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Subspaces of $GF(q)^\omega$ and Convolutional Codes

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The present paper is a self-contained treatment of subspaces of the space $GF(q)^\omega$ of all semi-infinite strings over $GF(q)$. Some necessary and sufficient conditions which characterize those subspaces of $GF(q)^\omega$ are derived which are convolutional codes, and the classes of subspaces defined by one or more of them are investigated. Moreover structural parameters of convolutional codes such as block length, rate, delay, and constraint length are considered as parameters of subspaces rather than parameters of an encoding device. As a conclusion it is obtained that for error-control purposes none of the investigated superclasses of the class of convolutional codes is better suited than the class of convolutional codes itself.

1. INTRODUCTION

Structural properties of convolutional codes have been investigated in different directions, on the one hand in an algebraic manner using the description of convolutional encoders by the D-transform matrix (cf. Massey, 1963; Forney, 1970) or on the other hand using the tree or trellis structure of these codes (cf. Viterbi, 1971; Forney, 1974). In general, it has been common to regard convolutional codes as spaces of semi-infinite vectors (i.e., subsets of $GF(q)^\omega$) (cf. Blahut, 1983). But then, in contrast to the case of block codes, where the spaces the codes are taken from are finite, in the case of the space of semi-infinite vectors there is a great variety of subsets, and it is difficult to select among this variety those subspaces of $GF(q)^\omega$ useful for coding theory.

To this end one wishes to introduce into subspaces an internal structure. This can be done from an algebraic point of view (cf. Piret, 1978), from a topological point of view (cf. Trachtenbrot and Barzdin, 1973, and Lindner and Staiger, 1977) and from the point of view of the theory of formal languages, where several classes of ω-languages (sets of semi-infinite strings) have been investigated, as regular (i.e., definable by finite automata) (cf. Trachtenbrot and Barzdin, 1973; Lindner and Staiger, 1977; Wagner, 1979), context-free (Linna, 1976), and recursive ω-languages (Wagner and Staiger, 1977).

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This latter point of view seems not to be very appropriate, since classes of ω-languages are usually defined by accepting devices rather than encoding circuits. Taking into account, however, the a priori knowledge that all sets of semi-infinite strings encodable by a sequential circuit are closed in the natural topology of $GF(q)^\omega$ (cf. Lindner and Staiger, 1977), we propose another approach to put convolutional codes in the context of language and automata theory.

We introduce the notion of a state of a subset of $GF(q)^\omega$ derived by a finite string over $GF(q)$ and investigate the interconnections between the structure of a subset of $GF(q)^\omega$ and the structure of the set of all its states (cf. also Staiger, 1983b). This seems not only to provide a deeper insight into the structure of linear codes, but also a tool for structurizing nonlinear codes as, for instance, nonlinear tree and trellis codes, and sliding block codes. A first attempt in this direction has been made by this author (Staiger, 1979).

We present here the state approach together with topological considerations in order to get an insight into the defining properties of convolutional codes. In particular, we are interested which superclasses of convolutional codes are defined by these properties taken each one alone. In order not to stress the matter too much, we have confined ourselves to the consideration of linear codes.

In the second section we introduce the necessary notation and some topological and algebraic apparatus. Then, in the third section, we analyze convolutional codes. Here we derive properties of convolutional codes which will be recognized in Section 6 as their defining properties. The fourth chapter deals with general properties of subspaces. Here the apparatus of states and a comparison method of sets of semi-infinite sequences based on finite part comparisons are introduced. Moreover, we obtain a first classification of subspaces according to the behavior (periodicity) of their family of zerostates. We show that the class of subspaces having a periodic family of zerostates is closely related to a class of subspaces defined by one of the properties of convolutional codes derived in the preceding section. This latter class of Σ-spaces is thoroughly investigated in Section 5. For this class such parameters of convolutional codes as clock length, rate, and delay are considered.

Then, in the sixth part we introduce two further properties of subspaces. It turns out that either of them defines convolutional codes among Σ-spaces. Further we deal with the relations of convolutional codes to the class of finite-state subspaces, this latter class being defined by one of the newly introduced properties. We conclude this part by investigating two parameters of subspaces which are related to the constraint length of codes. For the sake of completeness, Section 7 gives some results on remergable subspaces, the subspaces being defined by the second property introduced in the preceding section. Finally, in the eighth section we summarize the connections between

the properties (classes) of subspaces obtained in the earlier sections and give some independence results.

2. Preliminaries

Throughout this paper let Y be a finite Abelian group $(Y, +, 0)$ with zero element 0. The elements of Y will be also regarded as letters, and a usual $Y^*(Y^\omega)$ denotes the set of finite words (infinite sequences) on the alphabet Y. If $w \in Y^* \cup Y^\omega$, then $w \cdot b$ is the concatenation of w and b. This in an obvious way defines a product $W \cdot B$ of sets $W \subseteq Y^*$ and $B \subseteq Y^* \cup Y^\omega$.

The n-fold $(n \in N = \{0, 1, 2, ...\})$ concatenation of a word $w \in Y^*$ is denoted by w^n, and w^ω is the sequence in Y^ω formed by concatenating the word w infinitely many times, provided w is not the empty word. For convenience we shall write $w \cdot B$ and $W \cdot b$ instead of $\{w\} \cdot B$ and $W \cdot \{b\}$, respectively.

For any word $w \in Y^*$ let $|w|$ be its length, and, as sequences $\beta \in Y^\omega$ may be viewed as functions mapping $\omega = \{1, 2, 3, ...\}$ into Y, the nth letter of β is denoted by $\beta(n)$.

Finally, let $A(b) =_{\mathrm{df}} \{w: w \in Y^* \text{ and } b = w \cdot b' \text{ for some } b'\}$ be the set of all initial words of $b \in Y^* \cup Y^\omega$, and let $A(B) =_{\mathrm{df}} \bigcup_{b \in B} A(b)$.

Since Y is a group, the sets $Y^n =_{\mathrm{df}} \{w: w \in Y^* \text{ and } |w| = n\}$ and Y^ω will be considered also as groups where addition (also denoted by $+$) is defined componentwise. Their respective zero elements are $0^n \in Y^n$ and $0^\omega \in Y^\omega$. Additionally, the set Y^ω may be provided with a topological structure introducing the following metric ρ.

$$\begin{aligned} \rho(\beta, \xi) &=_{\mathrm{df}} 0, && \text{if } \beta = \xi, \\ &=_{\mathrm{df}} \max\left\{\frac{1}{n} : \beta(n) \neq \xi(n)\right\}, && \text{if } \beta \neq \xi. \end{aligned} \tag{1}$$

It is a well-known fact, that this metric space (Y^ω, ρ) is homeomorphic to Cantor's discontinuum (cf. Trachtenbrot and Barzdin, 1973), and hence is a complete and compact metric space. Moreover, ρ satisfies the ultrametric inequality

$$\rho(\beta, \zeta) \leqslant \max\{\rho(\beta, \eta), \rho(\xi, \eta)\} \tag{2}$$

for arbitrary $\beta, \xi, \eta \in Y^\omega$. Thus, as (Y^ω, ρ) is an ultrametric space, every open ball $K_\varepsilon(\beta) =_{\mathrm{df}} \{\xi: \xi \in Y^\omega \text{ and } \rho(\beta, \xi) < \varepsilon\}$ is also closed (cf. Dieudonné, 1960). From the defining equation (1) of the metric ρ one easily obtains that

$$K_\varepsilon(w \cdot \xi) = w \cdot Y^\omega \quad \text{for} \quad \frac{1}{|w|} > \varepsilon \geqslant \frac{1}{|w| + 1} \text{ and arbitrary } \xi \in Y^\omega. \tag{3}$$

Consequently, the open sets in (Y^ω, ρ), as unions of open balls, are easily characterized as the sets of form $W \cdot Y^\omega$, where $W \subseteq Y^*$. This yields the following characterization of the closure $C(F)$ of a set $F \subseteq Y^\omega$, i.e., of the smallest closed subset of Y^ω containing the set F.

LEMMA 1.1. (a) $C(F) = \bigcap_{n=0}^{\infty} (A(F) \cap Y^n) \cdot Y^\omega$,

(b) $C(F) = \{\beta: A(\beta) \subseteq A(F)\}$.

Proof. (a) Since Y^ω is a metric space, we have $C(F) = \bigcap_{\varepsilon>0} \bigcup_{\beta \in F} K_\varepsilon(\beta)$. In virtue of Eq. (3), for $1/n > \varepsilon \geqslant 1/(n+1)$ the union $\bigcup_{\beta \in F} K_\varepsilon(\beta)$ may be rewritten as $\{w: w \in A(F) \text{ and } |w| = n\} \cdot Y^\omega$. This proves our assertion.

(b) It suffices to show the equality $\{\beta: A(\beta) \subseteq A(F)\} = \bigcap_{n=0}(A(F) \cap Y^n) \cdot Y^\omega$. Let $A(\beta) \subseteq A(F)$. Then clearly, $\beta \in (A(F) \cap Y^n) \cdot Y^\omega$ for every $n \in N$. Now, if $\beta \in \bigcap_{n=0}^{\infty} (A(F) \cap Y^n) \cdot Y^\omega$ then for every $n \in N$ the initial word of length n of β belongs to $A(F)$. Hence, $A(\beta) \subseteq A(F)$. ∎

According to Lemma 1.1 a subset $F \subseteq Y^\omega$ is closed if and only if $A(\beta) \subseteq A(F)$ implies $\beta \in F$. Next, we connect the group theoretical with the topological properties of Y^ω. It is readily seen that the metric ρ is invariant under additive shift, i.e., $\rho(\beta + \eta, \xi + \eta) = \rho(\beta, \xi)$. Therefore, we introduce the norm $\|\beta\|$ of a sequence $\beta \in Y^\omega$ as

$$\|\beta\| =_{\mathrm{df}} \rho(\beta, 0^\omega).$$

Thus, $\rho(\beta, \xi) = \|\beta - \xi\|$, and from the ultrametric inequality (2) it follows

$$\|\beta + \xi\| \leqslant \max\{\|\beta\|, \|\xi\|\}. \tag{4}$$

This yields the following necessary and sufficient condition for the convergence of an infinite sum in Y^ω.

PROPOSITION 1.2. *The infinite sum $\sum_{i=0}^{\infty} \beta_i$ converges iff β_i tends to 0^ω as i approaches infinity.*

Proof. Let $\sum_{i=0}^{n} \beta_i \to_{n\to\infty} \sum_{i=0}^{\infty} \beta_i$. Then necessarily

$$\|\beta_n\| = \left\| \sum_{i=0}^{n} \beta_i - \sum_{i=0}^{n-1} \beta_i \right\| \xrightarrow[n\to\infty]{} 0.$$

Now let $\beta_i \to_{i\to\infty} 0^\omega$, and consider $\eta_{n,m} = \sum_{i=0}^{n} \beta_i - \sum_{i=0}^{m} \beta_i$ $(m < n)$. Then $\|\eta_{n,m}\| = \|\beta_{m+1} + \beta_{m+2} + \cdots + \beta_n\| \leqslant \max\{\|\beta_{m+1}\|, \ldots, \|\beta_n\|\}$ according to (4). Consequently $\|\eta_{n,m}\| \to_{n,m\to\infty} 0$, and $\sum_{i=0}^{n} \beta_i$ converges to some $\beta \in Y^\omega$, for Y^ω is a complete space. ∎

If Y is a Galois field GF(q) then Y^ω may be considered not only as a topological group but also as a normed space. Having this important case in mind, we will refer to Y^ω as a (normed linear) space and to closed subgroups $L \subseteq Y^\omega$ as subspaces. It should be mentioned that the set $\mathscr{L}$ of subspaces of Y^ω is closed under+ and $\cap$. Finally, for a countably infinite family $(L_i)_{i\in N}$ of subspaces we will write $\sum_{i=0}^{\infty} L_i$ to denote the subspace spanned by all spaces L_i.

3. Analysis of Convolutional Codes

In this section let Y be some Galois field GF(q). We shall consider a convolutional code as a subspace of GF$(q)^\omega$ as it is now usually done (cf. Blahut, 1983) in coding theory, and we derive conditions necessary for a subspace to be a convolutional code. Later on in Section 6 we shall show that these same conditions are also sufficient. A convolutional code is defined (cf. Costello, 1969) as the row space of its semi-infinite generator matrix $\mathfrak{G}$ of the following form

$$\mathfrak{G} = \begin{bmatrix} G_0 & G_1 & G_2 & \cdots & G_{\nu-1} & G_\nu & & & \\ & G_0 & G_1 & G_2 & \cdots & G_{\nu-1} & G_\nu & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & G_0 & G_1 & G_2 & \cdots \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where each G_i is a $k \times n$ matrix over GF(q), and the blank portions of the matrix $\mathfrak{G}$ are assumed to be filled with $k \times n$ zeromatrices. Consequently, the convolutional code L generated by $\mathfrak{G}$ is

$$L = \{\eta \otimes \mathfrak{G} : \eta \in Y^\omega\}, \tag{6}$$

where $\eta \otimes \mathfrak{G}$ denotes the product of the sequence $\eta \in Y^\omega$, regarded as a semi-infinite row vector, with the matrix $\mathfrak{G}$.

Theorem 3.1 (Staiger, 1980b). *Let $L \subseteq Y^\omega$ be a convolutional code. Then*

(a) *L is a linear and closed subset of Y^ω,*

(b) *$L \supseteq 0^n \cdot L$ for some $n > 0$,*

(c) *for every $w \in A(L)$ there is a $v \in Y^*$ such that $w \cdot v \cdot 0^\omega \in L$.*

Proof. Linearity follows easily from the defining equation (6). From the structure of the matrix $\mathfrak{G}$ follows, that if $\rho(\eta, \xi) \leqslant 1/(i \cdot k)$ then $\rho(\eta \otimes \mathfrak{G}, \xi \otimes \mathfrak{G}) \leqslant 1/(i \cdot n)$. Thus, $\Gamma: \eta \to \eta \otimes \mathfrak{G}$ is a continuous mapping from Y^ω to Y^ω. Since Y^ω is compact, the image $\Gamma(Y^\omega) = \{\eta \otimes \mathfrak{G}: \eta \in Y^\omega\}$ is closed.

The property (b) is easily obtained from

$$(0^k \cdot \eta) \otimes \mathfrak{G} = 0^n \cdot (\eta \otimes \mathfrak{G}).$$

Finally, let $w \in A(L)$. Without loss of generality we may assume $|w| = i \cdot n$ for some $i \geqslant 0$. Then there is an $\eta \in Y^\omega$ such that $w \in A(\eta \otimes \mathfrak{G})$. From the structure of $\mathfrak{G}$ follows that w depends only on the first $i \cdot k$ positions of η. Let u be the initial word of length $i \cdot k$ of η. Then w is an initial word of $(u \cdot 0^\omega) \otimes \mathfrak{G}$, and $(u \cdot 0^\omega) \otimes \mathfrak{G}$ has only finitely many nonzero entries. ∎

We conclude this section with some remarks on the parameters of a convolutional code. The matrix $\mathfrak{G}$ in Eq. (5) is designed from a k input, n output, feedback-free linear sequential circuit used as an (n, k)-encoder for the code L of Eq. (6). This encoder shifts out every time unit a block of n encoded symbols. Therefore, we will call n the block-length of the convolutional code. As one can design the encoder also as a $i \cdot k$ input, $i \cdot n$ output, circuit the block-length is not an invariant of a convolutional code. A sufficiently large block-length may be chosen in order to obtain a convolutional encoder in unit memory form (Lee, 1976). This will be illustrated by the following example.

EXAMPLE 3.1. Let L_1 be the binary convolutional code generated by $G_0 = (11)$, $G_1 = (01)$, and $G_2 = (11)$ $(v = 2)$. Figure 1 shows an encoding circuit for L_1 constructed according to (G_0, G_1, G_2). The same code L_1 may be generated by the matrices $G_0' = \left(\begin{smallmatrix} 11 & 01 \\ 00 & 11 \end{smallmatrix}\right)$ an $G_1' = \left(\begin{smallmatrix} 11 & 00 \\ 01 & 11 \end{smallmatrix}\right)$ where $v = 1$. The corresponding circuit is a unit-memory encoder (see Fig. 2).

A still open problem is to find the minimum block-length of a convolutional code (Conan, 1981). A solution, which however requires some refinement in order to be effectively applicable to generator matrices, to this problem is obtained in the next two sections.

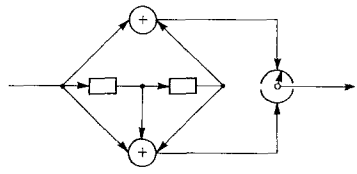

FIG. 1. A (2, 1)-minimum-block-length encoder for L_1.

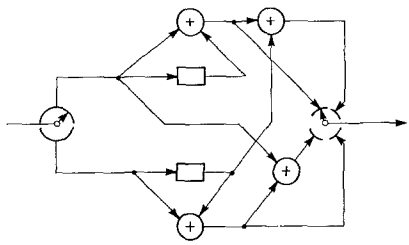

FIG. 2. A (4, 2)-unit-memory encoder for L_1.

The information rate of a convolutional code L is only k/n if the matrices $G_0, G_1, \ldots, G_\nu$ satisfy an independence condition (cf. Forney, 1970), but can be calculated as Shannon's (1948) channel capacity

$$\lim_{n\to\infty} \frac{\log s_L(n)}{n}$$

where $s_L(n)$ is the cardinality of $A(L) \cap Y^n$, or otherwise following the idea of Massey and Sain (1968) as k'/n, where k' is the limit of the difference of the ranks of the order $(i \cdot k) \times (i \cdot n)$ and $((i+1) \cdot k) \times ((i+1) \cdot n)$ left upper corner submatrices of the generator matrix $\mathfrak{G}$. These differences are also concerned with the delay of the code as it was pointed out in Massey and Sain (1968). We will return to these problems in the fifth section. Here we only add a simple example.

EXAMPLE 3.2. We regard the convolutional code $L_2 = Y \cdot 0 \cdot Y^\omega$ which can be obtained setting $k = n = 2$, $\nu = 1$, $G_0 = \binom{10}{00}$ and $G_1 = \binom{00}{01}$. Figure 3 displays the corresponding (2, 2)-encoder. Comparing the codes L_1 and L_2 with respect to the condition (b) of Theorem 3.1 we easily verify that L_1 satisfies the even stronger condition $0^n \cdot L_1 = L_1 \cap 0^n \cdot Y^\omega$, whereas L_2 does not, for $0^n \cdot L_2 \neq L_2 \cap 0^n \cdot Y^\omega = 0^n \cdot Y^\omega$ $(n = 2, 4, 6, \ldots)$.

Finally, what concerns the constraint length, there are different definitions of this term in use. We will point out in Section 6 which properties of the code (not the encoder) influence two of the possible values of constraint length.

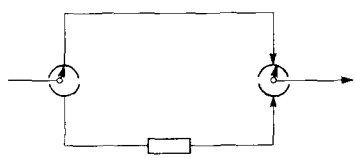

FIG. 3. A (2, 2) encoder for L_2.

4. Periodic Subspaces

In this section we deal with two main tools for comparing and classifying subspaces of Y^ω (not required $Y = \mathrm{GF}(q)$). For comparing we introduce the structure function of a subspace in a similar manner as it was done for languages (cf. Kuich, 1970). We provide subsets of Y^ω with an internal structure via the concept of states (cf. Lindner and Staiger, 1977; Staiger, 1983b). This enables one to obtain in an easy way a tree or a trellis describing a particular subspace of Y^ω, and thus may be a helpful tool in the theory of tree and trellis codes. Finally we classify the subspaces of Y^ω according to their behaviour (in terms of their state space) along the all zero sequence 0^ω.

In order to obtain the announced comparison method, we associate with each subset $W \subseteq Y^*$ its structure function s_W in the following way (cf. Kuich, 1970).

$$s_W(n) =_{\mathrm{df}} \mathrm{card}\, W \cap Y^n.$$

For $F \subseteq Y^\omega$ the structure function s_F is defined as $s_{A(F)}$. Structure functions of subsets of Y^ω have the following property: If $E \supseteq F$ and $s_E(n) > s_F(n)$, then there is a sequence $\beta \in E \setminus F$ whose initial word of length n does not belong to $A(F) \cap Y^n$. Consequently, $A(E) \cap Y^i \supset A(F) \cap Y^i$, and hence $s_E(i) > s_F(i)$ for every $i \geqslant n$.

Conversely, if $E \supseteq F$ and $s_E(n) \leqslant s_F(n)$ for infinitely many $n \in N$ then for every $i \in N$ the equality $A(E) \cap Y^i = A(F) \cap Y^i$ holds true. This in view of Lemma 1.1(a) implies $C(E) = C(F)$. Thus we have established the following comparison method.

LEMMA 4.1. *If $F \subseteq Y^\omega$ is closed, $E \supseteq F$ and $s_E(n) \leqslant s_F(n)$ for infinitely many n, then $E = F$.*

A further useful tool in the study of subsets is the concept of states. As demonstrated in Lindner and Staiger (1977) this gives a possibility to describe the internal structure of closed subsets of Y^ω.

Let $w \in Y^*$ and $F \subseteq Y^\omega$. The set $F/w =_{\mathrm{df}} \{\beta : w \cdot \beta \in F\}$ is called the state of F derived by the word $w \in Y^*$. The following properties of states are readily seen.

$$w \cdot (F/w) = F \cap w \cdot Y^\omega \tag{7}$$

$$(E \cup F)/w = E/w \cup F/w \tag{8}$$

$$(E \cap F)/w = E/w \cap F/w \tag{9}$$

$$E/w \subseteq F/w \quad \text{if} \quad E \subseteq F \tag{10}$$

$$(F/w)/v = F/w \cdot v \tag{11}$$

$$C(F)/w = C(F/w). \tag{12}$$

As a consequence of (12), a state of a closed subset of Y^ω is itself closed.

$$(F+E)/w = \bigcup_{u+v=w} (F/u + E/v). \tag{13}$$

The states of a subspace L have the following properties.

PROPOSITION 4.2. *Let $L \subseteq Y^\omega$ be a subspace and let $v, w \in A(L)$. Then*

(a) *every zerostate $L/0^k$ is also a subspace of Y^ω.*

(b) *L/w is a coset of $L/0^{|w|}$, i.e., there is a $\beta_w \in Y^\omega$ such that $L/w = L/0^{|w|} + \beta_w$.*

(c) *$L/w = L/0^{|w|}$ iff $0^\omega \in L/w$.*

(d) *If $L/w \supseteq L/v$ then $L/0^{|w|} \supseteq L/0^{|v|}$.*

Proof. (a) Trivially, $L/0^k$ is a subgroup, and as a state of a closed set $L/0^k$ is itself closed.

(b) Let $w \cdot \beta_w$ be an arbitrary sequence in L starting with the word w. Then $0^{|w|} \cdot \xi \in L$ iff $(0^{|w|} \cdot \xi + w \cdot \beta_w) = w \cdot (\xi + \beta_w) \in L$. Therefore, $\xi \in L/0^{|w|}$ iff $\xi + \beta_w \in L/w$.

(c) and (d) are immediate consequences of (b). ∎

As a further consequence of (b) we get

COROLLARY 4.3. *Let L be a subspace and let $w, v \in A(L) \cap Y^n$, $n \in N$. Then $L/w + L/v = L/(w+v)$.*

By Proposition 4.2 it is made apparent that in subspaces properties of the states L/w do depend heavily on the properties of their corresponding zerostates $L/0^{|w|}$. Therefore, in the remaining part of this section we deal with the sequence $(L/0^j)_{j \in N}$ of the zerostates of a particular space L. First, we derive some easily established properties.

PROPOSITION 4.4. *Let L be a subspace of Y^ω, and let $n, k > 0$.*

(a) *If $L \subseteq L/0^n$ then $L \subseteq L/0^{nm}$ for every $m \in N$.*

(b) *If $L \supseteq L/0^n$ then $L \supseteq L/0^{nm}$ for every $m \in N$.*

(c) *If $L/0^k \subseteq L \subseteq L/0^n$ then $L = L/0^k = L/0^n$.*

(d) *If $L \subseteq L/0^n$, $L/0^k \subseteq L/0^n$, and $n \leqslant k$ then $L/0^k = L/0^n$.*

(e) *If $L \supseteq L/0^n$, $L/0^k \supseteq L/0^n$, and $n \leqslant k$ then $L/0^k = L/0^n$.*

Proof. (a) and (b) are immediate consequences of (11) and (10).

(c) From (a) and (b) one obtains $L/0^{k\cdot n} \subseteq L/0^k \subseteq L/0^n \subseteq L/0^{n\cdot k}$.

(d) Consider $L' =_{df} L/0^n$. Then $L/0^k = L'/0^{k-n} \subseteq L' \subseteq L'/0^n$, and the assertion follows with(c).

(e) is the dual case of (d). ∎

This proposition throws some light on the behaviour of the sequence $(L/0^j)_{j\in N}$ of zerostates, namely, once we have $L/0^m \subseteq L/0^{m+n}$ (or $L/0^m \supseteq L/0^{m+n}$) then it is impossible to have a zerostate $L/0^j$ $(j \geqslant m)$ with $L/0^j \subset L/0^m$ $(L/0^j \supset L/0^m$, respectively). Thus the sequence $(L/0^j)_{j\in N}$

(1) consists of pairwise incomparable (with respect to set inclusion) subspaces, or there are $m, n \in N$, $n > 0$ such that for every $j \geqslant m$ one of the inclusions,

(2) $L/0^j \subseteq L/0^{j+n}$, or

(3) $L/0^j \supseteq L/0^{j+n}$, respectively, holds true.

Now, our aim is to show that in the latter cases inclusion can be replaced by equality, for a larger value of m possibly. To this end we use the structure function for comparing the zerostates. First we derive some properties. Let $F \subseteq Y^\omega$ and $\beta \in Y^\omega$, then

$$s_{F+\{\beta\}} = s_F. \tag{14}$$

This is trivial, for an additive shift does not change the number of initial words. One also has

$$s_{w\cdot F}(n + |w|) = s_F(n). \tag{15}$$

Now let L be a subspace of Y^ω. Then from (14) and Proposition 4.2(b) it follows that

$$s_{L/w} = s_{(L/0^{|w|})} \qquad \text{if} \quad w \in A(L). \tag{16}$$

PROPOSITION 4.5. *If L is a subspace of Y^ω; $i, j \in N$, then*

$$s_L(i+j) = s_L(i) \cdot s_{(L/0^i)}(j).$$

Proof. If we decompose $A(L) \cap Y^{i+j}$ with respect to the initial words of length i, we obtain

$$A(L) \cap Y^{i+j} = \bigcup_{w\in A(L)\cap Y^i} w \cdot (A(L/w) \cap Y^j).$$

Since $s_{L/w} = s_{(L/0^i)}$ for all $w \in A(L) \cap Y^i$, counting the number of words in the right-hand side of the equality yields $s_L(i+j) = s_L(i) \cdot s^{(j)}_{(L/0^i)}$. ∎

Now we can prove a theorem from which the above assertion immediately folows.

THEOREM 4.6. *If $L \subseteq Y^\omega$ is a subspace satisfying $L \subseteq L/0^n$ (or $L \supseteq L/0^n$) for some $n > 0$ then there is an $m \geqslant 0$ such that $L/0^{m \cdot n} = L/0^{m \cdot n + n}$.*

Proof. We prove the case $L \subseteq L/0^n$, the proof of the other case being nearly the same. For the sake of brevity let s_j denote the structure function of $L/0^{j \cdot n}$. We have $L \subseteq L/0^n \subseteq L/0^{2n} \subseteq \ldots$, and, consequently, $s_0(n) \leqslant s_1(n) \leqslant s_2(n) \leqslant \cdots \leqslant \operatorname{card} Y^n$. Hence, there is an $m \in N$ such that $s_m(n) = s_k(n)$ for all $k \geqslant m$. Applying Proposition 4.5 repeatedly, yields $s_k(j \cdot n) = \prod_{i=0}^{j-1} s_{m+i}(n)$, showing that $s_k(j \cdot n)$ is independent of k, for each $j \geqslant 1$. Now the spaces $L/0^{m \cdot n}$ and $L/0^{(m+1) \cdot n}$ satisfy the hypotheses of Lemma 4.1. Therefore, $L/0^{m \cdot n} = L/0^{m \cdot n + n}$. ∎

Thus we have shown that, whenever the sequence $(L/0^j)_{j \in N}$ contains two comparable (with respect to $\subseteq$) states, it is ultimately periodic. Hence

DEFINITION. A subspace $L \subseteq Y^\omega$ is called periodic provided there are natural numbers k, n $(k \neq n)$ such that $L/0^k = L/0^n$. The class of all periodic subspaces of Y^ω will be denoted by $\mathscr{L}_{\text{per}}$.

DEFINITION. If $L \in \mathscr{L}_{\text{per}}$ then

$$\text{per } L =_{\text{df}} \min\{n: n > 0 \text{ and } L/0^j = L/0^{j+n} \text{ for some } j \in N\}$$

is called the period of the subspace L.

The following lemma gives further insight into the structure of periodic spaces.

LEMMA 4.7. *A subspace $L \subseteq Y^\omega$ is periodic iff there are $w, v \in A(L)$, $|w| \neq |v|$ such that $L/w \supseteq L/v$. And, whenever $L/w \supseteq L/v$ then* per L *divides* $|w| - |v|$.

Proof. By definition, the condition is necessary. Now let $L/w \supseteq L/v$, $|w| = j$, $|v| = k$, and without loss of generality let $j > k$. By Proposition 4.2(d) we obtain $L/0^j \supseteq L/0^k$. Then Theorem 4.6 proves $L/0^m = L/0^{m+(j-k)}$ for some $m \in N$. Hence, L is periodic.

Suppose $j - k$ not to be a multiple of per L. Then there is an $i \in N$ such that $0 < (j-k) - i \cdot \text{per } L < \text{per } L$. This yields $L/0^{n+i \cdot \text{per } L} = L/0^{n+(j-k)}$ for some $n \in N$, contradicting the definition of per L. ∎

Since $L/0^j = L/0^k$ implies $L/0^{m+j} = L/0^{m+k}$, we obtain for $L \in \mathscr{L}_{\text{per}}$ the following formula

$$\text{per } L = \text{per } L/0^m$$

for all $m \in N$. This yields another tool for the calculation of the period.

PROPOSITION 4.8. *Let* $L/0^m = L'/0^n$. *Then* $L \in \mathscr{L}_{\text{per}}$ *implies* $L' \in \mathscr{L}_{\text{per}}$ *and, moreover* per $L =$ per L'.

Next we give some examples of periodic and aperiodic spaces.

EXAMPLE 4.1 (Lindner and Staiger, 1977). Consider the class $\mathscr{L}_{\text{fin}}$ of all finite subspaces. For every $B \in \mathscr{L}_{\text{fin}}$ there is an $m \in N$ such that $B \cap 0^m \cdot Y^\omega = \{0^\omega\}$, i.e., $B/0^m$ is the null-space $\{0^\omega\}$. Consequently, per $B = 1$.

EXAMPLE 4.2. If $L \subseteq Y^\omega$ is a convolutional code, then $L \in \mathscr{L}_{\text{per}}$, as the condition (b) of Theorem 3.1 is equivalent to $L/0^n \supseteq L$ for some $n > 0$.

EXAMPLE 4.3 (Staiger, 1982). Define

$$L_3 =_{\text{df}} \{\beta: \beta(p) \neq 0 \text{ only if } p \text{ is a prime number}\}.$$

One easily establishes that L_3 is indeed a subspace, and from the distribution of primes it follows that $L_3/0^j = L_3/0^k$ can never hold unless $j = k$. Thus L_3 is an aperiodic subspace.

Example 4.3 can be modified in several ways to obtain further aperiodic subspaces. One is given in the following.

EXAMPLE 4.4. $L_4 =_{\text{df}} \{\beta: \beta(p) = \beta(p^m)$ for all $m > 0$ and p prime, and $\beta(k) = 0$ otherwise$\}$ is also an aperiodic subspace of Y^ω.

Next we investigate the influence of an additional finite subspace on the ultimate behavior of the sequence $(L/0^j)_{j \in N}$.

LEMMA 4.9. *Let* $B \in \mathscr{L}_{\text{fin}}$ *and let* L *be an arbitrary subspace of* Y^ω. *Then there is an* $m \in N$ *such that* $(L + B)/0^m = L/0^m$.

Proof. Since B is finite and L is closed, $\varepsilon =_{\text{df}} \inf\{\|\beta - \eta\|: \beta \in B, \eta \in L$ and $\beta \notin L\} > 0$. Let $\varepsilon > 1/m$, and let $\eta \in L$ and $\beta \in B$ satisfy $\beta + \eta \in 0^m \cdot Y^\omega$. Hence $\|\beta + \eta\| = \|\beta - (-\eta)\| \leqslant 1/m < \varepsilon$, and, therefore, $(-\eta) \in L$ implies $\beta \in L$. Consequently, $(L + B) \cap 0^m \cdot Y^\omega \subseteq L$ which proves our assertion. ∎

LEMMA 4.10. *Let* L, L' *be subspaces of* Y^ω. *Then* $L/0^m = L'/0^m$ *for*

some $m \in N$ if and only if there are finite subspaces B and B' such that $(L \cap L') + B = L$ and $(L \cap L') + B' = L'$.

Proof. Sufficiency is proved by the application of the preceding lemma separately to the pairs $L \cap L'$, B and $L \cap L'$, B'. This yields numbers $j, k \in N$ satisfying $L/0^j = (L \cap L')/0^j$ and $L'/0^k = (L \cap L')/0^k$, respectively. Now set $m =_{\mathrm{df}} \max\{j, k\}$, and clearly $L/0^m = (L \cap L')/0^m = L'/0^m$.

In order to prove necessity, define $L'' =_{\mathrm{df}} 0^m \cdot (L/0^m) = L \cap 0^m \cdot Y^\omega = L' \cap 0^m \cdot Y^\omega \subseteq L \cap L'$ and consider the cosets of L'' in L. If $\eta, \xi \in L$ have the same initial word $w \in A(L) \cap Y^m$ then $\eta - \xi \in L \cap 0^m \cdot Y^\omega = L''$. Hence, there are $\operatorname{card} A(L) \cap Y^m$ cosets of L'' in L. If we choose in every such coset one element then the space B spanned by these elements is finite, and by construction $L = L'' + B = (L \cap L') + B$. In the same way we obtain a finite space B' with $L' = L'' + B' = (L \cap L') + B'$. ∎

We conclude this section giving another characterizing property of $\mathscr{L}_{\mathrm{per}}$. To this end we introduce the following class of subspaces.

DEFINITION. A subspace $L \subseteq Y^\omega$ is called a Σ-space provided $L \subseteq L/0^n$ for some $n > 0$.

The class of all Σ-subspaces of Y^ω will be denoted by $\mathscr{L}_\Sigma$. Theorem 4.6 proves that Σ-spaces are periodic. Also, as Example 4.2 shows, all convolutional codes are Σ-spaces, though, as we shall prove in the following sections, not every Σ-space is also a convolutional code. Among the finite spaces there is only one Σ-space, the null-space $\{0^\omega\}$.

PROPOSITION 4.11. *$\mathscr{L}_\Sigma$ is closed under $\cap$ and $+$.*

Proof. Let $L/0^n \supseteq L$ *and* $L'/0^m \supseteq L'$ for some $m, n > 0$. Equations (9) and (13) yield $L \cap L' \subseteq L/0^{nm} \cap L'/0^{mn} = (L \cap L')/0^{nm}$, and $L + L' \subseteq L/0^{nm} + L'/0^{mn} \subseteq (L + L')/0^{nm}$. ∎

Σ-spaces will be dealt with extensively in the next section. Here we only derive a result showing that Σ-spaces are in some sense the cores of periodic spaces.

THEOREM 4.12. *A subspace $L \subseteq Y^\omega$ is periodic iff there are a Σ-space L' and a finite space B such that $L = L' + B$.*

Proof. Sufficiency is readily seen by Lemma 4.9 and Proposition 4.8. Now let $L \in \mathscr{L}_{\mathrm{per}}$, i.e., there are $m, n \in N$, $n > 0$ such that $L/0^m = L/0^{m+n}$. Therefore $0^n \cdot (L/0^m) \subseteq L/0^m$. Without loss of generality we may assume $m \geqslant n$. Set $L' =_{\mathrm{df}} 0^m \cdot (L/0^m) = L \cap 0^m \cdot Y^\omega$. Since $L'/0^m = 0^{m-n} \cdot (L/0^m)$

$\supseteq 0^m \cdot (L/0^m) = L'$, L' is a Σ-space. Moreover $L' \subseteq L$ and $L'/0^m = L/0^m$. Hence, by Lemma 4.10, $L = L' + B$ for some finite space B. ∎

COROLLARY 4.13. *Every periodic subspace $L \subseteq Y^\omega$ contains a maximum Σ-space L_0.*

Proof. Consider the Σ-space $L' \subseteq L$ constructed in the proof of Theorem 4.12. L' has only finitely many cosets in L. Thus there is a maximum Σ-space L_0 satisfying $L' \subseteq L_0 \subseteq L$. If there were any other Σ-space $L'' \subseteq L$ not contained in L_0, then according to Proposition 4.11, $L_0 + L'' \supset L_0$ would be also a Σ-space in L containing L', which contradicts the maximality of L_0. ∎

For completeness we add

PROPOSITION 4.14. *$\mathscr{L}_{\text{per}}$ is closed under $\cap$ and $+$.*

Proof. Closure under $\cap$ is proved similar to Proposition 4.11, and closure under $+$ is an immediate consequence of Theorem 4.12 and Proposition 4.11. ∎

5. Σ-SPACES

As announced in the previous section, now we deal with the subclass $\mathscr{L}_\Sigma$ of $\mathscr{L}_{\text{per}}$. For Σ-spaces we shall investigate parameters as block-length, delay, and rate which are important features of convolutional codes and, in general, of periodic and Σ-spaces. We conclude this section estimating the cardinality of $\mathscr{L}_\Sigma$ and $\mathscr{L}_{\text{per}}$ via a topological density result.

First we derive a theorem which makes the term Σ-space more apparent. We could have called Σ-spaces likewise constant (time-invariant) spaces due to the similarity of their defining property $L \subseteq L/0^n$ to the constancy (time-invariance) of linear sequential circuits (Forney, 1970), but we prefer the above term to emphasize that a Σ-space is the infinite sum of shifted copies of a finite space.

According to property (b) in Theorem 3.1 and the discussion following this theorem we introduce the block-length of a Σ-space.

DEFINITION. Let L be a Σ-space. Any $n > 0$ satisfying $L \subseteq L/0^n$ will be called a block-length of the space L.

Since $L \subseteq L/0^n$ is equivalent to $0^n \cdot L \subseteq L$, every Σ-space L of block-length n containing a subspace B also contains $0^n \cdot B$, $0^{2n} \cdot B$,.... This yields the following result.

PROPOSITION 5.1. *Let B be an arbitrary subspace of Y^ω. Then*

$$L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B$$

is the smallest Σ-space of block-length n containing B.

Next, we carry out the main step in proving the announced result—the so-called B-construction.

LEMMA 5.2 (B-construction). *Let L be a Σ-space of block-legth n, and let $k \in N$ be such that $L/0^{k+n} = L/0^k$, then every subspace $B \subseteq L$ having $s_B(k+n) \geqslant s_L(k+n)$ satisfies*

$$L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B.$$

Proof. First we observe that $s_B(k+n) \geqslant s_L(k+n)$ and $B \subseteq L$ imply $A(B) \cap Y^{k+n} = A(L) \cap Y^{k+n}$ and hence $s_B(k) = s_L(k)$ and $s_{(B/0^k)}(n) = s_{(L/0^k)}(n)$.

Let L' be the smallest Σ-space of block-length n containing space B. Clearly, $L' \subseteq L$ and, moreover, $B/0^k \subseteq L'/0^k \subseteq L'/0^{k+j \cdot n}$ for all $j \in N$. By Proposition 4.5 we get

$$s_{L'} \cdot (k + i \cdot n) \geqslant s_B(k) \cdot (s_{(B/0^k)}(n))^i$$

for arbitrary $i \in N$. On the other hand, since $L/0^{k+n} = L/0^n$, $s_B(k) = s_L(k)$, and $s_{(L/0^k)}(n) = s_{(B/0^k)}(n)$, again Proposition 4.5 yields

$$s_L(k + i \cdot n) = s_B(k) \cdot (s_{(B/0^k)}(n))^i$$

for arbitrary $i \in N$. Now, the assertion follows from Lemma 4.1. ∎

In particular, we can choose in the above B-construction any suitable finite subspace $B \subseteq L$ to span L in the described manner. Thus Lemma 5.2 and Proposition 5.1 yield the announced theorem.

THEOREM 5.3. *A subset $L \subseteq Y^\omega$ is a Σ-space of block-length n if and only if there is a finite subspace B of Y^ω such that*

$$L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B.$$

Though we are not dealing extensively with the generation of subspaces L of Y^ω by arbitrary subsets $F \subseteq Y^\omega$, we derive a corollary to Theorem 5.3 which generalizes Theorem 2.5 in Piret (1978). To this end we observe that if

L is the smallest subspace containing a set $F \subseteq Y^\omega$, then for every $k \in N$ the space $A(L) \cap Y^k$ is spanned by $A(F) \cap Y^k$. Hence for every $k \in N$ there is a finite subset $E \subseteq F$ such that the space B spanned by E satisfies $A(B) \cap Y^k = A(L) \cap Y^k$. This proves the following corollary.

COROLLARY 5.4. *Let F be an arbitrary subset of Y^ω. Then for every $n > 0$ there is a finite subset $F_n \subseteq F$ such that the Σ-spaces of block-length n spanned by F_n and F, respectively, coincide.*

Σ-spaces are a special kind of periodic space. Thus, following Lemma 4.7, the period $\operatorname{per} L$ of a Σ-space L divides all its block-lengths, though as we see below the period itself need not be a block-length.

EXAMPLE 5.1. Consider the space $L_2 = Y \cdot 0 \cdot Y^\omega$ of Example 3.2. We have $L_2/0 = 0 \cdot Y^\omega \not\supseteq L_2$ and $L_2/0^n = Y^\omega$ for all $n \geqslant 2$. Hence every $n \geqslant 2$ is a block-length of L_2, but $\operatorname{per} L_2 = 1$ is not a block-length of L_2.

In order to obtain the minimum block-length of a Σ-space (or a convolutional code as mentioned in Section 3), we consider the subfamily $(L/0^{i \cdot \operatorname{per} L})_{i \in N}$ of the family of all zerostates of L and find the minimum value j such that $L \subseteq L/0^{j \cdot \operatorname{per} L}$. This can be done effectively (cf. Lindner and Staiger, 1977) whenever L is specified in a constructive way (e.g., by a finite state diagram, a generator matrix in the form of Eq. (5), a finite-state encoder) which is apparently the case for convolutional codes.

Next, we derive some further properties of the family $(L/0^{i \cdot \operatorname{per} L})_{i \in N}$. Observe that $L/0^m = L/0^{m + \operatorname{per} L}$ for any periodic space L if m is sufficiently large. Consequently, for any Σ-space L and any of its block-lengths n we have

$$L \subseteq L/0^{k \cdot n} = L/0^{k \cdot n + \operatorname{per} L} \tag{17}$$

for any sufficiently large k. This shows that the family $(L/0^{i \cdot \operatorname{per} L})_{i \in N}$ terminates with some state $\hat{L}$ satisfying $\hat{L} = \hat{L}/0^{\operatorname{per} L}$. The following lemma proves that this terminal state is the unique state appearing more than once in the family $(L/0^{i \cdot \operatorname{per} L})_{i \in N}$.

LEMMA 5.5. *Let L' be a subspace of Y^ω. It holds $L' = L'/0^n$ for some $n > 0$ iff L' is a Σ-space satisfying $L' = L'/0^{\operatorname{per} L'}$.*

Proof. Let $L' = L'/0^n$. Clearly, L' is a Σ-space of block-length n. Applying $L' = L'/0^n$ to Eq. (17) yields $L' = L'/0^{\operatorname{per} L}$. The reverse direction is trivial. ∎

Moreover, it follows from Eq. (17) that the terminal state $\hat{L}$ contains every state $L/0^{i \cdot \operatorname{per} L}$ $(i \in N)$, and, once chosen, the block-length n of the Σ-space

L, the family $(L/0^{i\cdot n})_{i\in N}$ has a certain delay in achieving the full capacity of $\hat{L}$. This leads to the following definition.

DEFINITION. For a Σ-space $L \subseteq Y^\omega$ of block-length n we will refer to

$$\delta_n =_{\mathrm{df}} \min\{j: L/0^{j\cdot n} = L/0^{(j+1)\cdot n}\}$$

as the delay of L relative to the block-length n. The number

$$\delta_L =_{\mathrm{df}} \min\{j: L/0^{j\cdot \mathrm{per}\, L} = L/0^{(j+1)\cdot \mathrm{per}\, L}\}$$

will be called the total delay of the space L.

Lemma 5.5 proves that these definitions are correct, i.e., the numbers δ_n and δ_L are uniquely specified. In both cases, the spaces $L/0^{\delta_n\cdot n}$ and $L/0^{\delta_L\cdot \mathrm{per}\, L}$ coincide with the terminal state $\hat{L}$, and moreover,

$$\delta_n = \min\{j: L/0^{j\cdot n} = \hat{L}\}$$

and

$$\delta_L = \min\{j: L/0^{j\cdot \mathrm{per}\, L} = \hat{L}\}$$

hold. Now, from the above studied behavior of the family $(L/0^{i\cdot \mathrm{per}\, L})_{i\in N}$ and its subfamilies $(L/0^{i\cdot n})_{i\in N}$, n being a block-length of L, one easily derives that the value δ_n is uniquely determined by the inequality

$$\delta_L \cdot \mathrm{per}\, L \leqslant \delta_n \cdot n < \delta_L \cdot \mathrm{per}\, L + n.$$

Consequently, $\delta_m \leqslant \delta_n \leqslant \delta_L$, if $m \geqslant n$, and $\delta_L = 0$ iff $\delta_n = 0$ iff $L = \hat{L}$. If $\delta_L \neq 0$, the delay δ_n of a Σ-space L depends on the chosen block-length n, and can range between its maximum values δ_L (which need not be achieved) and its minimum value 1 iff $n \geqslant \delta_L \cdot \mathrm{per}\, L$.

EXAMPLE 5.1 (Continued). We have $\mathrm{per}\, L_2 = 1$ and $L_2/0^n \supseteq L_2$ iff $n \geqslant 2$. This yields $\delta_L = 2$, but $\delta_n = 1$ for every block-length n of L.

Next, we will show that our definition of delay coincides with the definition given by Massey and Sain (1968), where the delay is proved to be the smallest number δ such that the differences of the ranks of the order $(i\cdot k)\times(i\cdot n)$ and $((i+1)\cdot k)\times((i+1)\cdot n)$ left upper corner submatrices of the generator matrix $\mathfrak{G}$ in Eq. (5) remain constant for $i \geqslant \delta$. In terms of our approach this difference remains constant iff the difference $s_L((i+1)\cdot n) - s_L(i\cdot n)$ of the cardinalities of the corresponding row spaces remains constant for $i \geqslant \delta$.

In virtue of Proposition 4.5 we have

$$s_L((i+1)\cdot n) - s_L(i\cdot n) = s_{(L/0^{i\cdot n})}(n).$$

For convenience we will temporarily abbreviate $s_{(L/0^{i\cdot n})}$ as $s_{i\cdot n}$. Our assertion is then proved by

LEMMA 5.6. *Let L be a Σ-space.*

(a) *For every block-length n of L the delay δ_n is the smallest number such that*

$$s_{k\cdot n}(n) = s_{(k+1)\cdot n}(n) \qquad \text{for all} \quad k \geqslant \delta_n.$$

(b) *The total delay δ_L is the smallest number such that*

$$s_{k\cdot \operatorname{per} L}(\operatorname{per} L) = s_{(k+1)\cdot \operatorname{per} L}(\operatorname{per} L) \qquad \text{for all } k \geqslant \delta_L.$$

Proof. (a) The equation $L/0^{m\cdot n} = L/0^{(m+1)\cdot n}$ implies $s_{k\cdot n}(n) = s_{(k+1)\cdot n}(n)$ for all $k \geqslant m$. Hence it remains to show that $s_{k\cdot n}(n) = s_{(k+1)\cdot n}(n)$ for $k \geqslant m$ implies $L/0^{m\cdot n} = L/0^{(m+1)\cdot n}$. But this is just a repetition of the last part of the proof of Theorem 4.6.

(b) Since in the proof of part (a) we have nowhere used $L \subseteq L/0^n$, the proof works as well for $n = \operatorname{per} L$. ∎

We have seen that $\delta_L = 0$ iff $\delta_n = 0$. Therefore, we will refer to a Σ-space L with $\delta_L = 0$ as a delay-free Σ-space. According to Lemma 5.5 this condition is equivalent to $L = L/0^n$ for some $n > 0$.

Here the following natural question arises: Is every Σ-space L contained in a unique minimum delay-free Σ-space? To answer this question, we first observe that from Lemma 4.7 it follows that for every Σ-space L and every $j \in N$ the set $\{L/0^{j+i} : 0 \leqslant i < \operatorname{per} L\}$ consists of mutually incomparable subspaces, and at most one of them contains L.

The behaviour of the family $(L/0^{i\cdot \operatorname{per} L})_{i\in N}$ studied above shows that there is one state $\hat{L} = L/0^{k\cdot n}$ such that $L \subseteq \hat{L} = \hat{L}/0^{\operatorname{per} L}$, whenever $L \subseteq L/0^n$. By Lemma 5.5, this state $\hat{L} = L/0^{k\cdot n}$ is a delay-free Σ-space containing L. Now consider any space $L' \supseteq L$ with $L' = L'/0^m$ for some $m > 0$. Then $L/0^{k\cdot n\cdot m} \subseteq L'/0^{k\cdot n\cdot m} = L'$. Since $L \subseteq L/0^n$, $L/0^{k\cdot n} \subseteq L/0^{k\cdot n\cdot m} \subseteq L'$. Thus we have proved

LEMMA 5.7. *For every Σ-space L there is a unique minimum delay free Σ-space $\hat{L}$ containing L.*

In what follows $\hat{L}$ will be called the delay-free closure of L. The above considerations have shown that $\hat{L}$ is the maximum zerostate of L containing L itself. By Proposition 4.8, $\operatorname{per} L = \operatorname{per} \hat{L}$, and moreover, Lemma 4.10 shows that L has only fintely many cosets in $\hat{L}$. Next, we regard some connections between the states of a Σ-space L and the states of its delay-free closure $\hat{L}$.

LEMMA 5.8. (a) *Let* $L \subseteq L/0^n$ *and* $\beta_w \in L/w$. *Then*

$$L/0^{j \cdot n} \cdot w = L/0^{j \cdot n + |w|} + \beta_w \qquad \text{for every } j \in N.$$

(b) *Suppose* $\hat{L} = L/0^k$, $w \in A(L)$, *and* $|w| \geqslant k$. *Then* $L/w = \hat{L}/w$.

Proof. (a) We have $L \subseteq L/0^{j \cdot n}$ and, therefore, $L/w \subseteq L/0^{j \cdot n} \cdot w$. Now the assertion follows from Proposition 4.2(b).

(b) From $L \subseteq \hat{L} = L/0^k$ it follows that k is a block-length of L and hence of $\hat{L}$. If $L/w = L/0^{|w|} + \beta_w$ then by (a) $L/0^k \cdot w = L/0^{k+|w|} + \beta_w$, i.e., $\hat{L}/w = \hat{L}/0^{|w|} + \beta_w$. Now $|w| \geqslant k$ implies $\hat{L}/0^{|w|} = L/0^{|w|}$, which proves the assertion. ∎

The second part of Lemma 5.8 shows that every state L/w of a Σ-space L is already a state of the delay-free closure $\hat{L}$ of L provided $|w|$ is not smaller than the smallest k satisfying $\hat{L} = L/0^k$. Therefore, the set of states of L consists of the set $\{\hat{L}/w: w \in A(\hat{L})\}$ plus an additional finite set $\{L/w: |w| < k\}$. This explains that replacing L, if possible, by its delay-free closure $\hat{L}$ simplifies the structure of Σ-space.

The last part of this section is concerned with the proof that there are uncountably many Σ-subspaces of Y^ω. To this end it is convenient to introduce a further parameter of subspaces, the rate. Here we follow the line, in which Shannon (1948) defined the channel capacity to be the quantity

$$\lim_{t \to \infty} \frac{\log s(t)}{t},$$

where $s(t)$ is the number of allowed messages transmitted by a discrete channel during a time interval of length t.

DEFINITION. We call

$$H_F =_{\mathrm{df}} \limsup_{n \to \infty} \frac{\log s_F(n)}{n}$$

the rate (or entropy) of a subset $F \subseteq Y^\omega$. (The base of the logarithm is always assumed to be the cardinality of the alphabet Y.)

We add some simple properties of H_F.

$$H_{F \cup E} = \max\{H_F, H_E\}. \tag{18}$$

This is easily derived by the inequality

$$\max\{s_F(n), s_E(n)\} \leqslant s_{F \cup E}(n) \leqslant 2 \cdot \max\{s_F(n), s_E(n)\}.$$

$$H_F = 0 \qquad \text{if} \quad F \neq \varnothing \text{ is finite.} \tag{19}$$

It has been shown (Lindner and Staiger, 1977) that for a restricted class of subspaces the rate H_L has a dimension-like behaviour. Here we mention only that if E and F are nonempty

$$H_{E\cup F} \leqslant H_{E+F} \leqslant H_E + H_F. \tag{20}$$

The first inequality follows easily from Eqs. (14) and (18), and the second inequality is proved by overbounding $s_{E+F}(n)$ via $s_E(n) \cdot s_F(n)$ using the equality $A(E+F) \cap Y^n = (A(E) \cap Y^n) + (A(F) \cap Y^n)$. If L is a subspace and $w \in A(L)$, then Eq. (16) and Proposition 4.5 yield

$$H_{L/w} = H_L. \tag{21}$$

As a consequence of Eq. (21) we obtain $H_L = H_{\hat{L}}$ for every Σ-space $L \subseteq Y^\omega$.

The following formula evaluates the rate of a periodic subspace. Let L be a periodic subspace such that $L/0^k = L/0^{k+n}$ for some $k, n \in N$, $n > 0$. Then

$$H_L = \lim_{j\to\infty} \frac{\log s_L(j)}{j} = \frac{\log s_{(L/0^k)}(n)}{n}. \tag{22}$$

This formula follows immediately from

$$s_L(k + i \cdot n + m) = s_L(k) \cdot (s_{L/0^k)}(n))^i \cdot s_{(L/0^k)}(m)$$

which is in turn a consequence of Proposition 4.5.

Equation (22) provides a method of calculating the rate of a periodic space. Here we shall give some examples.

EXAMPLE 5.2. According to (19) every finite subspace B has rate $H_B = 0$. Now consider a periodic space L having rate $H_L = 0$. Equation (22) shows that for every k such that $L/0^k = L/0^{k+n}$ for some $n > 0$ we have $A(L/0^k) \cap Y^n = \{0^n\}$. Hence $L/0^k = \{0^\omega\}$, and the space L is finite. Together with Example 4.1 we get the following characterization: $L \in \mathscr{L}_{\text{fin}}$ if and only if $L \in \mathscr{L}_{\text{per}}$ and $H_L = 0$.

EXAMPLE 5.3. Now consider the space L_3 of Example 4.3. One easily checks $\log s_{L_3}(n) = \text{card}\{p: p \leqslant n \text{ and } p \text{ is a prime}\}$. Hence $\log s_{L_3}(n) \approx n/\ln n$ according to the well-known result on the density of primes among natural numbers. Consequently, $H_{L_3} = 0$. This is just another proof that L_3 is not a periodic space.

EXAMPLE 5.4. One easily verifies that the structure functions of L_3 and L_4 (the space introduced in Example 4.4) are identical. Thus $H_{L_4} = 0$, and as an infinite space L_4 cannot be periodic.

Next, we mention a connection between rate and delay of Σ-spaces.

PROPOSITION 5.9. *A Σ-space L is delay-free iff $L \subseteq L/0^n$ implies* $\log s_L(n) = n \cdot H_L$.

Proof. If L is delay-free, then $L \subseteq L/0^n$ implies $L = L/0^n$, and Eq. (22) then proves $\log s_L(n) = n \cdot H_L$. Now let $L \subseteq L/0^n$ and $\log s_L(n) = n \cdot H_L$. Then $\hat{L} = \hat{L}/0^n$, and $H_{\hat{L}} = H_L$. Since $\hat{L}$ is delay-free, we have $\log s_{\hat{L}}(n) = n \cdot H_L$, and in

$$A(L) \cap Y^n \subseteq A(L/0^n) \cap Y^n \subseteq \cdots \subseteq A(\hat{L}) \cap Y^n$$

equality holds. By Lemma 5.6(a) L is delay-free. ∎

The following result gives a connection between the rate (entropy) and topological density of Σ-spaces. Similar and more general results linking together entropy and topological density can be found, e.g., in Lindner and Staiger (1977) or Staiger (1983a), respectively. As usual a set $F \subseteq Y^\omega$ is called nowhere dense iff its closure $C(F)$ does not contain any nonempty open subset.

LEMMA 5.10. *Let $L \subseteq Y^\omega$ be a Σ-space. Then the following conditions are equivalent.*

(a) *L is nowhere dense.*

(b) $H_L < 1$.

(c) $\hat{L} \neq Y^\omega$.

Proof. (b) → (a). If L is not nowhere dense then, since L is closed, L contains a nonempty open subset, i.e., $w \cdot Y^\omega \subseteq L$ for some $w \in Y^*$. Now, $H_{w \cdot Y^\omega} = 1$ is easily verified. This proves $H_L = 1$.

(a) → (c) If $\hat{L} = Y^\omega$ then there is a $j \in N$ such that $0^j \cdot Y^\omega \subseteq L$, and L is not nowhere dense.

(c) → (b) Let $H_L = 1$, then $H_{\hat{L}} = 1$ and $\hat{L}/0^{i \cdot \mathrm{per}\, L} = \hat{L}$. Consequently, from Eq. (22) it follows $\log s_{\hat{L}}(i \cdot \mathrm{per}\, L) = i \cdot \mathrm{per}\, L$. Thus

$$A(\hat{L}) \cap Y^{i \cdot \mathrm{per}\, L} = Y^{i \cdot \mathrm{per}\, L},$$

and, hence, $A(\hat{L}) = Y^*$. Since $\hat{L}$ is closed, we obtain $\hat{L} = Y^\omega$. ∎

In Theorem 4.12 we have shown that every periodic space L is the sum of a Σ-space L' and a finite space B. Thus, L is a finite union of cosets of the Σ-space L'. Since a finite union of nowhere dense subsets is nowhere dense (cf. Kuratowski, 1966) we obtain

COROLLARY 5.11. *Let $L \subseteq Y^\omega$ be a periodic space. Then L is nowhere dense iff $H_L < 1$.*

The following lemma gives an upper bound on the entropy of a Σ-space.

LEMMA 5.12. *Let $L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B$. Then $H_L \leqslant (\log \operatorname{card} B)/n$.*

Proof. Since $0^{j \cdot n}$ is the only initial word of length $j \cdot n$ of sequences $\beta \in \sum_{i=j}^{\infty} 0^{i \cdot n} \cdot B$, every word in $A(L) \cap Y^{j \cdot n}$ is an initial word of a sequence in $L'' = \sum_{i=0}^{j-1} 0^{i \cdot n} \cdot B$. This verifies the inequality

$$s_L(j \cdot n) \leqslant \operatorname{card} L'' \leqslant (\operatorname{card} B)^j,$$

which in turn implies

$$H_L \leqslant \lim_{j \to \infty} \frac{j \cdot \log \operatorname{card} B}{j \cdot n} = \frac{\log \operatorname{card} B}{n}. \quad \blacksquare$$

In particular Lemma 5.12 yields the upper bound on the rate of a convolutional code discussed in Section 3.

EXAMPLE 5.5. Consider the convolutional code (Σ-space) L generated by the matrix $\mathfrak{G}$ of Eq. (5) according to Eq. (6). Let B be the finite space spanned by the first k rows of the matrix $\mathfrak{G}$. Then $\log \operatorname{card} B \leqslant k$,

$$L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B,$$

and Lemma 5.12 yields $H_L \leqslant k/n$. ∎

We conclude this part by counting the number of Σ-subspaces of Y^ω. Let L_β be the finite space spanned by the sequence $\beta \in Y^\omega$. Then L_β has at most as many elements as Y has, and following Lemma 5.12,

$$L_n(\beta) =_{\mathrm{df}} \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot L_\beta \qquad \text{has entropy } H_{L_n(\beta)} \leqslant \frac{1}{n}.$$

By Lemma 5.10 for $n \geqslant 2$ the family $(L_n(\beta))_{\beta \in Y^\omega}$ is a family of nowhere dense Σ-spaces covering the entire space Y^ω. Since Y^ω is not nowhere dense this family cannot be countable (cf. Kuratowski, 1966). Thus we have proved

THEOREM 5.13. *There are uncountably many Σ-subspaces of Y^ω.*

6. Convolutional Codes as Subspaces

Hitherto we have considered two classes of subspaces which have some properties with convolutional codes in common. The following investigations are devoted to the defining properties of convolutional codes among the classes of Σ-spaces. To this end we introduce two classes of subspaces of Y^ω: remergable subspaces and finite-state subspaces. And we show that each one of these properties defines the class of convolutional codes among Σ-spaces, as well as that they define the class of delay-free convolutional codes among delay-free Σ-spaces. Though we are now dealing with convolutional codes, we will not suppose Y to be a Galois field unless explicitly stated otherwise. Finally, we introduce and compare two parameters of subspaces which resemble the constraint length of convolutional codes whichever reasonable definition of this length one would have chosen.

We start with the following definitions.

Definition. A subset $F \subseteq Y^\omega$ is called resynchronizable (or ultimately connected) iff for every $w \in A(F)$ there is a $v \in Y^*$ such that $F \subseteq F/w \cdot v$.

In the case of subspaces this property can be easily split, as we shall see later, into two mutually independent conditions. To this end we introduce the following notion, termed due to an effect which Forney (1974) calls remerging to the all-zero path.

Definition. A subspace $L \subseteq Y^\omega$ is referred to as a remergable space provided that for every $w \in A(L)$ there is a $v \in Y^*$ such that $w \cdot v \cdot 0^\omega \in L$.

Lemma 6.1 (Staiger 1980b). *A subspace $L \subseteq Y^\omega$ is resynchronizable iff L is a remergable Σ-space.*

Proof. Let L be resynchronizable. Then for every $w \in A(L)$ there is a $v \in Y^*$ such that $L \subseteq L/w \cdot v$. Since $0^\omega \in L$, we have $w \cdot v \cdot 0^\omega \in L$. Now regard the word $0 \in Y$. It holds $0 \in A(L)$. Consequently, there is a v_0 such that $L \subseteq L/0 \cdot v_0$. Then $0^\omega \in L/0 \cdot v_0$, and Proposition 4.2(c) yields $L \subseteq L/0 \cdot v_0 = L/0^{1+|v_0|}$.

Conversely, let $L \subseteq L/0^n$ for some $n > 0$, and without loss of generality let for every $w \in A(L)$ exist a $v \in Y^*$ such that $w \cdot v \cdot 0^\omega \in L$ and $|w \cdot v| = i \cdot n$ for a suitable $i \in N$. Then $0^\omega \in L/w \cdot v$. Hence again Proposition 4.2(c) shows that $L/w \cdot v = L/0^{|w \cdot v|}$. Since $|w \cdot v| = i \cdot n$, we obtain $L \subseteq L/0^{i \cdot n} = L/w \cdot v$. ∎

We regard still another property of subsets of Y^ω.

Definition. A subset $F \subseteq Y^\omega$ is finite-state provided the number of different nonempty states F/w ($w \in A(F)$) of F is finite.

Trachtenbrot (1962) was the first one who had investigated finite-state subsets of Y^ω. He put them into a connection with subsets of Y^ω definable by finite automata (cf. Trachtenbrot and Barzdin, 1973). In particular (see also Lindner and Staiger, 1977), he has shown that a closed subset $F \subseteq Y^\omega$ is finite-state iff F is definable by a finite automaton. We quote here some properties of closed finite-state subsets of Y^ω needed in the sequel.

PROPOSITION 6.2. *Let $F \subseteq Y^\omega$ be finite-state and closed, $w \in Y^*$. Then F/w is also finite-state and closed, and if $F/w \neq \emptyset$, then F/w contains an ultimately periodic sequence, i.e., there are $u, v \in Y^*$ such that $v \cdot u^\omega \in F/w$.*

Proof. Equation (12) shows that F/w is also closed and, clearly, the number of states of F/w does not exceed the number of states of F. Now, let $\beta \in F/w$. Then there are words v, u such that $v, v \cdot u \in A(\beta)$ and $\emptyset \neq F/w \cdot v = F/w \cdot v \cdot u$. Thus, $F/w \cdot v = F/w \cdot v \cdot u^j$ for every $j \in N$. Therefore, we have $v \cdot u^j \in A(F/w)$ for all $j \in N$. Since F/w is closed, $v \cdot u^\omega \in F/w$. ∎

A finite-state space cannot have infinitely many zerostates. This yields

PROPOSITION 6.3. *Every finite-state subspace of Y^ω is a periodic subspace.*

Now we are able to prove the main theorem of this section. This theorem gives the announced defining properties of convolutional codes, and shows that the conditions of Theorem 3.1 are also sufficient to specify convolutional codes among subspaces of Y^ω.

THEOREM 6.4 (Staiger 1980b; 1982). *Let L be a subspace of Y^ω. Then the following three conditions are equivalent:*

(a) *L is ultimately connected (i.e., resynchronizable).*

(b) *There are a finite subspace $B \subseteq Y^* \cdot 0^\omega$ and an $n > 0$ such that $L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B$.*

(c) *L is a finite-state Σ-space.*

Proof. (a) → (b). Since L is resynchronizable, L is a Σ-space. Hence there are $k \in N$ and $n > 0$ such that $L \subseteq L/0^n$ and $L/0^{k+n} = L/0^k$. According to Lemma 6.1 we choose for every $w \in A(L) \cap Y^{k+n}$ a $v \in Y^*$ such that $w \cdot v \cdot 0^\omega \in L$. Letting B be the finite space spanned by all these sequences, $B \subseteq Y^* \cdot 0^\omega$. By construction $s_B(k+n) \geqslant s_L(k+n)$, thus the B-construction (Lemma 5.2) proves that (b) is satisfied.

(b) → (c) Since B is a finite subspace of $Y^* \cdot 0^\omega$, there is an $m \in N$ such that $B \subseteq Y^m \cdot 0^\omega$. As mentioned in the proof of Lemma 5.12 every

$w \in A(L)$ with $|w| \leqslant j \cdot n$ already belongs to $A(\sum_{i=0}^{j-1} 0^{i \cdot n} \cdot B)$. Conequently, there is a $v \in Y^*$ with $|v| \leqslant m$ such that $w \cdot v \cdot 0^\omega \in L$, where the latter condition is equivalent to $L/w = L/0^{|w|} + v \cdot 0^\omega$. Thus $\{L/w: w \in A(L)\} \subseteq \{(L/0^j) + v \cdot 0^\omega : j \in N \text{ and } v \in Y^m\}$. The assertion now follows from the fact that, since L is a $\sum$-space, the number of distinct $L/0^j$ is finite.

(c) $\rightarrow$ (a) Let $L \subseteq L/0^n$. Since L is a finite-state subspace, in virtue of Proposition 6.2 for every $w \in A(L)$ there are $u, v \in Y^*$ such that $\beta = w \cdot v \cdot u^\omega \in L$. Without loss of generality we may assume $|u|$ to be a multiple of n greater than $|w|$. Then $\xi = 0^{|u|} \cdot w \cdot v \cdot u^\omega \in L$, too. Hence, $\beta - \xi = (w \cdot v \cdot u - 0^{|u|} \cdot w \cdot v) \cdot 0^\omega \in L$ and $w \in A(\beta - \xi)$. Now the assertion follows from Lemma 6.1. ∎

Theorem 6.4 yields several consequences. First, it gives three equivalent defining properties for subspaces of $\mathrm{GF}(q)^\omega$ to be a convolutional code. Second, if we compare the first and the third condition utilizing Lemma 6.1, we obtain

COROLLARY 6.5. *Let L be a Σ-space, then L is finite-state iff L is remergable.*

Moreover, as the (b) $\rightarrow$ (c)-part of the proof of Theorem 6.4 shows, in the case of a finite-state Σ-space L there is a universal bound on the length of the shortest word v_w satisfying $w \cdot v_w \cdot 0^\omega \in L$ when $w \in A(L)$. Thus we call

$$m_L =_{\mathrm{df}} \max_{w \in A(L)} \min\{|v|: w \cdot v \cdot 0^\omega \in L\} \tag{23}$$

the remerging length of a subspace $L \subseteq Y^\omega$. The reader will immediately observe that this definition is a slight modification of Massey's (1963) constraint length n_A. From Corollary 6.5, then, the natural question arises whether there is a connection between the remerging length and the number of states of a Σ-space L. Before proceeding to an answer to this question we shall regard a third consequence of Theorem 6.4 which considers delay-free subspaces. To this end we introduce the following notion.

DEFINITION. A subset $F \subseteq Y^\omega$ is called strongly connected, provided for every $w \in A(F)$ there is a $v \in Y^*$ such that $F/w \cdot v = F$.

Strongly connected subsets of Y^ω have been investigated before (Staiger 1980a) in parallel with ultimately connected (resynchronizable) subsets of Y^ω.

We shall prove next a strengthened version of Theorem 9.91 in Lindner and Staiger (1977) and a statement claimed there in connection with the theorem. First, we can restate Lemma 6.1 in the case of strongly connected subspaces.

LEMMA 6.6. (a) *A subspace $L \subseteq Y^\omega$ is strongly connected iff L is remergable and $L = L/0^n$ for some $n > 0$.*

(b) *A subspace $L \subseteq Y^\omega$ is strongly connected iff L is ultimately connected and $L = L/0^n$ for some $n > 0$.*

The proof of statement (a) is easily done by changing in the proof of Lemma 6.1 all inclusions $\subseteq$ to equality, and (b) is an immediate consequence of (a) and Lemma 6.1.

THEOREM 6.7. *Let L be a subspace of Y^ω. Then the following three conditions are equivalent.*

(a) *L is strongly connected.*

(b) *There are a finite subspace $B \subseteq Y^* \cdot 0^\omega$ and $n > 0$ such that $L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B$ and $\log s_B(n) = n \cdot H_L$.*

(c) *L is a finite-state delay-free Σ-space.*

Proof. Since our theorem is merely an extended version of Theorem 6.4, we confine ourselves to the proof of the additional parts.

(a) $\rightarrow$ (b). Let $L = L/0^n$ and $L = \sum_{i=0}^{\infty} 0^{i \cdot n} \cdot B$ for some finite $B \subseteq Y^* \cdot 0^\omega$. Then $A(B) \cap Y^n = A(L) \cap Y^n$, and Eq. (22) implies $n \cdot H_L = \log s_L(n) = \log s_B(n)$.

(b) $\rightarrow$ (c). From (b) it follows $L \subseteq L/0^n$ and $n \cdot H_L = \log s_B(n) \leqslant \log s_L(n)$. The proof is then completed by Proposition 5.9.

(c) $\rightarrow$ (a). This is a consequence of Lemma 6.6(b). ∎

One easily verifies that for delay-free convolutional codes the condition (b) of Theorem 3.1 can be strengthened to $L = L/0^n$ or, equivalently, $L \cap 0^n \cdot Y^\omega = 0^n \cdot L$. Thus, the code L_1 of Example 3.1 is delay-free whereas the code L_2 of Example 3.2 is not.

Next, we deal with the relations between finite-state spaces and finite-state Σ-spaces. We have seen that finite-state spaces are periodic spaces. Thus, our aim is to prove an analogue to Theorem 4.12 the latter relating together periodic spaces and Σ-spaces. Before we proceed to this goal, we need some further properties of finite-state subsets (cf. Lindner and Staiger, 1977; and Staiger 1983b).

PROPOSITION 6.8. *Let E, F be finite-state subsets of Y^ω. Then $E \cup F$, $E \cap F$, and $E + F$ are also finite-state.*

Proof. In virtue of Eqs. (8) and (9), $E \cup F$ and $E \cap F$ cannot have more than $\text{card}\{(E/v, F/u) : v, u \in Y^*\}$ states, and in virtue of Eq. (13), $F + E$ cannot have more than $2^{\text{card}\{(E/v, F/u) : u, v \in Y^*\}}$ states. ∎

PROPOSITION 6.9. *A finite subset $E \subseteq Y^\omega$ is finite-state iff it consists of only ultimately periodic sequences.*

Proof. Clearly $\{u \cdot v^\omega\}$ is finite-state. So the condition is sufficient. Let $\eta \in E$. Since E is finite, there is a $w \in Y^*$ such that $w \cdot (E/w) = E \cap w \cdot Y^\omega = \{\eta\}$. Then, by Proposition 6.2, E/w contains an ultimately periodic sequence $u \cdot v^\omega$. Consequently, $\eta = w \cdot u \cdot v^\omega$. ∎

THEOREM 6.10. *A subspace $L \subseteq Y^\omega$ is finite-state iff there are a finite-state Σ-space L' and a finite finite-state space B such that $L = L' + B$.*

Proof. In virtue of Proposition 6.8 the condition is sufficient. As in the proof of Theorem 4.12 we construct L' as $L \cap 0^m \cdot Y^\omega$, where $m \in N$ is sufficiently large. Then L' is a Σ-space and by Proposition 6.8 also finite-state. Now from every coset $L \cap w \cdot Y^\omega$ of L' in L choose an ultimately periodic sequence. Then the finite space B spanned by these sequences consists only of ultimately periodic sequences and is, hence, finite-state. Clearly, $L = L' + B$. ∎

Proposition 6.9 shows that there are only countably many finite finite-state subsets of Y^ω. Thus Theorem 6.10 together with Theorem 6.4 yields

COROLLARY 6.11. *There are countably many finite-state subspaces of Y^ω.*

Now, we investigate the connection between the remerging length m_L defined in Eq. (23) and the number of states of a finite-state Σ-space L. First, we give a relation between the remerging lengths m_L of the space and $m_{\hat{L}}$ of its delay-free closure:

$$m_{\hat{L}} \leqslant m_L \leqslant m_{\hat{L}} + \max\{\delta_L \cdot \operatorname{per} L - 1, 0\}. \tag{24}$$

Proof. According to (23), $m_{L'} = \max_{L'/w \neq \varnothing} \min\{|v|: 0^\omega \in (L'/w)/v\}$ for an arbitrary subspace L' of Y^ω. Now, the first inequality follows from $\{\hat{L}/w: w \in A(\hat{L})\} \subseteq \{L/w: w \in A(L)\}$. Since $\hat{L} = L/0^k$ for $k = \delta_L \cdot \operatorname{per} L$, Lemma 5.8(b) shows that $L/u = \hat{L}/u$ whenever $|u| \geqslant k$. Thus $w \in A(L)$ implies either $L/w = L$ or there is a u, $|u| < \delta_L \cdot \operatorname{per} L$ such that $w \cdot u \in A(L)$ and $L/w \cdot u = \hat{L}/w \cdot u$. This proves the second inequality. ∎

We add an example showing that the bounds in Eq. (24) are tight.

EXAMPLE 6.1. (a) Let $\hat{L}$ be a delay-free space and consider $L =_{\mathrm{df}} 0^k \cdot \hat{L}$. Clearly, $\delta_L = k$ and $m_{\hat{L}} = m_L$.

(b) Let $Y =_{\mathrm{df}} \mathrm{GF}(2)$, and define the convolutional code L via Eq. (6) through the generator matrix $\mathfrak{G}$: set in (5),

$$G_0 = \begin{pmatrix} 10 \\ 00 \end{pmatrix},\ G_1 = \begin{pmatrix} 00 \\ 10 \end{pmatrix},\ G_2 = \cdots = G_{v-1} = \begin{pmatrix} 00 \\ 00 \end{pmatrix},\quad \text{and}\quad G_v = \begin{pmatrix} 01 \\ 00 \end{pmatrix}.$$

Then $L/0^{2v+2} = \hat{L} = Y^\omega$ and $10^\omega \notin L/0^{2v+1}$. Hence per $L = 1$, $\delta_L = 2v + 2$; moreover, $m_{\hat{L}} = 1$ and $m_L = 2v + 1$ for $1 \cdot 0^{2v} \cdot 1 \cdot 0^\omega \in L$, but $1 \cdot v \cdot 0^\omega \notin L$ for any v with $|v| \leqslant 2v$.

The next lemma gives further insight into the remerging length and simplifies its evaluation.

LEMMA 6.12. *Let L be a delay-free Σ-space. Then*

$$m_L = \max_{w \in M} \min\{|v|: w \cdot v \cdot 0^\omega \in L\},$$

where $M =_{\mathrm{df}} \{w: w \in A(L)$ and $0 < |w| \leqslant n\}$.

Proof. Let m denote the right-hand side of the equation. Then, clearly, $m_L \geqslant m$. We show by induction on the word length $|w|$, that for every $w \in A(L)$ there is a $\beta_w \in Y^m \cdot 0^\omega$ such that $w \cdot \beta_w \in L$. Let this latter property be proved for all $w \in A(L)$ with $|w| \leqslant i \cdot n$. Now let $w \cdot y \in A(L)$ such that $|w| = i \cdot n$ and $|y| \leqslant n$. Then there is a $\beta_w \in Y^m \cdot 0^\omega$ such that $w \cdot \beta_w \in L$. Let $\beta_w = x \cdot \xi$, where $|x| = |y|$. Then $w \cdot x \in A(L)$ and also $(w \cdot y - w \cdot x) \in A(L)$. Since $L/0^{i \cdot n} = L$, we obtain $(y - x) \in A(L)$, where $|(y - x)| \leqslant n$. Thus $(y - x) \cdot \beta_{y-x} \in L$ for some $\beta_{y-x} \in Y^m \cdot 0^\omega$. This yields $0^{|w|} \cdot (y - x) \cdot \beta_{y-x} + w \cdot \beta_w = 0^{|w|} \cdot (y - x) \cdot \beta_{y-x} + w \cdot x \cdot \xi = w \cdot y \cdot (\beta_{y-x} + \xi) \in L$. It is now easy to see that $\beta_{y-x} + \xi \in Y^m \cdot 0^\omega$. ∎

The last part of this section is concerned with two parameters of convolutional codes (considered as subspaces of $\mathrm{GF}(q)^\omega$) related to the constraint length of the code (whichever reasonable definition one would choose). The first is the remerging length, and the second is the state space dimension, and hence closely related to Forney's (1970) definition of the constraint length which is the dimension of the state space of a reduced encoder for the code.

In what follows we confine ourselves to delay-free Σ-spaces for two reasons. First, the state set of a delayed Σ-space L consists, as was pointed out in the previous section, of the state set of its delay-free closure L plus an additional finite set of preliminary states and is therefore more difficult to treat and provides no more insight into the ultimate state structure of the subspace L.

Second, Forney's reduction of encoders includes the elimination of delay. Since we are dealing with convolutional codes, in further investigations we

will assume $Y = \mathrm{GF}(q)$ and, moreover, all subspaces $L \subseteq \mathrm{GF}(q)^\omega$ to be linear spaces over $\mathrm{GF}(q)$.

In the sequel let L be a delay-free Σ-space with period $\mathrm{per}\, L = n$. We define

$$Z_i =_{\mathrm{df}} \{L/w\colon w \in A(L) \cap Y^{i \cdot n}\}$$

and

$$Z_L =_{\mathrm{df}} \bigcup_{i=0}^{\infty} Z_i.$$

We will refer to Z_L as the state space of L.

PROPOSITION 6.13. (a) $Z_i \subseteq Z_{i+1}$,

(b) $Z_i = Z_L$ *iff* $Z_i = Z_{i+1}$.

Proof. (a) Follows from $L = L/0^n$.

(b) If $Z_i = Z_L$, then $Z_i = Z_{i+1}$. Now let $Z_i = Z_{i+1}$. It suffices to show that then $Z_{i+1} = Z_{i+2}$. Let $L/w \cdot v \in Z_{i+2}$, where $|w| = (i+1) \cdot n$ and $|v| = n$. Since $Z_{i+1} = Z_i$, we have $L/w = L/u$ for some $u \in Y^{i \cdot n}$. Consequently, $L/w \cdot v \in Z_{i+1}$. ∎

PROPOSITION 6.14. (a) *For every* $i \in N$, Z_i *is a linear space over* $\mathrm{GF}(q)$.

(b) Z_L *is a linear space over* $\mathrm{GF}(q)$.

Proof. (a) We show only that Z_i is closed under addition; multiplication by scalar is proved in a similar way. Let $w, v \in A(L) \cap Y^{i \cdot n}$. Then by linearity $w + v \in A(L) \cap Y^{i \cdot n}$. From Corollary 4.3, it follows $L/(w+v) = L/w + L/v$. Thus $L/w + L/v \in Z_i$ if $L/w, L/v \in Z_i$.

(b) follows from (a), Proposition 6.13(a), and the definition of Z_L. ∎

Next, we give some relations between the dimensions of the spaces Z_i and Z_L and the remerging length of the code L.

LEMMA 6.15. (a) *If* $i \geqslant \dim Z_L$ *then* $Z_i = Z_L$.

(b) *If* $i < \dim Z_L/(H_L \cdot \mathrm{per}\, L)$ *then* $Z_i \neq Z_L$.

Proof. In both cases we prove the assertion by contraposition.

(a) Let $Z_i \neq Z_L$. According to Proposition 6.13(b) or every $j \leqslant i$ the proper inclusion $Z_j \subset Z_{j+1}$ holds true. Consequently, $\dim Z_{j+1} \geqslant \dim Z_j + 1$, and hence $\dim Z_L \geqslant \dim Z_{i+1} > i$.

(b) Since $L = L/0^{i \cdot n}$, Eq. (22) implies $\mathrm{card}(A(L) \cap Y^{i \cdot n}) =$

card $Y^{i\cdot n\cdot H_L}$, whence $\dim Z_i \leqslant i\cdot n\cdot H_L$. Now, if $Z_i = Z_L$ then $\dim Z_L = \dim Z_i \leqslant i\cdot n\cdot H_L$. ∎

LEMMA 6.16. (a) *If* $Z_i = Z_L$ *then* $m_L < (i+1)\cdot \operatorname{per} L$.

(b) *If* $Z_i \neq Z_L$ *then* $m_L > i\cdot \operatorname{per} L$.

Proof. (a) In virtue of Lemma 6.12 it suffices to show that for every $w\in A(L)$, $|w|\leqslant n$ there is a $v\in Y^*$ such that $|v| < n\cdot(i+1)$ and $w\cdot v\cdot 0^\omega \in L$. Let $w\in A(L)$ and $|w|\leqslant n$. Then there is a v such that $w\cdot v\in A(L)\cap Y^{(i+1)\cdot n}$. Now, since $Z_i = Z_{i+1}$, we have a $u\in A(L)\cap Y^{i\cdot n}$ such that $L/w\cdot v = L/u$. Then $L/(w\cdot v - 0^n\cdot u) = L$ follows from $L = L/0^n$, and hence $(w\cdot v - 0^n\cdot u)\cdot 0^\omega = w\cdot(v - 0^{n-|w|}\cdot u)\cdot 0^\omega \in L$, where $|(v - 0^{n-|w|}\cdot u)| < (i+1)\cdot n$.

(b) From $m_L \leqslant i\cdot n$ we obtain that for every $w\in A(L)\cap Y^n$ there is a $v_w\in Y^{i\cdot n}$ such that $L/w\cdot v_w = L$.

Now consider $L/u'\in Z_{i+1}$, $|u'| = (i+1)\cdot n$. Then there are $w\in A(L)\cap Y^n$ and $u\in Y^{i\cdot n}$ such that $w\cdot u = u'$. Consequently, $L/w\cdot u = L/w\cdot u - L/w\cdot v_w = L/0^n\cdot(u - v_w)$. Since $L/0^n = L$, we have $L/w\cdot u = L/(u - v_w)\in Z_i$. This proves $Z_{i+1} = Z_i$ and, by Proposition 6.13(b), we obtain $Z_i = Z_L$. ∎

Now we can prove our theorem stating a general connection between Z_L and m_L.

THEOREM 6.17. *If* L *is a delay-free convolutional code. Then*

$$\frac{\dim Z_L}{H_L} - \operatorname{per} L < m_L < \operatorname{per} L\cdot(1+\dim Z_L).$$

Proof. First, we prove the second inequality using the (a)-parts of the Lemmas 6.15 and 6.16 If $i = \dim Z_L$, then $Z_i = Z_L$ and we obtain $m_L < (i+1)\cdot \operatorname{per} L$.

The proof of the other inequality uses likewise the (b)-parts of the just-mentioned lemmas. Let j be the largest integer smaller than $\dim Z_L/(H_L\cdot \operatorname{per} L)$. Then $Z_j \neq Z_L$ and hence $m_L > j\cdot \operatorname{per} L$. By the definition of the integer j, we have $j \geqslant \dim Z_L/(H_L\cdot \operatorname{per} L) - 1$, which proves the inequality. ∎

We conclude this section by showing that the bounds of Theorem 6.17 are tight.

EXAMPLE 6.2. We use the definition of L via Eq. (6) through the generator matrix $\mathfrak{G}$.

(a) Let $G_0 = (01)$, $G_1 = \cdots = G_{v-1} = (00)$, $G_v = (10)$. One easily verifies that $\operatorname{per} L = 2$, $H_L = \frac{1}{2}$, and that Z_L is spanned by the v states

$$L/010^{2\mu} = L + 0^{2(v-\mu-1)} \cdot 10^{\omega} \qquad (0 \leqslant \mu < v).$$

Thus $\dim Z_L = v$, and $m_L = 2v - 1$ satisfies

$$m_L = \frac{\dim Z_L}{H_L} - \operatorname{per} L + 1.$$

(b) If we exchange in (a) G_0 and G_v, we obtain a space L' having the same parameters except $m_{L'} = 2v + 1$, thus verifying

$$m_{L'} = \operatorname{per} L' \cdot (1 + \dim Z_L) - 1.$$

7. Remergable Subspaces

In Section 6 we have characterized convolutional codes as Σ-spaces having the remerging property. This property is important from a practical point of view: Linear codes not having the remerging property are susceptible to an unavoidable (not due to a bad choice of an encoder) infinite error propagation, i.e., decoding errors in a finite initial part of the received sequence can cause the decoder to make necessarily further on an infinite number of decoding errors. This effect can only be avoided if the code space is remergable. In the present section we will consider the class of all remergable subspaces (not only Σ-spaces).

Notation. By $\mathscr{L}_m$ we denote the class of all remergable subspaces of Y^{ω}.

The first important feature of remergable subspaces is the following.

Lemma 7.1. *Let L be an arbitrary subspace of Y^{ω}. Then $C(L \cap Y^* \cdot 0^{\omega})$ is the greatest remergable subspace contained in L.*

Proof. Clearly, $C(L \cap Y^* \cdot 0^{\omega})$ is a subspace of Y^{ω}. We have $A(C(L \cap Y^* \cdot 0^{\omega})) = A(L \cap Y^* \cdot 0^{\omega}) = \{w: w \cdot v \cdot 0^{\omega} \in L \text{ for some } v \in Y^*\}$ which implies that $C(L \cap Y^* \cdot 0^{\omega})$ is remergable. Moreover, if L' is any remergable subspace contained in L then $A(L') \subseteq A(L \cap Y^* \cdot 0^{\omega})$, and by Lemma 1.1 we get $L' \subseteq C(L \cap Y^* \cdot 0^{\omega})$, ∎

The following reformulation of the definition is an immediate consequence of the preceding lemma.

Corollary 7.2. *$L \in \mathscr{L}_m$ iff $C(L \cap Y^* \cdot 0^{\omega}) = L$.*

We obtain a closure property of $\mathscr{L}_m$.

LEMMA 7.3. *Let $(L_i)_{i\in N}$ be a family of remergable subspaces. Then $\sum_{i=0}^{\infty} L_i$ is also a remergable subspace.*

The proof of the lemma is readily verified by the inequality

$$C\left(\sum_{i=0}^{\infty} L_i \cap Y^* \cdot 0^\omega\right) \supseteq \sum_{i=0}^{\infty} C(L_i \cap Y^* \cdot 0^\omega) = \sum_{i=0}^{\infty} L_i.$$

This yields an estimate of the cardinality of $\mathscr{L}_m$.

COROLLARY 7.4. *There are uncountably many remergable subspaces of Y^ω.*

Proof. Clearly, $0^i \cdot Y \cdot 0^\omega \in \mathscr{L}_m$ for every $i \in N$. Hence

$$\sum_{i\in M} 0^i \cdot Y \cdot 0^\omega \in \mathscr{L}_m \qquad \text{for every } M \subseteq N. \quad \blacksquare$$

The class $\mathscr{L}_m$ is, unlike the other classes of subspaces hitherto considered, not closed under intersection. We give an example.

EXAMPLE 7.1. We start from the space L_4 of Example 4.4 and consider

$$L_{4,\text{even}} =_{\text{df}} \{\beta: \beta(p^i) = \beta(p^{i+1}) \text{ for } i > 0 \text{ even}, p \text{ prime, and } \beta(k) = 0, \text{ otherwise}\}$$

and $L_{4,\text{odd}}$, being defined similarly. By definition $L_{4,\text{even}}$ and $L_{4,\text{odd}}$ are remergable, but $L_4 = L_{4,\text{even}} \cap L_{4,\text{odd}}$ satisfies $C(L_4 \cap Y^* \cdot 0^\omega) = \{0^\omega\}$, thus $L_4 \notin \mathscr{L}_m$.

We get only a weaker property.

PROPOSITION 7.5. *Let L be a remergable subspace of Y^ω. Then $L \cap 0^k \cdot Y^\omega$ is remergable for arbitrary $k \in N$.*

The following example, however, shows that the space $0^k \cdot Y^\omega$ in Proposition 7.5 cannot be changed to an arbitrary remergable Σ-space.

EXAMPLE 7.2. Let $Y = \text{GF}(2)$. Consider $L = \{00, 11\}^\omega$ and $L' = Y \cdot L$. Both spaces are remergable and even finite-state. Moreover, L is a Σ-space. But $L \cap L' = \{0^\omega, 1^\omega\}$ is neither in $\mathscr{L}_m$ nor in $\mathscr{L}_\Sigma$.

In Theorem 6.4 we have seen that the property to be a Σ-space forces a remergable space to be finite-state. Next we shall prove that this is already valid for periodic remergable spaces.

THEOREM 7.6. *Let L be a periodic remergable subspace of Y^ω. Then L is finite-state.*

Proof. Let $L \in \mathscr{L}_{\text{per}} \cap \mathscr{L}_m$. According to the proof of Theorem 4.12 there is an $m \in N$ such that $L' =_{\text{df}} L \cap 0^m \cdot Y^\omega$ is a Σ-space. As $L' = L \cap 0^m \cdot Y^\omega$ is also a remergable space, Theorem 6.4 proves that L' is finite-state.

Now consider the finitely many cosets $L \cap w \cdot Y^\omega$ ($|w| = m$, $w \in A(L)$) of L' in L. Since L is remergable, every such coset contains a sequence $\beta \in Y^* \cdot 0^\omega$. Hence L is the sum $L' + E$ of a finite-state space L' and a finite set $E \subseteq Y^* \cdot 0^\omega$. Following Proposition 6.9 this latter set E is finite-state, and finally Proposition 6.8 proves that $L = L' + E$ is finite-state. ∎

We conclude with a remark on the closure properties of the class $\mathscr{L}_{\text{per}} \cap \mathscr{L}_m$. Both of the classes $\mathscr{L}_{\text{per}}$ and $\mathscr{L}_m$ are closed under $+$, hence $\mathscr{L}_{\text{per}} \cap \mathscr{L}_m$ is also closed under $+$. Example 7.2 shows that $\mathscr{L}_{\text{per}} \cap \mathscr{L}_m$ is not even closed under intersection with remergable Σ-spaces.

8. INTERCONNECTIONS BETWEEN THE CLASSES OF SUBSPACES

This last part of our paper summarizes the inclusion relations of the classes of subspaces hitherto mentioned. Moreover, we prove that except the inclusions and equalities presented in Fig. 4, no other inclusion result holds. First we introduce another abbreviation.

Notation. The class of all finite-state subspaces of Y^ω will be denoted by $\mathscr{L}_{\text{fs}}$.

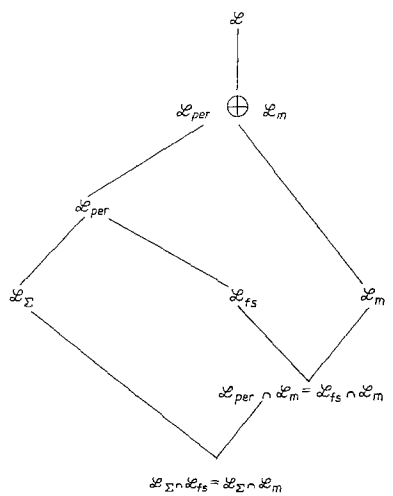

FIG. 4. Inclusion relations between the classes of subspaces.

We have to consider the classes $\mathscr{L}$ (of all subspaces), $\mathscr{L}_{\text{per}}$, $\mathscr{L}_{\Sigma}$, $\mathscr{L}_{\text{fs}}$, $\mathscr{L}_{m}$, and their intersections. According to Theorem 4.12 and Proposition 6.3 we have

$$\mathscr{L}_{\text{per}} \supseteq \mathscr{L}_{\Sigma} \cup \mathscr{L}_{\text{fs}}.$$

Thus, there remain the following intersection classes

$$\mathscr{L}_{\Sigma} \cap \mathscr{L}_{m} = \mathscr{L}_{\Sigma} \cap \mathscr{L}_{\text{fs}} \tag{25}$$

(cf. Theorem 6.4), and according to Theorem 7.6,

$$\mathscr{L}_{\text{per}} \cap \mathscr{L}_{m} = \mathscr{L}_{\text{fs}} \cap \mathscr{L}_{m}. \tag{26}$$

We have estimated the following cardinalities (Theorem 5.13, Corollary 7.4, and Corollary 6.11):

PROPOSITION 8.1. (a) *The classes* $\mathscr{L}$, $\mathscr{L}_{\text{per}}$, $\mathscr{L}_{\Sigma}$, *and* $\mathscr{L}_{m}$ *are uncountable.*

(b) $\mathscr{L}_{\text{fs}}$ *and its subclasses* $\mathscr{L}_{\text{per}} \cap \mathscr{L}_{m}$, *and* $\mathscr{L}_{\Sigma} \cap \mathscr{L}_{m}$ *are countable sets.*

Moreover, except $\mathscr{L}_{m}$ and $\mathscr{L}_{\text{per}} \cap \mathscr{L}_{m}$, all classes are closed under intersection. Hence

$$\mathscr{L}_{\Sigma} \cap \mathscr{L}_{m} \subset \mathscr{L}_{\text{per}} \cap \mathscr{L}_{m} \subset \mathscr{L}_{\text{fs}}.$$

This implies via (25) and (26),

$$\begin{aligned} \mathscr{L}_{\text{per}} \cap \mathscr{L}_{m} &\nsubseteq \mathscr{L}_{\Sigma}, \\ \mathscr{L}_{\text{fs}} &\nsubseteq \mathscr{L}_{m}, \end{aligned} \tag{27}$$

and

$$\mathscr{L}_{m} \nsubseteq \mathscr{L}_{\text{per}}. \tag{28}$$

Together with

$$\mathscr{L}_{\Sigma} \nsubseteq \mathscr{L}_{\text{fs}}$$

which holds for cardinality reasons, we have established the lower part of the diagram in Fig. 4.

Finally, we consider the class

$$\mathscr{L}_{\text{per}} \oplus \mathscr{L}_{m} =_{\text{df}} \{L + L' : L \in \mathscr{L}_{\text{per}} \text{ and } L' \in \mathscr{L}_{m}\},$$

being the smallest class of subspaces closed under $+$ and encompassing all

the spaces that we have investigated in this paper. From (27) and (28) we obtain

$$\mathscr{L}_{\text{per}} \cup \mathscr{L}_m \subset \mathscr{L}_{\text{per}} \oplus \mathscr{L}_m.$$

It is interesting to note that

$$\mathscr{L}_{\text{per}} \oplus \mathscr{L}_m \subset \mathscr{L}.$$

This is explained by the fact that $\mathscr{L}_{\text{per}} \oplus \mathscr{L}_m$ is not closed under intersection. To prove our assertion we consider the space L_4 of Example 7.1.

EXAMPLE 8.1. The space L_4 is the intersection of the two remergable spaces $L_{4,\text{even}}$ and $L_{4,\text{odd}}$ and has $\{0^\omega\}$ as its greatest remergable subspace. Thus $L_4 \in \mathscr{L}_{\text{per}} \oplus \mathscr{L}_m$ would imply $L_4 \in \mathscr{L}_{\text{per}}$. But according to Example 5.4, L_4 is not periodic.

9. CONCLUSIONS

In this paper we have investigated several classes of subspaces of $\text{GF}(q)^\omega$ in connection with the class of convolutional codes. Naturally, then the question arises whether there do exist subspaces having better error correction properties than convolutional codes. Since every periodic (finite-state) subspace is the sum of a Σ-space (finite-state Σ-space) and a finite space (Theorems 4.12 and 6.10), there is no use in utilizing periodic spaces or finite-state spaces instead of Σ-spaces or finite-state Σ-spaces, respectively. Neither do they achieve higher transmission rates, nor better distance properties than the maximum Σ-space contained in them.

Concerning the distance properties, it is really possible to construct Σ-spaces with $d_{\text{free}} = \infty$. Those spaces as well as all other infinite-state Σ-spaces cannot possess the remerging property.

These properties imply the following disadvantages: An infinite-state subspace cannot be encoded by a finite-state encoding device, since any finite-state machine can produce as output sets only finite-state subsets of Y^ω (cf. Lindner and Staiger, 1977). The second disadvantage consists in the lack of the remerging property mentioned in Section 7.

Thus, if one looks for codes with good decoding properties, one should always look for codes having the remerging property. This implies—in the case of linear codes—that one has to choose either finite-state or, otherwise, aperiodic subspaces. In the former case as pointed out, one is confined to ordinary convolutional codes, whereas in the latter case one has codes with a time-varying structure (cf. Forney, 1974; Zigangirov, 1974). This effect complicates the encoding circuit as well as the synchronization recovery and, therefore, makes such codes not useful for practical purposes.

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