Distortional hardening plasticity model for paperboard

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A B S T R A C T

A distortional hardening elasto-plastic model at finite strains suitable for modeling of orthotropic materials is presented. As a prototype material, paperboard is considered. An in-plane model is established. The model developed is motivated from non-proportional loading tests on paperboard where the paperboard is pre-strained in one direction and then loaded in the perpendicular direction. A softening effect is revealed in the pre-strained samples. The observed experimental findings cannot be accurately predicted by current models for paperboard. To be able to model the softening effects, a yield surface based on multiple hardening variables is introduced. It is shown that the model parameters can be obtained from simple uniaxial experiments. The model is implemented in a finite element framework which is used to illustrate the behavior of the model at some specific loading situations and is compared with strain fields obtained from Digital Image Correlation experiments.

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1. Introduction

Continuum-based constitutive models provide the macroscopic observable properties, e.g. force and stretch, resulting from the average behavior of the micro-structure. In this work, an anisotropic continuum-based material model suitable for fibrous materials is considered. Here focus is on paperboard materials, but the developed model can be used for a range of orthotropic materials. Paperboard is a heterogeneous material, where the heterogeneity stems from the manufacturing process where cellulose fibers placed on a traversing web are dried and pressed. The inhomogeneity and directional material dependence is due to the distribution and uneven drying of the fibers.

Modeling of paperboard is an active research area driven by the industry to improve converting and filling processes. In industrial converting operations, the paperboard experience complex load histories. To accurately predict the material behavior during converting operations is a challenging task. One important converting operation is the creasing process, which has been studied by several authors, and is critical for obtaining well formed liquid filled packages without defects. The creasing operation reduces the maximal bending moment and the deeper the scored line is creased, the more the maximal bending moment is reduced, cf. Cavlin (1988), Cavlin et al. (1997) and Nagasawa et al. (2003). The crease depth is limited by the occurrence of in-plane surface cracks.

Modeling of the in-plane fracture process in paperboard has been based on cohesive crack mechanisms in Tryding and Gustafsson (2001) and Mäkelä and Östlund (2012) and by continuum damage in Isaksson et al. (2004). During creasing the paperboard is stretched in one direction and then unloaded. In the subsequent forming process, the paperboard is stretched again, however in a direction perpendicular to the previous stretching direction. To evaluate the effect of non-proportional loading, simple non-proportional tests has been conducted on paperboard in the work herein.

Paperboard is classically characterized as an orthotropic material. The orthotropic directions are the Machine Direction (MD), Cross-machine Direction (CD) and out-of-plane direction (ZD), cf. Fig. 1. The MD and CD are referred to as the in-plane directions. The magnitude of the material properties in the MD direction are typically about 2–3 times larger compared with CD and about 100 times larger compared with ZD, cf. Stenberg (2002).

It has previously been demonstrated that continuum based approaches are able to represent the mechanical response of paperboard, cf. Xia et al. (2002), Harrysson and Ristinmaa (2008) and Mäkelä and Östlund (2003). It has been observed by e.g. Harrysson and Ristinmaa (2008), that after unloading from the non-linear region, non-recoverable strains are obtained, and therefore the use of plasticity theory is motivated. Since the paperboard is highly anisotropic, the constitutive model is inevitable required to be anisotropic. Moreover, large rotations and relatively large strains are present in industrial applications such as creasing and forming.
A number of anisotropic models are based on the Hill (1948) criterion with proportional expansion of the yield surface cf. Huang and Nygård (2010). Another well established yield surface for paperboard is the Tsai–Wu surface, cf. Tsai and Wu (1971) which also takes into account that the yielding in compression and tension differs. Accurate fit to uniaxial tests is usually obtained for these models, but investigations on how the actual yield surface develops in the stress space is usually not compared to experimental evidence. The experimental tests in this work reveals that the yield surface does not harden proportionally and therefore non-proportional hardening models are of importance for paperboard.

In Xia (2002), a continuum elasto-plastic model in combination with interfaces were used to model the creasing operation. This concept has been further studied in Huang and Nygård (2010), Nygård et al. (2009), Beex and Peerlings (2009) and Giampieri et al. (2011). The elasto-plastic model in this work is based upon the model in Xia et al. (2002) which is enhanced such that the laws of thermodynamics are fulfilled. The yield surface in Xia et al. (2002), is based upon the introduction of a set of subsurfaces in the stress-space. In this work, one internal variable is introduced for each subsurface. This is in contrast to the model by Xia et al. (2002) where only the effective plastic strain governs the hardening of all sub-surfaces. The concept of one internal variable to each subsurface allows for an anisotropic hardening of the yield surface, which is also known as distortion hardening. The derived model will allow non-proportional load histories to be taken into account, e.g. where the paperboard is stretched and unloaded in different directions. It is shown that the generalization of the yield criterion in Xia et al. (2002) to include distortion hardening can be made naturally in the thermodynamic framework. It is also shown that a calibration of the model parameters can be made using standard uniaxial tests.

To illustrate the predictive capabilities of the proposed model, the paperboard has been loaded at different angles in the plane, and also been compared with full-field measurements in two separate loading situations. The full field measurements have been obtained using Digital Image Correlation (DIC) equipment. DIC-measurements on paperboard have the potential to increase the understanding of the mechanisms present during loading of paperboard, cf. Hagman and Nygård (2012) for a recent contribution on the topic. As a particular load situation, a paperboard with a central hole has been considered in this work. The load has been applied parallel to both the MD and CD directions of the paperboard. The strain fields have been extracted from DIC-measurements and compared with simulation results using the derived material model.

The article is organized as follows, in Section 2 the experimental evidence on non-proportional tests is presented, Section 3 and 4 deals with the kinematic and thermodynamic considerations, where tensors will be considered in a Cartesian setting, i.e. following the work of Carlet (1988). Taking into account that the first and second laws of thermodynamics should be fulfilled, physical sound models and constitutive relations is developed. Section 5 presents the specific model and in Section 6 the calibration from uniaxial experiments is presented. In Sections 7 and 8, results and comparisons from uniaxial tests and DIC experiments are shown.

2. Experimental evidences

Non-proportional loading situations are present in many industrial process steps related to paperboard converting. To the authors knowledge, there is a lack of experimental results reported in the literature especially for non-proportional load situations of paperboard. As the present work is aimed to predict non-proportional hardening effects, non-proportional experimental test will be presented below.

A schematic illustration of the testing procedure of pre-straining the paperboard and then the subsequent uniaxial test is shown in Fig. 2. The large test sample is pre-strained (CD/MD) until failure, which corresponded to an average strain of: 6.5% in CD and 3.1% in MD. Several specimen are then cut out and loaded in the direction perpendicular to the original loading direction.

The pre-straining was done with a standard MTS-tensile test machine with a 160 mm wide paperboard and 145 mm clamped length. All the tests have been performed in a climate chamber with 50% moisture content and at room temperature 23 °C. The uniaxial tension tests were performed with the Th1 tensile tester (Lorentzon & Wetter Alwetron), which follows ISO 1924–3 standard, using w₀ = 15 mm wide paper and a clamped length of L₀ = 100 mm. The strain rate was 1.65%/s. The initial thickness, t₀, of the samples was determined to be t₀ = 0.38 mm. Sixteen uniaxial tension tests have been conducted in each direction for the pre-strained and non-prestrained samples.

The experimental results are shown in Fig. 3, both for the situation without pre-straining and when the samples were pre-strained. For the uniaxial MD and CD tests after pre-straining, a reduction in stiffness is observed as well as reduction in the hardening. The reduction of stiffness is approximately 25% for the MD-direction and 13% in CD. It is observed that pre-straining in MD changes the uniaxial stress–strain response in CD to a lesser extent compared with pre-straining in CD, which influences the MD response significantly. The effect of softening as shown in Fig. 3 is not well known in the literature. It should also be noted, that the thickness has been shown to remain almost constant during in-plane loading for several paperboard materials, cf. Stenberg (2002). Therefore a decoupling of the material response between the in-plane and the out-of-plane directions is assumed, i.e. zero Poisson’s ratio.
The effect of non-proportional loading, as described above, can not accurately be captured by elasto-plasticity using a single internal variable (often the effective plastic strain), cf. Xia et al. (2002) and Mäkelä and Östlund (2003). Orthotropic elasto-plastic models with a single internal variable, will overestimate the stress-strain response for samples that have been pre-strained in a perpendicular direction to the load direction. The overestimation stems from the fact that a single internal variable leads to an expansion of the yield surface and therefore an increased yield stress will be obtained upon reloading in a perpendicular direction. An attempt to simulate the pre-straining behavior using a standard orthotropic-elastic-plastic model with a Hill surface and isotropic hardening, cf. Abaqus (2012), is shown in Fig. 4.

The Hill-model clearly overestimates the response in MD for the pre-strained samples in CD. The experimental evidence shows a decreased yield stress for the pre-strained samples, whereas the Hill-model predicts an increased yield stress. Based on the experimental observations in Fig. 3, we propose a model in which the yield surface hardens non-isotropically in the stress-space. This effect is accomplished via the introduction of several internal variables, i.e. a form of distortion hardening. To reduce the complexity of the model, the reduction in elastic stiffness visible in the experimental tests will not be considered here.

Another approach for modeling the pre-straining is to use kinematic hardening for the evolution of the yield surface. Kinematic hardening has however not been considered here, but can also be introduced in the framework.

3. Kinematics

Consider a material body in the reference configuration $\Omega_0 \in \mathbb{R}^3$ at the time instance $t_0$ and at the deformed configuration $\Omega \in \mathbb{R}^3$ at time instance $t$. The non-linear map that defines the motion is given by $\phi(X,t) : \Omega_0 \times T \to \Omega$, in the time interval $T \in [t_0, t]$ and where $X$ denotes the position of a particle in the reference configuration and the position of the same particle in the current configuration is found as $x = \phi(X,t)$. The mapping of vectors in the reference configuration to the current configuration is given by the deformation gradient $F = \frac{dx}{dX}$. To model elasto-plasticity a multiplicative split of the deformation gradient into an elastic and a plastic part is assumed, i.e.

$$ F = F^e F^p, $$

where $F^e$ and $F^p$ are the elastic- and plastic-deformation gradients, respectively. The spatial velocity gradient defined as, $I = FF^{-1}$ can be additively be split into

$$ I = I^e + I^p, $$

where

$$ I^e = FF^{-1}, \quad I^p = F^p F^{-1}, $$

are referred to as the elastic and plastic velocity gradients. Further on, the polar decomposition of $F$ will be exploited and is given by

$$ F = V^e R^e, $$

where $R^e$ is the orthogonal elastic rotation tensor and $V^e$ is the symmetric positive definite left elastic stretch tensor. The elastic Finger tensor, $B^e$ is given by

$$ B^e = \frac{1}{2} (I^e + I^p) \frac{1}{2}.$$
\[ b' = F'(F')^T, \]

and will be used in the constitutive model. By using (2) and (5) the material time derivative of \( b' \) can be expressed as

\[ b^s = 2\text{sym}(bb') - 2\text{sym}(Fb'), \]

where \( \text{sym}() \) denotes the symmetric part of [ ].

The modeling framework for orthotropy will follow that outlined in Harrysson et al. (2007) and Harrysson and Ristinmaa, 2007. To model orthotropy, a set of director vectors, \( \{v_a^x, a = \{1, 2, 3\}\} \), which are aligned with the MD, CD and ZD directions and of unit length in the reference configuration are introduced. In this work, it is postulated that the director vectors evolve with the elastic rotation (see also Ask and Ristinmaa, 2008), i.e.

\[ v^x = R^x v_0^x, \]

where \( v^x \) is the director vector in the spatial configuration. The choice (7), will ensure that the director vectors will remain at unit length and orthogonal to each other during deformation. Note that in contrast to Harrysson and Ristinmaa (2007), the evolution of the director vectors are postulated in a total format instead of an incremental evolution.

A set of second order structural tensors defined as a dyadic product of the director vectors are introduced as

\[ m^x = v^x \otimes v^x, \]

and will be used in the constitutive model. By using (2) and (5) the material time derivative of \( m^x \) can be expressed as

\[ m^x = 2\text{sym}(\Omega^x m^x), \]

where

\[ \Omega^x = R^x R'^T, \]

was defined. The material time derivative of the elastic rotation tensor is found from the polar decomposition of \( F' \). Differentiation of (4) and making use of (2) results in

\[ R^x = \Omega^{-1} F' F - V^x R'^x. \]

The elastic stretch tensor can be written as \( V^e = \sqrt{b'} \) and the time derivative of \( V^e \) can be expressed as

\[ V^e = \frac{\partial \sqrt{b'}}{\partial b'} b' = \Omega^e b'. \]

The fourth order tensor \( \Omega^e \) can be computed by taking advantage of the spectral decomposition theorem, cf. Miehe (1998). In this work however, \( \sqrt{b'} \) is computed numerically with the Denman–Beaver square root iteration scheme, cf. Denman and Beavers (1976). The fourth order tensor \( \Omega^e \) is determined by first computing \( \sqrt{b'} \) with the Denman–Beaver scheme, followed by an analytical differentiation.

Furthermore for later purposes, the spin \( \Omega^e \) will be expressed in terms of \( l \) and \( \dot{l} \). By using (11), (12) and (6) in (10), the tensor \( \Omega^e \) can be rewritten as

\[ \Omega^e = V^{-1} (l - \dot{l}) V^e - 2V^{-1} \Omega^{-1} : (\text{sym}(\dot{b}') - \text{sym}(l b')). \]

The stress–strain relation will now be derived on the basis of thermodynamical arguments.

### 4. Thermodynamic considerations

Although the model is primarily intended for isothermal situations, it should fulfill the laws of thermodynamics. Ignoring the effects of temperature, the dissipation inequality in the spatial setting is defined as

\[ d = \tau : d - \rho_0 \dot{\psi} \geq 0, \]

where \( \tau \) is the symmetric part of the spatial velocity gradient, \( \tau \) is the Kirchhoff stress tensor and \( \psi \) is the Helmholtz free energy. The free energy is assumed to be a function of the elastic Finger tensor, \( b' \), the structural tensors, \( m^s \), and a set of internal variables, \( \kappa^{(j)} \), which accounts for irreversible effects, i.e. \( \rho_0 \dot{\psi} = \rho_0 \dot{\psi}^e (b', m^s, \kappa^{(j)}) \). The dissipation inequality then takes the form

\[ d = \tau : d - \rho_0 \frac{\partial \psi}{\partial b'} : b' - \rho_0 \frac{\partial \psi}{\partial m^s} : m^s - \rho_0 \frac{\partial \psi}{\partial \kappa^{(j)}} \kappa^{(j)} \geq 0. \]

The superscripts \( x \) and \( y \) in the expression above should be interpreted as a summation over the indices. Using the time derivative of the elastic Finger tensor (6) and the structural tensors (9), we arrive at

\[ d = \tau : d - \rho_0 \frac{\partial \psi}{\partial b'} : b' - 2\rho_0 \frac{\partial \psi}{\partial m^s} : m^s - \rho_0 \frac{\partial \psi}{\partial \kappa^{(j)}} \kappa^{(j)} \geq 0. \]

The Kirchhoff stress tensor (17) is symmetric, since the Helmholtz free energy is assumed to be an isotropic function of its arguments, cf. Harrysson and Ristinmaa (2007) and Menzel and Steinmann (2003). The remaining part of the dissipation inequality is given as,

\[ d = \tau : d' - K^{(j)} \kappa^{(j)} \geq 0, \]

where the energy conjugates to the internal variables was defined as

\[ K^{(j)} = \rho_0 \frac{\partial \psi}{\partial \kappa^{(j)}}. \]

The specific model will be discussed next.

### 5. The constitutive model

#### 5.1. Elasticity

The out-of-plane response is assumed to be decoupled from the in-plane response, and as a consequence only one structural tensor is needed to capture the in-plane behavior. Decoupling of the out-of-plane has been verified to be an accurate approximation for paperboard and used by several authors, cf. Stenberg (2002) and Nygård et al. (2009). Only the in-plane model will be considered here. The Helmholtz free energy is assumed to be split into an elastic and a plastic part,

\[ \rho_0 \dot{\psi} = \rho_0 \dot{\psi}^e (b', m^{(1)}) + \rho_0 \dot{\psi}^p (\kappa^{(j)}). \]

In the Helmholtz free energy, a general structural tensor for transverse isotropy will be utilized. It is defined as
\[ \mathbf{m} = p \mathbf{m}^{(1)} + q (I - \mathbf{m}^{(1)}) , \]  
(21)

where \( I \) is the second order identity tensor and \( p \) and \( q \) are material parameters. Note that \( \mathbf{m} \) can not be written as a dyadic product of a vector, however it can be Cholesky decomposed as \( \mathbf{m} = \mathbf{H}^T \mathbf{H} \), where \( \mathbf{H} \) is a lower triangular matrix of a general structural tensor \( \mathbf{m} \).

If the additional requirement \( p + 2q = 1 \) is imposed i.e. \( \text{tr}(\mathbf{m}_0) = 1 \), then it can be possible to relate \( q \) to a fiber distribution function with a normalizing condition, cf. Gasser et al. (2006). It turns out that enforcing \( p + 2q = 1 \) for the paperboard, that has been considered here, will give an accurate fit to experimental material data. See also Wahlström (2009) for a more thorough discussion on fiber distribution of paperboard.

A list of specific free energies that automatically fulfills stress-free reference configuration is given in Schröder et al. (2008). The free energy for the paperboard material has been chosen according to

\[ \rho_0 \mathbf{b}^\rho = A \left( \frac{1}{(x+1)(p+2q)} (f_1^{(x)} + f_2 - (p+2q)J) \right) , \]
(22)

where \( A \) and \( x \) a constitutive parameters. The strain invariants in (22) are defined as,

\[ f_1 = \text{tr} (\mathbf{b}' \mathbf{m}) \]
\[ f_2 = J^2 \text{tr} (\mathbf{b}'^{-1} \mathbf{m}) \]
\[ J = \sqrt{\det(b)} . \]

Polyconvexity implies that the free energy is a convex function in the arguments,

\( \{ \mathbf{F}, \text{cof}(\mathbf{F}), \det(\mathbf{F}) \} \), where \( \text{cof}(\cdot) \) is defined by \( \text{cof}(\mathbf{F}) = \det(\mathbf{F})^{-1} \mathbf{F}^T \mathbf{F} \). Polyconvexity together with the growth criterion guarantees the existence of at least one minimizer to the functional of the elastic boundary value problem, cf. Ball (1977). Since the second derivative of \( J \) with respect to \( J \) is zero, it is concluded that \( J \) is convex. The invariants \( f_1, f_2 \) can be split into terms involving the director vectors see Appendix A and then the proof found in Schröder and Neff (2002) can be used when \( p - q > 0 \). Alternatively the proof in Schröder et al. (2008) can be used, cf. also Ebbing (2010) for an extensive review of polyconvexity using crystallographic structural tensors and the Cholesky decomposition. Given that \( A, x, p \) and \( q \) are positive quantities, it is then concluded that (22) is a polyconvex free energy potential.

5.2. Plasticity

Many models are able to accurately predict the proportional stress-strain response for paperboard, whereas the predictive capability for non-proportional test are less accurate. Therefore a yield surface which hardens non-isotropically will be employed. Following the work in Xia et al. (2002), a set of yield sub-surface tensors \( \mathbf{n}^{(v)} \), which are normals to the yield planes, are introduced. For this purpose a set of dyadic products defined by the director vectors are introduced as,

\[ \mathbf{n}^{(v)} = n_1^{(v)} \mathbf{u}^{(1)} \otimes \mathbf{v}^{(1)} + n_2^{(v)} \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + n_3^{(v)} (\mathbf{v}^{(1)} \otimes \mathbf{v}^{(2)} + \mathbf{u}^{(2)} \otimes \mathbf{u}^{(1)} ) , \]
(24)

where \( n_1^{(v)}, n_2^{(v)} \) and \( n_3^{(v)} \) are constants. Six tensors, \( \mathbf{n}^{(v)} \), are introduced in the model, each associated to a yield plane. The yield planes are associated to the following stress states (in order from 1 to 6): MD tension, CD tension, positive oriented shear, MD compression, CD compression and negative oriented shear.

The conjugate variables, \( \mathbf{K}^{(v)} \), in (19), will be used as a measure of the distance in the stress-space, which a yield plane is translated. The plastic part of the free energy is postulated to be

\[ \rho_0 \mathbf{b}^\rho = \sum_{p=1}^6 \frac{1}{p} \left( b_p (K^{p(\mathbf{b})^\rho} + 1) \ln (b_p K^{p(\mathbf{b})^\rho} + 1) - b_p K^{p(\mathbf{b})^\rho} \right) . \]
(25)

Using (19), the conjugate quantities then takes the following form

\[ \mathbf{K}^{(\mathbf{n})} = a_1 \ln (b_i \mathbf{K}^{i(\mathbf{n})} + 1) . \]
(26)

Note that according to the decoupling present in (25), each hardening variable \( \mathbf{K}^{(\mathbf{n})} \) is associated with one internal variable, \( \mathbf{K}^{(\mathbf{n})} \). To allow for modeling of the non-proportional loading behavior revealed in the experimental tests, the yield surface proposed in Xia et al. (2002) will be enhanced. The enhanced part is related to the hardening behavior. The yield function is given as

\[ f(\mathbf{\tau}, \mathbf{n}^{(\mathbf{r})}, \mathbf{K}^{(\mathbf{n})}) = \sum_{i=1}^6 \chi_i (\mathbf{n}^{(\mathbf{n})} \cdot \mathbf{n}^{(\mathbf{r})})^{2k} - 1 , \]
(27)

where the stress \( \mathbf{n}^{(\mathbf{r})} \) is defined as

\[ \mathbf{n}^{(\mathbf{r})} = \mathbf{K}_0^{(\mathbf{n})} + \sum_{i=1}^6 \omega_i \mathbf{n}^{(i)} \mathbf{K}^{(\mathbf{n})} . \]
(28)

In (27), \( k \) is a constant natural number and \( \chi_i \) is a switch function, which determines if a yield plane is active for the current stress state and is defined as

\[ \chi_i = \begin{cases} 1 & \text{if } \mathbf{\tau} : \mathbf{n}^{(\mathbf{r})} > 0 \\ 0 & \text{otherwise}. \end{cases} \]
(29)

The quantity \( \omega_i \) in (28) defines a constant positive-definite matrix and it introduces a coupling between the hardening of the six different sub-yield surfaces, allowing for non-isotropic hardening. This will enable the model to capture the experimental observed behaviour shown in Fig. 3. Note that if \( \omega_{12} \) is chosen as the identity matrix, then the yield surface proposed in Xia et al. (2002) is retained. The yield surface for the situation when \( \tau_{12} = 0 \) is illustrated in Fig. 5.

The yield surface in Fig. 5 illustrates four yield plane gradients defined by the normals \( \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(4)} \) and \( \mathbf{n}^{(5)} \) together with a graphical interpretation of \( \tau^{(1)}, \tau^{(2)}, \tau^{(4)} \) and \( \tau^{(5)} \), i.e. the shortest distance to each yield plane from the origin. Increasing the exponent \( k \) in (27) will provide sharper corners in the yield surface, cf. Fig. 5. The value for \( k \) can be determined by considering biaxial stress states. The material parameters will however be derived from simple uniaxial tests and therefore a value for \( k \) has been chosen. The value \( k = 3 \) has been taken in this work, whereas in Xia et al. (2002) the choice \( k = 2 \) was made.

The evolution laws are given as

\[ \mathbf{d}^\mathbf{c} = \dot{\lambda} \frac{\partial f}{\partial \mathbf{b}}, \]
(30)

\[ \mathbf{K}^{(\mathbf{n})} = -\dot{\lambda} \omega \lambda^{-1} \frac{\partial f}{\partial \mathbf{K}^{(\mathbf{n})}} , \]

where \( \dot{\lambda} \) is a summation index and \( \lambda \) will be determined by enforcing \( f = 0 \) during elasto-plastic loading. It is further assumed that the plastic spin \( \mathbf{I}^p = 0 \). Note that the inverse of the coupling matrix, \( \omega \lambda^{-1} \), enters the evolution law for \( \mathbf{K}^{(\mathbf{n})} \) in (30). This format is chosen to obtain a physical interpretation of the internal variables in terms of the experimentally measured plastic strains. The dissipation inequality (18) with the evolution laws in (30) becomes

\[ d = \sum_{i=1}^6 2k \chi_i (\lambda \Lambda^{(\mathbf{n})})^{2k} (1 - K^{i(\mathbf{n})}) \geq 0 , \]
(31)

cf. Appendix B for a derivation and the definition of \( \Lambda \). A sufficient condition for the inequality (31) to be fulfilled, is that for all terms \( \chi \)
\[ K^{(2)} \leq K^{(3)} + 6 \sum_{\alpha = 1}^{6} \omega_{\beta\alpha} K^{(3)}, \quad \text{(32)} \]

where (28) was used. The criterion (32) will be discussed in detail for the specific \( \omega_{\beta\alpha} \) that has been employed in the calibration section.

6. Calibration procedure

The number of constitutive parameters involved in an orthotropic elastic, orthotropic plastic constitutive model is inevitable large. One strategy for finding the constitutive parameters is to make use of inverse modelling in conjunction with optimization methods, cf. Garbowski et al. (2011). In the present work we will present a simple approximate fitting procedure that enables the constitutive parameters to be determined from simple uniaxial tests. It turns out that the response obtained using the estimated parameters fits well to the experimental uniaxial curves present in the calibration process.

Five uniaxial tests are used to calibrate the in-plane model, i.e. tension tests in MD, CD and 45° compression tests in MD and CD. The following stress states are assumed to be valid in the uniaxial tests:

- MD-tension: \( \tau = \tau^{MD} m_0^{(1)} \)
- CD-tension: \( \tau = \tau^{CD} m_0^{(2)} \)
- MD-compression: \( \tau = \tau^{MD} m_0^{(3)} \)
- CD-compression: \( \tau = \tau^{CD} m_0^{(4)} \)
- 45°-tension: \( \tau = \frac{1}{2} (m_0^{(1)} + m_0^{(2)} + 2v_0^{(1)} \otimes v_0^{(2)} + 2v_0^{(2)} \otimes v_0^{(1)}) \) \( \text{45}° \)-tension.

The 45° tension stress state in (33) is obtained by rotating a uniaxial stress state 45 degrees. Note that the director vectors are assumed to be constant in the calibration procedure, i.e. the rotations are assumed negligible. It will turn out that this assumption will provide a good fit to the uniaxial curves. The Kirchhoff stresses \( \tau \) have been identified from the measured force \( F \) and the initial cross sectional area, \( A_0 \), and initial specimen length \( L_0 \), as

\[ \tau = \frac{F}{A_0} \left( 1 - \frac{u}{L_0} \right). \quad \text{(34)} \]

In the calibration procedure below, the elastic parameters will be considered first, and then the plastic part.

6.1. Elasticity

The initial (for small strains) orthotropic stiffness tensor, can be written as (in Voigt notation)

\[ \mathbf{D} = \frac{1}{1 - v_{12} v_{21}} \begin{bmatrix} E_{11} & v_{12} E_{22} & 0 \\ v_{21} E_{11} & E_{22} & 0 \\ 0 & 0 & (1 - v_{12} v_{21}) G_{12} \end{bmatrix}, \quad \text{(35)} \]

where \( E_{11} \) and \( E_{22} \) are the elastic modulus in MD and CD, \( v_{12} \) and \( v_{21} \) are the Poisson’s ratios and \( G_{12} \) is the shear modulus. Note that the symmetry condition \( v_{12} E_{22} = v_{21} E_{11} \) holds. The expression (35) will be used to relate the elastic parameters present in the model. The elastic moduli in MD, CD and 45° are deduced from the experimental uniaxial tension curves and \( G_{12} \), can be found from a standard expression found in Lekhnitskii (1968). The contraction has been measured from uniaxial DIC-tests and it was found that \( \sqrt{v_{12} v_{21}} \approx 0.30 \), \( \text{(36)} \)

where \( v_{12}, v_{21} \) are the Poisson’s ratios in MD and CD respectively. In the experimental investigation (Baum et al., 1981) the value \( \sqrt{v_{12} v_{21}} \approx 0.293 \) was found for a range of paperboards. The result (36) together with the symmetry condition of the compliance tensor (35) gives that the Poisson’s ratio can be found.

The stiffness tensor resulting from the strain energy (22), for \( \mathbf{F} = \mathbf{I} \), i.e. initial stiffness, has been computed numerically with the constraint \( q = (1 - p)/2 \). The difference of the matrix components in \( \mathbf{D} \) from (35) and from the stiffness resulting from the free energy (22) has been minimized in a least square sense to obtain the material data. The result of the fitting procedure is found in Table 1.

6.2. Plasticity

The calibration of the plastic parameters is a bit more involved. From the experimental evidences in Fig. 3, it can be concluded that the pre-strained samples display a softer response than

<table>
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<th>Elastic parameters</th>
<th>Value</th>
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<tr>
<td>( A ) (MPa)</td>
<td>950</td>
</tr>
<tr>
<td>( p ) (-)</td>
<td>0.49</td>
</tr>
<tr>
<td>( \sigma ) (-)</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Table 1 Numerical values of the elastic parameters.
non-prestrained samples. By decreasing the yield stress perpendicular to the pre-strained direction, a softening effect can be achieved, cf. Fig. 6.

In Fig. 6 the evolution of the yield surface is shown for uniaxial tension in CD. During loading in CD-tension, the distance to the yield sub-surface belonging to MD-tension is decreasing. It turns out that the distortion hardening illustrated in Fig. 6 can be captured by fitting the \( \omega_{ij} \) parameters present in the yield surface (27). However, first the components of the yield subsurfaces, \( n^{(i)}_j \), defined in (24) needs to be determined.

6.2.1. Yield subsurfaces, \( n^{(i)}_j \)

The yield plane normals \( n^{(i)}_j \) determines the shape of the yield surface. Consider first the MD-tension stress state given in (33). Insertion of (33) into (30) provides the plastic velocity gradient. Projection of the plastic velocity gradient on \( \mathcal{m}^{(i)}_j \) and \( \mathcal{m}^{(i)}_k \) gives

\[
\begin{align*}
\dot{\epsilon}^{(i)}_{11} &= \frac{2\lambda^{(i)} n^{(i)}_{11}}{\tau^{(i)}}, \\
\dot{\epsilon}^{(i)}_{22} &= \frac{2\lambda^{(i)} n^{(i)}_{22}}{\tau^{(i)}}.
\end{align*}
\]

The axial and lateral strain ratio \( d_{11}/d_{22} \) has been shown for many paperboard materials to remain approximately constant, cf. Xia et al. (2002) and Harrysson and Ristimaa (2008). For paperboard, the experimental data available for the yield-surface shape is limited and therefore the calibration of the \( n^{(i)}_j \) will be based on the assumption that the plastic strain rate equals the total strain rate,

\[
\dot{\epsilon}^{(i)}_{11} = \frac{n^{(i)}_{11}}{d_{22}} = v_{12},
\]

i.e. the approximation (38) that was adopted in cf. Xia et al. (2002) has been employed. Note also that (36) has been utilized in (38). Using the following normalizing condition

\[
\sqrt{n^{(i)}_{11}^2 + n^{(i)}_{22}^2 + 2n^{(i)}_{12}^2} = 1,
\]

and assuming no coupling to the shearing, i.e. \( n^{(i)}_{12} = n^{(i)}_{21} = 0 \), gives that \( n^{(i)}_{11} \) and \( n^{(i)}_{22} \) can be determined. A similar procedure for the CD-stress state can then be made. In summary, the yield plane normals for the MD- and CD-tension are obtained as,

\[
\begin{align*}
n^{(1)}_{11} &= \frac{1}{\sqrt{1 + \rho/\tau_{12}}}n^{(2)}_{11} = \frac{1}{\sqrt{1 + \rho/\tau_{12}}} 50 \quad \text{MD} \\
n^{(2)}_{22} &= -\sqrt{1 - n^{(2)}_{11}^2} = -\sqrt{1 - n^{(2)}_{11}^2} 10 \quad \text{CD}
\end{align*}
\]

For the sub-surfaces associated with the compression states in (33), it has been assumed that there is no coupling between the axial and lateral plastic strains due to lack of experimental evidence, therefore \( n^{(1)}_{11} = n^{(2)}_{22} = 1 \) and \( n^{(1)}_{12} = n^{(2)}_{21} = n^{(1)}_{21} = n^{(2)}_{12} = 0 \) is adopted. For the yield plane normals associated with the positive and negative shear, it is assumed \( n^{(1)}_{11} = n^{(2)}_{22} = n^{(1)}_{21} = n^{(2)}_{12} = 0 \), i.e. the shear sub-surfaces are assumed decoupled from the normal components. The normalizing condition (39) gives then \( n^{(2)}_{12} = 1/\sqrt{2} \) and \( n^{(2)}_{11} = -1/\sqrt{2} \). The numerical values for the yield plane subsurfaces are summarized in Table 2.

6.2.2. The coupling components \( \omega_{ij} \)

To achieve the distortional hardening as illustrated in Fig. 6, both \( \omega_{12} \) and \( \omega_{21} \) must be negative. Since no experimental data exists for the other directions, it is for simplicity assumed that the remaining cross terms \( \omega_{ij} = 0 \) \( \{ i \neq j \} \). Without loss of generality, the diagonal terms are further assumed to be normalized such that \( \omega_{11} = 1 \). For uniaxial tension loading in MD and CD, it follows from (33) and (27) that

\[
\begin{align*}
\tau^{(1)}(\kappa^{(1)}, \kappa^{(2)}) &= K^{(1)} + \omega_{12}K^{(2)} = \tau^{(MD)}n^{(1)}_{11} \\
\tau^{(2)}(\kappa^{(1)}, \kappa^{(2)}) &= K^{(2)} + \omega_{21}K^{(1)} = \tau^{(CD)}n^{(1)}_{11}.
\end{align*}
\]

Furthermore from (30) it follows that,

\[
\kappa^{(i)} = \lambda^{(i)} \frac{\kappa^{(i)}}{\tau^{(i)}}.
\]

**Table 2**

<table>
<thead>
<tr>
<th>Subsurface, ( v )</th>
<th>( n^{(i)}_{11} )</th>
<th>( n^{(i)}_{22} )</th>
<th>( n^{(i)}_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.92</td>
<td>-0.40</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>0.98</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0.71</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>-0.71</td>
</tr>
</tbody>
</table>
The coupling term $\omega_{12}$ is found by considering the yield function (27) for the pre-straining in CD-tension followed by MD-tension. During uniaxial pre-straining in CD, the evolution law (30) provides $K^{(1)}(K^{(1)}) = K^{(1)}(0) = 0$, due to $\chi^{(1)} = 0$, cf. (42). The state obtained after the pre-straining load in CD is given as

\[
\begin{align*}
\tau^{(1)}(0, K^{(2)}_{\text{pre}}) &= K^{(1)}_{\text{pre}} + \omega_{12}K^{(2)}_{\text{pre}} \\
\tau^{(2)}(0, K^{(2)}_{\text{pre}}) &= K^{(2)}_{\text{pre}} + K^{(2)}_{\text{pre}},
\end{align*}
\] (43)

where the subscript ‘pre’ denotes the value obtained during this loading. The experimental evidence when loading in MD, i.e. the perpendicular direction, indicates that $\tau^{(1)}(0, 0) \geq \tau^{(1)}(0, K^{(2)}_{MD})$ cf. also Fig. 6, which requires that yielding starts earlier for the pre-strained sample. The difference is denoted by

\[
\Delta \tau^{(1)}(0, 0) = \tau^{(1)}(0, 0, K^{(2)}_{\text{pre}}) = -\omega_{12}K^{(2)}_{\text{pre}}
\] (44)

where (43) was used. Since $\Delta \tau^{(1)}$ can be obtained from the experimental data it follows that

\[
\omega_{12} = \frac{\Delta \tau^{(1)}}{K^{(2)}_{\text{pre}}}. \tag{45}
\]

A similar procedure for $\omega_{21}$ is found by considering the yield function (27) for the MD-tension stress state followed by CD,

\[
\omega_{21} = \frac{\Delta \tau^{(2)}}{K^{(2)}_{\text{pre}}}. \tag{46}
\]

The parameters are identified as $\omega_{12} = -0.59$ and $\omega_{21} = -0.071$.

Returning to the condition for fulfilling the dissipation inequality, (32), it is required that

\[
\begin{align*}
K^{(1)} &\leq K^{(0)} + K^{(1)} + \omega_{12}K^{(2)} \\
K^{(2)} &\leq K^{(2)} + \omega_{21}K^{(1)},
\end{align*}
\] (47)

Consider now uniaxial tension in CD, i.e. (33b). For this stress state, the evolution laws provide $K^{(1)} = 0$, which implies that (47b) is automatically fulfilled and only (47a) needs to be considered. Rewriting (47a) and using (44) gives

\[
\Delta \tau^{(1)} \leq K^{(1)}
\] (48)

indicating that the decrease of MD-yield subsurface when loading in CD (left hand side), must be less than the initial distance to the MD-subsurface (right hand side). A similar interpretation can be made for $\omega_{21}$.

6.2.3. Hardening parameters

The hardening, $K^{(0)}$, can be identified from the experimental tests using the stress states in (33). Considering tensile loading in MD it follows from (27) that

\[
\tau^{(1)} = K^{(0)} + \Delta \tau^{(1)} = \tau^{MD}_{pre} n_{t1}^{(1)}.
\] (49)

From the evolution laws (30), the following relations are then obtained,

\[
\begin{align*}
d_{11}^{MD} &= 2K^{(0)}\frac{\Delta \tau^{(1)}}{\tau^{(1)}} \\
K^{(1)} &= \frac{2K^{(0)}\Delta \tau^{(1)}}{\tau^{(1)}},
\end{align*}
\] (50)

where $d_{11}^{MD}$ is the component of the symmetric plastic velocity gradient $d_{11}$ projected on $m_{11}^{(1)}$. Assuming negligible rotations enables the spatial velocity gradient to be expressed in terms of the plastic stretch tensor $\nabla p$, cf. (3), as

\[
\ln V_{MD}^{p} = d_{11}^{MD}
\] (51)

where $V_{MD}^{p}$ is the plastic stretch in the MD-tension stress state projected on $m_{11}^{(1)}$. Time integration of $d_{11}^{MD}$ will then give a relation to the logarithmic plastic stretch tensor. Then the ratio $d_{11}^{MD} / n_{t1}^{(1)}$ from (50) together with (51) and (49) enables the internal variables of $K^{(1)}$ to be determined as

\[
K^{(1)} = \frac{\ln V_{MD}^{p}}{n_{t1}^{(1)}}.
\] (52)

Thus allowing for $\tau^{MD} = \tau^{MD}(\ln V_{MD}^{p})$ to be established; which has been measured in the experimental tests. A similar procedure can be made for the remaining stress states in (33). To determine the hardening parameters $a_i$ and $b_i$, a least square fit in the MD, CD and 45° tension tests has been made. The numerical values are summarized in Table 3.

7. Numerical implementation

The backward Euler method is used for the update of the state variables. Consider a time interval $\Delta t \in [t_n, t_{n+1}]$ between loadstep $n$ and $n + 1$, where $F_{n+1}$ is given. First a trial step is made to check whether plasticity takes place in the elapsed time interval,

\[
\begin{align*}
F^{(n+1)}_{\text{trial}} &= F^{(n+1)} + \Delta t F^{(n+1)}_{\text{new}} \\
\n^{(n+1)}_{\text{trial}} &= R^{(n+1)}_{\text{trial}} p^{(n+1)} \\
K^{(n+1)}_{\text{trial}} &= K^{(n+1)}_{\text{trial}},
\end{align*}
\] (53)

where $R_{\text{trial}}$ is obtained from the polar decomposition of $F_{\text{trial}}$. Using (53), the trial value of the yield surface is computed according to $\tau^{\text{trial}}(F_{\text{trial}}, p^{(n+1)}), n^{(n+1)}_{\text{trial}}(n^{(n+1)}_{\text{trial}}, K^{(n+1)}_{\text{trial}})$. For the situation $\tau^{\text{trial}} < 0$, the updated variables are equal to the trial quantities otherwise an update is made according to (30). Using the backward Euler scheme, the discrete evolution equation becomes

\[
\begin{align*}
F_{n+1}^{(n+1)} &= F_{n+1}^{(n+1)} + \Delta t F_{n+1}^{(n+1)} F_{n+1}^{(n+1)} \frac{df}{df} F_{n+1}^{(n+1)} F_{n+1}^{(n+1)} \\
K_{n+1}^{(n+1)} &= K_{n+1}^{(n+1)} - \Delta t \frac{df}{df} K_{n+1}^{(n+1)} \frac{df}{df} F_{n+1}^{(n+1)} F_{n+1}^{(n+1)} \\
F_{n+1}^{(n+1)} &= f_{n+1}^{(n+1)}
\end{align*}
\] (54)

The equation system (54) is solved using the Newton–Raphson algorithm.

For the numerical treatment of the model, the Algorithmic Tangent Stiffness (ATS) matrix is needed. The algorithm allows us to derive an implicit expression for $F'$ as a function of $F$, i.e. the Kirchhoff stress can be written as $\tau = \tau(F') = \tau(F, F'(F))$. The ATS matrix is then given as

\[
\mathcal{D} = -[\nabla \tau(F')] - [\nabla \tau(F)] + \frac{\partial \tau}{\partial F} F'.
\] (55)

Table 3

<table>
<thead>
<tr>
<th>Subsurface, $v$</th>
<th>$K_{n+1}^{(1)}$ (MPa)</th>
<th>$a_i$ (MPa)</th>
<th>$b_i$ (-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>13</td>
<td>710</td>
</tr>
<tr>
<td>2</td>
<td>7.8</td>
<td>5.0</td>
<td>1200</td>
</tr>
<tr>
<td>3</td>
<td>9.3</td>
<td>5.1</td>
<td>1100</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>11.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>9.3</td>
<td>5.1</td>
<td>1100</td>
</tr>
</tbody>
</table>
8. Verification of calibration and uniaxial response

The model is fitted to uniaxial tension in the MD, CD and 45° directions, as well as uniaxial compression in MD and CD. The Long Compression Test (LCT) apparatus were used for compression tests, which has lateral support to prevent buckling, cf. Cavlin and Fellers (1975). The paperboard used in the compression tests were, \( w_0 = 25 \) mm wide and had a clamp length \( l_0 = 55 \) mm. The uniaxial tests and pre-straining where performed according to the description in Section 2. To test the calibration procedure, uniaxial finite element simulations have been performed. One element FEM-simulation where the internal force and the displacement has been extracted is shown in Fig. 7 for different angles.

As observed from Fig. 7, despite the approximative assumptions in the calibration procedure, an accurate fit to the uniaxial curves is obtained. Note too that no fitting has been made for the intermediate angles, 15°, 30°, 60°, 75° in tension and 45° in compression, indicating that the model provides realistic predictions in uniaxial loading situations. Note that the simplifying assumption of ideal plasticity for the compressive subsurfaces in MD and CD has been made, even though the slopes of the curves are not constant up to failure.

The predicted response for the non-proportional situations are shown in Fig. 8, where it is concluded that the presented model allows the hardening response of the pre-strained samples to be predicted. Note that the distortion hardening reduces the yield stress for the pre-strained samples and the simulated hardening response is then predicted by the model. The change in the initial stiffnesses present in the experiments are not captured by the proposed elasto-plastic model. Notice also the variation in the response increases when the samples are pre-strained. The pre-strained samples fail, however, at approximately the same displacement as the non-prestrained samples.

9. DIC comparison

Digital Image Correlation (DIC) measurements have been performed with a single camera on a sample with a central hole. Tests have been conducted when the loading direction is parallel to CD and when it is parallel to MD. The boundary conditions are given in Fig. 9(a). The resulting strain field from the experimental setup has been compared to the strain field obtained from the simulations.

The dimensions of the geometry are given by \( R = 10 \) mm, \( 2w = 50 \) mm, \( 2L = 80 \) mm. The tests have been conducted on a standard MTS-tensile machine with a displacement rate of 2 mm/min. The tests have been performed in climate chamber with moisture content \( \text{RH} = 50\% \) and temperature \( T = 23 \degree C \).

Four-node Lagrangian isoparametric elements have been used in the simulations. The DIC-field have been obtained by using the software VIC-2d (Correlated solutions Inc.). One high speed camera Gazelle GZL-22C5M-C (Point Grey Inc.), with a resolution of 2048 times 1088 at 280 frames per second has been used. Before examining the DIC results, the macroscopic load–displacement curve will be discussed.

The force–displacement curves in the MD and CD directions for both the experiments (blue) and simulations (red) are shown Fig. 10. The experimental force–displacement curves are recorded

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**Fig. 7.** Uniaxial stress–strain curves in CD and MD. Light blue and purple color is used for the experimental data obtained from samples that were not pre-strained and black for the simulations. Normalized force vs normalized displacement has been plotted, where \( A_0 \) is the initial cross section area and \( L_0 \) is the initial length. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 8.** Uniaxial stress–strain curves in CD and MD. Light blue color is used for the experimental data obtained from samples that were not pre-strained, purple color is used for the pre-strained samples and black for the simulations. Normalized force vs normalized displacement has been plotted, where \( A_0 \) is the initial cross section area and \( L_0 \) is the initial length. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
up to the state when a complete fracture occurs. During the softening part of the force–displacement curve fracture occurs in the samples, and since fracture is not considered in the current model, the simulation are stopped when the softening is initiated. A total of 14 experiments were performed in CD and 12 in MD. The boxplots in the Fig. 10 indicate the variation of the global-force response for the different samples that have been tested. A visual comparison between the experiment and simulation curves, shows that the simulations provide a good prediction of experiments within expected experimental variation.

Contour plots of the largest principal Green–Lagrange strain from the simulated strain fields and experimental DIC-strain fields are shown in Figs. 11(a) and 12(a), respectively. Typical DIC-fields when loading is applied parallel to CD and MD are shown. The comparisons are made at the displacement levels, \( u_y = \{0.75, 1.00, 1.25\} \) in CD and \( u_y = \{0.45, 0.60, 0.75\}\) in MD. In Fig. 10 the displacement levels are marked from (1) to (3) in the force–displacement curves.

From the DIC-strain fields in Figs. 11(b) and 12(b) the black contour indicates the full geometry of the samples. The experimental strain fields in Figs. 11(b) and 12(b) has been extracted from a single test, and thus some variations in the strain-field arising from the inherent inhomogeneity of the paperboard are visible in the figures. It is concluded from Figs. 11 and 12 that the overall strain fields obtained between the simulations and the experiments are similar. The strain level and distribution at the different displacements are about the same in the experiments and simulations. In MD, it is noticed that the strain-field is smeared out more in the vertical direction, whereas in CD the strain field is smeared more in the horizontal direction. These effects are noticed both in the simulations and in the experiments.

The inhomogeneity of paperboard was investigated in Hagman and Nygård (2012) using DIC for uniaxial testing, where localized strain fields were observed for the uniaxial load tests. The DIC tests conducted here shows that a continuum model can capture the overall strain fields, even though the paperboard is heterogeneous. The inherent inhomogeneity of paperboard does not appear to be crucial for the overall strain field in the load cases considered here, when comparing the simulations and experiments. Note that typical fiber lengths are around 1–3 mm with a width and thickness around 10–50 μm.

The error between the DIC-samples and the simulations have also been investigated. The absolute error and a relative error have been defined as

\[
e_{abs} = e^{(sim)}_1 - e^{(DIC)}_1, \quad e_{rel} = \frac{\left| e^{(sim)}_1 - e^{(DIC)}_1 \right|}{e^{(DIC)}_1}
\]

where \(e^{(sim)}_1\) and \(e^{(DIC)}_1\) are the largest principal strains from the simulations and DIC respectively. The principal strains are compared at approximately the same positions in the DIC and simulations by averaging the strains at the nodes from simulations within a radius of \(r = 1\) mm from the corresponding coordinates in the DIC test. The error have been plotted at the displacement level (3) marked in Fig. 10 for MD and CD (see Fig. 13).

A similar tendency for the error are observed for MD as well as for CD. The largest absolute error occurs at the horizontal sides of the holes. This implies that the strain field at the hole is not perfectly captured. The error plots suggests that at the sides of the hole along a horizontal central line, the simulation overestimates the principal strains. The relative error at the same positions, indicates that the relative error next to the hole along the horizontal line is relatively small. Considering now the relative error along the vertical symmetry line, it is at maximum at the top and bottom of the hole. From the error plots it can be concluded that the simulations predicts smaller magnitude of the strains at the top and bottom location of the hole. This might possible be due to out-of-plane
deformation (buckling) which is not visible when using a single camera for the DIC tests or boundary effects.

10. Conclusions

A distortional hardening elasto-plastic model at finite strains applicable for paperboard has been presented within a thermodynamically consistent frame work. Non-proportional experiments have been performed, which shows that paperboard pre-loaded in a perpendicular direction displays a softened response. It is shown that this effect can be modeled by introducing coupling effects such that a softening takes place in the direction perpendicular to the loading direction.

The elastic part of the model, utilizes an polyconvex free energy. This introduces physical parameters $p$ and $q$, related to the fiber distribution of the material. In this work however, the parameters was determined by fitting to the uniaxial force–displacement response.

![Contour plots of largest principal Lagrangian strain for loading in the CD-direction at the displacement levels](image1)

![Contour plots of largest principal Lagrangian strain for loading in the MD direction at the displacement levels](image2)
For the description of the plastic part of the model, the yield surface in Xia et al. (2002) is chosen. To allow for a general coupling between the hardening response, multiple hardening variables together with a coupling matrix between the hardening variables is introduced. The coupling matrix allows the yield surface to harden distortionally. The yield surface remains convex despite large distortion of the yield surface during hardening. It is shown that the dissipation inequality sets constraints on the choice of parameters for the coupling matrix which are physically logical.

The calibration procedure for the plastic part of the model is shown to be straightforward and after that some approximations are introduced, the experimental comparison does not compromise the predicted response. Only uniaxial tests is needed in the calibration which significantly reduce the experimental complexity. The subsurface parameters has been calibrated by assuming the total strain ratio equals the plastic strain ratio. This assumption needs to be experimentally verified and more investigations are needed to determine the exact shape of the yield surface.

Validation experiments of a sample with a central hole has been performed to investigate performance of the model, i.e. a non-homogeneous strain field. Digital Image correlation measurements were performed on the samples to allow comparisons of the strain fields between the simulations and the experiments. The results revealed that a qualitative match between the simulated strain field and the experimental DIC-field was obtained. The error between the simulated strain field and the DIC strain field were also compared. The error was largest at the top and bottom of the holes at the final stages of loading. The deviation in the strain field can be due to the constitutive model but it can also be explained by out-of-plane behavior or boundary effects. However, the overall shape of the strain field from a single plate-hole test is captured. It is concluded that the continuum approach for the modeling of the in-plane behavior, is able to represent the inhomogeneous strain field from the DIC-measurements, despite the inherent inhomogeneous structure of paperboard.

Acknowledgments

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Appendix A. Polyconvexity of the free energy

First, it is noticed that $I_1$ can be written as

$$I_1 = \text{tr}(\mathbf{b}' \mathbf{m}) = p\text{tr}(\mathbf{b}' (m^{(1)})) + q\text{tr}(\mathbf{b}' (m^{(2)})) + r\text{tr}(\mathbf{b}' (m^{(3)})).$$  \hspace{1cm} (A.1)

A similar expression can be obtained for $I_2$ and since a sum of convex functions is convex, it is therefore sufficient to prove that

$$\tilde{I}_1^x = \text{tr}(\mathbf{b}' m^{(x)})$$

$$\tilde{I}_2^x = j^2\text{tr}(\mathbf{b}' m^{(x)})$$  \hspace{1cm} (A.2)

are convex for $x = 1, 2, 3$. From the polar decomposition theorem, (7) and the symmetry of $\mathbf{V}$, it follows that the invariants can be expressed as

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig13.png}
\caption{Contour plot of the error in CD and MD at the displacement level (3) (a) Absolute error, $\varepsilon_{\text{abs}}$, for CD loading (b) Relative error, $\varepsilon_{\text{rel}}$, for CD loading (c) Absolute error, $\varepsilon_{\text{abs}}$, for MD loading (d) Relative error, $\varepsilon_{\text{rel}}$, for MD loading.}
\end{figure}
Appendix B. Derivation of dissipation inequality

Inserting the evolution laws (30) into the dissipation inequality (18) provides

\[ d = \mathbf{\Gamma} : \mathbf{\dot{\varepsilon}} + K^{(\gamma)} \lambda \mathbf{\omega}_1 : \mathbf{\dot{\gamma}} + \mathbf{\Pi} \geq 0 \]  

(B.1)

where summation is done over \( \gamma \) and \( \nu \). To simplify the notation \( \Lambda_\nu \) is defined as

\[ \Lambda_\nu = \frac{\mathbf{\tau} \cdot \mathbf{H}}{\tau^{(0)}}. \]  

(B.2)

The derivatives are then given by

\[ \frac{\partial}{\partial \mathbf{T}} = 2k \chi_1^{(0)} A^{2k-1} \mathbf{m} \]

\[ \frac{\partial}{\partial K^{(0)}} = -2k \chi_1^{(0)} \omega_1 \Lambda^{2k} \]  

(B.3)

Insertion into (B.1) gives

\[ d = \mathbf{\tau} : 2k \chi_1^{(0)} \Lambda^{2k-1} \mathbf{m} - 2k \chi_1^{(0)} \frac{K^{(0)} \Lambda^{2k}}{\tau^{(0)}} \]

\[ = 2k \chi_1^{(0)} \Lambda^{2k} \left( 1 - \frac{K^{(0)}}{\tau^{(0)}} \right). \]  

(B.4)

where summation is done over \( \gamma \).

References


