3.37}
\]

where

\[
c_{nM}(z) = \sum_{j=-2n-2M+3}^{2n+2M-3} c_{nM}(j) z^j, \quad z \in \mathbb{C}.
\]
Then $f$ is interpolatory.

Proof. Lemma 3.3 and Eq. (3.29) imply that $|C_{n,M}(e^{-iu})| > 0$ for $-\pi < u < \pi$. It follows from Proposition 2.1 that $\hat{f}(u) \neq 0$ for $u \in [0, 2\pi)$. By the Poisson summation formula,

$$\sum_{j \in \mathbb{Z}} \langle f, f(\cdot - j) \rangle e^{-iju} = \sum_{j \in \mathbb{Z}} |\hat{f}(u + 2\pi j)|^2 > 0, \quad u \in [0, 2\pi).$$

Hence $f$ is stable. Therefore, the cascade algorithm

$$f_k(x) = \sum_{j=-2n-2M+3}^{2n+2M-3} 2C_{n,M}(j)f_{k-1}(2x - j), \quad k = 1, 2, \ldots,$$

with starting vector

$$f_0(x) = \begin{cases} x & 0 \leq x < 1 \\ 1 - x & 1 \leq x \leq 2, \end{cases}$$

converges uniformly to $f$ (see [2, 14]). It follows by induction that $f_n|_{x} = \delta$ for all $n = 0, 1, \ldots$. Hence $f$ is interpolatory. 

Equation (3.29) provides a natural factorization of $C_{n,M}(z)$ in three ways.

3.1. Case 1: CQF and Orthonormal Wavelets

Since $\mathbb{B}_{n,M}(e^{-iu})\Phi_n(e^{-iu}) \neq 0$ for $-\pi < u < \pi$, we may assume without loss of generality that

$$C_{n,M}(e^{-iu}) > 0, \quad \text{for all } -\pi < u < \pi.$$

Therefore, by the Fejer–Riesz theorem, $C_{n,M}(e^{-iu})$ can be factored as

$$C_{n,M}(e^{-iu}) = |D_{2M+2n-3}(e^{-iu})|^2,$$

where

$$D_{2M+2n-3}(z) = \sum_{j=0}^{2M+2n-3} d_{n,M}(j)z^j, \quad z \in \mathbb{C} \setminus \{0\},$$

is a polynomial of degree $2M + 2n - 3$. The sequence $d_{n,M}$ is a CQF. The following result is a corollary of Theorem 3.4.

Theorem 3.5. Suppose that $f$ is a continuous function satisfying (3.37). Then the refinable function $\varphi_{n,M}$ with the filter sequence $d_{n,M}$ is orthonormal.

We now consider an example which shows the relation between the filter sequences for Daubechies’ compactly supported orthonormal wavelets and the fundamental function constructed above.
Choose the filter sequence $h_3$ with $Z$-transform

$$H_3(z) = \left(\frac{1 + z}{2}\right)^3. \quad (3.40)$$

The corresponding refinable function $\phi = Q_3$ is the quadratic $B$-spline

$$\phi(x) := \begin{cases} 
\frac{1}{2} x^2 & 0 \leq x < 1 \\
\frac{1}{2} (-2x^2 + 6x - 3) & 1 \leq x < 2 \\
\frac{1}{2} (3 - x)^2 & 2 \leq x \leq 3,
\end{cases}$$

so that

$$\Phi_3(z) = z\left(\frac{1 + z}{2}\right), \quad z \in \mathbb{C} \setminus \{0\}.$$

By (2.5), the sequence $a_3$ is

$$a_3(1) = \frac{1}{8}, \ a_3(2) = \frac{4}{8}, \ a_3(3) = \frac{6}{8}, \ a_3(4) = \frac{4}{8}, \ a_3(5) = \frac{1}{8},$$

and its $Z$-transform

$$A_3(z) = z\left(\frac{1 + z}{2}\right)^4.$$

For any nonnegative integer $M$,

$$A_{3,M}(z) = \left(\frac{1 + z}{2}\right)^{2M} A(z) = z\left(\frac{1 + z}{2}\right)^{2M+4}. \quad (3.41)$$

The filter sequence $C_{3,M}(z)$ in (3.29) becomes

$$C_{3,M}(z) = z^{-2M-5} \left(\frac{1 + z}{2}\right)^{2M+4} \hat{B}_{3,M}(z), \quad (3.42)$$

where $\hat{B}_{3,M}(z)$ is the unique solution of (3.24). Equivalently,

$$\left(\frac{1 + z}{2}\right)^{2M+4} B_{3,M}(z) - \left(\frac{1 - z}{2}\right)^{2M+4} B_{3,M}(-z) = z^{2M+5}. \quad (3.43)$$
Since the sequence \((\beta_{M,j}(j))_{j=-2}^{2M+2}\) is symmetric, restricting \(z\) to the unit circle, Eq. (3.43) becomes

\[
(1 - y)^{M+2} \tilde{\Phi}_{M+1}(y) + y^{M+2} \tilde{\Phi}_{M+1}(1 - y) = 1,
\]

where \(\tilde{\Phi}_{M+1}(y)\) is a polynomial in \(y := \sin^2 u/2\) of degree \(M + 1\) with real coefficients. Equation (3.44), which is (3.36) with \(m = 1\), is the familiar Bezout equation obtained by Daubechies [9]. It has a unique solution in the class of polynomials of degree \(\leq M + 1\), and the solution is given by

\[
\tilde{\Phi}_{M}(y) = \sum_{j=0}^{M+1} \binom{M + j + 1}{j} y^j.
\]

This example shows the relationship between the filter sequences for Daubechies’ compactly supported orthonormal wavelets and the interpolatory function constructed in Section 2.

In general, the filter sequence for a compactly supported orthonormal refinable function is obtained from the filter sequence of the corresponding interpolatory function by square root factorization using the Fejer–Riesz theorem. This requires that \(C(e^{-iu})\) be nonnegative on the unit circle, which is a serious constraint. Further, for multiwavelet construction this factorization is complicated, and for multidimensional wavelet construction this factorization is not available. The problem is much simpler if we do not insist on orthonormality, but require only that the integer shifts of a pair of refinable functions form biorthogonal bases. The next two cases are devoted to the applications of the above results in the construction of refinable functions which generate biorthogonal wavelet bases.

### 3.2. Case 2: Dual Filter Sequences with One B-spline

We have

\[
C_{n,M}(z) = \left(\frac{1 + z}{2}\right)^k B_{n,M,k}(z^{-1}), \quad z \in \mathbb{C} \setminus \{0\},
\]

where

\[
B_{n,M,k}(z^{-1}) := z^{-2(n+M)} \left(\frac{1 + z}{2}\right)^{2M+n-k} \tilde{\Phi}_{n,M}(z) \Phi(z), \quad z \in \mathbb{C} \setminus \{0\},
\]

for some \(1 \leq k < 2M + n\).

This factorization gives a pair of dual filter sequences leading to two refinable functions with biorthogonal shifts, one of which is the forward B-spline \(Q_k\) of degree
Let $\hat{Q}_{n,M,k}$ be the other refinable function corresponding to $B_{n,M,k}(z)$; i.e., $\hat{Q}_{n,M,k}$ is the solution of the refinement equation

$$\hat{Q}_{n,M,k}(x) = \sum_j \hat{b}_{n,M,k}(j) \hat{Q}_{n,M,k}(2x - j),$$

where

$$B_{n,M,k}(z) = \sum_j b_{n,M,k}(j) z^j, \quad z \in C \setminus \{0\}.$$ 

Following Strang [28], a matrix is said to satisfy Condition \(E\) if 1 is a simple eigenvalue and all its other eigenvalues lie inside the unit circle. Let $\phi$ be a refinable function satisfying (1.2), and let $\hat{W}_{r_h}$ be the restricted transition operator corresponding to the autocorrelation $r_h$ of $h$. It is known that if $\phi$ is stable, i.e., its integer shifts form a Riesz basis of its closed linear span, then $\hat{W}_{r_h}$ satisfies Condition \(E\).

**Theorem 3.6.** Suppose that $C_{n,M}(z)$ is factored as in (3.46), and $\hat{Q}_{n,M,k}$ is a refinable function in $L^2(\mathbb{R})$ with filter sequence $\hat{b}_{n,M,k}$. Then $\hat{Q}_k$ and $\hat{Q}_{n,M,k}$ generate a pair of biorthogonal Riesz bases of compactly supported scaling functions.

**Proof.** Since the restricted transition operator for the autocorrelation of the filter sequence for $\hat{Q}_k$ satisfies Condition \(E\), we need only show that the transition operator $\hat{W}_{r_h}$ for the autocorrelation of the filter sequence $\hat{b}_{n,M,k}$ satisfies Condition \(E\). By Lemma 3.3, \( |B_{n,M,k}(e^{-i\omega})| > 0 \) for $-\pi < \omega < \pi$. By the same argument as in the proof of Theorem 3.4,

$$\sum_{j \in \mathbb{Z}} \langle \hat{Q}_{n,M,k}, \hat{Q}_{n,M,k}(\cdot - j) \rangle e^{-iu} = \sum_{j \in \mathbb{Z}} \langle \hat{Q}_{n,M,k}(u + 2\pi j) \rangle^2 > 0, \quad u \in [0, 2\pi).$$

Hence $\hat{Q}_{n,M,k}$ is stable. It follows that $\hat{W}_{r_h}$ satisfies Condition \(E\).

### 3.3. Case 3: Dual Filter Sequences with No B-splines

We have

$$C_{n,M}(z) = P_{n,k}(z) P_{n,M,k}(z^{-1}), \quad z \in C \setminus \{0\}, \quad \text{(3.48)}$$

where

$$P_{n,k}(z) := z^{-1} \left( \frac{1 + z}{2} \right)^k \Phi_n(z), \quad z \in C \setminus \{0\}, \quad \text{(3.49)}$$

$$\hat{P}_{n,M,k}(z^{-1}) := z^{-2(n+M+1)} \left( \frac{1 + z}{2} \right)^{2M+n-k} \hat{B}_{n,M}(z), \quad z \in C \setminus \{0\}, \quad \text{(3.50)}$$

for some $0 \leq k < 2M + n$. Let

$$P_{n,k}(z) = \sum_j p_{n,k}(j) z^j, \quad \text{(3.51)}$$
and
\[ \hat{p}_{n,M,k}(z) = \sum_j p_{n,M,k}(j)z^j, \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.52) \]

**Theorem 3.7.** If \( \phi_{n,k} \) and \( \hat{\phi}_{n,M,k} \) are refinable functions in \( L^2(\mathbb{R}) \) with filter sequences \( p_{n,k} \) and \( \hat{p}_{n,M,k} \), respectively, then \( \phi_{n,k} \) and \( \hat{\phi}_{n,M,k} \) generate a pair of biorthogonal Riesz bases of compactly supported scaling functions.

**Proof.** We need only show that the transition operators \( W_{r_p} \) and \( W_{r_{\hat{p}}} \) corresponding to the autocorrelations of \( p_{n,k} \) and \( \hat{p}_{n,M,k} \), respectively, satisfy Condition E. The proof is the same as that of Theorem 3.6. We shall omit the details. \( \blacksquare \)

**Remark 2.** A standard construction (see [18]) gives the corresponding wavelets.

### 4. Computation of Filter Sequences

For the purpose of computation, we summarize the methods and results developed in Section 3 in the form of algorithms. These algorithms lead to the solution of a cardinal interpolation problem and the computation of filter sequences and the corresponding scaling functions introduced in Subsections 3.2 and 3.3. These filter sequences are useful in discrete wavelet transforms.

For both Case 2 and Case 3, some common results are required for the computation of the corresponding filter sequences. They are the values of the forward B-splines \( Q_n \) at the integers, the solution \( \beta_{n,M} \) of the linear system (3.24), and the sequence \( \gamma_{n,M} \) which is the convolution of \( Q_n \) and \( \beta_{n,M} \). These three sequences can be computed by Algorithms 4.1, 4.2, and 4.3 below.

**Algorithm 4.1 (Computation of \( Q_n(j), j = 1, 2, \ldots, n - 1 \)).**

*Input: \( n \geq 1, j \in \mathbb{Z} \).*  
*Step 1: If \( n = 1 \) then*  
\[ Q_n(j) := \begin{cases} 1, & j = 0 \\ 0, & \text{otherwise.} \end{cases} \]  
*Else*  
\[ Q_n(j) := \begin{cases} (jQ_{n-1}(j) + (n - j)Q_{n-1}(j - 1))/(n - 1), & 1 \leq j \leq n - 1 \\ 0, & \text{otherwise.} \end{cases} \]  
*Output: \( Q_n(j) \).*

**Algorithm 4.2 (Computation of the \( \beta_{n,M} \)).**

*Input: \( n \geq 3, M \geq 0, Q_n(j), j = 1, \ldots, n - 1 \).*  
*Step 1: Compute the vector \( a_{n,M} \)*  
\[ a_{n,M}(l) := \sum_{j=0}^{2M+n} \frac{1}{2^{2M+n}} \binom{2M+n}{j} Q_n(l-j), \quad l = 1, \ldots, 2M + 2n - 1. \]
Step 2: Form the matrix $\tilde{W}_{n,M}$

$$\tilde{W}_{n,M} := 2(a_{n,M}(2l - j))_{i,j=1}^{2M+2n-1}. $$

Step 3: Solve for $\beta_{n,M}$ in the linear system

$$\tilde{W}_{n,M}\beta_{n,M} = \delta_{n+M}$$

where $\delta_{n+M}$ is the $(n + M)$-th unit vector.

Output: $\beta_{n,M}(j), j = 1, 2, \ldots, 2M + 2m - 1.$

Algorithm 4.3 (Computation of $\gamma_{n,M}$).

Input: $Q_n$, $\beta_{n,M}$.

Step 1: Compute the convolution

$$\gamma_{n,M}(j) := \sum_{l=1}^{n} Q_n(l)\beta_{n,M}(j - l), \quad j = 3, 4, \ldots, 3n + 2M - 3.$$ 

Output: $\gamma_{n,M}(j), j = 3, 4, \ldots, 3n + 2M - 3.$

We now consider the computation of filter sequences for Case 2. From Section 3, the scaling function $\phi$ is the forward $B$-spline $Q$. From (3.1), we have

$$Q_n(x) = 2 \sum_{j=0}^{k} b_k(j)Q_n(2x - j),$$

where

$$b_k(j) = \frac{1}{2^k} \binom{k}{j}, \quad j = 0, 1, \ldots, k.$$ 

The dual scaling function $\hat{\phi}$ is $\hat{Q}_{n,k}$, which satisfies the refinement equation

$$\hat{Q}_{n,k}(x) = 2 \sum_{j=-2(n+M)+3+k}^{2(n+M)-3} \hat{b}_{n,k}(j)\hat{Q}_{n,k}(2x - j),$$

where

$$\hat{b}_{n,k}(j) := \sum_{l=0}^{2M+n-k} \frac{1}{2^{2M+n-k}} \binom{2M+n-k}{l} \gamma_{n,M}(2(n + M) - j - l),$$

$$j = -2(n + M) + 3 + k, \ldots, 2(n + M) - 3. \quad (4.1)$$

Now we consider the computation of filter sequences for Case 3. Here the scaling function $\phi_{n,k}$ satisfies

$$\phi_{n,k}(x) = 2 \sum_{j=0}^{n+k-2} p_{n,k}(j)\phi_{n,k}(2x - j),$$

where

$$p_{n,k}(j) := \sum_{l=0}^{k} \frac{1}{2^k} \binom{k}{l} Q_n(j + 1 - l), \quad j = 0, 1, \ldots, n + k - 2. \quad (4.2)$$
TABLE 1
Case 2: Ranges of Values of $k$ for the $(n, M)$ Tuples Shown for Which $\tilde{b}_{n,M}$ Satisfy Condition E

<table>
<thead>
<tr>
<th>$M$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
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<tbody>
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<td>1-3</td>
<td>1-3</td>
<td>1-4</td>
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<td>1-5</td>
<td>1-5</td>
<td>1-6</td>
<td>1-6</td>
<td>1-7</td>
</tr>
<tr>
<td>5</td>
<td>1-3</td>
<td>1-3</td>
<td>1-4</td>
<td>1-4</td>
<td>1-5</td>
<td>1-5</td>
<td>1-6</td>
<td>1-6</td>
<td>1-7</td>
<td>1-7</td>
</tr>
<tr>
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<td>1-4</td>
<td>1-4</td>
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<tr>
<td>9</td>
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<td>1-5</td>
<td>1-6</td>
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<td>1-7</td>
<td>1-7</td>
<td>1-8</td>
<td>1-8</td>
</tr>
</tbody>
</table>

The dual scaling function $\tilde{\phi}_{n,M,k}$ satisfies

$$\tilde{\phi}_{n,M,k}(x) = 2 \sum_{j = -2M + n + k + 1}^{2(n+M)-3} \tilde{p}_{n,M,k}(j) \tilde{\phi}_{n,M,k}(2x - j),$$

where

$$p_{n,M,k}(j) := \sum_{l=0}^{2M+n-k} \frac{1}{2M+n-k} \binom{2M+n-k}{l} \beta_{n,M}(2(n+M) - j - l - 1),$$

$$j = -2M - n + k + 1, \ldots, 2(n+M) - 3. \quad (4.3)$$

Note that for both Case 2 and Case 3, the lengths of the two filter sequences add to $4(n + M - 1)$, which is independent of $k$.

For each $(n, M)$ tuple, where $n \geq 3$ and $M \geq 0$, the filter sequences for both Case 2 and Case 3 satisfy Condition E only for a certain range of values of $k$. These ranges of values of $k$ for $3 \leq n \leq 9$ and $0 \leq M \leq 9$ are shown in Tables 1 and 2.

TABLE 2
Case 3: Ranges of Values of $k$ for the $(n, M)$ Tuples Shown for Which $p_{n,k}$ and $\tilde{p}_{n,M,k}$ Both Satisfy Condition E

<table>
<thead>
<tr>
<th>$M$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
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<td>1-2</td>
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<td>0-0</td>
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</tbody>
</table>
Remark 3. If \( n = 3 \), the biorthogonal Riesz bases of compactly supported wavelets of Case 2 are the biorthogonal wavelets bases of Cohen, Daubechies, and Feauveau (see [6, 8]).

Remark 4. If \( n = 3 \), the biorthogonal Riesz bases of compactly supported wavelets of Case 3 reduce to those of Case 2. Specifically, when \( n = 3 \), the wavelets generated by the triplet \((n, M, k)\) of Case 3 are the same as the wavelets generated by \((n, M, k + 1)\) of Case 2.

Remark 5. Although the results in Section 4 are established for \( n \geq 3 \), it is easy to show that the results for \( n = 2 \) are the same as for \( n = 3 \) for both Case 2 and Case 3.

5. NUMERICAL RESULTS

In this section, we present results of numerical experiments with the filter sequences for Case 2 and Case 3 in Section 4. The results shown are compared with those obtained by Daubechies’ compactly supported wavelet coefficients, \( D_4 \), and Chui and Wang’s semi-orthogonal spline wavelets of order 4.

The numerical experiment considered in this section involves multiplying an \( N \times N \) matrix \( A \) with an \( N \)-vector \( v \). Both the matrix and the vector are first transformed by discrete wavelet transform to \( \tilde{A} \) and \( \tilde{v} \), respectively. The matrix \( \tilde{A} \) is then compressed by discarding entries below a certain threshold \( \epsilon \), giving \( \tilde{A}^* \). The compressed matrix \( \tilde{A}^* \) is then multiplied with \( \tilde{v} \) to give \( \tilde{u} \). Discrete wavelet transform is then applied to \( \tilde{u} \) to reconstruct \( u \), which is an approximation to the product \( Au \). (See [19] for details of forming \( \tilde{A} \) and \( \tilde{v} \), the discrete wavelet transform used, and a class of matrices which can be compressed by wavelet transforms.)

Two matrices are considered in the experiments.

**Example 1.**

\[
A_{i,j} = \begin{cases} 
1/|i-j| & i \neq j \\
0 & i = j.
\end{cases}
\]

**Example 2.**

\[
A_{i,j} = \begin{cases} 
|i-j|\log |i-j| & i \neq j \\
0 & i = j.
\end{cases}
\]

The \( N \)-vector \( v \) used in the numerical experiments is given by \( v_j = j/N, j = 1, 2, \ldots, N \).

The numerical results are presented only for the filters \((3, 1, 2), (3, 2, 2), (4, 1, 1), \) and \((5, 1, 1)\) for Case 2, and \((4, 1, 1)\) and \((5, 1, 1)\) for Case 3. These filters have relatively short length (for time efficiency) and comparatively large number of vanishing moments (for good data compression). Tables 3 and 4 give numerical results on these cases. Three values of \( N \) are considered in the examples. For each value of \( N \), the compression ratio (defined to be the number of nonzero entries divided by \( N^2 \)) is fixed, and the corresponding value of \( \epsilon \) required for the compression is...
TABLE 3
Results for Example 1

| Method | $N$  | $t_f$ | $t_r$ | $t_m$ | $||Av - u||_2/N$ | $C_c$ | $\epsilon$ |
|--------|------|-------|-------|-------|------------------|------|----------|
| DW     | 256  | 6.46  | 0.02  | 0.03  | 8.7968x10^{-8}   | 30   | 8.705x10^{-7}|
|        | 512  | 24.91 | 0.06  | 0.06  | 1.2892x10^{-8}   | 20   | 1.016x10^{-7}|
|        | 1024 | 98.98 | 0.11  | 0.11  | 4.1887x10^{-8}   | 10   | 5.389x10^{-7}|
| CW     | 256  | 6.22  | 0.50  | 0.02  | 5.9428x10^{-8}   | 30   | 1.739x10^{-10}|
|        | 512  | 25.11 | 0.90  | 0.05  | 5.3630x10^{-8}   | 20   | 6.633x10^{-13}|
|        | 1024 | 98.81 | 1.66  | 0.11  | 4.7236x10^{-8}   | 10   | 1.325x10^{-11}|
| Case 2 (3,1,2) | 256 | 4.79  | 0.02  | 0.02  | 1.1156x10^{-7}   | 30   | 1.502x10^{-4} |
|        | 512  | 19.62 | 0.04  | 0.05  | 1.6076x10^{-8}   | 20   | 4.577x10^{-5}  |
|        | 1024 | 75.52 | 0.08  | 0.12  | 1.9030x10^{-8}   | 10   | 8.755x10^{-5}  |
| Case 2 (3,2,2) | 256 | 6.21  | 0.03  | 0.02  | 5.5964x10^{-8}   | 30   | 1.199x10^{-5}  |
|        | 512  | 24.92 | 0.06  | 0.05  | 2.3586x10^{-9}   | 20   | 1.505x10^{-6}  |
|        | 1024 | 98.46 | 0.10  | 0.12  | 3.6505x10^{-9}   | 10   | 3.229x10^{-6}  |
| Case 2 (4,1,1) | 256 | 6.31  | 0.02  | 0.02  | 1.9779x10^{-8}   | 30   | 2.055x10^{-7}  |
|        | 512  | 24.75 | 0.05  | 0.06  | 7.3092x10^{-10}  | 20   | 8.343x10^{-9}  |
|        | 1024 | 98.47 | 0.10  | 0.11  | 1.4096x10^{-9}   | 10   | 2.191x10^{-8}  |
| Case 2 (5,1,1) | 256 | 7.87  | 0.03  | 0.03  | 9.9425x10^{-8}   | 30   | 1.313x10^{-6}  |
|        | 512  | 30.84 | 0.06  | 0.06  | 8.2974x10^{-10}  | 20   | 1.327x10^{-8}  |
|        | 1024 | 121.99 | 0.13  | 0.11  | 2.7791x10^{-9}   | 10   | 5.625x10^{-8}  |
| Case 3 (4,1,1) | 256 | 6.50  | 0.03  | 0.02  | 3.1624x10^{-7}   | 30   | 3.427x10^{-6}  |
|        | 512  | 25.28 | 0.05  | 0.05  | 4.0415x10^{-9}   | 20   | 3.897x10^{-7}  |
|        | 1024 | 100.20 | 0.10  | 0.11  | 3.2990x10^{-7}   | 10   | 2.968x10^{-6}  |
| Case 3 (5,1,1) | 256 | 8.26  | 0.03  | 0.02  | 7.9563x10^{-7}   | 30   | 6.089x10^{-6}  |
|        | 512  | 31.56 | 0.06  | 0.06  | 4.3516x10^{-8}   | 20   | 4.576x10^{-7}  |
|        | 1024 | 124.02 | 0.13  | 0.11  | 9.8519x10^{-9}   | 10   | 1.029x10^{-6}  |

obtained for each filter. This allows us to compare the wavelet methods in terms of the accuracy of the compressed matrix products at equal compression ratios.

In Tables 3 and 4, DW denotes Daubechies’ wavelet of order 4, and CW denotes Chui and Wang’s spline wavelet of order 4. Column 2 contains the values of $N$, the order of the matrix $A$. Column 3 gives the CPU time, $t_f$, taken for performing wavelet transforms in obtaining the compressed matrix. Column 4 gives the CPU time, $t_r$, taken for decomposing the vector $v$ and reconstructing from the vector $u$. Column 5 gives the CPU time, $t_m$, taken to multiply the compressed matrix and the vector $v$. All CPU times are given in seconds. Column 6 gives the two-norm of the error vector $u - Av$ divided by $N$. Column 7 gives the compression ratio $C_c$ and Column 8 gives the corresponding $\epsilon$ required to achieve the compression $C_c$.

From both tables, one can see that selected filters for Case 2 perform generally better than both DW and CW methods in both accuracy and time efficiency. On the other hand, the selected filters for Case 3 only perform better than the DW and CW methods in Example 2, and worse than those filters for Case 2 in both examples.
TABLE 4
Results for Example 2

<table>
<thead>
<tr>
<th>Method</th>
<th>$N$</th>
<th>$t_f$</th>
<th>$t_e$</th>
<th>$t_m$</th>
<th>$|Av-u|_2/N$</th>
<th>$C_e$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DW</td>
<td>256</td>
<td>6.34</td>
<td>0.02</td>
<td>0.02</td>
<td>5.2169×10^{-6}</td>
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<td>5.609×10^{-5}</td>
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<tr>
<td></td>
<td>512</td>
<td>25.19</td>
<td>0.05</td>
<td>0.06</td>
<td>2.0335×10^{-3}</td>
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<td>1.703×10^{-5}</td>
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<tr>
<td></td>
<td>1024</td>
<td>98.84</td>
<td>0.10</td>
<td>0.11</td>
<td>7.6883×10^{-3}</td>
<td>10</td>
<td>4.848×10^{-5}</td>
</tr>
<tr>
<td>CW</td>
<td>256</td>
<td>6.24</td>
<td>0.51</td>
<td>0.02</td>
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<td>30</td>
<td>2.509×10^{-5}</td>
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<tr>
<td></td>
<td>512</td>
<td>24.64</td>
<td>0.89</td>
<td>0.05</td>
<td>1.2132×10^{-2}</td>
<td>20</td>
<td>1.595×10^{-6}</td>
</tr>
<tr>
<td></td>
<td>1024</td>
<td>98.46</td>
<td>1.65</td>
<td>0.11</td>
<td>4.8472×10^{-2}</td>
<td>10</td>
<td>3.229×10^{-6}</td>
</tr>
<tr>
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<td>4.86</td>
<td>0.02</td>
<td>0.02</td>
<td>2.4810×10^{-3}</td>
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<tr>
<td>(3,1,2)</td>
<td>512</td>
<td>18.75</td>
<td>0.04</td>
<td>0.05</td>
<td>7.0524×10^{-6}</td>
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<tr>
<td></td>
<td>1024</td>
<td>75.16</td>
<td>0.08</td>
<td>0.11</td>
<td>1.6879×10^{-5}</td>
<td>10</td>
<td>8.755×10^{-5}</td>
</tr>
<tr>
<td>Case 2</td>
<td>256</td>
<td>6.23</td>
<td>0.03</td>
<td>0.02</td>
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<td>512</td>
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<td>0.05</td>
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<tr>
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<td>0.11</td>
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<td>Case 3</td>
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<td>0.02</td>
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<td>7.6375×10^{-6}</td>
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REFERENCES


21. D. X. Qi, A class of local explicit many knot spline interpolation scheme, University of Wisconsin, MRC TSR #2238.


