Invariant Hyperfunction Solutions to Invariant Differential
Equations on the Space of Real Symmetric
Matrices

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The real special linear group of degree $n$ naturally acts on the vector space of $n \times n$ real symmetric matrices. How to determine invariant hyperfunction solutions of invariant linear differential equations with polynomial coefficients on the vector space of $n \times n$ real symmetric matrices is discussed in this paper. We prove that every invariant hyperfunction solution is expressed as a linear combination of Laurent expansion coefficients of the complex power of the determinant function with respect to the parameter of the power. Then the problem is reduced to the determination of Laurent expansion coefficients.

Key Words: invariant hyperfunction; symmetric matrix space; linear differential equations

0. INTRODUCTION

Let $V := \text{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\text{SL}_n(\mathbb{R})$ be the special linear group over $\mathbb{R}$ of degree $n$. Then the group $G := \text{SL}_n(\mathbb{R})$ acts on the vector space $V$ by the representation

$$\rho(g) : x \mapsto g \cdot x := g x^t g,$$

with $x \in V$ and $g \in G$. Let $D(V)$ be the algebra of linear differential operators on $V$ with polynomial coefficients and let $\mathcal{H}(V)$ be the space of hyperfunctions on $V$. We denote by $D(V)^G$ and $\mathcal{H}(V)^G$ the subspaces of $G$-invariant linear differential operators and of $G$-invariant hyperfunctions.

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on $V$, respectively. For a given invariant differential operator $P(x, \partial) \in D \times (V)^G$ and an invariant hyperfunction $v(x) \in \mathcal{H}(V)^G$, we consider the linear differential equation

$$P(x, \partial)u(x) = v(x), \quad (2)$$

where the unknown function $u(x)$ is in $\mathcal{H}(V)^G$.

The main problem of this paper is the construction of invariant hyperfunction solutions to the linear differential equation (2). In particular, when $v(x)$ is a delta-function $\delta(x)$ on $V$, this is a problem of the existence and the construction of $G$-invariant fundamental solution for $P(x, \partial)$. However, it is difficult to solve these problems for all $G$-invariant differential operators $P(x, \partial)$ on $V$. In this paper, we assume that all the homogeneous degrees of the monomial components of $P(x, \partial)$ are equal to a certain integer $k$. Then we say that $P(x, \partial)$ is homogeneous and call the integer $k$ the total degree of $P(x, \partial)$. Furthermore, we assume that the $G$-invariant hyperfunction $v(x)$ is annihilated by a homogeneous $G$-invariant differential operator. Then we can prove that the solutions to (2) are expressed in terms of the Laurent expansion coefficients of the determinant functions. Thus we can apply the author’s result in [14].

We explain the organization of this paper. In Section 1, we describe the problem in a general setting and give some notions and notations we use in this paper. The important notions are homogeneous differential operators and quasi-homogeneous hyperfunctions. In Section 2, we introduce $G$-invariant differential equations on the real symmetric matrix space $\text{Sym}_n(\mathbb{R})$ and hyperfunctions $P^{[\alpha, \delta]}(x)$ given as linear combinations of complex powers of the determinant function on $\text{Sym}_n(\mathbb{R})$. An important fact of this section is Proposition 2.1, that gives generators of the algebra of $G$-invariant differential operators. In Section 3, we define $b_P$-function that will play an important role in this paper and clarify its properties. In Section 4, we prove the first main theorem (Theorem 4.1), which shows that every $G$-invariant solution to $P(x, \partial)u(x) = 0$ is given as a linear combination of quasi-homogeneous hyperfunctions under suitable conditions. In Section 5, we examine the properties of the complex powers $P^{[\alpha, \delta]}(x)$ more precisely and, especially prove that every $G$-invariant quasi-homogeneous hyperfunction is given by a linear combination of Laurent expansion coefficients of $P^{[\alpha, \delta]}(x)$ at on point $s = \lambda$ and the converse is true. In Section 6, by applying the results in Section 5, we prove that there exists a $G$-invariant solution $u(x)$ of $P(x, \partial)u(x) = v(x)$ for a $G$-invariant quasi-homogeneous $v(x)$ and that it is determined only by its $b_P$-function.

The aim of this paper is to prove that we can construct all the solutions for given differential equations $P(x, \partial)u(x) = 0$ or $P(x, \partial)u(x) = v(x)$ using the Laurent expansion coefficients of the complex power function $|\text{det}(x)|^s (s \in \mathbb{C})$. This is not an abstract existence theorem. In order to accomplish our
purpose, we prove Theorem 4.1 in Section 4, Corollary 5.1 in Section 5, Theorems 6.1 and 6.2 in Section 6, which are main results of this paper. They guarantee that every $G$-invariant hyperfunction solution for $P(x, \partial)u \times (x) = 0$ or $P(x, \partial)u(x) = v(x)$ can be written as a finite sum of the Laurent expansion coefficients of $|\det(x)|^s$ and that the solution space is determined by the $b_P$-function of $P(x, \partial)$ (see Definition 3.1).

Methée’s papers [8–10] are pioneer works on this area. He solved the problem in the case that the indefinite rotation group acts on the real vector space. The problem of “construction of invariant hyperfunction solutions for invariant differential operators” seems to have been first considered by Methée [8] in the framework of Schwartz’s distribution theory. The book by Bogoliubov et al. [2] on quantum field theory took up his works in the first chapter and present his results precisely. However, Methée’s method seems to be difficult to extend to other cases. The author would like to propose more generally applicable method using holonomic system theory of $D$-modules in this paper. The author thinks that the method employed in this paper is more universal and applicable to the wide range of the actions of Lie groups to real vector spaces.

**Notation.** In this paper, for a square matrix $x$, we denote by $^t x$, $\text{tr}(x)$ and $\det(x)$ the transpose of $x$, the trace of $x$ and the determinant of $x$, respectively. The complex numbers, the real numbers and the integers are denoted by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively. The subscripts signify the properties of the sets. For example, $\mathbb{Z}_{\geq 0}$ means the non-negative integers and $\mathbb{Z}_{> 0}$ means the positive integers.

### 1. FUNDAMENTAL DEFINITIONS AND PROBLEMS

In this section we explain some definitions we shall use in this paper and describe the problem at a general setting.

Let $V$ be a finite dimensional real vector space of dimension $m$ with a linear coordinate $(x_1, \ldots, x_m)$. Then a polynomial with complex coefficients on $V$ is given as a complex finite linear combination of monomials $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ with $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_m^\alpha$. We denote by $\partial_i$ the partial derivative $\frac{\partial}{\partial x_i}$ with respect to the variable $x_i$. We define a monomial of $\partial_j$’s by $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}$ with $\beta := (\beta_1, \ldots, \beta_m) \in \mathbb{Z}_m^\beta$. We define the degrees of multi-index by $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $|\beta| := \beta_1 + \cdots + \beta_m$.

The generators $x_1, \ldots, x_m$ and $\partial_1, \ldots, \partial_m$ are commutative, respectively, and hence their algebras are polynomial algebras $\mathbb{C}[x_1, \ldots, x_m]$ and $\mathbb{C}[\partial_1, \ldots, \partial_m]$, respectively. However, $x_i$ and $\partial_j$ are not commutative in general. They have a commutation relation

$$\partial_j x_i = x_i \partial_j + \delta_{ij},$$

(3)
where $\delta_{ij}$ is Kronecker's delta. The $C$-algebra generated by $x_1, \ldots, x_m$ and $\partial_1, \ldots, \partial_m$ with the commutation relations (3) is a non-commutative $C$-algebra. We denote it by $D(V)$ and call an element of $D(V)$ a differential operator on $V$. A differential operator on $V$ is uniquely expressed as a finite linear combination of monomial differential operators

$$a_{z^\beta}x^\alpha \partial^\beta := a_{z^\beta}(x_1^{z_1} \cdots x_m^{z_m})(\partial_1^{\beta_1} \cdots \partial_m^{\beta_m}) \quad (4)$$

with $a_{z^\beta} \in \mathbb{C}$. We call the expression of a differential operator using the monomial forms (4) a normal form of the differential operator.

We shall give definitions of a homogeneous differential operator in $D(V)$ and its homogeneous degree.

**Definition 1.1 (Homogeneous Differential Operators).** For a given monomial differential operator $a_{z^\beta}x^\alpha \partial^\beta$, we call $|z| - |\beta|$ (resp. $|\beta|$) a homogeneous degree (resp. an order) of the monomial differential operator $a_{z^\beta}x^\alpha \partial^\beta$. A homogeneous differential operator of homogeneous degree $k$ in $D(V)$ is a differential operator given as a finite linear combination of monomial differential operators of homogeneous degree $k$. We denote by $D_k(V)$ the $\mathbb{C}$-vector space of homogeneous differential operators of homogeneous degree $k$.

Let $P(x, \partial)$ be a differential operator in $D(V)$. Then $P(x, \partial)$ is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}} \sum_{|z| - |\beta| = k} a_{z^\beta}x^\alpha \partial^\beta. \quad (5)$$

Then each term

$$P_k(x, \partial) := \sum_{|z| - |\beta| = k} a_{z^\beta}x^\alpha \partial^\beta$$

is a homogeneous differential operator of degree $k$. Thus, we see that

$$D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V),$$

where $D_k(V)$ is a $\mathbb{C}$-vector subspace of homogeneous differential operators of homogeneous degree $k$ in $D(V)$. Note that $D_k(V)$ is invariant under the linear coordinate transformation of $V$ and a linear coordinate transformation of $V$ gives a $\mathbb{C}$-algebra isomorphism of $D(V)$ that preserves each $D_k(V)$.  


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On the other hand, $P(x, \partial)$ is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\substack{x, \beta \in \mathbb{Z}_{\geq 0}^m \mid |\beta| = k}} a_{x\beta} x^\beta \partial^\beta. \quad (6)$$

We call the order of $P(x, \partial)$ the highest number $k$ in sum (6). Let $q$ be the order of $P(x, \partial)$. Then the differential operator

$$\sigma(P)(x, \partial) := \sum_{\substack{x, \beta \in \mathbb{Z}_{\geq 0}^m \mid |\beta| = q}} a_{x\beta} x^\beta$$

is called the principal part of $P(x, \partial)$ and the polynomial

$$\sigma(P)(x, \xi) := \sum_{\substack{x, \beta \in \mathbb{Z}_{\geq 0}^m \mid |\beta| = q}} a_{x\beta} x^\beta \xi^\beta \quad (8)$$

is called the principal symbol of $P(x, \partial)$. Here $\xi$ is the coordinate of the dual space of $V$ corresponding to $\partial$.

From the definition, $D_k(V)$ is closed under the additive operation, but not closed under the multiplicative operation. However, we can easily check that

$$(a_{x\beta} x^\beta \partial^\beta)(b_{\gamma\delta} x^\gamma \partial^\delta) = \sum_{|\mu| - |\nu| = r} c_{\mu\nu} x^\mu \partial^\nu, \quad (9)$$

where $r = |x| - |\beta| + |\gamma| - |\delta|$ and $c_{\mu\nu} \in \mathbb{C}$ are zero except for a finite number of them. Namely, we have

$$D_k(V) \times D_l(V) \ni (P, Q) \mapsto P \cdot Q \in D_{k+l}(V) \quad (10)$$

and $\bigoplus_{k \in \mathbb{Z}} D_k(V)$ gives a gradation of $D(V)$.

Next, we shall consider the differential operators invariant under the action of a subgroup $G \subset \text{GL}(V)$, where $\text{GL}(V)$ is the general linear group on the vector space $V$. The action of $g \in G$ to $V$ leads to an algebra automorphism on $D(V)$ since $g \in G$ gives a linear coordinate transformation on $V$. We say that a differential operator invariant under the action of all $g \in G$ a $G$-invariant differential operator on $V$. We denote $D(V)^G$ the totality of $G$-invariant differential operators on $V$. We can easily check that $D(V)^G$ a subalgebra of $D(V)$ and $D(V)^G = \bigoplus_{k \in \mathbb{Z}} D_k(V)^G := \bigoplus_{k \in \mathbb{Z}} D_k(V) \cap D(V)^G$ gives a natural gradation induced from the gradation $D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V)$.

Remark 1.1. Let $P(x, \partial) \in D(V)$ be a homogeneous differential operator of degree $k$ and let $Q(x)$ be a homogeneous polynomial of degree $l$. Then the polynomial $P(x, \partial)Q(x)$ is a homogeneous polynomial of degree $k + l$. 

Namely, the gradation \( D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V) \) is consistent with the gradation on the polynomial algebra by the homogeneous degree. Similarly, we see that the gradation \( D(V)^G = \bigoplus_{k \in \mathbb{Z}} D_k(V)^G \) is consistent with the gradation on the algebra of \( G \)-invariant polynomials by the homogeneous degree.

Let \( \mathcal{B}(V) \) be the space of hyperfunctions on \( V \) and let \( \mathcal{B}(V)^G \) be the space of \( G \)-invariant hyperfunctions on \( V \). One of the important notions of this paper is \( G \)-invariant quasi-homogeneous hyperfunctions.

**Definition 1.2 (Quasi-Homogeneous Hyperfunctions).** We say that \( v(x) \in \mathcal{B}(V) \) is quasi-homogeneous if and only if there exist a complex number \( \lambda \in \mathbb{C} \) and a non-negative integer \( k \in \mathbb{Z}_{\geq 0} \) satisfying

\[
F_{r,\lambda} \circ F_{r,\lambda} \circ \cdots \circ F_{r,\lambda}(v) = 0
\]

for all \( r \in \mathbb{R}_{>0} \) where \( F_{r,\lambda}(v) := v(r \cdot x) - r^\lambda v(x) \). We call \( \lambda \in \mathbb{C} \) the homogeneous degree (or simply degree) of \( v(x) \) and \( k \in \mathbb{Z}_{\geq 0} \) the quasi-degree of \( v(x) \). It is easily checked that (11) is equivalent to

\[
(\partial - \lambda)^{k+1} v(x) = 0
\]

with \( \partial := \sum_{i=1}^m x_i \partial_i \). In particular, when a quasi-homogeneous function \( v(x) \) is of quasi-degree \( k \) and not \( k - 1 \), we say that \( v(x) \) is quasi-homogeneous of proper quasi-degree \( k \).

For example, let \( P(x) \) be a homogeneous polynomial of degree \( n \) and let \( \lambda \) be a complex number with sufficiently large real part. Then \( |P(x)|^\lambda \) is a quasi-homogeneous hyperfunction of degree \( \lambda n \) and quasi-degree 0. More generally, \( |P(x)|^\lambda (\log |P(x)|)^k \) is a quasi-homogeneous hyperfunction of degree \( \lambda n \) and quasi-degree \( k \).

We use the following notations in this paper:

1. \( \text{QH}(\lambda) := \{ u(x) \in \mathcal{B}(V) \mid u(x) \text{ is quasi-homogeneous of degree } \lambda \in \mathbb{C} \} \).
2. \( \text{QH}(\lambda)^G := \text{QH}(\lambda) \cap \mathcal{B}(V)^G \).
3. \( \text{QH} := \bigoplus_{\lambda \in \mathbb{C}} \text{QH}(\lambda) \).
4. \( \text{QH}^G := \bigoplus_{\lambda \in \mathbb{C}} \text{QH}(\lambda)^G \).

**Proposition 1.1.** Let \( P(x, \partial) \in D(V) \) (resp. \( \in D(V)^G \)) be a non-zero homogeneous differential operator of homogeneous degree \( \mu \). If \( f(x) \in \mathcal{B}(V) \) (resp. \( \in \mathcal{B}(V)^G \)) is quasi-homogeneous of degree \( \lambda \in \mathbb{C} \), then \( P(x, \partial) f(x) \in \mathcal{B}(V) \) (resp. \( \in \mathcal{B}(V)^G \)) is quasi-homogeneous of degree \( \lambda + \mu \in \mathbb{C} \).
Proof. Let \( P(x, \partial) = \sum_{|\beta| = \mu} a_{\beta} x^\beta \partial^\beta \in D(V) \) be a homogeneous differential operator of degree \( \mu \) and let \( \partial := \sum_{i=1}^m x_i \partial_i \). We prove that
\[
P(x, \partial)(\partial - \lambda) = (\partial - \lambda - \mu)P(x, \partial). \tag{13}
\]
For a monomial term \( a_{\beta} x^\beta \partial^\beta \) in \( P(x, \partial) \), we have
\[
a_{\beta} x^\beta \partial^\beta (\partial - \lambda) = a_{\beta} x^\beta (\partial - \lambda + |\beta|)\partial^\beta
\]
\[
= a_{\beta}(\partial - \lambda + |\beta| - |\alpha|)x^\beta \partial^\beta
\]
\[
= (\partial - \lambda + |\beta| - |\alpha|)a_{\beta} x^\beta \partial^\beta = (\partial - \lambda - \mu)a_{\beta} x^\beta \partial^\beta,
\]
and hence we have (13). Thus for a quasi-homogeneous \( f(x) \in \mathcal{B}(V) \) of degree \( \lambda \), we have
\[
(\partial - \lambda - \mu)^k P(x, \partial)f(x) = P(x, \partial)(\partial - \lambda)^k f(x) = 0
\]
for some \( k \in \mathbb{Z}_{>0} \). Then we see that \( P(x, \partial)f(x) \) is a quasi-homogeneous hyperfunction of degree \( \lambda + \mu \).

For \( P(x, \partial) \in D(V)^G \) and \( f(x) \in \mathcal{B}(V)^G \), we can prove it in the same way.

Remark 1.2. The notion of quasi-homogeneous hyperfunctions is the same as that of associated homogeneous generalized functions introduced by Gelfand and Shilov [4, Chap. 1, Section 4] when we consider the functions of one variable. In other words, as far as we only consider the case of one-variable function, “associated homogeneous generalized functions of order \( k \) and of degree \( \lambda \)” defined in the Gelfand–Shilov book is just the same as “quasi-homogeneous hyperfunctions of degree \( \lambda \) and of quasi-degree \( k \)” defined in this paper. Gelfand and Shilov introduced this notion to characterize Laurent expansion coefficients of the complex power \( x^s \) of homogeneous function \( x \) with respect to the complex variable \( s \in \mathbb{C} \). We see later (in Section 5) that \( G \)-invariant quasi-homogeneous hyperfunctions are obtained as Laurent expansion coefficients of the complex powers \( |P(x)|^s \) of \( G \)-invariant polynomial \( P(x) \) with respect to the complex variable \( s \in \mathbb{C} \) in the case of \( V = \text{Sym}_n(\mathbb{R}) \) and \( G = \text{SL}_n(\mathbb{R}) \).

2. COMPLEX POWERS OF DETERMINANT FUNCTIONS AND INVARIANT DIFFERENTIAL OPERATORS ON THE SYMMETRIC MATRIX SPACE

From now on, we shall deal with the symmetric matrix space \( \text{Sym}_n(\mathbb{R}) \) on which the special linear group \( \text{SL}_n(\mathbb{R}) \) acts naturally. Let \( V := \text{Sym}_n(\mathbb{R}) \) be the space of \( n \times n \) symmetric matrices over the real field \( \mathbb{R} \) and let \( \text{SL}_n(\mathbb{R}) \) be
the special linear group over $\mathbb{R}$ of degree $n$. Then the group $G := \text{SL}_n(\mathbb{R})$ acts on the vector space $V$ by the representation

$$\rho(g) : x \mapsto g \cdot x := gx^i g,$$

with $x \in V$ and $g \in G$. The pair $(G, V) = (\text{SL}_n(\mathbb{R}), \text{Sym}_n(\mathbb{R}))$ is the object that we shall study in this paper.

The vector space $V$ decomposes into a finite number of $\text{GL}_n(\mathbb{R})$-orbits;

$$V := \bigcup_{0 \leq i \leq n} \bigcup_{0 \leq j \leq n-i} S^j_i,$$  \hspace{1cm} (14)

where

$$S^j_i := \{ x \in \text{Sym}_n(\mathbb{R}) \mid \text{sgn}(x) = (j, n - i - j) \}$$  \hspace{1cm} (15)

with integers $0 \leq i \leq n$ and $0 \leq j \leq n - i$. We put

$$S := \bigcup_{1 \leq i \leq n} \bigcup_{0 \leq j \leq n-i} S^j_i \subset V$$

and call it the singular set. In particular, each orbit in $S$ is an $\text{SL}_n(\mathbb{R})$-orbit. An $\text{SL}_n(\mathbb{R})$-orbit in $S$ is called a singular orbit. The subset $S_i := \{ x \in V \mid \text{rank}(x) = n - i \}$ is the set of elements of rank $n - i$. It is easily seen that $S := \bigsqcup_{i=1}^n S_i$ and $S_i = \bigsqcup_{j=0}^{n-i} S^j_i$. The strata $\{S^j_i\}_{1 \leq i \leq n, 0 \leq j \leq n-i}$ have the following closure inclusion relation:

$$\overline{S^j_i} \supset S^{j-1}_{i+1} \cup S^j_{i+1},$$  \hspace{1cm} (16)

where $\overline{S^j_i}$ means the closure of the stratum $S^j_i$.

We denote $P(x) := \det(x)$ and we set $S := \{ x \in V \mid \det(x) = 0 \}$. We call $S$ the singular set of $V$. The subset $V - S$ decomposes into $n + 1$ connected components,

$$V_i := \{ x \in \text{Sym}_n(\mathbb{R}) \mid \text{sgn}(x) = (i, n - i) \}$$  \hspace{1cm} (17)

with $i = 0, 1, \ldots, n$. Here, $\text{sgn}(x)$ for $x \in \text{Sym}_n(\mathbb{R})$ is the signature of the quadratic form $q_x(\mathbf{v}) := \mathbf{v} \cdot x \cdot \mathbf{v}$ on $\mathbf{v} \in \mathbb{R}^n$. We define the complex power function of $P(x)$ by

$$|P(x)|^s_i := \begin{cases} |P(x)|^s & \text{if } x \in V_i, \\ 0 & \text{if } x \notin V_i \end{cases}$$  \hspace{1cm} (18)

for a complex number $s \in \mathbb{C}$. These functions are well defined on $V - S$ but it is not clear whether they are extended to the whole space $V$. In order to
make $|P(x)|^s_i$ well defined as a hyperfunction on $V$, we use the analytic continuation with respect to $s \in \mathbb{C}$. Let $\mathcal{S}(V)$ be the space of rapidly decreasing smooth functions on $V$. For $f(x) \in \mathcal{S}(V)$, the integral

$$Z_i(f, s) := \int_V |P(x)|^s_i f(x) \, dx,$$

is convergent if the real part $\Re(s)$ of $s$ is sufficiently large and is meromorphically extended to the whole complex plane. Thus, we can regard $|P(x)|^s_i$ as a tempered distribution and hence a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$. We consider a linear combination of the hyperfunctions $|P(x)|^s_i$

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^n a_i |P(x)|^s_i$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a}, s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

**Remark 2.1.** We say that a hyperfunction $f(x)$ on $V$ is **singular** if the support of $f(x)$ is contained in the singular set $S$. In particular, any singular invariant hyperfunction is written as a finite sum of quasi-homogeneous hyperfunctions. In addition, if $f(x)$ is $\text{SL}_n(\mathbb{R})$-invariant, i.e., $f(g \cdot x) = f(x)$ for all $g \in \text{SL}_n(\mathbb{R})$, we call $f(x)$ a **singular invariant** hyperfunction on $V$. Any negative-order coefficient of a Laurent expansion of $P^{[\vec{a}, s]}(x)$ is a singular invariant hyperfunction, since the integral

$$\int f(x) P^{[\vec{a}, s]}(x) \, dx = \sum_{i=0}^n a_i Z_i(f, s)$$

is an entire function with respect to $s \in \mathbb{C}$ if $f(x) \in C_0^\infty(V - S)$, where $C_0^\infty \times (V - S)$ is the space of compactly supported $C^\infty$-functions on $V - S$. Conversely, we have the following proposition. Any singular $G$-invariant hyperfunction on $V$ is given as a linear combination of some negative-order coefficients of Laurent expansions of $P^{[\vec{a}, s]}(x)$ at various poles and for some $\vec{a} \in \mathbb{C}^{n+1}$. See [12, 13]. Thus, we see that any singular invariant hyperfunction is written as a linear combination of quasi-homogeneous hyperfunctions.

As defined in Definition 1.1, homogeneous differential operator of degree $k \in \mathbb{Z}$ is given by

$$P(x, \partial) = \sum_{x, \beta \in \mathbb{Z}_{\geq 0}^n \atop |x| - |eta| = k} a_{x, \beta} x^\beta \partial^\beta,$$
where \( m = n(n + 1)/2 \) in the case of symmetric matrix space. The notations here are written as

\[
x = (x_{ij})_{n \geq j \geq i \geq 1}, \quad \partial = (\partial_{ij})_{n \geq j \geq i \geq 1},
\]

\[
x^\alpha = \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta = \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}
\]

with

\[
x = (x_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |x| = \sum_{n \geq j \geq i \geq 1} x_{ij}
\]

and

\[
\beta = (\beta_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}.
\]

We define \( \partial^* \) by

\[
\partial^* = (\partial_{ij}^*) = \left( \varepsilon_{ij} \frac{\partial}{\partial x_{ij}} \right), \quad \text{and} \quad \varepsilon_{ij} := \begin{cases} 1 & i = j, \\ \frac{1}{2} & i \neq j. \end{cases}
\] (22)

We shall give some examples of \( G \)-invariant homogeneous differential operators.

**Example 2.1.** We give here *fundamental* invariant homogeneous differential operators in the sense that they form a complete set of generators of \( D(V)^{SL_n(\mathbb{R})} \) and \( D(V)^{GL_n(\mathbb{R})} \), which will be obtained by the fact in Proposition 2.1.

1. Let \( h \) and \( n \) be positive integers with \( 1 \leq h \leq n \). A sequence of increasing integers \( p = (p_1, \ldots, p_h) \in \mathbb{Z}^h \) is called an increasing sequence in \([1, n] \) of length \( h \) if it satisfies \( 1 \leq p_1 < \cdots < p_h \leq n \). We denote by \( \text{IncSeq}(h, n) \) the set of increasing sequences in \([1, n] \) of length \( h \).

2. For two sequences \( p = (p_1, \ldots, p_h) \) and \( q = (q_1, \ldots, q_h) \in \text{IncSeq}(h, n) \) and for an \( n \times n \) symmetric matrix \( x = (x_{ij}) \in \text{Sym}_n(\mathbb{R}) \), we define an \( h \times h \) matrix \( x_{(p,q)} \) by

\[
x_{(p,q)} := (x_{p_i,q_j})_{1 \leq i \leq j \leq h}.
\]

In the same way, for an \( n \times n \) symmetric matrix \( \partial = (\partial_{ij}) \) of differential operators, we define an \( h \times h \) matrix \( \partial_{(p,q)} \) of differential operators by

\[
\partial_{(p,q)}^* := (\partial_{p_i,q_j}^*)_{1 \leq i \leq j \leq h}.
\]
3. For an integer $h$ with $1 \leq h \leq n$, we define
\[
P_h(x, \partial) := \sum_{p,q \in \text{IncSeq}(h,n)} \det(x_{p,q}) \det(\partial_{(p,q)}^*). \tag{23}
\]

4. In particular, $P_n(x, \partial) = \det(x) \det(\partial^*)$ and Euler’s differential operator $\mathcal{E}$ is given by
\[
\mathcal{E} := \sum_{n \geq j \geq i \geq 1} x_{ij} \frac{\partial}{\partial x_{ij}} = \text{tr}(x \cdot \partial^*) = P_1(x, \partial). \tag{24}
\]

These are all homogeneous differential operators of degree 0 and invariant under the action of $G := \text{SL}_n(\mathbb{R})$, and hence it is also invariant under the action of $G := \text{GL}_n(\mathbb{R})$.

5. $\det(x)$ and $\det(\partial^*)$ are homogeneous differential operators of degree $n$ and $-n$, respectively. They are invariant under the action of $G := \text{SL}_n(\mathbb{R})$, and relatively invariant differential operators under the action of $G := \text{GL}_n(\mathbb{R})$, with characters $\chi(g) = \det(g)^2$ and $\chi^{-1}(g) = \det(g)^{-2}$, respectively.

Proposition 2.1. (1) Every $\text{GL}_n(\mathbb{R})$-invariant differential operator in $D(V)$ can be expressed as a polynomial in $P_i(x, \partial)$ ($i = 1, \ldots, n$) defined in (23). The algebra $D(V)^{\text{GL}_n(\mathbb{R})}$ is isomorphic to the polynomial algebra $\mathbb{C}[P_1, \ldots, P_n]$.

(2) Every $\text{SL}_n(\mathbb{R})$-invariant differential operator in $D(V)$ can be expressed as a polynomial in $P_i(x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$ (see Proposition 2.1). The algebra $D(V)^{\text{SL}_n(\mathbb{R})}$ is generated by $P_i(x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$ but is not isomorphic to the polynomial algebra.

Remark 2.2. The differential operators $\det(x)$ and $\det(\partial^*)$ are not commutative. Then the polynomial expression of an $\text{SL}_n(\mathbb{R})$-invariant differential operator $P(x, \partial)$ in terms of $P_i(x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$ is not unique. In this paper, by “polynomial” expression of $P(x, \partial)$ in terms of $P_i(x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$, we mean an expression as a finite sum of monomial terms of the form
\[
P_1(x, \partial)^{h_1} \cdots P_{n-1}(x, \partial)^{h_{n-1}} (\det(x)^{h_n} (\det(\partial^*))^{h_{n+1}}
\]
with non-negative integers $h_i$ ($i = 1, \ldots, n + 1$).

Proof. The proof of Proposition 2.1(1) is given in [7, pp. 66–67]. We go to the proof of Proposition 2.1(2).
Let $Q(x, \partial)$ be an $\text{SL}_n(\mathbb{R})$-invariant differential operator in $D(V)$. We want to prove that $Q(x, \partial)$ can be expressed as a polynomial in $P_i \times (x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$. We first show that it is sufficient to prove it when $Q(x, \partial)$ is a homogeneous differential operator. Indeed, any $\text{SL}_n(\mathbb{R})$-invariant differential operator $Q(x, \partial)$ can be decomposed as

$$Q(x, \partial) = \sum_{k \in \mathbb{Z}} Q^{(k)}(x, \partial),$$

where $Q^{(k)}(x, \partial)$ is the homogeneous part of degree $k$, i.e., the sum of all the monomial terms of degree $k$. Let $c \in \mathbb{R}$ and $g \in \text{SL}_n(\mathbb{R})$. Then we have

$$\sum_{k \in \mathbb{Z}} c^k Q^{(k)}(x, \partial) = \sum_{k \in \mathbb{Z}} Q^{(k)}(c \cdot x, c^{-1} \cdot \partial) = Q(cx, c^{-1} \partial) = Q(cv, c^{-1} t g^{-1} \partial)$$

and hence we have

$$Q^{(k)}(x, \partial) = Q^{(k)}(gx, t g^{-1} \partial),$$

for each $k \in \mathbb{Z}$. This means that each $Q^{(k)}(x, \partial)$ is $\text{SL}_n(\mathbb{R})$-invariant. Then if we prove that $Q(x, \partial)$ can be expressed as a polynomial in $P_i(x, \partial)$ ($i = 1, \ldots, n - 1$), $\det(x)$ and $\det(\partial^*)$ when $Q(x, \partial)$ is a homogeneous $\text{SL}_n(\mathbb{R})$-invariant differential operator, then it is valid for any $\text{SL}_n(\mathbb{R})$-invariant differential operator.

Now we suppose that $Q(x, \partial)$ is a homogeneous $\text{SL}_n(\mathbb{R})$-invariant differential operator of degree $k \in \mathbb{Z}$. If $k = 0$, then $Q(x, \partial)$ is $\text{GL}_n(\mathbb{R})$-invariant, and hence we have proved it by Proposition 2.1(1). Then we suppose that $k \neq 0$. Since $Q(x, \partial)$ is homogeneous and $\text{SL}_n(\mathbb{R})$-invariant, $Q(x, \partial)$ is relatively invariant under the action of $\text{GL}_n(\mathbb{R})$, and hence we have

$$Q(gx, t g^{-1} \partial) = \det(g)^{2k'} Q(x, \partial)$$

(25)

for all $g \in \text{GL}_n(\mathbb{R})$ with $k' = k/n \in \mathbb{Z} - \{0\}$.

In fact, since $Q(x, \partial)$ is relatively invariant under the action of $\text{GL}_n(\mathbb{R})$, there exists $r \in \mathbb{Z}$ satisfying

$$Q(gx, t g^{-1} \partial) = \det(g)^r Q(x, \partial)$$

for all $g \in \text{GL}_n(\mathbb{R})$. We shall prove that $r$ is an even integer. Since $Q(x, \zeta)$ is a non-zero polynomial on $V \times V^*$. There exists a suitable point $(x_0, \zeta_0) \in V \times V^*$.
$V \times V^*$ such that $Q(x_0, \xi_0) \neq 0$. In particular, we may take $x_0$ to be positive definite. By moving the point $(x_0, \xi_0)$ by the action of $\text{GL}_n(\mathbb{R})$, we may assume that $x_0$ and $\xi_0$ have the forms

$$
x_0 = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

and

$$
\xi_0 = \begin{bmatrix}
y_1 & 0 & \cdots & 0 & 0 \\
0 & y_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & y_{n-1} & 0 \\
0 & 0 & \cdots & 0 & y_n
\end{bmatrix}.
$$

If $r$ is odd, then by taking $g = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & -1
\end{bmatrix}$, we have $\det(g) = -1$. Then we have

$$
Q(x_0, \xi_0) = Q(gx_0, g^{-1}\xi_0) = \det(g)^r Q(x_0, \xi_0) = (-1)^r Q(x_0, \xi_0) = (-1)Q(x_0, \xi_0).
$$

From the assumption that $Q(x_0, \xi_0) \neq 0$, this is a contradiction. Then we have $r$ is an even integer. On the other hand, since $Q(x, \partial)$ is homogeneous of degree $k$, the character $\det(g)^r$ is a homogeneous rational function on $\text{GL}_n \times (\mathbb{R})$ of degree $2k$. Then we have $2k = rn$. Since $r$ is even, $k$ is divisible by $n$ and $r = 2(k/n) = 2k'$. Thus we have (25).

We shall prove that the order of $Q(x, \partial)$ is zero. Then $Q(x, \partial)$ is a polynomial in $x$. Since $Q(x, \partial)$ is $\text{SL}_n(\mathbb{R})$-invariant, it is expressed as a polynomial in $\det(x)$, and hence the proposition is valid.

Next, we suppose that any $Q(x, \partial)$ is expressed as a polynomial of $P_i(x, \partial)$ $(i = 1, \ldots, n-1)$, $\det(x)$ and $\det(\partial^*)$ if the order of $Q(x, \partial)$ is less than $q - 1$ and if $Q(x, \partial)$ is homogeneous of degree $k \in \mathbb{Z} - \{0\}$ and $\text{SL}_n(\mathbb{R})$-invariant. Then we take one $Q(x, \partial)$ whose order is $q$ and which is supposed to be homogeneous of degree $k \in \mathbb{Z} - \{0\}$ and $\text{SL}_n(\mathbb{R})$-invariant. Note that $k$ is divisible by $n$. We put $k' := k/n$ and

$$
F(x, \partial) := \begin{cases}
Q(x, \partial) \det(\partial)^{k'} & \text{if } k' > 0, \\
\det(x)^{-k'} Q(x, \partial) & \text{if } k' < 0
\end{cases}
$$
Then $F(x, \partial)$ is homogeneous of degree 0 and $\text{SL}_n(\mathbb{R})$-invariant. Thus, by Proposition 2.1(1), $F(x, \partial)$ is written as a polynomial of $P_i(x, \partial)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\partial^*)$. Therefore, the principal symbol $\sigma(F)$ $(x, \xi)$ is a polynomial of $P_i(x, \xi)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\xi^*)$. Here $\xi$ is the dual coordinate corresponding to $\partial$. Then

$$\sigma(Q)(x, \xi) = \begin{cases} \sigma(F)(x, \xi) \det(\xi)^{-k'} & \text{if } k' > 0, \\ \det(x)^{k'} \sigma(F)(x, \xi) & \text{if } k' < 0 \end{cases}$$

is not only a rational function of $P_i(x, \xi)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\xi^*)$ but also a polynomial of them since $P_i(x, \xi)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\xi^*)$ are algebraically independent. Thus we can write

$$\sigma(Q)(x, \xi) = R(P_1(x, \xi), \ldots, P_{n-1}(x, \xi), \det(x), \det(\xi^*)),$$

where $R$ is a polynomial. Then by putting

$$Q_1(x, \partial) := Q(x, \partial) - R(P_1(x, \partial), \ldots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)),$$

the order of $Q_1(x, \partial)$ is less than $q - 1$ and $Q_1(x, \partial)$ is homogeneous of degree $k \in \mathbb{Z} - \{0\}$ and $\text{SL}_n(\mathbb{R})$-invariant. Therefore, form the induction hypothesis, $Q_1(x, \partial)$ is expressed as a polynomial of $P_i(x, \partial)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\partial^*)$ and so is

$$Q(x, \partial) = Q_1(x, \partial) - R(P_1(x, \partial), \ldots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)).$$

Thus, by induction of the order, we have proved that $Q(x, \partial)$ is expressed as a polynomial of $P_i(x, \partial)$ $(i = 1, \ldots, n - 1)$, $\det(x)$ and $\det(\partial^*)$ if $Q(x, \partial)$ is homogeneous of degree $k \in \mathbb{Z} - \{0\}$ and $\text{SL}_n(\mathbb{R})$-invariant. \[\square\]

3. $B_p$-FUNCTIONS OF INVARIANT DIFFERENTIAL OPERATORS

As we will see later (Theorem 4.1), the most important object for our problems is the $b_p$-function (Definition 3.1) of the invariant differential operator $P(x, \partial)$ and its homogeneous degree. In this section we shall define $b_p$-functions and give some examples.

**Proposition 3.1.** Let $P(x, \partial) \in D(V)^G$ be a homogeneous differential operator.

1. The homogeneous degree of $P(x, \partial)$ is in $(n \cdot \mathbb{Z})$. Namely the homogeneous degree is divisible by $n$. If the homogeneous degree of $P(x, \partial)$ is $nk$, then it is relatively invariant under the action of $g \in \text{GL}_n(\mathbb{R})$.
corresponding to the character $\text{det}(g)^{2k}$, i.e.,

$$P(g \cdot x, g^{-1} \cdot \partial) = \text{det}(g)^{2k} P(x, \partial).$$

(2) If the homogeneous degree of $P(x, \partial)$ is $nk$ with $k \in \mathbb{Z}$, then we have

$$P(x, \partial)(\text{det } x)^s = b_P(s)(\text{det } x)^{s+k},$$

where $b_P(s)$ is a polynomial in $s \in \mathbb{C}$ and $x \in \text{Sym}_n(\mathbb{R})$ is positive definite. We have also

$$P(x, \partial)P^{[\vec{a}, s]}(x) = b_P(s) \text{det}(x)^k P^{[\vec{a}, s]}(x)$$

$$= b_P(s) \text{sgn}(\text{det}(x))^k P^{[\vec{a}, s+k]}(x)$$

$$= b_P(s)P^{[\vec{a}^#, s+k]}(x)$$

for all $x \in V - S$. Here we put

$$\vec{a}^#k := ((-1)^n a_0, (-1)^{(n-1)}a_1, \ldots, a_n) \in \mathbb{C}^{n+1}.$$

(3) If the homogeneous degree of $P(x, \partial)$ is $nk$ with $k<0$, then we have $b^{-k}(s-1)b_P(s)$ where $b^{-k}(s-1) := b(s-1)b(s-2) \cdots b(s-(-k))$ with $b(s) := \prod_{i=1}^{n} (s + i+1)$.

Proof. (1) By Proposition 2.1, any $\text{SL}_n(\mathbb{R})$-invariant $P(x, \partial)$ is written as a polynomial of $P_i(x, \partial)$ ($i = 1, \ldots, n-1$), $\text{det}(x)$ and $\text{det}(\partial^*)$. The homogeneous degrees of $P_i(x, \partial)$ ($i = 1, \ldots, n-1$) are 0 and those of $\text{det}(x)$ and $\text{det}(\partial^*)$ are $n$ and $-n$, respectively. Therefore the homogeneous degree of $P(x, \partial)$ is a multiple of $n$. On the other hand, the operators $P_i \times (x, \partial)$ ($i = 1, \ldots, n-1$) are absolutely invariant under the action of $g \in \text{GL}_n(\mathbb{R})$ and the operators $\text{det}(x)$ and $\text{det}(\partial^*)$ are relatively invariant under the action of $g \in \text{GL}_n(\mathbb{R})$ corresponding to the character $\text{det}(g)^2$ and $\text{det}(g)^{-2}$, respectively. Then each monomial of $P_i(x, \partial)$ ($i = 1, \ldots, n-1$), $\text{det}(x)$ and $\text{det}(\partial^*)$ in $P(x, \partial)$ is relatively invariant and the corresponding character is determined by its homogeneous degree. Then, if $P(x, \partial)$'s homogeneous degree is $nk$, it is relatively invariant under the action of $g \in \text{GL}_n(\mathbb{R})$ corresponding to the character $\text{det}(g)^{2k}$.

(2) Note that $P^{[\vec{a}, s]}(x) = \sum_{i=1}^{n} a_i |P(x)|_n^s$. For $x \in V_n$, $x$ is positive definite matrix and $|P(x)|_n^s = (\text{det}(x))^s$. Then there exists a polynomial $b_P(s)$ satisfying

$$P(x, \partial)|P(x)|_n^s = P(x, \partial)(\text{det}(x))^s$$

$$= b_P(s)(\text{det}(x))^{s+k}$$

$$= b_P(s)|P(x)|_n^{s+k},$$
since $P(x, \partial)|P(x)|^s_i$ is a relatively invariant function under the action of $g \in \text{GL}_n(\mathbb{R})$ corresponding to the character $(\det(g))^{-(s+k)}$ and since $V_n$ is a $\text{GL}_n(\mathbb{R})$-orbit. Here, note that the equation

$$P(x, \partial)(\det(x))^s = b_p(s)(\det(x))^{s+k}$$

is extended to any $x \in V - S$ by an analytic continuation through the complex domain $V \otimes \mathbb{C}$.

Next, for $x \in V_i$, we have

$$|P(x)|^s_i = |\det(x)|^s = ((-1)^{n-i}(\det(x)))^s = (-1)^{(n-i)s}(\det(x))^s.$$

However, note that the value of the complex power $(-1)^{(n-i)s}$ is determined by taking a suitable branch of analytic continuation, but it must be compatible with the branch of analytic continuation of $(\det(x))^s$. Then, for $x \in V_i$, we have

$$P(x, \partial)|P(x)|^s_i = P(x, \partial)((-1)^{n-i}(\det(x)))^s$$

$$= (-1)^{(n-i)s} P(x, \partial)(\det(x))^s$$

$$= (-1)^{(n-i)s} b_p(s)(\det(x))^{s+k} \quad \text{(by (29))}$$

$$= (-1)^{(n-i)s} b_p(s)(-1)^{(n-i)(s+k)} |P(x)|^s_i^{s+k} \quad \text{(by (30))}$$

$$= (-1)^{(n-i)k} b_p(s) |P(x)|^s_i^{s+k}$$

$$= (-1)^{(n-i)k} b_p(s) |P(x)|^s_i^{s+k}.$$

Then we have

$$P(x, \partial)P[\overline{\alpha}, \overline{s}](x) = b_p(s)P[\overline{\alpha}^{nk}, -s+k](x)$$

for all $x \in V - S$.

(3) Let $P(x, \partial)$ be a homogeneous $\text{SL}_n(\mathbb{R})$-invariant differential operator of degree $nk$ with $k < 0$. From the result in Remark 2.2, each monomial in $P(x, \partial)$ has $(\det(\partial^*))^r$ with $r > (k)$. Namely, for a monomial in $P(x, \partial)$

$$\prod_{h=1}^{n-1} P_h(x, \partial)^{p_h} (\det(x))^q (\det(\partial^*))^r$$

with $p_h(h = 1, \ldots, n-1), q \in \mathbb{Z}_{>0}$, $r$ must be greater than $-k$. Since

$$(\det(\partial^*))^r(\det(x))^q = (\text{const.}) \times b(s-1)b(s-2) \cdots b(s-r)(\det(x))^{s-r},$$
the \( b_p \)-function of \( P(x, \partial) \) must contain \( b^{-k}(s - 1) := b(s - 1)b(s - 2) \cdots b(s - (-k)) \) as a divisor.

Now we can give the definition of \( b_p \)-function for a given \( \text{SL}_n(\mathbb{R}) \)-invariant differential operator \( P(x, \partial) \).

**Definition 3.1 (\( b_p \)-Function).** Let \( P(x, \partial) \in D(V)^G \) be a homogeneous differential operator of homogeneous degree \( nk \). We call \( b_p(s) \) in (26) the \( b_p \)-function of \( P(x, \partial) \).

**Remark 3.1.** The explicit computation of \( b_p \)-functions for a given invariant differential operator \( P(x, \partial) \) is an important problem. Muro [15] gives an algorithm to compute it explicitly. The method employed in [15] is to give a procedure to rewrite \( P(x, \partial) \) in terms of the invariant differential operators \( P_i(x, \partial) \) \( (i = 1, \ldots, n - 1) \), \( \det(x) \) and \( \det(\partial^*) \) defined in Example 2.1. Then, since we have computed the \( b_p \)-functions of \( P_i(x, \partial) \) \( (i = 1, \ldots, n - 1) \), \( \det(x) \) and \( \det(\partial^*) \) in Example 2.1, we obtain the \( b_p \)-function of the given \( P(x, \partial) \).

The algorithm in [15] is possible to be implemented on some computer algebra system. But the possibility of completion of the calculation fully depends on the performance of the computer.

4. **FIRST MAIN THEOREM AND ITS PROOF**

The purpose of this section is to prove the following theorem.

**Theorem 4.1.** Let \( P(x, \partial) \in D(V)^G \) be a non-zero homogeneous differential operator with homogeneous degree \( kn \). We suppose that

\[
\text{the degree of } b_p(s) = \text{the order of } P(x, \partial). \tag{32}
\]

The space of \( G \)-invariant hyperfunction solutions of the differential equation \( P(x, \partial)u(x) = 0 \) is finite dimensional. The solutions \( u(x) \) are given as finite linear combinations of quasi-homogeneous \( G \)-invariant hyperfunctions.

**Remark 4.1.** There are many invariant differential operators that violate condition (32). For example, consider the invariant differential operator \( P(x, \partial) := \det(x) \det(\partial^*) - c(\frac{1}{n}\partial)^n \). Here, \( c \) is the constant satisfying \( \det(\partial^*) \det(x)^s = c \prod_{k=0}^{n-1} (s + \frac{k}{n}) \det(x)^{s-1} \) and \( \partial = \text{tr}(x \cdot \partial^*) \). Note that \( \frac{1}{n} \partial \det(x)^s = s \det(x)^s \). Then we have \( b_p(s) = c \prod_{k=0}^{n-1} (s + \frac{k}{n}) - s^n \), and hence the degree of \( b_p(s) \) is \( n - 1 \) while the order of \( P(x, \partial) \) is \( n \). Indeed, the principal symbol of \( P(x, \partial) \) is \( \sigma(P(x, \partial)) = \det(x) \det(\xi^*) - c(\frac{1}{n} \text{tr}(x \xi^*))^n \) and it is \( \neq 0 \) since \( \det(x) \) and \( \det(\xi^*) \) are irreducible polynomials on \( (x, \xi) \in V \times V^* \). Furthermore, by
putting $P(x, \partial) := \det(x) \det(\partial^*) - c \prod_{k=0}^{n-1} (1_n + \frac{k}{2})$, we have a non-zero invariant differential operator $P(x, \partial)$ of degree $n$ satisfying $b_P(s) \equiv 0$.

**Proof.** Note that the functional equation

$$\mathfrak{M}_1 : \begin{cases} P(x, \partial)u(x) = 0, \\ u(x) \text{ is } \text{SL}_n(\mathbb{R})\text{-invariant}, \end{cases}$$

and the system of linear differential equation

$$\mathfrak{M}_2 : \begin{cases} P(x, \partial)u(x) = 0, \\ \langle A \cdot x, \partial \rangle u(x) = 0 \text{ for all } A \in \mathfrak{sl}_n(\mathbb{R}), \end{cases}$$

are equivalent. Here, $\mathfrak{sl}_n(\mathbb{R})$ is the Lie algebra of $\text{SL}_n(\mathbb{R})$, the action of $A \in \mathfrak{sl}_n(\mathbb{R})$ to $x \in V = \text{Sym}_n(\mathbb{R})$ is $A \cdot x := Ax + x^tA$ and $\langle x, \xi \rangle := \text{tr}(x \cdot \xi)$ is a canonical bilinear form on $(x, \xi) \in T^*V = V \times V^*$, which is automatically extended to the complexification to $(x, \xi) \in T^*V_{\mathbb{C}} = V_{\mathbb{C}} \times V_{\mathbb{C}}^*$. We shall use $\mathfrak{M}_2$ instead of $\mathfrak{M}_1$ in the following.

**Lemma 4.1.** Suppose condition (32). Then the system of linear differential equation $\mathfrak{M}_2$ is a holonomic system and the hyperfunction solution space of $\mathfrak{M}_2$ is finite dimensional.

**Proof.** It is well known that the hyperfunction solution space of a holonomic system is finite dimensional. For the proof, refer [6, Theorem 5.1.7, p. 115].

In order to show that $\mathfrak{M}_2$ is a holonomic system, we have only to prove that the characteristic variety of $\mathfrak{M}_2$ is a complex Lagrangian subvariety in $T^*V_{\mathbb{C}}$ where $V_{\mathbb{C}}$ is a complexification of $V$. From the definition, the characteristic variety of $\text{ch}\mathfrak{M}_2$ of $\mathfrak{M}_2$ is given by

$$\text{ch}(\mathfrak{M}_2) := \left\{ (x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \sigma(P)(x, \xi) = 0 \text{ and } \langle A \cdot x, \xi \rangle = 0 \right\}$$

for all $A \in \mathfrak{sl}_n(\mathbb{R})$.

since the differential operators in (34) form an involutive basis of the differential equation $\mathfrak{M}_2$. Let

$$W := \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \mathfrak{sl}_n(\mathbb{R})\},$$

$$W_0 := \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \mathfrak{gl}_n(\mathbb{R})\},$$

(36)
where $\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra of $\text{GL}_n(\mathbb{R})$. From the definition, we have

$$W_0 = W \cap \{(x, \xi) \in V_C \times V_C^* | \langle x, \xi \rangle = 0\}.$$  

(38)

Let $T^*_S V_C$ be the conormal bundle of $S_i := \{x \in \text{Sym}_n(\mathbb{C}) | \text{rank}(x) = n - i\}$ and let $\overline{T^*_S V_C}$ be its Zariski-closure. Then, we have

$$W_0 = \bigcup_{i=0}^{n} \overline{T^*_S V_C}$$  

(39)

and

$$W \cap \{(x, \xi) \in V_C \times V_C^* | \det(x) = 0\} = \bigcup_{i=1}^{n} \overline{T^*_S V_C} \subset W_0,$$

$$W \cap \{(x, \xi) \in V_C \times V_C^* | \det(\xi) = 0\} = \bigcup_{i=0}^{n-1} \overline{T^*_S V_C} \subset W_0.$$  

(40)

Moreover, we can prove that

$$W - W_0 \text{ is a Zariski open dense subset in } W.$$  

(41)

Results (39)–(41) are obtained by computing the $\text{GL}_n(\mathbb{C})$-orbit structure of $W$ explicitly (see the author’s result [11, p. 400]). Since each $\Lambda_i := \overline{T^*_S V_C}$ is an irreducible Lagrangian subvariety in $T^*V_C$, $W_0$ is a Lagrangian subvariety in $T^*V_C$.

We prove Lemma 4.1 by showing that the characteristic variety $\text{ch}(\mathfrak{M}_2)$ coincides with $W_0$. Before proving this, we need some arguments on the subvariety $W$, $W_0$ and $W^\circ$. Let

$$W^\circ := \{(x, s\partial^* \log \det(x)) \in V_C \times V_C^* | s \in \mathbb{C} - \{0\}, x \in V - S\},$$  

(42)

and let $\overline{W^\circ}$ be its Zariski-closure. Here, $\partial^*$ is a symmetric matrix of differential operator defined by (22). We shall prove that

$$W^\circ = W - W_0 \quad \text{and} \quad \overline{W^\circ} = W.$$  

(43)

It is clear that $\overline{W^\circ} = W$ if $W^\circ = W - W_0$ is valid since $W - W_0$ is a Zariski open dense subset in $W$. So we have only to prove that $W^\circ = W - W_0$.

We first show that $W^\circ \subset W - W_0$. If $(x_0, \xi_0) \in W^\circ$, then $\det(x_0) \neq 0$ and

$$\xi_0 = s_0 \partial^* \log \det(x)|_{x=x_0} = s_0(x_0)^{-1}.$$
with some constant $s_0 \in \mathbb{C}$. Then for any $A \in \text{sl}_n(\mathbb{R})$, we have

$$
\langle A \cdot x_0, \xi_0 \rangle = \text{tr}(A \cdot x_0 \xi_0) = s_0 \text{tr}((A \cdot x_0)(x_0)^{-1})
$$

$$
= s_0 \text{tr}((Ax_0 + x_0^tA)(x_0)^{-1})
$$

$$
= s_0(\text{tr}(Ax_0(x_0)^{-1}) + \text{tr}((x_0^tA)(x_0)^{-1}))
$$

$$
= s_0(\text{tr}(A) + \text{tr}(A^t)) = 0,
$$

and hence $(x_0, \xi_0) \in W$. On the other hand, since

$$
\langle x_0, \xi_0 \rangle = \text{tr}(x_0 \xi_0) = s_0 \text{tr}(x_0(x_0)^{-1}) = \text{tr}(I_n) \neq 0,
$$

we have $(x_0, \xi_0) \notin W_0$. Then $W^\circ \subset W - W_0$ follows.

Next we prove that $W^\circ \supset W - W_0$. Suppose that $(x_0, \xi_0) \in W - W_0$. Then we have $\det(x_0) \neq 0$. In order to prove it, we assume that $\det(x_0) = 0$. Then there exists $A \in \text{sl}_n(\mathbb{R})$ satisfying $A \cdot x_0 = x_0$. Therefore, we have

$$
0 = \langle A \cdot x_0, \xi_0 \rangle = \langle x_0, \xi_0 \rangle,
$$

since $(x_0, \xi_0) \in W = \{(x, \xi) \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \text{sl}_n(\mathbb{R})\}$. This means that $(x_0, \xi_0) \in W_0$ and it violates the assumption that $(x_0, \xi_0) \in W - W_0$. Then $\det(x_0) \neq 0$.

Since $\xi_0$ is not zero and contained in the orthogonal complement of the tangent subspace

$$
\text{sl}_n(\mathbb{C}) \cdot x_0 = \left\{ \text{the complex vector space generated by } A \cdot x_0 \text{ with } A \in \text{sl}_n(\mathbb{R}) \right\} \subset T V_\mathbb{C},
$$

it is a non-constant multiple of $x_0^{-1}$. In fact, $x_0^{-1}$ is contained in the orthogonal complement of $\text{sl}_n(\mathbb{C}) \cdot x_0$ by the same argument in (44). On the other hand, the dimension of $\text{sl}_n(\mathbb{C}) \cdot x_0$ is $n(n + 1)/2 - 1$ since it is the tangent space at $x_0$ of the subvariety $\{x \in V_\mathbb{C} \mid \det(x) = \det(x_0)\}$, which is an $\text{SL}_n(\mathbb{C})$-orbit of $x_0$ in $V_\mathbb{C}$. Therefore, the orthogonal complement is one dimensional and it is generated by $x_0^{-1}$ and hence $\xi_0 = c(x_0)^{-1}$ with a non-zero constant $c$. Then we have

$$(x_0, \xi_0) = (x_0, c(x_0)^{-1}) \in W^\circ$$

if $(x_0, \xi_0) \in W - W_0$. This means $W^\circ \supset W - W_0$. Then, by combining the fact that $W^\circ \subset W - W_0$ proved in the preceding paragraph, we have $W^\circ = W - W_0$. 

We show that

\[ s = \frac{1}{n} \langle x, \xi \rangle \bigg|_{W^\circ} \quad (45) \]

on the subvariety \( W^\circ = W - W_0 \). Since

\[ \langle x, \xi \rangle = (x, s\partial^* \log \det(x)) = (x, sx^{-1}) \]

on \( W^\circ = W - W_0 \), we have

\[ \langle x, \xi \rangle = \langle x, sx^{-1} \rangle = \tr(sxx^{-1}) = \tr(sI_n) = sn, \]

and hence we have (45). The function \( s = \frac{1}{n} \langle x, \xi \rangle |_{W^\circ} \) can be naturally extended to \( W = W/W_0 = W^\circ \) and

\[ W_0 = W \cap \{ (x, \xi) | \langle x, \xi \rangle = 0 \} = W \cap \{ (x, \xi) | s = 0 \}. \quad (46) \]

Now we go back to the proof of the fact that the characteristic variety \( \text{ch}(W_2) \) coincides with \( W_0 \). Let \( nk(k \in \mathbb{Z}) \) be the homogeneous degree of \( P(x, \partial) \) and Let \( q(q \in \mathbb{Z}_{\geq 0}) \) be the order of \( P(x, \partial) \). We denote by \( \sigma(P)(x, \xi) \) the principal symbol of \( P(x, \partial) \). By restricting \( P(x, \partial) \) to \( W^\circ \), we have

\[ \sigma(P)(x, s\partial^* \log \det(x)) = \sigma(P)(x, sx^{-1}) = s^q \sigma(P)(x, x^{-1}). \]

On the other hand, we have

\[ P(x, \partial) \det(x)^s \]

\[ = s^q \sigma(P)(x, \partial^* \det(x)) \det(x)^{s-q} + (\text{lower degree terms in } s) \]

\[ = s^q \sigma(P)(x, \det(x)^{-1}\partial^* \det(x)) \det(x)^s + (\text{lower degree terms in } s) \]

\[ = s^q \det(x)^{-k} \sigma(P)(x, x^{-1}) \det(x)^{s+k} + (\text{lower degree terms in } s) \]

\[ = b_P(s) \det(x)^{s+k}. \]

From assumption (32), the \( b_P\)-function is given by

\[ b_P(s) = b_0 s^q + b_1 s^{q-1} + \cdots + b_q \]

with \( b_0 \neq 0 \). Then we have \( \det(x)^{-k} \sigma(P)(x, x^{-1}) = b_0 \neq 0 \) and hence

\[ \sigma(P)(x, x^{-1}) = b_0 \det(x)^k. \]
Then by considering \( \sigma(P)(x, \xi) \) on \( W^\circ \), we have \( (x, \xi) = (x, sx^{-1}) \) and

\[
\sigma(P)(x, \xi)|_{W^\circ} = s^q \sigma(P)(x, x^{-1})|_{W^\circ} = s^q b_0 \det(x)^k|_{W^\circ}.
\]

If \( k \geq 0 \), then \( \sigma(P)(x, \xi) \) is extended to \( W \) naturally as \( s^q b_0 \det(x)^k \). Then

\[
\text{ch}(\mathfrak{m}_2) = W \cap \{(x, \xi) \mid \sigma(P)(x, \xi) = 0\} = W \cap \{(x, \xi) \mid s^q b_0 \det(x)^k = 0\}
\]

\[
= (W \cap \{(x, \xi) \mid s = 0\}) \cup (W \cap \{(x, \xi) \mid \det(x) = 0\}),
\]

and, by (46) and (40), we have \( \text{ch}(\mathfrak{m}_2) = W_0 \). If \( k \leq 0 \), then \( q \geq -nk \) and

\[
\sigma(P)(x, \xi)|_{W^\circ} = s^q \sigma(P)(x, x^{-1})|_{W^\circ} = s^q b_0 \det(x)^k|_{W^\circ}
\]

\[
= s^q b_0 \det(s^{-1})^k|_{W^\circ} = s^{q + nk} b_0 \det(\xi)^{-k}|_{W^\circ}
\]

since \( (x, \xi) = (x, sx^{-1}) \) on \( W^\circ \). Then \( \sigma(P)(x, \xi) \) is extended to \( W \) naturally as \( s^{q + nk} b_0 \det(\xi)^{-k} \) and

\[
\text{ch}(\mathfrak{m}_2) = W \cap \{(x, \xi) \mid \sigma(P)(x, \xi) = 0\} = W \cap \{(x, \xi) \mid s^{q + nk} b_0 \det(\xi)^{-k}\}
\]

\[
= (W \cap \{(x, \xi) \mid s = 0\}) \cup (W \cap \{(x, \xi) \mid \det(\xi) = 0\}),
\]

and, by (46) and (40), we have \( \text{ch}(\mathfrak{m}_2) = W_0 \). Thus we complete the proof. \[\square\]

**Lemma 4.2.** Let \( \mathcal{S}ol(\mathfrak{m}_2) \) be the hyperfunction solution space to the system of linear differential equation \( \mathfrak{m}_2 \). Then the Euler operator \( \mathfrak{e} := \text{tr}(x\partial^*) \) is a linear endomorphism on the finite dimensional complex vector space \( \mathcal{S}ol(\mathfrak{m}_2) \).

**Proof.** This is clear since \( \mathfrak{e} \) is commutative with the differential operators \( P(x, \partial) \) and \( \langle A \cdot x, \partial \rangle(A \in \text{sl}_n(\mathbb{R})) \). \[\square\]

Now we go back to the proof of Theorem 4.1. Let \( f \) be the dimension of the vector space \( \mathfrak{m}_2 \) and consider the linear map

\[
\mathfrak{e} : \mathcal{S}ol(\mathfrak{m}_2) \to \mathcal{S}ol(\mathfrak{m}_2).
\]

We can choose a basis \( \{u_i(x)\}_{i = 1, \ldots, f} \) of \( \mathcal{S}ol(\mathfrak{m}_2) \) so that the matrix expression of the linear map \( \mathfrak{e} \) with respect to \( \{u_i(x)\}_{i = 1, \ldots, f} \) is a Jordan’s canonical form. Then, for each \( u_i(x) \), there exist an eigenvalue \( \lambda_i \) and a
non-negative integer $k_i$ satisfying

\[
\begin{bmatrix}
  u_i(x) \\
  u_{i+1}(x) \\
  \vdots \\
  u_{i+k_i-1}(x) \\
  u_{i+k_i}(x)
\end{bmatrix}
= \begin{bmatrix}
  \lambda_i & 1 & 0 & \cdots & \cdots & 0 \\
  0 & \lambda_i & 1 & \cdots & \cdots & \vdots \\
  \vdots & 0 & 0 & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \vdots & \cdots & 1 & 0 \\
  0 & \cdots & 0 & \lambda_i & 1 & \vdots
\end{bmatrix}
\begin{bmatrix}
  u_i(x) \\
  u_{i+1}(x) \\
  \vdots \\
  u_{i+k_i-1}(x) \\
  u_{i+k_i}(x)
\end{bmatrix}.
\]

From this equation, we have

\[(\vartheta - \lambda_i)^{k_i+1} u_i(x) = 0,
\]

which means that $u_i(x)$ is a $G$-invariant quasi-homogeneous hyperfunction. This is what we have to prove (see Definition 1.2). [1]

5. SOME PROPERTIES OF LAURENT EXPANSION

COEFFICIENTS OF COMPLEX POWERS OF DETERMINANT

FUNCTION

The following theorem is well known, see, for example, [13]. The hyperfunction $P^{[\tilde{a},s]}(x)$ with a meromorphic parameter $s \in \mathbb{C}$ has the following functional equation (47).

**Proposition 5.1.** Let $\tilde{\sigma}^\#$ be the symmetric matrix of differential operators defined by (22).

1. We have

\[
(\det(\tilde{\sigma}^\#)) P^{[\tilde{a},s+1]}(x) = b(s) P^{[\tilde{a}^\#,s]}(x)
\]

\[
= b(s) (\det(x)) P^{[\tilde{a},s-1]}(x)
\]

with $\tilde{a}^\# = \tilde{a}^#_1 := ((-1)^n a_0, \ldots, -a_{n-1}, a_n)$ and

\[
b(s) = c(s+1) \left( s + \frac{3}{2} \right) \cdots \left( s + \frac{n+1}{2} \right),
\]

where $c$ is a constant.

2. $P^{[\tilde{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -(k+1)/2$ with $k = 1, 2, \ldots$. The possible highest order of the pole of
\[ P^{[\bar{a},s]}(x) \text{ at } s = -(k + 1)/2 \text{ is} \]

\[
\begin{cases}
\frac{k+1}{2} & (k = 1, 2, \ldots, n - 1), \\
\frac{n}{2} & (k = n, n + 1, \ldots, \text{ and } k + n \text{ is odd}), \\
\frac{n+1}{2} & (k = n, n + 1, \ldots, \text{ and } k + n \text{ is even}).
\end{cases}
\] (49)

**Proof.** (1) This is a special case of Proposition 3.1(2), and the \( b_P \)-function for \( \text{det}(\hat{\varrho}^*) \) in (48) is well known. See, for example, [22].

(2) This is also well known. See also [14].

Here we give two definitions.

**Definition 5.1 (Possible Highest Order).** Let \( \lambda \in \mathbb{C} \) be a fixed complex number.

(1) We denote by \( \text{PHO}(\lambda) \) the possible highest order of the pole of \( P^{[\bar{a},s]}(x) \) at \( s = \lambda \). Namely, we define

\[
\text{PHO}(\lambda) := \begin{cases}
\frac{k+1}{2} & (k = 1, 2, \ldots, n - 1), \\
\frac{n}{2} & (k = n, n + 1, \ldots, \text{ and } k + n \text{ is odd}), \\
\frac{n+1}{2} & (k = n, n + 1, \ldots, \text{ and } k + n \text{ is even}), \\
0 & \text{otherwise}.
\end{cases}
\] (50)

(2) Let \( q \in \mathbb{Z} \). We define a vector subspace \( A(\lambda, q) \) of \( \mathbb{C}^{n+1} \) by

\[
A(\lambda, q) := \{ \bar{a} \in \mathbb{C}^{n+1} \mid P^{[\bar{a},s]}(x) \text{ has a pole of order } \leq q \text{ at } s = \lambda \}. \] (51)

Then we have \( A(\lambda, q - 1) \subset A(\lambda, q) \) by definition. We define \( A(\lambda, q) \) by

\[
\overline{A(\lambda, q)} := A(\lambda, q)/A(\lambda, q - 1)
\] (52)

It is easily verified that \( \overline{A(\lambda, q)} = \{0\} \) if \( q > \text{PHO}(\lambda) \) or \( q < 0 \). We have

\[
\bigoplus_{q \in \mathbb{Z}} \overline{A(\lambda, q)} = \bigoplus_{0 \leq q \leq \text{PHO}(\lambda)} \overline{A(\lambda, q)} \approx \mathbb{C}^{n+1}.
\] (53)

In particular, \( \bar{a} = 0 \) if \( \bar{a} \in A(\lambda, q) \) for some \( q < 0 \) since \( A(\lambda, q) = \{0\} \) for \( q < 0 \).

**Definition 5.2 (Laurent Expansion Coefficients).** Let \( \lambda \in \mathbb{C} \) be a fixed complex number.

(1) We define \( o(\bar{a}, \lambda) \in \mathbb{Z} \) by

\[
o(\bar{a}, \lambda) := \text{the order of pole of } P^{[\bar{a},s]}(x) \text{ at } s = \lambda.
\] (54)
Then $o(\bar{a}, \lambda) \in \mathbb{Z}_{\geq 0}$. We have $p = o(\bar{a}, \lambda)$ if and only if $\bar{a} \in A(\lambda, p)$ and $[\bar{a}] \in A(\lambda, p)$ is not zero.

(2) Let $\bar{a} \in \mathbb{C}^{n+1}$ and let $r = o(\bar{a}, \lambda) \in \mathbb{Z}_{\geq 0}$. This means that $P^{[\bar{a}, s]}(x)$ has a pole of order $r$ at $s = \lambda$. Then we have the Laurent expansion of $P^{[\bar{a}, s]}(x)$ at $s = \lambda$,

$$P^{[\bar{a}, s]}(x) = \sum_{w=-r}^{\infty} P^{[\bar{a}, \lambda]}(x)(s - \lambda)^w.$$  

(55)

We often denote by

$$\text{Laurent}^{(w)}(s = \lambda)(P^{[\bar{a}, s]}(x)) := P^{[\bar{a}, \lambda]}(x)$$

the $w$th Laurent expansion coefficient of $P^{[\bar{a}, s]}(x)$ at $s = \lambda$ in (55). It is easily checked that $P^{[\bar{a}, \lambda]}(x)$ is linear with respect to $\bar{a} \in \mathbb{C}^{n+1}$.

We shall investigate some properties of $P^{[\bar{a}, s]}(x)$ and their Laurent expansion coefficients $P^{[\bar{a}, \lambda]}(x)$ at $s = \lambda$. First we show the following lemma.

**Lemma 5.1.** For $\bar{a} \in \mathbb{C}^{n+1}$, let $r = o(\bar{a}, \lambda) \in \mathbb{Z}_{\geq 0}$ be the order of pole of $P^{[\bar{a}, s]}(x)$ at $s = \lambda$ and let

$$P^{[\bar{a}, s]}(x) = \sum_{w \in \mathbb{Z}_{\geq 0}} P^{[\bar{a}, \lambda]}(x)(s - \lambda)^w$$

be the Laurent expansion of $P^{[\bar{a}, s]}(x)$ at $s = \lambda$. Then we have

$$\frac{1}{n} (\mathcal{G} - n\lambda) P^{[\bar{a}, \lambda]}(x) = P^{[\bar{a}, \lambda]}(x)$$

for all $w \in \mathbb{Z}$ and hence $P^{[\bar{a}, \lambda]}(x) \neq 0$ for all $w \geq -r$ and $P^{[\bar{a}, \lambda]}(x) = 0$ for all $w < -r$. In addition, we have $(\mathcal{G} - n\lambda)^{i+1} P^{[\bar{a}, \lambda]}(x) = 0$ and $(\mathcal{G} - n\lambda)^i P^{[\bar{a}, \lambda]}(x) \neq 0$ for $i = 1, 2, \ldots$, where $\mathcal{G} := \text{tr}(x \partial^p)$.

**Proof.** Note that

$$\frac{1}{n} (\mathcal{G} - n\lambda) P^{[\bar{a}, \lambda]}(x) = (s - \lambda) P^{[\bar{a}, s]}(x).$$

Then we have

$$\sum_{w \in \mathbb{Z}} \frac{1}{n} (\mathcal{G} - n\lambda) P^{[\bar{a}, \lambda]}(x)(s - \lambda)^w = \sum_{w \in \mathbb{Z}} P^{[\bar{a}, \lambda]}(x)(s - \lambda)^{w+1},$$

and hence

$$\frac{1}{n} (\mathcal{G} - n\lambda) P^{[\bar{a}, \lambda]}(x) = P^{[\bar{a}, \lambda]}(x)$$
for all $w \in \mathbb{Z}$. Therefore, if $P^{[\bar{a}, \lambda]}_w(x) = 0$, then $P^{[\bar{a}, \lambda]}(x) = 0$, and if $P^{[\bar{a}, \lambda]}_w(x) \neq 0$, then $P^{[\bar{a}, \lambda]}_{w+1}(x) \neq 0$. Since $P^{[\bar{a}, \lambda]}_{-r}(x) = 0$ and $P^{[\bar{a}, \lambda]}(x) \neq 0$ from the assumption, we have the results by applying (57) repeatedly.

Then we have the following proposition.

**Proposition 5.2.** Let $\bar{a}, \bar{b} \in \mathbb{C}^{n+1}$ and let $r = \text{PHO}(\lambda)$.

(1) Let $q$ be an integer in $q \leq r$. We have

$$\bar{a} - \bar{b} \in A(\lambda, q)$$

if and only if

$$\text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{a}, \lambda]}(x)) = \text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{b}, \lambda]}(x))$$

for $w = -r, -r + 1, \ldots, -q - 1$. In particular,

$$P^{[\bar{a}, \lambda]}(x) = P^{[\bar{b}, \lambda]}(x)$$

if $\bar{a} - \bar{b} \in A(\lambda, q)$ for some $q < 0$.

(2) Let $\bar{a}_1, \ldots, \bar{a}_k \in \mathbb{C}^{n+1}$ be the vectors satisfying that they are linearly independent in the quotient space $\mathbb{C}^{n+1}/A(\lambda, q - 1)$ with a positive integer $q$. Then, for an integer $w$ with $w \geq -q$, the hyperfunctions

$$\{\text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{a}_i, \lambda]}(x))\}_{i=1,2,\ldots,k}$$

are linearly independent.

**Proof.** (1) If $\bar{a} - \bar{b} \in A(\lambda, q)$, then $P^{[\bar{a} - \bar{b}, \lambda]}(x)$’s order of pole at $s = \lambda$ is less than $q$. By expanding both the sides of

$$P^{[\bar{a} - \bar{b}, \lambda]}(x) = P^{[\bar{a}, \lambda]}(x) - P^{[\bar{b}, \lambda]}(x)$$

as Laurent expansions, we have

$$\text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{a} - \bar{b}, \lambda]}(x)) = 0 \text{ if } w < - q,$$

and hence

$$\text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{a}, \lambda]}(x)) = \text{Laurent}_{s=\lambda}^{(w)}(P^{[\bar{b}, \lambda]}(x))$$

for $w < -q$. In particular, if $q < 0$ and $\bar{a} - \bar{b} \in A(\lambda, q)$, then $P^{[\bar{a} - \bar{b}, \lambda]}(x)$ has a zero at $s = \lambda$, which means $\bar{a} - \bar{b} = 0$. Then we have $\bar{a} = \bar{b}$. 

Then we have the following proposition.
(2) For an integer \( w \geq -q \), if
\[
\sum_{i=1}^{k} c_i P_{w}^{[\bar{a}_i, \lambda]}(x) = \sum_{i=1}^{k} P_{w}^{[c, \bar{a}_i, \lambda]}(x) = 0,
\]
then \( \sum_{i=1}^{k} P_{w}^{[c, \bar{a}_i, \lambda]}(x) \)'s order of pole at \( s = \lambda \) is strictly less than \( q \) by Lemma 5.1. Then \( \sum_{i=1}^{k} c_i \bar{a}_i \in A(\lambda, q - 1) \) and hence \( \sum_{i=1}^{k} c_i \bar{a}_i \) is zero in the quotient space \( \mathbb{C}^{n+1}/A(\lambda, q - 1) \). Since \( \bar{a}_i(i = 1, \ldots, k) \) are linearly independent in \( \mathbb{C}^{n+1}/A(\lambda, q - 1) \), we have \( c_1 = \cdots = c_k = 0 \). Then
\[
\{\text{Laurent}_{s=\lambda}^{(w)}(P_{w}^{[\bar{a}_i, \lambda]}(x)) = P_{w}^{[\bar{a}_i, \lambda]}(x)\}_{i=1,2,\ldots,k}
\]
are linearly independent. ■

For each \( \lambda \in \mathbb{C} \), if \( P_{w}^{[\bar{a}, \lambda]}(x) \) does not have a pole at \( s = \lambda \), then \( P_{w}^{[\bar{a}, \lambda]}(x) = \hat{P}_{w}^{[\bar{a}, \lambda]}(x) \) is well-defined and a non-zero homogeneous hyperfunction of homogeneous degree \( \lambda n \). If \( P_{w}^{[\bar{a}, \lambda]}(x) \) has a pole at \( s = \lambda \) of order \( p \), then \( (s - \lambda)^p P_{w}^{[\bar{a}, \lambda]}(x) \) is a non-zero homogeneous hyperfunction of homogeneous degree \( \lambda n \). Furthermore, as we have remarked in Remark 1.2, we can prove that Laurent expansion coefficients of \( P_{w}^{[\bar{a}, \lambda]}(x) \) are quasi-homogeneous hyperfunctions and the converse is also true. We shall prove it in Theorem 5.1. Before proving the theorem, we show Lemma 5.2. This is a consequence of the author's paper [13].

We define a standard basis of \( \mathbb{C}^{n+1} \).

**Definition 5.3 (Standard Basis).** Let
\[
\text{SB} := \{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n\}
\]
be a basis of \( \mathbb{C}^{n+1} \). We say that SB is a **standard basis** of \( \mathbb{C}^{n+1} \) at \( s = \lambda \) if the following property holds: there exists an increasing integer sequence
\[
0 < k(0) < k(1) < \cdots < k(\text{PHO}(\lambda)) = n
\]
such that
\[
\text{SB}_q := \{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k(q)}\}
\]
is a basis of \( A(\lambda, q) \) for each \( q \) in \( 0 \leq q \leq \text{PHO}(\lambda) \). It is easily seen that the representatives of \( \text{SB}_q - \text{SB}_{q-1} \) form a basis of the quotient vector space \( A(\lambda, q) := A(\lambda, q)/A(\lambda, q - 1) \).

We need the following lemma which is essentially proved in [13].

**Lemma 5.2.** Let \( v(x) \) be a \( G \)-invariant homogeneous hyperfunction of degree \( n\lambda \), i.e., quasi-homogeneous of degree \( n\lambda \) and of quasi-degree 0 and let \( \{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n\} \) be a standard basis of \( \mathbb{C}^{n+1} \) at \( s = \lambda \). Then \( v(x) \) can be
expressed uniquely as

\[ v(x) = \sum_{i=0}^{n} c_i \text{Laurent}_{s=\lambda}^{(-\sigma(a_i, \lambda))}(P^{[a_i, s]}(x)) \]

with suitable \( c_i \in \mathbb{C} \) \((i = 0, \ldots, n)\) where \( \sigma(a_i, \lambda) \) is the order of pole of \( P^{[a_i, s]}(x) \) at \( s = \lambda \). In other words, the elements

\[ \{\text{Laurent}_{s=\lambda}^{(-\sigma(a_i, \lambda))}(P^{[a_i, s]}(x))\}_{i=0, \ldots, n} \]

are linearly independent and form a basis of the space of hyperfunctions that are \( G \)-invariant and homogeneous of degree \( n\lambda \).

**Proof.** In Muro’s paper [13, Theorem 5.6], he proved that

1. the dimension of \( G \)-invariant homogeneous hyperfunctions of homogeneous degree \( n\lambda \) is \( n + 1 \);
2. any \( G \)-invariant homogeneous hyperfunction of homogeneous degree \( n\lambda \) is written as

\[
\sum_{i=0}^{n} c_i(s)|P(x)|_{i}^{s},
\]

where \( c_i(s) \) are meromorphic functions defined at \( s = \lambda \).

Then we can write as

\[ v(x) = \sum_{i=0}^{n} c_i(s)|P(x)|_{i}^{s}, \]

with \( c_i(s) = \sum_{j \in \mathbb{Z}} c_{ij}(s - \lambda)^j \) are meromorphic functions near \( s = \lambda \).

We see that \( c_i(s) \)'s are assumed to be holomorphic near \( s = \lambda \). Indeed, the Laurent expansion of \( |P(x)|_{i}^{s} \) is given by

\[
\sum_{w \in \mathbb{Z}} P^{[\bar{e}_i, \lambda]}(x)(s - \lambda)^w,
\]

where \( \bar{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the unit vector only whose \( i \)th entry is 1. Then we have

\[
\sum_{i=0}^{n} c_i(s)|P(x)|_{i}^{s} = \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} c_{ij} P^{[\bar{e}_i, \lambda]}(x)(s - \lambda)^{j+w} = \sum_{i=0}^{n} \sum_{k \in \mathbb{Z}} \sum_{k=j+w} c_{ij} P^{[\bar{e}_i, \lambda]}(x)(s - \lambda)^k
\]
\[ \sum_{i=0}^{n} \sum_{k \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} c_{i,k-w} p_{w}^{[\bar{\epsilon}_{i},\lambda]}(x)(s - \lambda)^{k} \]

\[ = \sum_{k \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} \left( \sum_{i=0}^{n} c_{i,k-w} \bar{\epsilon}_{i} \right) p_{w}^{[\bar{\epsilon}_{i},\lambda]}(x)(s - \lambda)^{k}. \]

By putting \( \bar{b}_{k-w} := \sum_{i=0}^{n} c_{i,k-w} \bar{\epsilon}_{i} \). Hence we have

\[ \sum_{w \in \mathbb{Z}} p_{w}^{[\bar{b}_{k-w},\lambda]}(x) = 0, \]

for all \( k < 0 \) and

\[ v(x) = \sum_{w \in \mathbb{Z}} p_{w}^{[\bar{b}_{k-w},\lambda]}(x). \tag{61} \]

If \( w < 0 \), then \( \text{Supp}(p_{w}^{[\bar{a},\lambda]}(x)) \subseteq S \) (see Remark 2.1), and hence we have

\[ \sum_{w \in \mathbb{Z}} p_{w}^{[\bar{b}_{k-w},\lambda]}(x)|_{V-S} = \sum_{w \in \mathbb{Z}} p_{w}^{[\bar{b}_{k-w},\lambda]}(x)|_{V-S} \]

\[ = \sum_{w \in \mathbb{Z}} \sum_{i=0}^{n} c_{i,k-w} p_{w}^{[\bar{\epsilon}_{i},\lambda]}(x)|_{V-S} \]

\[ = \sum_{w \in \mathbb{Z}} \sum_{i=0}^{n} c_{i,k-w} |P(x)|_{\bar{\epsilon}}(\log |P(x)|)^{w}|_{V-S} = 0 \]

for any \( k < 0 \). Since the hyperfunctions in

\[ \{ |P(x)|_{\bar{\epsilon}}(\log |P(x)|)^{w} \}_{i=0,...,n}^{w=0,1,...} \]

are linearly independent, we have \( c_{i,k-w} = 0 \) for all \( i = 0, \ldots, n, \; k = -1, -2, \ldots \) and \( w = 0, 1, \ldots \). This means that

\[ c_{i,j} = 0 \quad \text{for all } i = 0, \ldots, n \quad \text{and } j = -1, -2, \ldots. \]

Therefore, we may assume that each \( c_{i}(s) \) is holomorphic at \( s = \lambda \) and \( \bar{b}_{j} = 0 \) for \( j = -1, -2, \ldots. \)
By \((61)\), we have

\[
v(x) = \sum_{\omega \leq 0 \atop \omega \in \mathbb{Z}} P_{\omega}^{[\hat{b}, \lambda]}(x) = \sum_{-\text{PHO}(\lambda) \leq \omega \leq 0} P_{\omega}^{[\hat{b}, \lambda]}(x).
\]

We shall show that each \(P_{\omega}^{[\hat{b}, \lambda]}(x)\) is homogeneous of degree \(n\lambda\). Indeed, since \(v(x)\) is homogeneous of degree \(n\lambda\) by definition, we have

\[
\frac{1}{n}(\vartheta - n\lambda) \sum_{-\text{PHO}(\lambda) \leq \omega \leq 0} P_{\omega}^{[\hat{b}, \lambda]}(x) = \sum_{-\text{PHO}(\lambda) \leq \omega \leq 0} P_{\omega}^{[\hat{b}, \lambda]}(x) = 0.
\]

by \((57)\). The non-zero hyperfunctions in

\[
\{ P_{\omega}^{[\hat{b}, \lambda]}(x) | -\text{PHO}(\lambda) \leq \omega \leq 0 \}
\]

are linearly independent since their support are dimensionally different, i.e.,

\[
\text{dim (Supp}(P_{\omega}^{[\hat{a}, \lambda]}(x))) < \text{dim (Supp}(P_{\omega}^{[\hat{a}, \lambda]}(x)))
\]

are if \(\text{Supp}(P_{\omega}^{[\hat{a}, \lambda]}(x)) \neq 0\), \(\text{Supp}(P_{\omega}^{[\hat{a}, \lambda]}(x)) \neq 0\) and \(-\text{PHO}(\lambda) \leq \omega_1 < \omega_2 \leq 0\), by the results of Theorem 2.3 in [14]. Then we have

\[
\frac{1}{n}(\vartheta - n\lambda) P_{\omega}^{[\hat{b}, \lambda]}(x) = P_{\omega}^{[\hat{b}, \lambda]}(x) = 0
\]

for each \(-\text{PHO}(\lambda) \leq \omega \leq 0\). Therefore, if \(P_{\omega}^{[\hat{b}, \lambda]}(x) \neq 0\), then \(o(\omega, \lambda) = -w\).

Using the standard basis \(\text{SB} := \{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n\}\) defined by \((58)\), \(\text{SB}_q = \{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{k(q)}\}\) is a basis of \(A(\lambda, q)\) and \(\text{SB}_q - \text{SB}_{q-1} = \{\hat{a}_{k(q)-1} + 1, \ldots, \hat{a}_{k(q)}\}\) is a basis of \(A(\lambda, q)\). In the sum

\[
v(x) = \sum_{-\text{PHO}(\lambda) \leq \omega \leq 0} P_{\omega}^{[\hat{b}, \lambda]}(x),
\]

if \(P_{\omega}^{[\hat{b}, \lambda]}(x) \neq 0\), then \(o(\omega, \lambda) = -w\) and \(\hat{b}^{-w} \in A(\lambda, -w)\), and hence we can write

\[
\hat{b}^{-w} = \sum_{i=k(-w)-1}^{k(-w)} c_i \hat{a}_i + (a \text{ linear sum of } \hat{a}_i \text{ in } i = 0, \ldots, k(-w-1)).
\]
Since $\tilde{a}_i \in A(\lambda, -w - 1)$ for $i = 0, \ldots, k(-w - 1)$ and $w = o(\tilde{a}_i, \lambda)$ for $i = k(-w - 1) + 1, \ldots, k(-w)$, we have

$$P^{[\tilde{a}_i, \lambda]}_w(x) = P^{[\tilde{a}_i, \lambda]}_w(\sum_{i=0}^{k(-w)} c_{i\tilde{a}_i, \lambda}(x)) = \sum_{i=k(-w)-1}^{k(-w)} c_i P^{[\tilde{a}_i, \lambda]}_w(\lambda(x)).$$

Then we have

$$v(x) = \sum_{-\text{PHO}(\lambda) \leq w \leq 0} P^{[\tilde{a}_i, \lambda]}_w(x)$$

$$= \sum_{w=0}^{\text{PHO}(\lambda)} \sum_{i=k(-w)-1}^{k(-w)} c_i P^{[\tilde{a}_i, \lambda]}_w(\lambda(x))$$

$$= \sum_{i=0}^{n} c_i P^{[\tilde{a}_i, \lambda]}_w(\lambda(x))$$

by defining $k(-1) = -1$ and $c_i = 0$ for $i = k(-w - 1) + 1, \ldots, k(-w)$ if $P^{[\tilde{a}_i, \lambda]}_w(x) = 0$. This is what we want to prove.

By using standard basis of $\mathbb{C}^{n+1}$, we have the following proposition.

**Proposition 5.3.** Let $SB := \{\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_n\}$ be a standard basis of $\mathbb{C}^{n+1}$ at $s = \lambda$ and let $r_j := o(\tilde{a}_j, \lambda) \in \mathbb{Z}_{\geq 0}$. Then the Laurent expansion coefficients at $s = \lambda$

$$\{\text{Laurent}_{s = \lambda}^{(-r_j+i)}(P^{[\tilde{a}_j, s]}(x))\}_{i=0, 1, 2, \ldots, n}$$

are linearly independent.

**Proof.** We have only to show that the elements of the finite subset

$$\{\text{Laurent}_{s = \lambda}^{(-r_j+i)}(P^{[\tilde{a}_j, s]}(x))\}_{i=0, 1, 2, \ldots, k} \text{ and } j=0, 1, 2, \ldots, n$$

of (62) are linearly independent. We shall prove it by induction on the number $k$. If $k = 0$, we see that the elements of (63) are linearly independent by Lemma 5.2. Next, we suppose that it is true when $k \geq 0$ and that

$$\sum_{i=0}^{k+1} \sum_{j=0}^{n} c_{ij} \text{Laurent}_{s = \lambda}^{(-r_j+i)}(P^{[\tilde{a}_j, s]}(x)) = 0,$$
where \( c_{ij} \) are constants. Then we have

\[
\left( \frac{1}{n} (\beta - n\lambda) \right) \sum_{i=0}^{k} \sum_{j=0}^{n} c_{ij} \text{Laurent}_{s=\lambda}^{(-r_i)} (P[^{[\bar{a},\lambda]}]_i(x))
\]

\[
= \sum_{i=1}^{k+1} \sum_{j=0}^{n} c_{ij} \text{Laurent}_{s=\lambda}^{(-r_i+i-1)} (P[^{[\bar{a},\lambda]}]_i(x))
\]

\[
= \sum_{i=0}^{k} \sum_{j=0}^{n} c_{i+1,j} \text{Laurent}_{s=\lambda}^{(-r_i+i)} (P[^{[\bar{a},\lambda]}]_i(x)) = 0
\]

by Lemma 5.1. Then, by the induction hypothesis, we have

\[
c_{i+1,j} = 0 \quad \text{for all } i = 0, \ldots, k \quad \text{and} \quad j = 0, \ldots, n.
\]

Then, by (64), we have

\[
\sum_{j=0}^{n} c_{0,j} \text{Laurent}_{s=\lambda}^{(-r_j)} (P[^{[\bar{a},\lambda]}]_j(x)) = 0,
\]

and hence, by Lemma 5.2, we have

\[
c_{0,j} = 0 \quad \text{for all } j = 0, \ldots, n.
\]

Thus we complete the proof by induction. ■

**Theorem 5.1.** Let \( r := o(\bar{a}, \lambda) \in \mathbb{Z}_{\geq 0} \) be the order of the pole of \( P[^{[\bar{a},\lambda]}]_i(x) \) at \( s = \lambda \).

1. Then the Laurent expansion coefficient of \( P[^{[\bar{a},\lambda]}]_i(x) \) at \( s = \lambda \), defined by (56)

\[
\text{Laurent}_{s=\lambda}^{(w)} (P[^{[\bar{a},\lambda]}]_i(x)) = P[^{[\bar{a},\lambda]}]_w (x)
\]

is a quasi-homogeneous hyperfunction of degree \( n\lambda \) of quasi-degree \( r + w \). Conversely, let \( v(x) \in \text{QH}(n\lambda) \), the space of \( \mathcal{G} \)-invariant quasi-homogeneous hyperfunctions (Definition 1.2). Then \( v(x) \) is written as a linear combination of Laurent expansion coefficients of \( |P(x)|_i^s \) at \( s = \lambda \).

2. Let

\[
\text{LC}(\lambda, w) := \text{the vector space generated by}
\]

\[
\{ \text{Laurent}_{s=\lambda}^{(w)} (P[^{[\bar{a},\lambda]}]_i(x)) \mid \bar{a} \in \mathbb{C}^{n+1} \}
\]
i.e., the vector space of wth Laurent expansion coefficients of $P_{w}^{[\dot{\alpha}, \dot{\lambda}]}(x)$ at $s = \lambda$. Then we have the direct sum decomposition

$$
QH(n\lambda)^G = \bigoplus_{w \in \mathbb{Z}, \ w \geq -\text{PHO}(\lambda)} \text{LC}(\lambda, w).
$$

(67)

**Proof.** (1) It is clear that $P_{w}^{[\dot{\alpha}, \dot{\lambda}]}(x)$ is a quasi-homogeneous hyperfunction because we have

$$(\mathfrak{g} - n\lambda)^{r + w + 1} P_{w}^{[\dot{\alpha}, \dot{\lambda}]}(x) = 0$$

by Lemma 5.1.

We prove the converse by induction on the quasi-degree of $v(x) \in QH(n\lambda)^G$. First we suppose that $v(x)$’s quasi-degree is 0, i.e., $v(x)$ is homogeneous of degree $n\lambda$. Then, by Lemma 5.2, $v(x)$ is written as a linear combination of Laurent expansion coefficients of $|P(x)|_w^G$ at $s = \lambda$.

Next we suppose that $v(x)$ is written as a linear combination of Laurent expansion coefficients of $|P(x)|_w^G$ at $s = \lambda$ if $v(x) \in QH(n\lambda)^G$ is of quasi-degree is $q - 1$. We shall prove this is true even if $v(x)$ is of quasi-degree is $q$. Let

$$
v_0(x) := \left( \frac{1}{n} (\mathfrak{g} - n\lambda) \right)^q v(x).
$$

Then, by Definition 1.2, we have $(\frac{1}{n} (\mathfrak{g} - n\lambda))v_0(x) = 0$, and hence, by Lemma 5.2, $v_0(x)$ is written as

$$
v_0(x) = \sum_{i=0}^{n} c_i \text{Laurent}_{v = \lambda}^{-o(\dot{\alpha}_i, \dot{\lambda})}(P_{\dot{\alpha}_i, \dot{\lambda}}(x)) = \sum_{i=0}^{n} c_i P_{w}^{[\dot{\alpha}_i, \dot{\lambda}]}(x)
$$

by using a standard basis $\{\dot{\alpha}_0, \dot{\alpha}_1, \ldots, \dot{\alpha}_n\}$ of $\mathbb{C}^{n+1}$ at $s = \lambda$ and constants $c_i \in \mathbb{C}$. By putting

$$
v_1(x) := v(x) - \sum_{i=0}^{n} c_i P_{w}^{[\dot{\alpha}_i, \dot{\lambda}]}(x),
$$

we have

$$
\left( \frac{1}{n} (\mathfrak{g} - n\lambda) \right)^q v_1(x) = v_0(x) - \sum_{i=0}^{n} c_i P_{w}^{[\dot{\alpha}_i, \dot{\lambda}]}(x) = 0
$$

by applying (57) $q$ times. Then $v_1(x) \in QH(n\lambda)^G$ and it is of quasi-degree is $q - 1$. By the induction hypothesis, $v_1(x)$ is written as a linear combination
of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda$, and so is

$$v(x) = v_1(x) + \sum_{i=0}^{n} c_i P_{-0(\lambda,\lambda)+q}(x).$$

Thus we complete the proof by induction on the quasi-degree.

(2) We have seen that the vector spaces

$$\text{LC}(\lambda, w) \quad (w \in \mathbb{Z} \quad \text{and} \quad w \geq - \text{PHO}(\lambda))$$

are linearly independent by Proposition 5.3 since $\text{LC}(\lambda, w)$ is generated by non-zero Laurent expansion coefficients $P_{\alpha}(\lambda)$ where $\{\alpha_0, \ldots, \alpha_n\}$ is a standard basis of $\mathbb{C}^{n+1}$ at $s = \lambda$. Then we have the result.

By combining Theorems 4.1 and 5.1, we have the following corollary.

**Corollary 5.1.** Let $P(x, \partial) \in D(V)^G$ be a non-zero homogeneous differential operator with homogeneous degree $kn$ satisfying condition (32). Then $G$-invariant hyperfunction solutions $u(x)$ to the differential equation $P(x, \partial)u(x) = v(x)$, which is given as a sum of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda - k$ and hence is quasi-homogeneous of degree $n(\lambda - k)$.

6. SECOND MAIN THEOREMS AND THEIR PROOFS

The purpose of this section is to prove the following theorems.

**Theorem 6.1.** Let $P(x, \partial) \in D(V)^G$ be a non-zero homogeneous differential operator with homogeneous degree $kn$ and let $v(x)$ be a quasi-homogeneous $G$-invariant hyperfunction of homogeneous degree $n\lambda$. We suppose that

$$b_P(s) \neq 0. \quad (68)$$

Then:

(1) We can construct a $G$-invariant hyperfunction solution $u(x) \in \mathcal{B}(V)^G$ to the differential equation $P(x, \partial)u(x) = v(x)$, which is given as a sum of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda - k$ and hence is quasi-homogeneous of degree $n(\lambda - k)$.

(2) Any $G$-invariant hyperfunction solution $u(x)$ is given as finite linear combinations of quasi-homogeneous $G$-invariant hyperfunctions, and hence it is written as a finite linear combinations of Laurent expansion coefficients of $|P(x)|_i^s$ at a finite number of points in $\mathbb{C}$. 
Remark 6.1. In Theorem 6.1, we assume condition (68), but we do not need condition (32) in Theorem 4.1. Condition (32) is obviously stronger than condition (68). In fact, Theorem 6.1 guarantees only the existence of invariant hyperfunction solutions to the invariant differential equation $P \times (x, \partial)u(x) = v(x)$ while Theorem 4.1 asserts that all the invariant hyperfunction solutions to $P(x, \partial)u(x) = 0$ come from Laurent expansion coefficients. Namely, Theorem 4.1 is rather delicate theorem than Theorem 6.1, and it needs a stronger condition (32). Furthermore, there exists a non-zero invariant differential operator that violate condition (68) as we have seen in Remark 4.1.

Proof. The second statement is derived from the first statement by Theorem 4.1 and Lemma 5.2. Indeed, we see that any $G$-invariant hyperfunction solution to the differential equation $P(x, \partial)u(x) = v(x)$ is a sum of several quasi-homogeneous $G$-invariant hyperfunctions, and hence it is written as a finite linear combinations of Laurent expansion coefficients of $[P(x)]^G_w$ at a finite number of points.

We shall prove the first statement. Let $P(x, \partial)$ be a $G$-invariant homogeneous differential operator of homogeneous degree $nk$. For a $G$-invariant quasi-homogeneous hyperfunction $v(x)$ of homogeneous degree $n\lambda$, we have

$$v(x) \in \text{QH}(n\lambda)^G = \bigoplus_{w \in \mathbb{Z}} \bigoplus_{w \geq \text{PHO}(\lambda)} \text{LC}(\lambda, w).$$

By Theorem 5.1, $v(x)$ is written as a finite sum of the hyperfunctions which are given as Laurent expansion coefficients of $P^{(a, s)}(x)$ at $s = \lambda$: $P^{(a, \lambda)}(x)$ with some $w \in \mathbb{Z}$ and some $\tilde{a} \in C^{n+1}$. Thus we have only to show Theorem 6.1 when $v(x) = P^{(a, \lambda)}(x)$.

By (27), we have

$$P(x, \partial)P^{(a, s)}(x) = b_P(s)P^{(a', s+k)}(x),$$

where $b_P(s)$ is the $b_P$-function of $P(x, \partial)$. By expanding both the sides of (69) to Laurent series at $s = \lambda$, we have

$$P(x, \partial)P^{(a, s)}(x) = P(x, \partial) \sum_{w \in \mathbb{Z}} P^{(a, \lambda)}(x)(s - \lambda)^w$$

$$= b_P(s)P^{(a', s+k)}(x)$$

$$= b_P(s) \sum_{w' \in \mathbb{Z}} P^{(a', \lambda', s+k)}(x)(s - \lambda)^{w'}.$$
Since \( b_P(s) \neq 0 \), we can divide it as
\[
b_P(s) = (s - \lambda)^p \tilde{b}(s) \quad \text{with} \quad \tilde{b}(\lambda) \neq 0.
\]
Then \( \tilde{b}(s)^{-1} \) is holomorphic at \( s = \lambda \) and expanded to Taylor series
\[
\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i (s - \lambda)^i.
\]
We have
\[
P(x, \partial) \sum_{w \in \mathbb{Z}} \left( \sum_{i+j=w} b_i P_j^{[\alpha, \lambda]}(x) \right) (s - \lambda)^w
\]
\[
= P(x, \partial) \sum_{j \in \mathbb{Z}} \tilde{b}(s)^{-1} P_j^{[\alpha, \lambda]}(x)(s - \lambda)^j
\]
\[
= (s - \lambda)^p \sum_{w' \in \mathbb{Z}} P_{w'-p}^{[\alpha, \lambda]}(x)(s - \lambda)^{w'}
\]
\[
= \sum_{w \in \mathbb{Z}} P_{w'-p}^{[\alpha, \lambda, \lambda+k]}(x)(s - \lambda)^w.
\]
Comparing both the sides of (70), we obtain
\[
P(x, \partial) \left( \sum_{i+j=w} b_i P_j^{[\alpha, \lambda]}(x) \right) = P_{w'-p}^{[\alpha, \lambda, \lambda+k]}(x)
\]
for each \( w \in \mathbb{Z} \). By arranging the indices we have
\[
P(x, \partial) \left( \sum_{i+j=w+p} b_i P_j^{[\alpha-k, \lambda'-k]}(x) \right) = P_{w}^{[\alpha, \lambda'-k]}(x),
\]
with \( \lambda' = \lambda + k \), which is what we have to prove. Then we can construct a \( \mathbf{G} \)-invariant hyperfunction solution \( u(x) \) to
\[
P(x, \partial) u(x) = P_{w}^{[\alpha, \lambda']}(x),
\]
which is written as finite linear combinations of Laurent expansion coefficients of \( |P(x)|_w^p \) at \( s = \lambda' - k = \lambda \). This is true for any \( \mathbf{G} \)-invariant solution to
\[
P(x, \partial) u(x) = v(x),
\]
where \( v(x) \) is a \( \mathbf{G} \)-invariant quasi-homogeneous hyperfunction.
Next, we consider the construction of $G$-invariant hyperfunction solutions to $P(x, \partial)u(x) = 0$. By Theorem 4.1, a $G$-invariant hyperfunction solution $u(x)$ is written as

$$u(x) = u_1(x) + \cdots + u_p(x),$$  \hspace{1cm} (71)

where each $u_i(x)$ is a quasi-homogeneous hyperfunctions of homogeneous degree $n\lambda_i$ and $\lambda_i \in \mathbb{C}$ are mutually different complex numbers. Then we have

$$P(x, \partial)u_i(x) = 0 \quad \text{for each} \quad i = 1, 2, \ldots, p. \hspace{1cm} (72)$$

Indeed, we see that

$$P(x, \partial)u(x) = P(x, \partial)u_1(x) + \cdots + P(x, \partial)u_p(x) = 0,$$

where the homogeneous degree of each $P(x, \partial)u_i(x)$ is $n\lambda_i + nk$. If some of $P(x, \partial)u_i(x)$ ($i = 1, \ldots, p$) are not zero, then they are zero since they are linearly independent. This is a contradiction. Then we have (72). Then we have only to construct quasi-homogeneous $G$-invariant hyperfunction solution of homogeneous degree $n\lambda$, which is written as a finite linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ ($i = 0, \ldots, n$) at $s = \lambda$.

**Theorem 6.2.** Let $P(x, \partial) \in D(V)^G$ be a non-zero homogeneous differential operator of homogeneous degree $kn$ satisfying condition (32). Then we can construct the $G$-invariant quasi-homogeneous hyperfunction solution of homogeneous degree $n\lambda$ to the differential equation $P(x, \partial)u(x) = 0$ as a finite linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ ($i = 0, \ldots, n$) at $s = \lambda$. It is determined by the homogeneous degree $kn$ and $b_P(s)$ and does not depend on $P(x, \partial)$ itself.

**Proof.** Let $P(x, \partial)$ be a non-zero homogeneous differential operator of homogeneous degree $kn$ and whose $b_P$-function is $b_P(s)$. Then we have

$$P(x, \partial)p_{[a, s]}(x) = b_P(s)p_{[a, s + k]}(x).$$

We expand both the sides into the Laurent series. By the Laurent expansions

$$b_P(s) = \sum_{i \in \mathbb{Z}} b_i(s - \lambda)^i,$$

$$P^{[a, s]}(x) = \sum_{w \in \mathbb{Z}} p^{[a, \lambda]}(x)(s - \lambda)^w,$$

$$p^{[a, n, s + k]}(x) = \sum_{j \in \mathbb{Z}} p^{[a, n, \lambda + k]}(x)(s - \lambda)^j,$$
we have
\[ P(x, \partial) \sum_{w \in \mathbb{Z}} P_w^{[\bar{a}, \lambda]}(x)(s - \lambda)^w = \sum_{w \in \mathbb{Z}} \sum_{i+j=w} b_i P_j^{[a^w_{\bar{k}}, \lambda+k]}(x)(s - \lambda)^w. \]

Then we have
\[ P(x, \partial) P_w^{[\bar{a}, \lambda]}(x) = \sum_{i+j=w} b_i P_j^{[a^w_{\bar{k}}, \lambda+k]}(x). \]

When \( u(x) \) is given as a quasi-homogeneous hyperfunction of degree \( n\lambda \), it is written as a finite sum
\[ u(x) = \sum_{p=1}^{q} P_{w_p}^{[a_p, \lambda]}(x) \tag{73} \]

with \( w_p \in \mathbb{Z} \) and \( \bar{a}_p \in \mathbb{C}^{n+1} \). Then
\[
P(x, \partial)u(x) = \sum_{p=1}^{q} \left( \sum_{i+j=w_p} b_i P_j^{[a^w_{\bar{k}}, \lambda+k]}(x) \right) = \sum_{p=1}^{q} \sum_{j \in \mathbb{Z}} b_{w_p-j} P_j^{[a^w_{\bar{k}}, \lambda+k]}(x)
\]

\[ = \sum_{j \in \mathbb{Z}} \sum_{p=1}^{q} b_{w_p-j} P_j^{[a^w_{\bar{k}}, \lambda+k]}(x) = \sum_{j \in \mathbb{Z}} \sum_{p=1}^{q} b_{w_p-j} P_j^{[a^w_{\bar{k}}, \lambda+k]}(x)
\]

\[ = \sum_{j \in \mathbb{Z}} P_{[\bar{c}_j, \lambda+k]}(x), \]

where \( \bar{c}_j := \sum_{p=1}^{q} b_{w_p-j} a^w_{\bar{k}} \). This is a finite sum since \( \bar{c}_j = 0 \) for sufficiently large \( |j| \). By Theorem 5.1, non-zero elements in \( \{ P_j^{[\bar{c}_j, \lambda+k]}(x) | j \in \mathbb{Z} \} \) are linearly independent. Then \( P(x, \partial)u(x) = 0 \) is equivalent to that
\[ P_{[\bar{c}_j, \lambda+k]}(x) = 0 \quad \text{for all} \ j \in \mathbb{Z}. \tag{74} \]

Thus we can construct a solution \( u(x) \) as a function of the form (73) satisfying condition (74). Condition (74) depends only on \( k \) and \( b_p(s) \). Then the condition for \( G \)-invariant \( u(x) \) to be annihilated by \( P(x, \partial) \) depends only on \( k \) and \( b_p(s) \). \( \square \)

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