Linear Algebra and its Applications 432 (2010) 259–274

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Connected hyperplanes in binary matroids

Manoel Lemos *, T.R.B. Melo

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Pernambuco 50740-540, Brazil

ARTICLE INFO

Article history:
Received 21 November 2004
Accepted 4 August 2009
Available online 4 September 2009

Submitted by R.A. Brualdi

AMS classification:
05B35

Keywords:
Binary matroid
Cocircuit
Connected hyperplane
Hyperplane
Matroid
Non-separating cocircuit

ABSTRACT

For a 3-connected binary matroid M, let $\dim_A(M)$ be the dimension of the subspace of the cocycle space spanned by the non-separating cocircuits of M avoiding A, where $A \subseteq E(M)$. When $A = \emptyset$, Bixby and Cunningham, in 1979, showed that $\dim_A(M) = r(M)$. In 2004, when $|A| = 1$, Lemos proved that $\dim_A(M) = r(M) - 1$. In this paper, we characterize the 3-connected binary matroids having a pair of elements that meets every non-separating cocircuit. Using this result, we show that $2\dim_A(M) \geqslant r(M) - 3$, when M is regular and $|A| = 2$. For $|A| = 3$, we exhibit a family of cographic matroids with a 3-element set intersecting every non-separating cocircuit. We also construct the matroids that attains McNulty and Wu's bound for the number of non-separating cocircuits of a simple and cosimple connected binary matroid.

1. Introduction

We say that a cocircuit C^* of a matroid M is *non-separating* when $M \backslash C^*$ is connected. Note that a cocircuit of a matroid M is non-separating if and only if its complement is a connected hyperplane of M. For a connected graphic matroid, a non-separating cocircuit corresponds to the star of a vertex whose deletion from the associated graph yields a 2-connected matroid.

Non-separating circuits and cocircuits play an important role in the understanding of the structure of graphic matroids. For example, with the aid of these cocircuits, Kelmans [6] gave an elegant proof of Whitney's 2-Isomorphism Theorem (see [18]) and Tutte [16] obtained a nice characterization of the 3-connected graphs which are planar. Bixby and Cunningham [2] generalized Tutte's result for the class of binary matroids by proving Edmonds's Conjecture, namely: a 3-connected binary matroid is graphic if and only if each element belongs to exactly two (or at most two) non-separating cocircuits.

* Corresponding author.
E-mail addresses: manoel@dmat.ufpe.br (M. Lemos), trbm@dmat.ufpe.br (T.R.B. Melo).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.08.007

Moreover, Bixby and Cunningham also proved that each element of a 3-connected binary matroid belongs to at least two non-separating cocircuits. Kelmans [5] and, independently, Seymour (see [12]) proved that every simple and cosimple connected binary matroid has a non-separating cocircuit. It is somewhat striking that every connected binary matroid which is simple and cosimple has at least four non-separating cocircuits as proved by McNulty and Wu [10]. Moreover, McNulty and Wu's result is sharp: there is an infinite family of matroids that attains the bound. In general, even a 3-connected matroid may not have a non-separating cocircuit. This is true, for example, for $U_{r,n}$ provided $2 < r < n$.

Theorem 1.2 reduces the problem of finding non-separating cocircuits of a simple and cosimple connected binary matroid to the problem of finding non-separating cocircuits of some 3-connected binary matroids avoiding some elements. We present its proof in Section 2. To state it, we need to describe a decomposition for a connected matroid due to Cunningham and Edmonds [3].

For matroid notation and terminology, we follow Oxley [14]. Let M be a connected matroid such that $|E(M)| \geqslant 3$. A *tree decomposition* of M is a tree T with edges labeled by $e_1, e_2, \ldots, e_{k-1}$ and vertices labeled by matroids $M_1, M_2, \ldots, M_k$ such that

(i) each M_i is 3-connected with at least four elements or is either a circuit or cocircuit with at least three elements;
(ii) $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k) = E(M) \cup \{e_1, e_2, \ldots, e_{k-1}\}$;
(iii) if the edge e_i joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$;
(iv) if no edge joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_2})$ is empty;
(v) M is the matroid that labels the single vertex of the tree $T/e_1, e_2, \ldots, e_{k-1}$ at the conclusion of the following process: contract the edges $e_1, e_2, \ldots, e_{k-1}$ of T one by one in order; when e_i is contracted, its ends are identified and the vertex formed by this identification is labeled by the 2-sum of the matroids that previously labeled the ends of e_i.

Cunningham and Edmonds [3] proved the following result.

Theorem 1.1. *Every connected matroid M has a tree decomposition T_M in which no two adjacent vertices are both labeled by circuits or are both labeled by cocircuits. Furthermore, the tree T_M is unique to within relabeling of its edges.*

We call T_M the *canonical tree decomposition of* M. Let $\Lambda_2^u(M)$ be the set of matroids that label vertices of T_M. We set

$$\Lambda_2^t(M) = \{H \in \Lambda_2^u(M) : H \text{ is not a circuit or a cocircuit}\}.$$

For a connected matroid M and $A \subseteq E(M)$, we denote the set of non-separating cocircuits of M avoiding A by $\mathcal{R}_A^*(M)$. When $A = \emptyset$, we use $\mathcal{R}^*(M)$ instead of $\mathcal{R}_A^*(M)$.

Theorem 1.2. *Suppose that M is a simple and cosimple connected binary matroid. If M is not 3-connected, then*

$$\mathcal{R}^*(M) = \mathcal{R}_{A_1}^*(M_1) \cup \mathcal{R}_{A_2}^*(M_2) \cup \cdots \cup \mathcal{R}_{A_n}^*(M_n),$$

where $\Lambda_2^t(M) = \{M_1, M_2, \ldots, M_n\}$ and, for $i \in \{1, 2, \ldots, n\}$, A_i is the set of edges of T_M incident to M_i. (That is, $A_i = E(M_i) - E(M)$.)

From this result, when M is a simple and cosimple connected binary matroid,

$$|\mathcal{R}^*(M)| = \sum_{H \in \Lambda_2^t(M)} |\mathcal{R}_{E(H)-E(M)}^*(H)|. \tag{1.1}$$

To obtain a lower bound for $|\mathcal{R}^*(M)|$, it is enough to find one for $|\mathcal{R}_{E(H)-E(M)}^*(H)|$, for some $H \in \Lambda_2^t(M)$. When $|E(H) - E(M)| = 1$, that is, when H labels a terminal vertex of T_M, Lemos [8] got a lower bound for $|\mathcal{R}_{E(H)-E(M)}^*(H)|$. Moreover, when $|E(H) - E(M)| \geqslant 2$, he constructed an infinite family of matroids to show that the best general lower bound that one obtains for $|\mathcal{R}_{E(H)-E(M)}^*(H)|$ is 0.

Let M be a 3-connected binary matroid. For a subset A of $E(M)$, we denote by $\dim_A(M)$ the dimension of the subspace of the cocycle space spanned by the non-separating cocircuits of M avoiding A. When $A = \emptyset$, we use $\dim(M)$ instead of $\dim_A(M)$. The main result of Lemos [8] is:

Theorem 1.3. *Let M be a* 3*-connected binary matroid such that $r(M) \geqslant 1$. If A is an 1-element subset of $E(M)$, then*

$$|\mathcal{R}^*_A(M)| \geqslant \dim_A(M) = r(M) - 1.$$

Using (1.1) and Theorem 1.3, one can obtain McNulty and Wu [10] bound for the number of non-separating cocircuits of a matroid M that satisfies the hypothesis of Theorem 1.2, since there is at least two $H \in \Lambda^t_2(M)$ such that $|E(H) - E(M)| = 1$. (That is, T_M has at least two terminal vertices.) When M is a 3-connected binary matroid, Bixby and Cunningham [2] proved that M has at least $r(M) + 1$ non-separating cocircuits. To accomplish this task, they showed that $\dim(M) = r(M)$. Observe that there is a huge gap between the bounds for the number of non-separating cocircuits obtained by McNulty and Wu [10], for a simple and cosimple connected binary matroid, and by Bixby and Cunningham [2], for a 3-connected binary matroid. Lemos [8] proved that M has at least $r(M) - 1$ non-separating cocircuits, when M is a simple and cosimple connected binary matroid having just one 2-separation. He also constructed an infinite family of matroids to show that the bound given by McNulty and Wu [10] is sharp when the matroid has exactly two 2-separations.

Now, we resume the known bounds for the number of non-separating cocircuits that have been discussed in the previous paragraph. If M is a simple and cosimple connected binary matroid, then

$$|\mathcal{R}^*(M)| \geqslant \begin{cases} r(M)+1, & \text{when } M \text{ is 3-connected;}\\ r(M)-1, & \text{when } M \text{ has just one 2-separation;}\\ 4, & \text{when } M \text{ has just two 2-separations;}\\ 4, & \text{in general.}\end{cases}$$

Moreover, any of these bounds is sharp.

In this paragraph, we describe a family of matroids that plays an important role in the theory of non-separating cocircuits in binary matroids. For a positive integer n, let O_n be the vector matroid of the matrix $[I_n|A_n]$ over $GF(2)$, where $A_n = (a_{ij})$ is an $n \times n$ matrix such that

$$a_{ij} = \begin{cases} 0, & \text{when } i = j \text{ and } 1 \leqslant i \leqslant \mathrm{n}-1;\\ 1, & \text{when } i \neq j \text{ or } i = j = n.\end{cases}.$$

For example, the matrix $[I_5|A_5]$ is equal to:

$$\begin{bmatrix} 1&0&0&0&0&0&1&1&1&1\\ 0&1&0&0&0&1&0&1&1&1\\ 0&0&1&0&0&1&1&0&1&1\\ 0&0&0&1&0&1&1&1&0&1\\ 0&0&0&0&1&1&1&1&1&1 \end{bmatrix}.$$

If the $2n$ columns of $[I_n|A_n]$ are labeled by $a_1, a_2, \dots, a_{n-1}, a, b_1, b_2, \dots, b_{n-1}, b$ respectively, then, when $n \geqslant 2$, $O_n/a_{n-1}\backslash b_{n-1} = O_{n-1}$. As $A_n^T = A_n$, it follows that $O_n \cong O_n^*$. More precisely, $(a_1b_1)(a_2b_2)\cdots(a_{n-1}b_{n-1})(ab)$ is an isomorphism between O_n and O_n^*. We say that b is the *tip* and a is the *cotip* of O_n. When $n \geqslant 4$, O_n/a is the binary spike having $n-1$ legs (see [13]). When $n \geqslant 3$, O_n is 3-connected. (Observe that $O_3 \cong M(K_4)$.) Moreover, for $i \in \{1, 2, \dots, n-1\}$, $\{a_i, b_i, b\}$ is a triangle of O_n and $\{a_i, b_i, a\}$ is a triad of O_n. At last, for any 2-subset $\{i, j\}$ of $\{1, 2, \dots, n-1\}$, $(a_ia_j)(b_ib_j)$ is an automorphism of O_n known as a *t-automorphism*. (The matrix $[I_n|A_n]$ remains invariant after the permutation of: the columns labeled by a_i and a_j; the columns labeled by b_i and b_j; and the ith and jth lines.)

Lemos [8] observed that every non-separating cocircuit of O_n meets $\{a, b\}$, where b and a are respectively the tip and cotip of O_n. In this paper, we show that O_n is the only obstruction to the existence of non-separating cocircuits avoiding a 2-element set of a 3-connected binary matroid, namely:

Theorem 1.4. *Let M be a* 3*-connected binary matroid without a minor isomorphic to O_n, for an integer n exceeding two. If A is a* 2*-subset of $E(M)$, then*

$$|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant \frac{r(M)+1-n}{2}.$$

For an integer n exceeding two, label the maximal stable sets of $K_{3,n}$ by V_1 and V_2 so that $|V_1| = 3$. Let $K''_{3,n}$ and $K'''_{3,n}$ be graphs obtained from $K_{3,n}$ by adding respectively two or three pairwise non-parallel edges joining vertices in V_1. (An edge of $K''_{3,n}$ or $K'''_{3,n}$ which is incident only to vertices belonging to V_1 is called an *added edge*.)

Now, we give examples to show that the bound for $\dim_A(M)$ given in Theorem 1.4 is the best possible. Let α and β be constants such that

$$\dim_A(M) \geqslant \alpha r(M) + \beta, \tag{1.2}$$

for every 3-connected binary matroid M without a minor isomorphic to O_n and 2-subset A of $E(M)$. First, we show that $\alpha \leqslant \frac{1}{2}$. Take $M = M^*(K''_{3,m})$, for an integer m exceeding two. If A is the set of added edges of $K''_{3,m}$, then $\dim_A(M) = m - 1$. As $r(M) = 2m$, it follows that

$$\alpha + \frac{\beta}{r(M)} \leqslant \frac{\dim_A(M)}{r(M)} = \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}.$$

Therefore $\alpha \leqslant \frac{1}{2}$. If $\alpha = \frac{1}{2}$, then $\beta \leqslant -\frac{n-1}{2}$ because (1.2) holds for O_{n-1}.

The next result is a consequence of Theorem 1.4, for $n = 4$, because O_4 is not regular. (Observe that $O_4/a \cong F_7$, where a is the cotip of O_4.)

Corollary 1.1. *Let M be a* 3*-connected regular matroid. If A is a* 2*-subset of $E(M)$, then*

$$|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant \left\lceil \frac{r(M)-3}{2} \right\rceil.$$

Observe that Corollary 1.1 is sharp, since $M^*(K''_{3,m})$ attains the bound for $\dim_A(M)$, when $m \geqslant 3$. It may be possible to obtain a better bound for $|\mathcal{R}_A^*(M)|$.

The bound obtained in Theorem 1.4 can be substantially improved provided we assume that M does not have also a minor isomorphic to $M^*(K''_{3,n-1})$. The bound given in Theorem 1.5 is very closed to the bound for the class of graphic matroids, namely: $|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant r(M) - 3$, when M is a 3-connected graphic matroid and A is a 2-subset of $E(M)$. (This happens because the non-separating cocircuits of a 3-connected graphic matroid are the stars of the vertices of the associated graph.)

Theorem 1.5. *Let M be a* 3*-connected binary matroid without a minor isomorphic to O_n or to $M^*(K''_{3,n-1})$, for an integer n exceeding two. If A is a* 2*-subset of $E(M)$, then*

$$|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant r(M) + 1 - n.$$

Now, we give an example to show that Theorem 1.5 is sharp. Let m be an integer such that $m \geqslant 3$. Choose a triangle T of O_{n-1}, for $n \geqslant 4$. If b and a are respectively the tip and the cotip of O_{n-1}, then $b \in T$, say $T = \{b, c, d\}$. Suppose that T is also a triangle of the rank-m wheel W_m so that b and d are spokes. Moreover, assume that $E(O_{n-1}) \cap E(W_m) = T$. Let M be the matroid obtained from the generalized parallel connection of O_{n-1} and $M(W_m)$ by deleting c. Observe that $r(M) = r(O_{n-1}) + r(W_m) - 2 = n + m - 3$. Note that every triad of $M(W_m)$ that does not include c is a non-separating cocircuit of M. Moreover, these are the only non-separating cocircuits of M avoiding $A = \{a, b\}$. That is, $|\mathcal{R}_A^*(M)| = \dim_A(M) = m - 2$ and both bounds of Theorem 1.5 are attained by M.

The next result was proved by Ding et al. in [4]. It gives a list of unavoidable minors of a large 3-connected binary matroid. Observe that O_n and $M^*K''_{3,n}$ are the only non-graphic matroids in their list. It is intriguing that we need to exclude the non-graphic matroids in their list to obtain a lower bound to the number of non-separating cocircuits of a 3-connected binary matroid avoiding a 2-element set that is close to the bound for 3-connected graphs.

Theorem 1.6. *For every integer n greater than two, there is an integer* $N(n)$ *such that every* 3*-connected binary matroid with more than* $N(n)$ *elements contains a minor isomorphic to one of* $M(K''_{3,n})$, $M^*(K''_{3,n})$, $M(W_n)$ *and* O_n.

The next result plays a very important role in the proofs of Theorems 1.4 and 1.5. It characterizes the 3-connected binary matroids without a non-separating cocircuit avoiding a fixed 2-element set. We prove it in Section 3.

Theorem 1.7. *Suppose that M is a* 3*-connected binary matroid such that* $r(M) \geqslant 3$. *Then, for a* 2*-subset A of* $E(M)$, *the following statements are equivalent*:

(i) *Every non-separating cocircuit of M meets A.*
(ii) *For an integer n exceeding two, there is an isomorphism* Ψ *of M into* O_n *such that* $\Psi(A) = \{a, b\}$, *where b is the tip and a the cotip of* O_n.

For a 3-subset A of $E(M)$, it seems difficult to obtain a similar characterization of the matroids satisfying (i) of Theorem 1.7. In this case, we have an example belonging to the class of cographic matroids. For an integer n exceeding one, if $M = M^*(K'''_{3,n})$ and A is the set of added edges of $K'''_{3,n}$, then every non-separating cocircuit of M meets A.

In Section 3, we use Theorems 1.2, 1.3, and 1.7 to construct all the simple and cosimple connected binary matroids that have just four non-separating cocircuits. That is, we construct all the matroids that attain the bound obtained by McNulty and Wu [10]. This paper is based on part of the Ph.D. Thesis of Melo [11].

2. Reduction to the 3-connected case

In this section, we prove Theorem 1.2. The next lemma plays a fundamental role in its proof. Its dual describes the behavior of a non-separating cocircuit of a connected matroid with respect to a 2-separation.

Lemma 2.1. *Let C be a circuit of a connected matroid M such that* M/C *is also connected. If* $\{X, Y\}$ *is a* 2*-separation of M, then there is* $Z \in \{X, Y\}$ *such that* $Z \subseteq C$ *or* $C \subseteq Z$. *Moreover, when M is cosimple,* $C \subseteq Z$, *for some* $Z \in \{X, Y\}$.

Proof. For $Z \in \{X, Y\}$, let $C_Z = C \cap Z$. If $C_Z = \emptyset$, for some $Z \in \{X, Y\}$, say $Z = X$, then $C \subseteq Y$ and the result follows. We may assume that $C_Z \neq \emptyset$, for each $Z \in \{X, Y\}$. In particular, C_Z is a proper subset of C and so C_Z is independent in M. Hence there is a basis B_Z of $M|Z$ such that $C_Z \subseteq B_Z$. Because $\{X, Y\}$ is a 2-separation of M, $B_X \cup B_Y$ contains just one circuit of M which must be C. Moreover, $(B_X \cup B_Y) - e$ is a basis of M, for every $e \in C$.

Now, we prove that, for $Z \in \{X, Y\}$,

$$r_{M/C}(Z - C_Z) = r(Z) - |C_Z|. \tag{2.1}$$

Without loss of generality, we may assume that $Z = X$. First, we show that

$$r(X \cup C_Y) = r(X) + |C_Y| - 1. \tag{2.2}$$

As $B_X \cup C_Y$ spans $X \cup C_Y$ and contains C, it follows that $(B_X \cup C_Y) - e$ spans $X \cup C_Y$, when $e \in C_Y$. But $(B_X \cup C_Y) - e$ is contained in the basis $(B_X \cup B_Y) - e$ of M. Hence $(B_X \cup C_Y) - e$ is a basis of $M|(X \cup C_Y)$ and so (2.2) follows. By (2.2),

$$r_{M/C}(X - C_X) = r(X \cup C_Y) - r(C) = [r(X) + |C_Y| - 1] - [|C| - 1] = r(X) - |C_X|.$$

Thus (2.1) holds. As $r(M/C) = r(M) - [|C| - 1]$, it follows, by (2.1), that

$$r_{M/C}(X - C_X) + r_{M/C}(Y - C_Y) - r(M/C) = r(X) + r(Y) - r(M) - 1 = 0.$$

By hypothesis, M/C is connected and so $\{X - C_X, Y - C_Y\}$ is not a 1-separation of M/C. Thus there is a $Z \in \{X, Y\}$ such that $Z - C_Z = \emptyset$. That is, $Z \subseteq C$. Moreover, Z is contained in a non-trivial series class of M. □

In the next lemma, we use the following definition for 2-sum of matroids. Let M_1 and M_2 be matroids such that $|E(M_1) \cap E(M_2)| = 1$, say $E(M_1) \cap E(M_2) = \{e\}$. (Note that M_i may have less than three elements and e may be a loop or coloop of M_i.) When e is not a loop of both M_1 and M_2, we define the 2-*sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 as the matroid over $[E(M_1) \cup E(M_2)] - e$ having

$$\mathcal{C}(M_1 \backslash e) \cup \mathcal{C}(M_2 \backslash e) \cup \{(C_1 \cup C_2) - e : e \in C_i \in \mathcal{C}(M_i), \text{ for } i \in \{1, 2\}\}$$

as its family of circuits. When e is a loop of M_1 or M_2, say M_1, we define the 2-*sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 as $(M_1 \backslash e) \oplus (M_2/e)$. (Observe that our definition is different from that given in [14].) This new definition for 2-sum keeps all the nice properties that the usual definition of 2-sum has: it is commutative, it commutes with the operations of duality, deletion and contraction; and, when each factor has at least two elements, its result is connected if and only if each factor is connected.

Lemma 2.2. *Suppose that* M *is a simple and cosimple connected matroid. If* M *is not* 3-*connected, then*

$$\mathcal{R}^*(M) \subseteq \mathcal{R}^*_{A_1}(M_1) \cup \mathcal{R}^*_{A_2}(M_2) \cup \cdots \cup \mathcal{R}^*_{A_n}(M_n),$$

where $\Lambda_2^t(M) = \{M_1, M_2, \ldots, M_n\}$ *and, for* $i \in \{1, 2, \ldots, n\}$, $A_i = E(M_i) - E(M)$.

Proof. If $C^* \in \mathcal{R}^*(M)$, then, by the dual of Lemma 2.1, $C^* \subseteq E(H)$, for some $H \in \Lambda_2^u(M)$. Hence C^* is a cocircuit of H. We have two cases to consider:

Case 1. $|E(H) - C^*| \geqslant 2$.

As the operation of 2-sum commutes with the operation of deletion, it follows that $M \backslash C^*$ is the 2-sum of the matroids belonging to the set $[\Lambda_2^u(M) - \{H\}] \cup \{H \backslash C^*\}$. By hypothesis, $M \backslash C^*$ is connected and so $H \backslash C^*$ is connected. Therefore $C^* \in \mathcal{R}^*(H)$. The result follows provided $H \in \Lambda_2^t(M)$. We may assume that H is a circuit or a cocircuit. In both possibilities for H, we arrive at a contradiction, since $H \backslash C^*$ is a connected matroid having at least two elements.

Case 2. $|E(H) - C^*| \leqslant 1$.

As M is not 3-connected, it follows that $E(H) - E(M) \neq \emptyset$. (That is, H must have as an element a label of an edge of $T(M)$.) Thus $E(H) = C^* \cup b$, where b labels an edge of $T(M)$. Hence $r(H) = r(\{b\}) + 1 = 2$. As M is simple and cosimple, it follows that $H \cong U_{2,n}$, for some $n \geqslant 4$. In particular, $H \in \Lambda_2^t(M)$ and $C^* \in \mathcal{R}^*_{\{b\}}(H)$. □

Proof of Theorem 1.2. By Lemma 2.2, we need to show only that $\mathcal{R}^*_{A_i}(M_i) \subseteq \mathcal{R}^*(M)$, for every $i \in \{1, 2, \ldots, n\}$. As M_i is a 3-connected binary matroid having at least four elements, it follows that $r(M_i) \geqslant 3$. If $C^* \in \mathcal{R}^*_{A_i}(M_i)$, then $M_i \backslash C^*$ is a connected matroid having at least three elements. Hence $M \backslash C^*$ is the 2-sum of the matroids belonging to the set $[\Lambda_2^u(M) - \{M_i\}] \cup \{M_i \backslash C^*\}$. Therefore $M \backslash C^*$ is connected because every matroid in this set is connected. That is, $C^* \in \mathcal{R}^*(M)$ and so $\mathcal{R}^*_{A_i}(M_i) \subseteq \mathcal{R}^*(M)$. □

3. A characterization of O_n

In this section, we show that M is isomorphic to O_n, for some integer n exceeding two, provided M is a 3-connected binary matroid such that $\mathcal{R}^*_A(M) = \emptyset$, for some 2-subset A of $E(M)$. Moreover, this isomorphism maps the elements of A into the tip and the cotip of O_n.

In the next lemma, we show that every 3-connected 1-element binary lift or extension of O_n, for $n \geqslant 3$, has a non-separating cocircuit avoiding both the tip and cotip of O_n.

Lemma 3.1. *Let n be an integer exceeding two. If M is a* 3*-connected binary* 1*-element extension of* O_n *and* $A = \{a, b\}$, *where b and a are respectively the tip and the cotip of* O_n, *then*

(i) *there is a cocircuit* C^* *of M such that* $M \backslash C^*$ *is connected and* $C^* \cap A = \emptyset$; *and*
(ii) *there is a circuit C of M such that* M/C *is connected and* $C \cap A = \emptyset$.

Proof. By definition, O_n is the vector matroid of the matrix $[I_n|A_n]$ over $GF(2)$. Moreover, the $2n$ columns of this matrix are labeled respectively by $a_1, a_2, \ldots, a_{n-1}, a, b_1, b_2, \ldots, b_{n-1}, b$. If $M\backslash e = O_n$, then there is a column vector $v = (v_1, v_2, \ldots, v_n)^T$ such that M is the vector matroid of the matrix $[I_n|A_n|v]$ over $GF(2)$ and e labels the last column of this matrix. Applying an appropriate sequence of t-automorphisms of O_n to the matrix $[I_n|A_n|v]$, we may assume that there is a integer s such that $v_1 = v_2 = \cdots = v_s = 1$ and $v_{s+1} = v_{s+2} = \cdots = v_{n-1} = 0$. Now, we divide the proof into two cases:

Case 1. $v_n = 0$.
Observe that $C_1 = \{a_1, a_2, \ldots, a_s, e\}$ and $C_2 = \{a_{s+1}, a_{s+2}, \ldots, a_{n-1}, a, b, e\}$ are circuits of M. (If $s = n-1$, then $C_2 = \{a, b, e\}$.) In particular, $s \geqslant 2$. As $C^* = \{a_1, b_1, a_2, b_2\} = \{a_1, b_1, a\} \triangle \{a_2, b_2, a\}$ is a cocircuit of O_n, $e \in C_1$ and $|C^* \cap C_1|$ is even, it follows that C^* is a cocircuit of M. Observe that $M\backslash C^*$ is connected because: for each integer i such that $3 \leqslant i \leqslant n-1$, $\{a_i, b_i, b\}$ is a circuit of $M\backslash C^*$; and C_2 is a circuit of $M\backslash C^*$. Thus (i) follows. To conclude (ii), we need to prove only that M/C_1 is connected. But $M/C_1 = (M\backslash e)/\{a_1, a_2, \ldots, a_s\} = O_n/\{a_1, a_2, \ldots, a_s\}$. As $O_n/\{a_1, a_2, \ldots, a_s\} \cong O_n/\{a_{n-s}, a_{n-(s-1)}, \ldots, a_{n-1}\}$ and $O_n/\{a_{n-s}, a_{n-(s-1)}, \ldots, a_{n-1}\}$ is obtained from O_{n-s} by adding all the elements belonging to $\{b_{n-s}, b_{n-(s-1)}, \ldots, b_{n-1}\}$ in parallel to b, it follows that $O_n/\{a_1, a_2, \ldots, a_s\}$ is connected. Hence M/C_1 is connected and (ii) also follows.

Case 2: $v_n = 1$.
Observe that $C_3 = \{a_1, a_2, \ldots, a_s, a, e\}$ is a circuit of M. In particular, $s \geqslant 1$. Note that $s \leqslant n-3$ because v is not equal to the column of A_n labeled by b_{n-1} or by b. As $D^* = \{a_{n-2}, a_{n-1}, b_{n-2}, b_{n-1}\}$ is a cocircuit of O_n, $e \in C_3$ and $|D^* \cap C_3|$ is even, it follows that D^* is a cocircuit of M. Observe that $M\backslash D^*$ is connected because: for each integer i such that $1 \leqslant i \leqslant n-3$, $\{a_i, b_i, b\}$ is a circuit of $M\backslash D^*$; and C_3 is a circuit of $M\backslash D^*$. Thus (i) follows. Observe that $C_4 = \{b_{s+1}, a_{s+2}, a_{s+3}, \ldots, a_{n-1}, e\}$ is a circuit of M. To conclude (ii), we need to prove only that M/C_4 is connected. But $M/C_4 = (M\backslash e)/\{b_{s+1}, a_{s+2}, a_{s+3}, \ldots, a_{n-1}\} = [O_n/\{a_{s+2}, a_{s+3}, \ldots, a_{n-1}\}]/b_{s+1}$. As $O_n/\{a_{s+2}, a_{s+3}, \ldots, a_{n-1}\}$ is obtained from O_{s+2} by adding all the elements belonging to $\{b_{s+2}, b_{s+3}, \ldots, b_{n-1}\}$ in parallel to b and O_{s+2} is 3-connected, it follows that $[O_n/\{a_{s+2}, a_{s+3}, \ldots, a_{n-1}\}]/b_{s+1}$ is connected. Hence M/C_4 is connected and (ii) also follows. $\square$

It may be possible that the next lemma is already known but we do not have a reference for it.

Lemma 3.2. *If* T^* *is a triad of a 3-connected matroid M that meets a triangle T, then,*

(i) T^* *is a non-separating cocircuit of M; and*
(ii) $si(M/e)$ *is 3-connected, for* $e \in T^* - T$.

Proof. The proof of (i) will be omitted since it is straightforward. To prove (ii), it is enough to show that every 2-separation of M/e is trivial. This is the case, when $|E(M)| \leqslant 6$. We may assume that $|E(M)| \geqslant 7$. As $\{T^* - e, E(M) - T^*\}$ is a 2-separation of $M\backslash e$, $T^* - e \subseteq T$ and $T^* - e$ spans T in M, it follows that $\{T, E(M) - (T \cup e)\}$ is a 2-separation of $M\backslash e$. Hence $M\backslash e$ has a non-trivial 2-separation. By the main result of Bixby [1], every 2-separation of M/e is trivial. $\square$

To prove Theorem 1.7, we need the next two lemmas from Bixby and Cunningham [2].

Lemma 3.3. *Suppose that M is a* 3*-connected binary matroid such that* $r^*(M) \geqslant 3$. *If* $e \in E(M)$, $M\backslash e$ *is* 3*-connected and* $C^* \in \mathcal{R}^*(M\backslash e)$, *then* C^* *or* $C^* \cup e$ *belongs to* $\mathcal{R}^*(M)$.

Lemma 3.4. *Suppose that M is a 3-connected binary matroid such that $r^*(M) \geqslant 3$. If $e \in E(M), M/e$ is 3-connected and $C^* \in \mathcal{R}^*(M/e)$, then*

(i) *$C^* \in \mathcal{R}^*(M)$; or*
(ii) *there are $C_1^*, C_2^* \in \mathcal{R}^*(M)$ such that $C_1^* \cap C_2^* = \{e\}, C_1^* \cup C_2^* = C^* \cup e$ and $C_1^* \triangle C_2^* = C^*$.*

The next result shows that (i) implies (ii) in Theorem 1.7. Therefore Theorem 1.7 follows because Lemos [8] pointed out that (ii) implies (i). To prove Proposition 3.1, we use a structure called fan. (For definitions and results about chains and fans, we use Oxley and Wu [15].)

Proposition 3.1. *Suppose that M is a 3-connected binary matroid such that $r(M) \geqslant 3$. If, for a 2-subset A of $E(M)$, every non-separating cocircuit of M meets A, then there is an integer n exceeding two and an isomorphism Ψ of M into O_n such that $\Psi(A) = \{a', b'\}$, where b' is the tip and a' the cotip of O_n.*

Proof. Suppose this result is not true and choose a counter-example M such that $|E(M)|$ is minimum. First, we show that $r(M) \geqslant 4$. If $r(M) = 3$, then M is isomorphic to $M(K_4)$ or to F_7. In both cases, we arrive at a contradiction. If $M \cong F_7$, then there is a triangle T of M such that $A \subseteq T$. Hence $C^* = E(M) - T$ is a cocircuit of M such that $M \backslash C^* = M|T$ is connected and $C^* \cap A = \emptyset$; a contradiction. If $M \cong M(K_4)$, then A meets the star of each vertex of K_4. Therefore A is a matching of K_4; a contradiction because there is an isomorphism between $M(K_4)$ and O_3 that maps the edges of this matching into the tip and the cotip of O_3. Thus $r(M) \geqslant 4$ and so $|E(M)| \geqslant 7$.

In this paragraph, we show that every element belonging to $E(M) - A$ is essential. Suppose that $e \in E(M) - A$ is non-essential. By definition, $M \backslash e$ or M/e is 3-connected. In both cases, we will arrive at a contradiction. If $M \backslash e$ is 3-connected, then, by Lemma 3.3, every non-separating cocircuit of $M \backslash e$ meets A. By the choice of M, there is an isomorphism of $M \backslash e$ into O_n, for some $n \geqslant 3$, that maps the elements of A into the tip and cotip of O_n; a contradiction to Lemma 3.1(i) and so M/e is 3-connected. By Lemma 3.4, every non-separating cocircuit of M/e meets A. By the choice of M, there is an isomorphism between M/e and O_n, for some $n \geqslant 3$, that maps A into $\{a, b\}$, where b and a are respectively the tip and cotip of O_n. Taking the dual, there is an isomorphism between $M^* \backslash e$ and O_n that maps A into $\{a, b\}$. By Lemma 3.1(ii), there is a circuit C of M^* such that $C \cap A = \emptyset$ and M^*/C is connected. But $(M^*/C)^* = M \backslash C$; a contradiction because C is a cocircuit of M. Thus every element belonging to $E(M) - A$ is essential.

As $r(M) \geqslant 4$ and $|A| = 2$, it follows that M is not graphic. In particular, M is not isomorphic to a wheel. By Theorem 1.6 of [15], M has at least two non-essential elements. Hence A is the set of non-essential elements of M. By Theorem 1.6 of [15], for each $e \in E(M) - A$, there is a fan $\mathcal{F}_e$ of M having e as an element. Moreover, A is the set of terminal elements of $\mathcal{F}_e$. By Lemma 3.2(i), every triad of $\mathcal{F}_e$ meets A. If n_e denotes the number of links of $\mathcal{F}_e$, then $n_e \leqslant 3$. Moreover,

(i) $n_e = 1$ and the link of $\mathcal{F}_e$ is a triangle; or
(ii) $n_e = 1$ and the link of $\mathcal{F}_e$ is a triad; or
(iii) $n_e = 2$; or
(iv) $n_e = 3$ and two links of $\mathcal{F}_e$ are triads.

By orthogonality, when (i) happens for some $e \in E(M) - A$, then (i) occurs for every $e \in E(M) - A$. Therefore A spans $E(M) - A$ in M; a contradiction. Hence (i) never happens.

By orthogonality, when (iii) occurs for some $e \in E(M) - A$, then (iii) happens for every $e \in E(M) - A$. By Corollary 1.7 of [15], there is a partition $A_1, A_2, \ldots, A_{n-1}$ of $E(M) - A$ such that, for every $i \in \{1, 2, \ldots, n-1\}, A_i \cup b$ is a triangle of M and $A_i \cup a$ is a triad of M, where $A = \{a, b\}$. Let C be a circuit of M such that $\{a, b\} \subseteq C$. As, by orthogonality, $|C \cap (A_i \cup a)|$ is even, for every $i \in \{1, 2, \ldots, n-1\}$, it follows that $|C \cap A_i| = 1$, say $C \cap A_i = \{a_i\}$. So $C = \{a_1, a_2, \ldots, a_{n-1}, a, b\}$. If $A_i = \{a_i, b_i\}$, for $i \in \{1, 2, \ldots, n-1\}$, then $\{a_i, b_i, b\}$ is a triangle of M. So C spans b_i in M. Hence C spans M. In particular, $r(M) = r(M^*) = n$. Let $[I_n|B]$ be a matrix that represents M in $GF(2)$. Suppose that the columns of $[I_n|B]$ are labeled by $a_1, a_2, \ldots, a_{n-1}, a, b_1, b_2, \ldots, b_{n-1}, b$. Using respectively the circuits $\{a_1, b_1, b\}$, $\{a_2, b_2, b\}, \ldots, \{a_{n-1}, b_{n-1}, b\}$ and $\{a_1, a_2, \ldots, a_{n-1}, a, b\}$, one concludes that $B = A_n$. Hence $M \cong O_n$

and b and a are mapped by this isomorphism into the tip and cotip of O_n; a contradiction. Thus (iii) does not occur.

Suppose that (iv) occurs for some $e \in E(M) - A$. If $\mathcal{F}_e$ is equal to T_1^*, T, T_2^*, then $(T_1^* - T) \cup (T_2^* - T) = A$. Now, we show that T_1^* and T_2^* are the unique triads of M meeting T. If T_3^* is a triad of M such that $T_3^* \notin \{T_1^*, T_2^*\}$ and $T_3^* \cap T \neq \emptyset$, then $T_3^* \cap A = \emptyset$ because $T_3^* - T \neq T_i^* - T$, for $i \in \{1, 2\}$. By Lemma 3.2(i), T_3^* is a non-separating cocircuit of M; a contradiction. Thus T_3^* does not exist. As T_1^* and T_2^* are the unique triads of M meeting T, it follows, by the dual of Theorem 1.8(i) of [15], that $M/f\backslash g$ is 3-connected, where $f \in T_1^* \cap T_2^*$ and $g \in T - f$. Observe that $(T_1^* \cup T_2^*) - \{f, g\}$ is a triad of $M/f\backslash g$ that contains A. Therefore there is no isomorphism between $M/f\backslash g$ and O_n, for some $n \geqslant 3$, that maps the elements belonging to A into the tip and cotip of O_n. By the choice of M, there is a non-separating cocircuit C^* of $M/f\backslash g$ avoiding A. As M/f is obtained from $M/f\backslash g$ by adding g in parallel with the element belonging to $T - \{f, g\}$, it follows that $D^* = C^*$, when $C^* \cap T = \emptyset$, or $D^* = C^* \cup (T - f)$, when $C^* \cap T \neq \emptyset$, is a non-separating cocircuit of M/f avoiding A. By hypothesis, $M\backslash D^*$ is not connected. So $M\backslash D^*$ has two connected components, namely: $(M/f)\backslash D^*$ and $M|f$. Hence $D^* = C^* \cup (T - f)$. We arrive at a contradiction because $A \cup f$ is contained in a series class of $M\backslash(T - f)$. Therefore (iv) also does not occur.

Hence (ii) happens for every $e \in E(M) - A$. Thus A spans $E(M) - A$ in M^*. We arrive at a contradiction and the result follows. □

We say that a simple and cosimple connected binary matroid M is *MW* provided T_M is a path such that:

(i) If H labels a terminal vertex of T_M, then H is isomorphic to $M(K_4)$.
(ii) If H does not label a terminal vertex of T_M and $H \in \Lambda_2^t(M)$, then there is an isomorphism Ψ of H into O_n, for some integer n exceeding two, that maps the elements belonging to $E(H) - E(M)$ into the tip and cotip of O_n.

(Observe that every circuit or cocircuit that labels a vertex of T_M must have three elements because M is simple and cosimple and T_M is a path.)

Now, we characterize which simple and cosimple connected binary matroids attain the lower bound for the number of non-separating cocircuits given by McNulty and Wu [10].

Corollary 3.1. *Let M be a simple and cosimple connected binary matroid. Then, the following statements are equivalent:*

(i) *M has exactly four non-separating cocircuits.*
(ii) *M is MW.*

Proof. First, we show that (i) implies (ii). If M is 3-connected, then, by a theorem of Bixby and Cunningham [2], M has at least $r(M) + 1$ non-separating cocircuits. Hence $r(M) = 3$ and M is isomorphic to $M(K_4)$ or F_7. Therefore M is isomorphic to $M(K_4)$ because F_7 has seven non-separating cocircuits. Thus M is MW and the result follows. We may assume that M is not 3-connected. If $M_1, M_2, \ldots, M_n$ are the terminal vertices of T_M, then $n \geqslant 2$. As M is simple and cosimple, it follows that $\{M_1, M_2, \ldots, M_n\} \subseteq \Lambda_2^t(M)$. Moreover, $r(M_i) \geqslant 3$, for $i \in \{1, 2, \ldots, n\}$. By (1.1) and Theorem 1.3, $|\mathcal{R}^*(M)| \geqslant r(M_1) + r(M_2) + \cdots + r(M_n) - n$. Therefore $n = 2$ and $r(M_1) = r(M_2) = 3$. Moreover, M_1 and M_2 must be isomorphic to $M(K_4)$ because F_7 has three non-separating cocircuits avoiding an element. By (1.1), $|\mathcal{R}^*_{E(H)-E(M)}(H)| = 0$, for every $H \in \Lambda_2^t(M) - \{M_1, M_2\}$. By Theorem 1.7, $H \cong O_n$, for some $n \geqslant 3$, and this isomorphism maps the elements of $E(H) - E(M)$ into the tip and cotip of O_n. Thus M is MW. Observe that (ii) implies (i) by (1.1). □

4. Some auxiliary lemmas

The next lemma generalizes Lemma 3.4.

Lemma 4.1. *Suppose that M is a 3-connected binary matroid such that $r(M) \geqslant 4$. Let e be an element of M such that $si(M/e)$ is 3-connected. If $A \subseteq E(M)$ and $X \subseteq E(M) - e$ is chosen so that $si(M/e) = M/e\backslash X$ and $A' = A - (X \cup e)$ has maximum cardinality, then, for each $C^* \in \mathcal{R}^*_{A'}(si(M/e))$, there is $D^* \in \mathcal{C}(M^*)$ such that $C^* \subseteq D^* \subseteq C^* \cup X$ and*

(i) $D^* \in \mathcal{R}^*_A(M)$; *or*
(ii) $D^* = D^*_1 \triangle D^*_2$, *for some* $D^*_1, D^*_2 \in \mathcal{R}^*_{A-e}(M)$ *so that* $D^*_1 \cap D^*_2 = \{e\}$.

Moreover, when (ii) *happens,* $M\backslash D^*$ *has just one coloop, namely* e.

Proof. Suppose that $C^* \in \mathcal{R}^*_{A'}(si(M/e))$. Let D^* be the cocircuit of M/e obtained from C^* by replacing each one of its elements by the parallel class of M/e that contains it. Note that $H = E(M/e) - D^*$ is a connected hyperplane of M/e because $C^* \in \mathcal{R}^*_{A'}(si(M/e))$. As $C^* \cap A' = \emptyset$, it follows that $D^* \cap A = \emptyset$, by the choice of X. Note also that $C^* \subseteq D^* \subseteq C^* \cup X$. If $M|(H \cup e)$ is connected, then $D^* \in \mathcal{R}^*_A(M)$ and (i) follows. Assume that $M|(H \cup e)$ is not connected. As $[M|(H \cup e)]/e = (M/e)|H$ is connected, it follows that e is a coloop of $M|(H \cup e)$. Thus D^* spans e in M^* and, since M is binary, there is a partition $\{X_1, X_2\}$ of D^* such that $D^*_i = X_i \cup e$ is a cocircuit of M, for $i \in \{1, 2\}$. As M is 3-connected, it follows that $\min\{|X_1|, |X_2|\} \geqslant 2$. Now, we show that $M\backslash D^*_i$ is connected, for $i \in \{1, 2\}$, say $i = 1$. As e is a coloop of $M|(H \cup e)$, it follows that $M|H$ is connected and $M\backslash D^*_1$ has a connected component N such that $H \subseteq E(N)$. If $f \in X_2$, then $r(H \cup f) = r(H) + 1 = r(M) - 1$ and so $H \cup f$ spans X_2. For $g \in X_2 - f$, let C_g be a circuit of M such that $g \in C_g \subseteq H \cup \{f, g\}$. Note that $C_g \cap H \neq \emptyset$ because $|C_g| \geqslant 3$. Hence $E(N) = H \cup X_2$, since $X_2 - f \neq \emptyset$. That is, $M\backslash D^*_1$ is connected. To conclude (ii), we need to prove only that e is the unique coloop of $M\backslash D^*$. This follows because $M\backslash D^*$ has just two connected components, one of them being $M|H$, and $r(H) \geqslant r(M) - 2 \geqslant 2$. $\square$

Lemma 4.2. *Suppose that M is a 3-connected binary matroid such that $r(M) \geqslant 4$. Let e be an element of M such that $si(M/e)$ is 3-connected. If $A \subseteq E(M)$ and $X \subseteq E(M) - e$ is chosen so that $si(M/e) = M/e\backslash X$ and $A' = A - (X \cup e)$ has maximum cardinality, then:*

(i) *If $e \notin A$, then*
$\dim_A(M) \geqslant \dim_{A'}(si(M/e))$,
with equality only if $\mathcal{R}^*_A(M) = \mathcal{R}^*_{A\cup e}(M)$.
(ii) *If there is a chain T_1, T_2, T_3 of M such that $e \in T_1 \cap T_2 \cap T_3$, T_2 is a triangle of M and $e \in A$, then*
$\dim_A(M) \geqslant \dim_{A'}(si(M/e)) - 1$.

Proof. Fix a basis $\mathcal{B}$ for the subspace of the cocycle space of $si(M/e)$ spanned by $\mathcal{R}^*_{A'}(si(M/e))$. For $C^* \in \mathcal{B}$, we define two subsets Ω'_{C^*} and Ω''_{C^*} of $\mathcal{C}^*(M/A)$. Let D^* be a cocircut satisfying the conclusions of Lemma 4.1. We set $\Omega'_{C^*} = \{D^*\}$. When Lemma 4.1(i) happens, we define $\Omega''_{C^*} = \{D^*\}$. When Lemma 4.1(ii) happens, we define $\Omega''_{C^*} = \{D^*_1, D^*_2\}$, when $e \notin A$, or $\Omega''_{C^*} = \emptyset$, when $e \in A$. (Observe that Ω''_{C^*} spans Ω'_{C^*} in the cocycle space of M, when $e \notin A$ or Lemma 4.1(i) holds for C^*.) If

$$\mathcal{B}' = \bigcup_{C^* \in \mathcal{B}} \Omega'_{C^*} \quad \text{and} \quad \mathcal{B}'' = \bigcup_{C^* \in \mathcal{B}} \Omega''_{C^*},$$

then

$$\mathcal{B}' \text{ is a set of linearly independent cocircuits in the cocycle space of } M \text{ and} \tag{4.1}$$
$$\mathcal{B}'' \subseteq \mathcal{R}^*_A(M). \tag{4.2}$$

Now, we prove (i). As $e \notin A$, it follows that $\mathcal{B}''$ spans $\mathcal{B}'$ in the cocycle space of M. By (4.1) and (4.2),

$$\dim_A(M) \geqslant |\mathcal{B}'| = |\mathcal{B}| = \dim_{A'}(si(M/e)). \tag{4.3}$$

If there is $D'^* \in \mathcal{R}^*_A(M)$ such that $e \in D'^*$, then the equality in (4.3) does not occurs, since $e \notin D^*$, for every $D^* \in \mathcal{B}'$. Thus, when equality holds in (4.3), $\mathcal{R}^*_A(M) = \mathcal{R}^*_{A\cup e}(M)$.

To prove (ii), we need to show that:

$$\text{If Lemma 4.1(ii) happens for } C^* \in \mathcal{R}^*_{A'}(si(M/e)), \text{ then } C^* \subseteq T_1 \triangle T_3. \tag{4.4}$$

Let D_1^* and D_2^* be as described in Lemma 4.1(ii). By orthogonality, $T_2 - e \subseteq D_1^* \cup D_2^*$. So $T_2 - e \subseteq D^*$. But $S = (T_1 \cup T_3) - (T_2 - e)$ is a series class or a set of coloops of $M\backslash(T_2 - e)$. By Lemma 4.1(ii), e is a coloop of $M\backslash D^*$. So $S - D^*$ is a set of coloops of $M\backslash D^*$. By Lemma 4.1(ii), $M\backslash D^*$ has just one coloop, namely e, and so $S - e \subseteq D^*$. That is, $T_1 \triangle T_3$ is a subset of D^*. Thus $D^* = T_1 \triangle T_3$ and (4.4) follows. By (4.4), $\mathcal{B}''$ spans $\mathcal{B}' - \{T_1 \triangle T_3\}$ in the cocycle space of M. The result follows by (4.1) and (4.2). □

The results proved in this section are used in [9].

5. A minimal counter-example

Throughout this section, we suppose that:

(a) α and β are real numbers such that $0 < \alpha \leqslant 1$ and $\beta \leqslant -3\alpha$;
(b) A is a 2-element set; and
(c) $\mathcal{F}$ is a class of binary matroids closed under minors.

We define

$$\mathcal{F}_{\alpha,\beta} = \{M \in \mathcal{F} : A \subseteq E(M), M \text{ is 3-connected and } \dim_A(M) < \alpha r(M) + \beta\}.$$

Our goal is to prove the following result:

Theorem 5.1. *Let M be a member of $\mathcal{F}_{\alpha,\beta}$ with the minimum number of elements. If $\mathcal{R}^*_A(M) \neq \emptyset$, then there is an isomorphism from M to $M^*(K''_{3,n})$, for some $n \geqslant 2$, mapping the elements of A into the added edges of $K''_{3,n}$.*

Proof. Observe that $r(M) \geqslant 5$ because

$$1 \leqslant \dim_A(M) < \alpha r(M) + \beta \leqslant \alpha[r(M) - 3] \leqslant r(M) - 3. \tag{5.1}$$

We divide the proof of this result in a sequence of lemmas.

Lemma 5.1. *If $e \in E(M) - A$, then $M\backslash e$ is not 3-connected.*

Proof. Suppose that $M\backslash e$ is 3-connected. By hypothesis, $M\backslash e \notin \mathcal{F}_{\alpha,\beta}$ and so

$$\dim_A(M\backslash e) \geqslant \alpha r(M\backslash e) + \beta = \alpha r(M) + \beta > \dim_A(M).$$

If $C^* \in \mathcal{R}^*_A(M\backslash e)$, then, by Lemma 3.3, $\mathcal{R}^*_A(M) \cap \{C^*, C^* \cup e\} \neq \emptyset$. Thus $\dim_A(M) \geqslant \dim_A(M\backslash e)$; a contradiction. □

Lemma 5.2. *Suppose that T is a triangle of M such that $A \nsubseteq T$. If $e \in T - A$, then there is a triad T^* of M such that $e \in T^*$.*

Proof. By Lemma 5.1, $M\backslash f$ is not 3-connected, for every $f \in T - A$. As $|T - A| \geqslant 2$, it follows, by Tutte's Triangle Lemma [17], that every element of $T - A$ belongs to a triad of M. □

Lemma 5.3. *If $e \in E(M) - A$ and $si(M/e)$ is 3-connected, then $\mathcal{R}^*_A(M) = \mathcal{R}^*_{A\cup e}(M)$.*

Proof. Choose the simplification of M/e so that $A' = A \cap E(si(M/e))$ has maximum cardinality. Suppose that $si(M/e) = M/e\backslash X$. There are two cases to be considered. We deal with them simultaneously. First, we determine a lower bound for $\dim_{A'}(si(M/e))$. If $|A'| = 1$, then, by Theorem 1.3 and (5.1),

$$\dim_{A'}(si(M/e)) \geqslant r(si(M/e)) - 1 = r(M) - 2 \geqslant \alpha r(M) + \beta + 1.$$

If $|A'| = 2$, then, by hypothesis, $si(M/e) \notin \mathcal{F}_{\alpha,\beta}$ and so

$$\dim_{A'}(si(M/e)) \geqslant \alpha r(si(M/e)) + \beta = \alpha r(M) + \beta - \alpha \geqslant \alpha r(M) + \beta - 1.$$

By Lemma 4.2(i),

$$\dim_A(M) \geqslant \dim_{A'}(si(M/e)).$$

We have a contradiction, unless $|A'| = |A|$ and

$$\dim_A(M) = \dim_{A'}(si(M/e)).$$

Again, by Lemma 4.2(i), we have that $\mathcal{R}^*_A(M) = \mathcal{R}^*_{A\cup e}(M)$. $\square$

Lemma 5.4. *If T^* is a triad of M that meets at least one triangle of M, then $T^* \cap A \neq \emptyset$.*

Proof. Suppose that $T^* \cap A = \emptyset$. Let T be a triangle of M such that $T \cap T^* \neq \emptyset$. If $e \in T^* - T$, then, by Lemma 3.2(ii), $si(M/e)$ is 3-connected. So, by Lemma 5.3, $\mathcal{R}^*_A(M) = \mathcal{R}^*_{A\cup e}(M)$. Hence $T^* \notin \mathcal{R}^*(M)$; a contradiction to Lemma 3.2(i). Thus $T^* \cap A \neq \emptyset$. $\square$

Lemma 5.5. *If T_1^*, T, T_2^* is a chain of M such that T is a triangle, then*

(i) $T_1^* \cap T \cap T_2^* \nsubseteq A$; *or*
(ii) *there is a triad T_3^* of M such that $T_3^* \notin \{T_1^*, T_2^*\}$, $T_3^* \cap T \neq \emptyset$ and $A \subseteq T \cup T_3^*$.*

Proof. Let a be the element belonging to $T_1^* \cap T \cap T_2^*$. Suppose that $a \in A$. If $si(M/a)$ is 3-connected, then, by Theorem 1.3 and (5.1),

$$\dim_{A-a}(si(M/a)) \geqslant r(si(M/a)) - 1 = r(M) - 2 \geqslant \alpha r(M) + \beta + 1;$$

a contradiction because, by Lemma 4.2(ii),

$$\dim_A(M) \geqslant \dim_{A-a}(si(M/a)) - 1.$$

Hence $si(M/a)$ is not 3-connected. By the dual of Theorem 1.8 of [15], T meets a third triad of M, say T_3^*. By Lemma 5.4, $T_3^* \cap A \neq \emptyset$. Thus $A - a \subseteq T_3^*$, since $a \notin T_3^*$. We have (ii). $\square$

Let $\mathcal{T}$ be the set of triangles T of M such that $T \cap C^* \neq \emptyset$, for some $C^* \in \mathcal{R}^*_A(M)$.

Lemma 5.6. *If $T \in \mathcal{T}$, then there are triads T_1^* and T_2^* of M such that T_1^*, T, T_2^* is a fan $\mathcal{F}_T$ of M. Moreover,*

(i) $T \cap A = \emptyset$;
(ii) $A = (T_1^* - T) \cup (T_2^* - T)$;
(iii) *T meets exactly two triads of M; and*
(iv) *t does not belong to a non-separating cocircuit of M avoiding A, where $t \in T_1^* \cap T \cap T_2^*$.*

Proof. Let C^* be a non-separating cocircuit of M avoiding A so that $T \cap C^* \neq \emptyset$. By Lemma 5.4, $|C^*| \geqslant 4$. By orthogonality, $|C^* \cap T| = 2$. Observe that there is no triad T^* of M such that $T^* \cap T = C^* \cap T$, otherwise $T^* - T$ is a coloop of $M\backslash C^*$. (In particular, there is at most two triads of M meeting T.) By Lemma 5.2, there are different triads T_1^* and T_2^* of M meeting T and (iii) follows. If $t \in T_1^* \cap T \cap T_2^*$, then $t \in T - C^*$. By Lemma 5.5, $t \notin A$, since T meets exactly two triads of M. Hence $T \cap A = \emptyset$ and (i) follows. By Lemma 5.4, $T_i^* \cap A \neq \emptyset$, for each $i \in \{1, 2\}$, say $a_i \in T_i^* \cap A$. Hence $a_i \in T_i^* - T$ because $T \cap A = \emptyset$, by (i). Therefore $A = \{a_1, a_2\}$, since $a_1 \neq a_2$, by Theorem 1.6 of [15]. We have (ii).

Now, we show that T_1^*, T, T_2^* is a fan of M. If T_1^*, T, T_2^* is not a fan of M, then there is a triangle T' of M such that $T \neq T'$ and $T' \cap T_i^* \neq \emptyset$, for some $i \in \{1, 2\}$, say $i = 1$. By orthogonality, $a_1 \in T'$ and $t \notin T'$. Hence $T' \cap (T - t) \neq \emptyset$ and so $T' \cap C^* \neq \emptyset$. That is, $T' \in \mathcal{T}$ and $T' \cap A \neq \emptyset$; a contradiction to (i). Hence T_1^*, T, T_2^* is a fan of M. To finish the proof of this lemma, we need to show that $t \notin D^*$, when $D^* \in \mathcal{R}_A^*(M)$. If $t \in D^*$, then, by the previous paragraph applied to the pair (T, D^*) instead of (T, C^*), T meets a third triad of M; a contradiction to (iii). □

Lemma 5.7. *If $C^* \in \mathcal{R}_A^*(M)$, then there are triangles T_1 and T_2 of M meeting C^* such that $C^* = (T_1 \cap C^*) \cup (T_2 \cap C^*)$. Moreover, $|C^*| = 4$ and $|\mathcal{T}| \geqslant 2$.*

Proof. By Lemma 5.3, M/e is not 3-connected, for every $e \in C^*$. By the dual of Theorem 1 of [7], there are different triangles T_1 and T_2 of M meeting C^*. In particular, $\{T_1, T_2\} \subseteq \mathcal{T}$ and so $|\mathcal{T}| \geqslant 2$. By Lemma 5.6, for $i \in \{1, 2\}$, there are triads T_{1i}^* and T_{2i}^* of M such that T_{1i}^*, T_i, T_{2i}^* is a fan of M. Moreover, if $t_i \in T_i - C^*$, then $t_i \in T_{1i}^* \cap T_i \cap T_{2i}^*$ and $A = (T_{1i}^* - T_i) \cup (T_{2i}^* - T_i)$. Hence

$$[T_{11}^* \; \triangle \; T_{21}^*] \; \triangle \; [T_{12}^* \; \triangle \; T_{22}^*] = [A \cup (T_1 \cap C^*)] \; \triangle \; [A \cup (T_2 \cap C^*)] = (T_1 \cap C^*) \; \triangle \; (T_2 \cap C^*)$$

is a union of cocircuits of M. Therefore $C^* = (T_1 \cap C^*) \; \triangle \; (T_2 \cap C^*)$. As M is 3-connected, it follows that $|C^*| \geqslant 3$ and so $(T_1 \cap C^*) \cap (T_2 \cap C^*) = \emptyset$. Thus $|C^*| = 4$. □

For $T \in \mathcal{T}$, let X_T be the set of elements of $\mathcal{F}_T$. If $X = \cup_{T \in \mathcal{T}} X_T$, then, by Lemma 5.7, $C^* \subseteq X$, for every $C^* \in \mathcal{R}_A^*(M)$. Assume that $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$. If, for $i \in \{1, 2, \dots, n\}$, $T_i = \{e_i, f_i, g_i\}$, then, by Lemma 5.6, we can suppose that $T_{1i}^* = \{a, e_i, g_i\}$ and $T_{2i}^* = \{g_i, f_i, b\}$ are triads of M, where $A = \{a, b\}$.

Lemma 5.8. *If $n = |\mathcal{T}|$, then there is an isomorphism from $M^*|X$ into $M(K_{3,n}'')$ mapping the elements of A into the added edges of $K_{3,n}''$. Moreover, $B^* = \{a, b, g_1, g_2, \dots, g_n\}$ is a basis of $M^*|X$ and, when $E(M) - X \neq \emptyset$,*

(i) *$\{X - A, E(M) - (X - A)\}$ is a 3-separation of M^*; and*
(ii) *$\{X - A, E(M) - X\}$ is a 1-separation of M^*/A.*

Proof. Observe that B^* spans X in M^* and $B = \{e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n\}$ spans $X - A$ in M. Therefore

$$r_M(X - A) + r_{M^*}(X - A) - |X - A| \leqslant [2n] + [n + 2] - [3n] = 2 \tag{5.2}$$

and so $\{X - A, E(M) - (X - A)\}$ is a 3-separation of M^* and (i) follows. Moreover, when $E(M) - X \neq \emptyset$, we have equality in (5.2). Therefore $r_{(M^*/A)^*}(X - A) = r_{M \backslash A}(X - A) = r_M(X - A) = 2n$. We also have that $r_{M^*/A}(X - A) = r_{M^*}(X) - r_{M^*}(A) = n$. (The last equality follows by the equality in (5.2).) Hence

$$r_{M^*/A}(X - A) + r_{(M^*/A)^*}(X - A) - |X - A| = [n] + [2n] - [3n] = 0$$

and so (ii) also holds.

Now, we show that B^* is a basis of $M^*|X$. If B^* contains a cocircuit C^* of M, then $g_i \in C^*$, for some $i \in \{1, 2, \dots, n\}$, since $|C^*| \geqslant 3$. Observe that $C^* \cap T_i = \{g_i\}$; a contradiction to orthogonality. Thus B^* is a basis of $M^*|X$.

Let $[I_{n+2}|B_{n+2}]$ be a matrix that represents $M^*|X$ over $GF(2)$, where B_{n+2} is a $(n + 2) \times 2n$ matrix. Label the columns of I_{n+2} by $a, b, g_1, g_2, \dots, g_n$. If $v = (v_1, v_2, \dots, v_{n+2})^T$ and $w = (w_1, w_2, \dots, w_{n+2})^T$ are respectively the columns of B_{n+2} labeled by e_i and f_i, for some $i \in \{1, 2, \dots, n\}$, then $v_j = 1$ if and only if $j \in \{1, i + 2\}$ and $w_j = 1$ if and only if $j \in \{2, i + 2\}$. Hence $[I_{n+2}|B_{n+2}]$ also represents $M(K_{3,n}'')$ over $GF(2)$, where the edges incidents to the vertex of degree $n + 2$ label the columns of I_{n+2} (with the first two columns labeled by the added edges of $K_{3,n}''$). □

Lemma 5.9. *If T^* is a triad of M, then $A \not\subseteq T^*$.*

Proof. Assume that $A \subseteq T^*$. By Lemma 5.7, $|\mathcal{T}| \geqslant 2$. If $T \in \mathcal{T}$, then, by Lemma 5.6, T meets exactly two triads T_1^* and T_2^* of M. If $D^* = T_1^* \triangle T_2^*$, then $|D^*| = 4$ and so D^* is a cocircuit of M. Observe that $A \subseteq D^*$. As $D^* \triangle T^*$ has cardinality three, it follows that $D^* \triangle T^*$ is a third triad of M meeting T; a contradiction to Lemma 5.4. Therefore $A \nsubseteq T^*$. □

Lemma 5.10. *Theorem 5.1 follows unless $E(M) - X \neq \emptyset$. Moreover, $|E(M) - X| \geqslant 3$.*

Proof. If $E(M) - X = \emptyset$, then, by Lemma 5.8, Theorem 5.1 follows. We may assume that $E(M) - X \neq \emptyset$. Now, we show that $\mathrm{cl}_{M^*}(X) = X$. Suppose that $e \in \mathrm{cl}_{M^*}(X) - X$. There is a cocircuit C^* of M such that $e \in C^* \subseteq B^* \cup e$ because $B^* = \{a, b, g_1, g_2, \ldots, g_n\}$ is a basis of $M^*|X$. By orthogonality with $\{e_i, f_i, g_i\}$, $g_i \notin C^*$. Hence $C^* = A \cup e$; a contradiction to Lemma 5.9. Thus $\mathrm{cl}_{M^*}(X) = X$. In particular, X is contained in a hyperplane of M^*. Therefore $|E(M) - X| \geqslant 3$, since $E(M) - X$ contains a circuit of M. □

By Lemma 5.10, we may assume that $|E(M) - X| \geqslant 3$. Let N be an 1-element binary extension of M^* such that A is contained in a triangle T of N, say $T = A \cup t$. By Lemma 5.9, N is 3-connected and $M^* = N \backslash t$. If $N_1 = N|[(X - A) \cup T]$ and $N_2 = N|[(E(M) - X) \cup T]$, then, by Lemma 5.8(i) and (ii), N is the generalized parallel connection of N_1 and N_2. Moreover, by Lemma 5.8, $N_1 \cong M(K_{3,n}''')$, where $n = |\mathcal{T}|$, and N_2 is 3-connected. Now, we need an auxiliary lemma whose proof is omitted because it is straightforward.

Lemma 5.11. *Let T be a triangle of a connected binary matroid H. If $X \subseteq E(H) - T$ and $r_{H/X}(T) \leqslant 1$, then H/X is not connected.*

Lemma 5.12. *If C is a circuit of N_2 such that N_2/C is connected, then $C \cap A \neq \emptyset$.*

Proof. Assume that $C \cap A = \emptyset$. First, we show that $t \in C$. If $t \notin C$, then $C \cap T = \emptyset$. By Lemma 5.11, $r_{N_2/C}(T) = 2$. Hence N/C is the generalized parallel connection of N_1 and N_2/C. As both N_1 and N_2/C are connected, it follows that N/C is connected. Observe that $[N/C]\backslash t$ is connected because $t \in T \subseteq E(N_1)$ and $[N/C]|E(N_1) = N_1$ is 3-connected. But $[N/C]\backslash t = [N\backslash t]/C = M^*/C$. Thus C is a non-separating cocircuit of M avoiding A; a contradiction because $C \cap X = \emptyset$. Therefore $t \in C$.

If $T' \in \mathcal{T}$, then, by Lemma 5.6, there is $t' \in T'$ such that t' does not belong to a non-separating cocircuit of M avoiding A. Moreover, $(T' - t') \cup A$ is a cocircuit of M. Hence $[(T' - t') \cup A] \triangle (A \cup t) = (T' - t') \cup t$ is a circuit of N_1 and so $D = (T' - t') \cup (C - t)$ is a circuit of N. In particular, D is a cocircuit of M.

Choose $e \in T' - t'$ and $f \in C - t$. Note that t is a loop of $N_2/(C - t)$ and $N_2/(C - t)\backslash t = N_2/C$ is connected. Therefore: (a) $N_2/(C - \{f, t\})\backslash t$ is connected; or (b) f is a loop or coloop of $N_2/(C - \{f, t\})\backslash t$. But $\{f, t\}$ is a circuit of $N_2/(C - \{f, t\})$ and so, when (b) occurs, $\{f, t\}$ is a connected component of $N_2/(C - \{f, t\})$; a contradiction because N_2 is cosimple. Hence (a) occurs. By Lemma 5.11, T is a triangle and $\{f, t\}$ is a parallel class of $N_2/(C - \{f, t\})$. Note that T is a triangle and $\{e, t\}$ a parallel class of $N_1/(T' - \{e, t'\})$. Thus $N/(D - \{e, f\})$ is the generalized parallel connection of $N_1/(T' - \{e, t'\})$ and $N_2/(C - \{f, t\})$. Moreover, $\{e, f, t\}$ is a parallel class of $N/(D - \{e, f\})$. Hence $N/(D - \{e, f\})/\{e, f, t\}$ is the parallel connection of $N_1/(T' - \{e, t'\})/\{e, t\}$ and $N_2/(C - \{f, t\})/\{f, t\}$. (These two matroids have $T - t$ as a pair of parallel elements in common.) As both $N_1/(T' - \{e, t'\})/\{e, t\}$ and $N_2/(C - \{f, t\})/\{f, t\} = N_2/C$ are connected, it follows that $N/(D - \{e, f\})/\{e, f, t\} = (N/D)/t$ is connected. But t is a loop of N/D and so $(N/D)/t = (N/D)\backslash t = (N\backslash t)/D = M^*/D$. That is, D is a non-separating cocircuit of M avoiding A; a contradiction because $D \nsubseteq X$.

By Lemma 5.12, every non-separating cocircuit of N_2^* meets A. By Theorem 1.7, there is an isomorphism Ψ of N_2^* into O_m, for some $m \geqslant 3$, mapping the elements belonging to A into the tip and cotip of O_m. We arrive at a contradiction because A is contained in a triad of N_2^* and there is no triad of O_m that contains both the tip and the cotip of O_m. Therefore Theorem 5.1 follows. □

6. Proofs of Theorems 1.4 and 1.5

In this section, we prove the other two main results of this paper. Theorems 1.4 and 1.5 are, respectively, consequences of Theorems 6.1 and 6.2.

Theorem 6.1. *Suppose that M is a 3-connected binary matroid. Let A be a 2-subset of $E(M)$. If there is no isomorphism between a minor H of M using A and O_n, for $n \geqslant 3$, mapping the elements of A into the tip and cotip of O_n, then*

$$|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant \frac{r(M)+1-n}{2}.$$

Proof. First, we define a class of matroids $\mathcal{F}$. A matroid N does not belong $\mathcal{F}$ if and only if N has a minor H such that $A \subseteq E(H)$ and there is an isomorphism of H into O_n mapping the elements of A into the tip and cotip of O_n. Observe that $\mathcal{F}$ is closed under minors. Take $\alpha = \frac{1}{2}$ and $\beta = \frac{1-n}{2}$.

If $\mathcal{F}_{\alpha,\beta} = \emptyset$, then the result follows. Assume that $\mathcal{F}_{\alpha,\beta} \neq \emptyset$. Choose $N \in \mathcal{F}_{\alpha,\beta}$ such that $|E(N)|$ is minimum. If $\mathcal{R}_A^*(N) = \emptyset$, then, by Theorem 1.7, there is an isomorphism of N into O_m, for $m \geqslant 3$, mapping the elements of A into the tip and cotip of O_m. As $N \in \mathcal{F}_{\alpha,\beta}$, it follows that $m < n$. We arrive at a contradiction, since

$$0 = \dim_A(N) < \alpha r(N) + \beta = m\alpha + \beta = \frac{m+1-n}{2} \leqslant 0.$$

Therefore $\mathcal{R}_A^*(N) \neq \emptyset$. By Theorem 5.1, there is an isomorphism between N and $M^*(K''_{3,m})$, for some $m \geqslant 2$, mapping the elements of A into the added edges of $K''_{3,m}$. But

$$m-1 = \dim_A(N) < \alpha r(N) + \beta = 2m\alpha + \beta = m + \frac{1-n}{2}$$

and so $n < 3$; a contradiction. □

Theorem 6.2. *Suppose that M is a 3-connected binary matroid. Let A be a 2-subset of $E(M)$ and let n be an integer exceeding two. If there is no isomorphism between a minor H of M using A and*

(i) *O_n mapping the elements of A into the tip and cotip of O_n; or*
(ii) *$M^*(K''_{3,n-1})$ mapping the elements of A into the added edges of $K''_{3,n-1}$,*
then,

$$|\mathcal{R}_A^*(M)| \geqslant \dim_A(M) \geqslant r(M) + 1 - n.$$

Proof. First, we define a class of matroids $\mathcal{F}$. A matroid N does not belong $\mathcal{F}$ if and only if N has a minor H such that $A \subseteq E(H)$ and there is an isomorphism of H into

(i) O_n mapping the elements of A into the tip and cotip of O_n; or
(ii) $M^*(K''_{3,n-1})$ mapping the elements of A into the added edges of $K''_{3,n-1}$.

Observe that $\mathcal{F}$ is closed under minors. Take $\alpha = 1$ and $\beta = 1 - n$.

If $\mathcal{F}_{\alpha,\beta} = \emptyset$, then the result follows. Assume that $\mathcal{F}_{\alpha,\beta} \neq \emptyset$. Choose $N \in \mathcal{F}_{\alpha,\beta}$ such that $|E(N)|$ is minimum. If $\mathcal{R}_A^*(N) = \emptyset$, then, by Theorem 1.7, there is an isomorphism of N into O_m, for $m \geqslant 3$, mapping the elements of A into the tip and cotip of O_m. As $N \in \mathcal{F}_{\alpha,\beta}$, it follows that $m < n$. We arrive at a contradiction, since

$$0 = \dim_A(N) < \alpha r(N) + \beta = m\alpha + \beta = m + 1 - n \leqslant 0.$$

Therefore $\mathcal{R}_A^*(N) \neq \emptyset$. By Theorem 5.1, there is an isomorphism between N into $M^*(K''_{3,m})$, for some $m \geqslant 2$, mapping the elements of A into the added edges of $K''_{3,m}$. As $N \in \mathcal{F}_{\alpha,\beta}$, it follows that $m < n-1$ or $m + 1 < n$. But

$$m - 1 = \dim_A(N) < \alpha r(N) + \beta = 2m\alpha + \beta = 2m + 1 - n$$

and so $n < m + 2$; a contradiction. □

Acknowledgments

Lemos is partially supported by CNPq (Grants No. 302195/02-5 and 664107/97-4) and FAPESP/CNPq (Grant No. 2003/09925-5). Melo is supported by CNPq (Grant No. 140133/00-3) and FACEPE (Grant No. BFD03981.01/2004).

References

[1] R.E. Bixby, A simple theorem on 3-connectivity, Linear Algebra Appl. 45 (1982) 123–126.
[2] R.E. Bixby, W.H. Cunningham, Matroids, graphs, and 3-connectivity, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, pp. 91–103.
[3] W.H. Cunningham, A Combinatorial Decomposition Theory, Ph.D. Thesis, University of Waterloo, 1973.
[4] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected binary matroids, J. Combin. Theory Ser. B 66 (1996) 334–360.
[5] A.K. Kelmans, The concepts of a vertex in a matroid, the non-separating circuits and a new criterion for graph planarity, in: Algebraic Methods in Graph Theory, vol. 1, Colloq. Math. Soc. János Bolyai, vol. 25, Szeged, Hungary, 1978, North Holland, Amsterdam, 1981, pp. 345–388.
[6] A.K. Kelmans, Graph planarity and related topics, in: N. Robertson, P.D. Seymour (Eds.), Graph Structure Theory, Contemporary Mathematics, vol. 147, 1991, pp. 635–667.
[7] M. Lemos, On 3-connected matroids, Discrete Math. 73 (1989) 273–283.
[8] M. Lemos, Non-separating cocircuits in binary matroids, Linear Algebra Appl. 382 (2004) 171–178.
[9] M. Lemos, T.R.B. Melo, Non-separating cocircuits in matroids, Discrete Appl. Math. 156 (2008) 1019–1024.
[10] J. McNulty, H. Wu, Connected hyperplanes in binary matroids, J. Combin. Theory Ser. B 79 (2000) 87–97.
[11] T.R.B. Melo, Hiperplanos Conexos em Matróides Binárias, Tese de Doutorado, Universidade Federal de Pernambuco, Recife, 2004.
[12] J.G. Oxley, Cocircuit coverings and packings for binary matroids, Math. Proc. Cambridge Phil. Soc. 83 (1978) 347–351.
[13] J.G. Oxley, The binary matroids with no 4-wheel minor, Trans. Amer. Math. Soc. 301 (1987) 63–75.
[14] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[15] J. Oxley, H. Wu, On the structure of 3-connected matroids and graphs, Eur. J. Combin. 21 (2000) 667–688.
[16] W.T. Tutte, How to draw a graph, Proc. London Math. Soc. 13 (1963) 734–768.
[17] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.
[18] H. Whitney, 2-Isomorphic graphs, Amer. J. Math. 55 (1933) 245–254.