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Torsion theories induced from commutative subalgebras

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ABSTRACT

We begin a study of torsion theories for representations of finitely generated algebras U over a field containing a finitely generated commutative Harish-Chandra subalgebra Γ. This is an important class of associative algebras, which includes all finite W-algebras of type A over an algebraically closed field of characteristic zero, in particular, the universal enveloping algebra of gl_n (or sl_n) for all n. We show that any Γ-torsion theory defined by the coheight of the prime ideals of Γ is liftable to U. Moreover, for any simple U-module M, all associated prime ideals of M in $\mathrm{Spec}\,\Gamma$ have the same coheight. Hence, the coheight of these associated prime ideals is an invariant of a given simple U-module. This implies the stratification of the category of U-modules controlled by the coheight of the associated prime ideals of Γ. Our approach can be viewed as a generalization of the classical paper by Block (1981) [4]; it allows, in particular, to study representations of gl_n beyond the classical category of weight or generalized weight modules.

1. Introduction

A classical, very difficult and intriguing problem in the representation theory of Lie algebras is the classification of simple modules over complex simple finite dimensional Lie algebras. Such a classification is only known for the Lie algebra sl_2 due to the results of Block [4]. It remains an open problem in general, even in the subcategory of weight modules with respect to a fixed Cartan subalgebra. On the other hand, due to the results of Fernando [10] and Mathieu [25], the classification of simple weight modules with finite dimensional weight spaces is well known for any simple finite dimensional Lie algebra.

The basic idea proposed in [4] in the case of sl_2 can be explained as follows. First, we consider a maximal commutative subalgebra $\Gamma \subset U(\mathrm{sl}_2)$ (in our terms, a Gelfand–Tsetlin subalgebra), which is generated by a Cartan subalgebra and the center of $U(\mathrm{sl}_2)$. Then one fixes a central character χ of $U(\mathrm{sl}_2)$. After that, all simple modules with central character χ are divided into torsion (or generalized weight) and torsion-free modules with respect to $\Gamma/(\mathrm{Ker}\,\chi)$. Thereafter, the investigation of both classes of modules is reduced to the investigation of the simple modules over a (skew) group algebra of the group $\mathbb{Z}$. An analogous idea works in the more general context of generalized Weyl algebras of rank 1 [2,3], which allows a complete classification of the simple modules.

A similar approach applied in the case of a Lie algebra $\mathrm{gl}(n)$ (or sl_n) allows one to go beyond the category of weight modules with finite dimensional spaces. Namely, one considers the full subcategory of weight Gelfand–Tsetlin gl_n-modules with respect to a *Gelfand–Tsetlin subalgebra* Γ (certain maximal commutative subalgebra of $U(\mathrm{gl}_n)$) [7,15]. That is, one considers those modules V that have a decomposition as a Γ-module

$$V = \bigoplus_{\mathbf{m}\in \mathrm{Specm}\,\Gamma} V(\mathbf{m}),$$

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where $V(\mathbf{m}) = \{v \in V | \exists N, \mathbf{m}^N v = 0\}$. This class is based on natural properties of a Gelfand–Tsetlin basis for finite dimensional representations of simple classical Lie algebras [18,29,26]. Gelfand–Tsetlin subalgebras were considered in various connections in [11,28,20,21,19]. The theory developed in [14,15] was an attempt to unify the representation theories of the universal enveloping algebra of gl_n and of the generalized Weyl algebras. We underline that Gelfand–Tsetlin modules over gl_n are weight modules with respect to some Cartan subalgebra of gl_n but they are allowed to have infinite dimensional weight spaces.

In this paper we begin a study of general torsion theories for representations of a class of associative algebras which includes all finite W-algebras of type A over an algebraically closed field of characteristic zero. In particular, the universal enveloping algebra of gl_n (or sl_n) is an example of such algebra for all n, where Γ is a Gelfand–Tsetlin subalgebra.

In the rest of the paper K will be a fixed arbitrary field. Only in the applications of the final section we will require it to be algebraically closed of characteristic zero. We shall consider the following situation.

Setup 1.1. *We shall assume that U is a finitely generated K-algebra having a commutative (not necessarily central) subalgebra Γ satisfying the following properties:*

(1) *Γ is finitely generated as a K-algebra*
(2) *Γ is a* Harish-Chandra subalgebra, *i.e., for each $u \in U$ the Γ-bimodule $\Gamma u \Gamma$ is a finitely generated both as a left and as a right Γ-module.*

For our purposes, we shall fix a finite subset $\{u_1, \ldots, u_n\} \subset U$ (e.g. a finite set of generators) such that U is generated as a K-algebra by $\Gamma \cup \{u_1, \ldots, u_n\}$.

If M is a Gelfand–Tsetlin U-module with respect to Γ then the associated prime ideals of M in Spec Γ, which form the assassin Ass(M), are maximal. Our goal is to understand torsion categories of modules over U that are more general than Gelfand–Tsetlin categories. Such modules have associated primes in Spec Γ which are not maximal.

Our main result is the following theorem. We refer to Section 2 for definitions.

Theorem 1.2. *Let U be an algebra and Γ be a commutative subalgebra as in Setup 1.1, and let $i \geq 0$ be a natural number. Then*

(1) *The Γ-torsion theory associated to the subset $Z_i \subset \mathrm{Spec}\,\Gamma$ of prime ideals of coheight $\leq i$ is liftable to U.*
(2) *For any simple U-module M all associated prime ideals of M in* Spec Γ *have the same coheight.*

Theorem 1.2 provides a stratification of the module category U-Mod with respect to the coheight of the associated primes. In classical cases as finite W-algebras it happens that the endomorphism algebra of any simple U-module is one dimensional, the center $Z = Z(U)$ of U is an integral domain (polynomial ring) contained in Γ and Γ is also an integral domain (polynomial ring), which is flat over Z. Under these circumstances (see Proposition 5.1), all simple objects in the module category U-Mod are exhausted by simple U-modules whose associated primes have a fixed coheight $0 \leq i \leq K\,dim(\Gamma) - K\,dim(Z)$, where $K\,dim$ denotes the Krull dimension. The case $i = 0$ corresponds to Gelfand–Tsetlin modules (with respect to Γ) and the case $i = K\,dim(\Gamma) - K\,dim(Z)$ corresponds to the simple U-modules which are torsion-free with respect to some central character $\chi : Z \longrightarrow K$.

Our second main result provides information about the assassin of a simple U-module.

Theorem 1.3. *Let U, Γ, $u_1, \ldots, u_n$ be as in Setup 1.1, $M = Ux$ be a cyclic U-module generated by an element x such that $\mathrm{ann}_\Gamma(x) = \mathbf{p}$ is a prime ideal of Γ and suppose that all ideals in* Ass(M) *have equal coheight. If $\mathbf{q} \in$* Ass(M) *then there is a sequence $\mathbf{q} = \mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_s = \mathbf{p}$ of prime ideals of equal coheight and a sequence of indices $k_1, \ldots, k_s \in \{1, \ldots, n\}$ such that*

$$\frac{\Gamma u_{k_i} \Gamma}{\mathbf{q}_{i-1} u_{k_i} \Gamma + \Gamma u_{k_i} \mathbf{q}_i} \neq 0,$$

for all $i = 1, \ldots, s$.

All these results can be applied to the class of Galois orders over finitely generated Noetherian domains [14]. In particular, the results are valid for all finite W-algebras of type A, e.g. $U(\mathrm{gl}_n)$ for all n.

2. Torsion theories over a commutative Noetherian ring

In this section we collect some facts concerning torsion theories over commutative Noetherian rings. Recall that, given a not necessarily commutative ring R, a *torsion theory* over R is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of R-Mod satisfying the following two conditions:

(1) $\mathcal{T} = ^{\perp}\mathcal{F}$ consists of those R-module T such that $\mathrm{Hom}_R(T, F) = 0$, for all $F \in \mathcal{F}$
(2) $\mathcal{F} = \mathcal{T}^{\perp}$ consists of those R-module F such that $\mathrm{Hom}_R(T, F) = 0$, for all $T \in \mathcal{T}$

Note that any of the component classes of a torsion theory determines the other. In the above situation, for every R-module M there exists a (unique up to isomorphism) exact sequence

$$0 \to T \longrightarrow M \longrightarrow F \to 0,$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Then the assignments $M \rightsquigarrow t(M) := T$ and $M \rightsquigarrow F =: M/t(M)$ are functorial and yield a right adjoint and a left adjoint, respectively, to the inclusion functors $\mathcal{T} \hookrightarrow R\text{-Mod}$ and $\mathcal{F} \hookrightarrow R\text{-Mod}$. The functor $t : R\text{-Mod} \longrightarrow \mathcal{T}$ is called the *torsion radical* associated to $\mathcal{T}$. The torsion theory is called *hereditary* when $\mathcal{T}$ is closed under taking submodules, which is equivalent to say that $\mathcal{F}$ is closed under taking injective envelopes (see chapter VI of [27] for all details and terminology concerning torsion theories).

In this paper we are mainly interested in torsion theories over commutative Noetherian rings. In this section, unless otherwise stated, *Γ will be a commutative Noetherian ring*, We shall denote by $\mathrm{Spec}\,\Gamma$ (resp. $\mathrm{Specm}\Gamma$) the prime (resp. maximal) spectrum of Γ. Given a Γ-module M and a prime ideal $\mathbf{p} \in \mathrm{Spec}\Gamma$, we shall denote by $M_{\mathbf{p}}$ the localization of M at $\mathbf{p}$. We shall consider two important subsets of $\mathrm{Spec}\Gamma$ associated to M. Namely the *support* of M, $\mathrm{Supp}(M) = \{\mathbf{p} \in \mathrm{Spec}\,\Gamma \mid M_{\mathbf{p}} \neq 0\}$, and the so-called *assassin* of M, $\mathrm{Ass}(M)$, which consists of those $\mathbf{p} \in \mathrm{Spec}(\Gamma)$ such that $\mathbf{p} = \mathrm{ann}_{\Gamma}(x) := \{g \in \Gamma : gx = 0\}$, for some $x \in M$.

We now recall some properties of these sets. In the statement and in what follows, for every subset $X \subset \mathrm{Spec}\Gamma$, we denote by $\mathrm{Min}\,X$ (resp. $\mathrm{Max}X$) the set of minimal (resp. maximal) elements of X.

Proposition 2.1. *Let $X \subseteq \mathrm{Spec}\Gamma$ be any nonempty subset and M be a Γ-module. The following assertions hold:*

(1) *Every element of X contains a minimal element of X*
(2) $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ *and* $\mathrm{Min}\,\mathrm{Ass}(M) = \mathrm{Min}\,\mathrm{Supp}(M)$.

Proof. The set $\mathrm{Spec}\Gamma$ satisfies DCC with respect to inclusion. Indeed if $\mathbf{p} = \mathbf{p}_0 \supsetneq \mathbf{p}_1 \supsetneq \cdots$ is a descending chain of prime ideals, then the number of nonzero terms in it is bounded above by the height of $\mathbf{p}$, which is always finite (cf. [24][Theorem 13.5]).

If $X \subseteq \mathrm{Spec}\Gamma$ is any nonempty subset and $\mathbf{p} \in X$, then, by the DCC property, the set $\{\mathbf{q} \in X : \mathbf{q} \subseteq \mathbf{p}\}$ has a minimal element which is then a minimal element of X.

Let now take $\mathbf{p} \in \mathrm{Ass}(M)$, so that $\mathbf{p} = \mathrm{ann}_{\Gamma}(\Gamma x)$, for some $x \in M$. Then $\mathbf{p} \in \mathrm{Ass}(\Gamma x) \subseteq \mathrm{Supp}(\Gamma x)$ (see [24][Theorem 6.5]). Putting $N = \Gamma x$, we get that $N_{\mathbf{p}} \neq 0$, which implies that $M_{\mathbf{p}} \neq 0$ due to the exactness of the localization functor. Then $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$.

Since M is the direct union of its finitely generated submodules and the localization functor is exact and preserves direct unions it follows that $\mathrm{Supp}(M) = \bigcup_{N<M} \mathrm{Supp}(N)$, where the union is taken over all finitely generated submodules N of M. In particular, if $\mathbf{p} \in \mathrm{Min}\,\mathrm{Supp}(M)$ then $\mathbf{p} \in \mathrm{Min}\,\mathrm{Supp}(N)$, for some $N < M$ finitely generated. But then $\mathbf{p} \in \mathrm{Ass}(N)$ (cf. [24][Theorem 6.5]), and so $\mathbf{p} \in \mathrm{Ass}(M)$. From the inclusion $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ we conclude that $\mathbf{p} \in \mathrm{Min}\,\mathrm{Ass}(M)$, thus proving that $\mathrm{Min}\,\mathrm{Supp}(M) \subseteq \mathrm{Min}\,\mathrm{Ass}(M)$.

Conversely, if $\mathbf{p} \in \mathrm{Min}\,\mathrm{Ass}(M)$ then we fix a cyclic submodule $N = \Gamma x$ such that $\mathbf{p} = \mathrm{ann}_{\Gamma}(N)$. Then we have $\mathbf{p} \in \mathrm{Ass}(N) \subset \mathrm{Supp}(N) \subset \mathrm{Supp}(M)$. By assertion 1, there exists $\mathbf{q} \in \mathrm{Min}\,\mathrm{Supp}(M)$ such that $\mathbf{q} \subseteq \mathbf{p}$. But equality must hold since we already know that $\mathrm{Min}\,\mathrm{Supp}(M) \subseteq \mathrm{Min}\,\mathrm{Ass}(M)$ and $\mathbf{p}$ is minimal in $\mathrm{Ass}(M)$. Therefore $\mathbf{p} \in \mathrm{Min}\,\mathrm{Supp}(M)$ and we get that $\mathrm{Min}\,\mathrm{Ass}(M) = \mathrm{Min}\,\mathrm{Supp}(M)$. □

Definition 1. A subset $Z \subseteq \mathrm{Spec}\,\Gamma$ is called *closed under specialization* when the following property holds:
(*) If $\mathbf{p} \subseteq \mathbf{q}$ are prime ideals with $\mathbf{p} \in Z$, then $\mathbf{q}$ belongs to Z.

The prototypical examples of closed under specialization subsets of $\mathrm{Spec}\,\Gamma$ are the Zariski-closed subsets and those of the form $\mathrm{Supp}(M)$, where M is a Γ-module. The following is a crucial result from [16].

Theorem 2.2. *Let Γ be a commutative Noetherian ring. The assignments $Z \rightsquigarrow (\mathcal{T}_Z, \mathcal{T}_Z^{\perp})$, where $\mathcal{T}_Z = \{T \in \Gamma\text{-Mod} : \mathrm{Supp}(T) \subseteq Z\}$, and $(\mathcal{T}, \mathcal{F}) \rightsquigarrow Z_{(\mathcal{T},\mathcal{F})} = \{\mathbf{p} \in \mathrm{Spec}\,\Gamma : \Gamma/\mathbf{p} \in \mathcal{T}\}$ define mutually inverse order-preserving one-to-one correspondences between the closed under specialization subsets of $\mathrm{Spec}\,\Gamma$ and the hereditary torsion theories in Γ-Mod.*

For a given module M, it is important to identify the *torsion submodule $t_Z(M)$* with respect to the torsion theory $(\mathcal{T}_Z, \mathcal{T}_Z^{\perp})$.

Proposition 2.3. *Let $Z \subseteq \mathrm{Spec}\,\Gamma$ be a closed under specialization subset and M be a Γ-module. For an element $x \in M$, the following assertions are equivalent:*

(1) *x belongs to $t_Z(M)$*
(2) $\mathrm{Ass}(\Gamma x) \subseteq Z$ *(resp.* $\mathrm{Min}\,\mathrm{Ass}(\Gamma x) \subseteq Z$*)*
(3) *If $\mathbf{p}$ is a prime ideal such that $\mathrm{ann}_{\Gamma}(x) \subseteq \mathbf{p}$, then $\mathbf{p} \in Z$*
(4) *There are prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_r \in Z$ (resp. $\mathbf{p}_1, \ldots, \mathbf{p}_r \in \mathrm{Min}\,Z$) and integers $n_1, \ldots, n_r > 0$ such that $\mathbf{p}_1^{n_1} \cdot \cdots \cdot \mathbf{p}_r^{n_r} x = 0$.*

Proof. (1) $\iff$ (2) $\iff$ (3) Due to the fact that $\mathcal{T}_Z$ is closed under taking submodules, assertion (1) is equivalent to say that $\Gamma x \in \mathcal{T}_Z$, i.e., to say that $\mathrm{Supp}(\Gamma x) \subseteq Z$. But $\mathrm{Supp}(\Gamma x)$ is precisely the set of prime ideals containing $\mathrm{ann}_\Gamma(x)$ (cf. Proposition III.4.6 in [23]). Moreover, Z being closed under specialization, Proposition 2.1 implies that $\mathrm{Supp}(\Gamma x) \subseteq Z$ holds exactly when $(\mathrm{Min})\mathrm{Ass}(\Gamma x) \subseteq Z$.

(3) $\Longrightarrow$ (4) Let $\{\mathbf{p}_1, \dots, \mathbf{p}_r\}$ be the (finite) set of prime ideals of Γ which are minimal among those containing $\mathrm{ann}_\Gamma(x)$. In particular, they belong to Z. Then we have $\mathbf{p}_1 \cdot \dots \cdot \mathbf{p}_r \subseteq \mathbf{p}_1 \cap \dots \cap \mathbf{p}_r = \sqrt{\mathrm{ann}_\Gamma(x)}$, where $\sqrt{I}$ denotes the radical of I, for every ideal I of Γ. Thus there is a positive integer $n > 0$ such that $\mathbf{p}_1^n \cdot \dots \cdot \mathbf{p}_r^n = (\mathbf{p}_1 \cdot \dots \cdot \mathbf{p}_r)^n \subseteq \mathrm{ann}_\Gamma(x)$. By Proposition 2.1(1), replacing each $\mathbf{p}_i$ by a minimal element of Z contained in it if necessary, we can find the needed $\mathbf{p}_i$ in $\mathrm{Min}\,Z$.

(4) $\Longrightarrow$ (3) Let $\mathbf{p}_1, \dots, \mathbf{p}_r \in Z$ and $n_1, \dots, n_r > 0$ be as in condition (4). Then we have $\mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_r^{n_r} \subseteq \mathrm{ann}_\Gamma(x)$. If $\mathbf{p}$ is a prime ideal such that $\mathrm{ann}_\Gamma(x) \subseteq \mathbf{p}$ then there is some $j = 1, \dots, r$ such that $\mathbf{p}_j \subseteq \mathbf{p}$. It follows that $\mathbf{p} \in Z$ since Z is closed under specialization. □

The following example of closed under specialization subsets of $\mathrm{Spec}\,\Gamma$ will be the most interesting for us.

Example 2.4. One defines a transfinite ascending chain of subsets $(Z_i)_{i \text{ ordinal}}$ as follows. We put $Z_0 = \mathrm{Specm}\,\Gamma$. If $i > 0$ is any ordinal and Z_j has been defined for all $j < i$, then $Z_i = \bigcup_{j<i} Z_j$, in case i is a limit ordinal, and $Z_i = Z_{i-1} \cup \mathrm{Max}(\mathrm{Spec}\,\Gamma \setminus Z_{i-1})$ in case i is nonlimit. It is not difficult to see that there is a minimal ordinal δ such that $\mathrm{Spec}\,\Gamma = Z_\delta$ and that all Z_i are closed under specialization. In particular, for each $\mathbf{p} \in \mathrm{Spec}\,\Gamma$, there is a minimal ordinal $i_{\mathbf{p}}$ such that $\mathbf{p} \in Z_{i_{\mathbf{p}}}$. This ordinal is nonlimit and we put $\mathrm{cht}(\mathbf{p}) = i_{\mathbf{p}}$ and call it the *coheight* of $\mathbf{p}$.

Using Theorem 2.2, we get a corresponding transfinite ascending chain of torsion classes $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \dots \subseteq \mathcal{T}_i \subseteq \cdots$ such that $\Gamma\text{-Mod} = \mathcal{T}_\delta = \bigcup_{i \le \delta} \mathcal{T}_i$. Then, for every Γ-module M, there is uniquely determined (not necessarily nonlimit) ordinal i such that $M \in \mathcal{T}_i$ and $M \notin \mathcal{T}_j$, for all $j < i$. We also have $t_i(M) \subseteq t_j(M)$, for all $i \le j$, where t_i denotes the torsion radical associated to $\mathcal{T}_i$.

Corollary 2.5. *Let Γ be a commutative Noetherian ring, M be a nonzero Γ-module and i be a nonlimit ordinal. The following assertions are equivalent:*

(1) $t_i(M) = M$ *but* $t_{i-1}(M) = 0$
(2) *The next two conditions hold:*
  (a) *For every $x \in M$ there are prime ideals $\mathbf{p}_1, \dots, \mathbf{p}_r$ of coheight exactly i and positive integers $n_1, \dots, n_r > 0$ such that* $\mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_r^{n_r} x = 0$
  (b) *If $\mathbf{p}$ is a prime ideal of coheight $< i$ and $x \in M$ is an element such that $\mathbf{p}x = 0$, then $x = 0$.*
(3) *The prime ideals in $\mathrm{Ass}(M)$ have coheight exactly i.*

Proof. (1) $\iff$ (3) By Proposition 2.3 and the fact that the torsion theories $\mathcal{T}_i$ are hereditary, we have that $t_i(M) = M$ iff $\mathrm{Ass}(M) \subseteq Z_i$ and $t_{i-1}(M) = 0$ iff $\mathrm{Ass}(M) \cap Z_{i-1} = \emptyset$. Therefore assertion (1) holds if and only if $\mathrm{Ass}(M) \subseteq Z_i \setminus Z_{i-1}$, which is equivalent to assertion (3).

(2) $\Longrightarrow$ (1) From Proposition 2.3 and condition (2)(a) we get that $t_i(M) = M$. On the other hand, if we had $0 \neq x \in t_{i-1}(M)$ that same proposition would give that $\emptyset \neq \mathrm{Ass}(\Gamma x) \subseteq Z_{i-1}$. We then get $g \in \Gamma$ such that $gx \neq 0$ and $\mathrm{ann}_\Gamma(gx) = \mathbf{p}$ is a prime ideal in Z_{i-1}. That would contradict condition (2)(b).

(1), (3) $\Longrightarrow$ (2) Let us prove condition (2)(b) by way of contradiction. Suppose that there are $0 \neq x \in M$ and $\mathbf{p} \in Z_{i-1}$ such that $\mathbf{p}x = 0$. Taking a maximal element in the set $\{\mathrm{ann}_\Gamma(gx) : g \in G \text{ and } gx \neq 0\}$, we obtain a $\mathbf{q} \in \mathrm{Ass}(\Gamma x) \subseteq \mathrm{Ass}(M)$ (cf. [24][Theorem 6.1]) such that $\mathbf{p} \subseteq \mathbf{q}$. Since Z_{i-1} is closed under specialization we get that $\mathbf{q} \in Z_{i-1}$, which contradicts assertion (3).

We next prove condition (2)(a). Let us take $0 \neq x \in M$. Then, by Proposition 2.3, we have prime ideals $\mathbf{p}_1, \dots, \mathbf{p}_r \in Z_i$ (hence of coheight $\le i$) and positive integers $n_1, \dots, n_r > 0$ such that $\mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_r^{n_r} x = 0$. It is not restrictive to choose the $\mathbf{p}_i$ and the n_i in such a way that the latter ones are minimal, i.e., that $\mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_k^{n_k - 1} \cdot \dots \cdot \mathbf{p}_r^{n_r} x \neq 0$ for all $k = 1, \dots, r$. That immediately implies the existence of elements $g_k \in \Gamma$ such that $g_k x \neq 0$ and $\mathbf{p}_k \subseteq \mathrm{ann}_\Gamma(g_k x)$, for all $k = 1, \dots, r$. By [24][Theorem 6.1], we find $\mathbf{q}_k \in \mathrm{Ass}(\Gamma x) \subseteq \mathrm{Ass}(M)$ such that $\mathbf{p}_k \subseteq \mathbf{q}_k$, for all $k = 1, \dots, r$. But then, by assertion (3), we have $i = \mathrm{cht}(\mathbf{q}_k) \le \mathrm{cht}(\mathbf{p}_k) \le i$ for $k = 1, \dots, r$. Therefore we have $\mathrm{cht}(\mathbf{p}_k) = i$, for $k = 1, \dots, n$. □

Our next goal is to give the precise structure of the Γ-modules in $\mathcal{T}_0$, which is actually given by a more general result, Proposition 2.7 below, which will follow from the following version of the Chinese Remainder Theorem:

Lemma 2.6. *Let $I_1, \dots, I_r$ ($r > 1$) be pairwise distinct ideals of Γ. The following assertions are equivalent:*

(1) *I_i and I_j are coprime, for all $i \neq j$*
(2) *The canonical ring homomorphism $\Gamma \longrightarrow \prod_{1 \le i \le r} \Gamma / I_i$ is surjective.*

In such case $\bigcap_{1 \le i \le r} I_i = I_1 \cdot \dots \cdot I_r$.

Proof. See [1], Proposition 1.10(i). □

In the rest of the paper, if $\mathbf{p} \in \mathrm{Spec}\,\Gamma$ and M is a Γ-module, we shall denote by $M(\mathbf{p})$ the submodule consisting of those $x \in M$ such that $\mathbf{p}^n x = 0$, for some $n \ge 0$. Note that, in such case, if $\mathbf{p} \in \mathrm{Ass}(M(\mathbf{p}))$ then $\mathrm{Min}\,\mathrm{Ass}(M(\mathbf{p})) = \{\mathbf{p}\}$.

Proposition 2.7. *Let M be a Γ module such that* $\operatorname{Min}\operatorname{Ass}(M)$ *consists of pairwise coprime ideals (e.g. if* $\operatorname{Ass}(M) \subseteq \operatorname{Specm}\Gamma$*). Then* $\operatorname{Min}\operatorname{Ass}(M) = \operatorname{Ass}(M)$ *and* $M = \oplus_{\mathbf{p}\in\operatorname{Ass}(M)} M(\mathbf{p})$.

Proof. We shall prove that $M = \oplus_{\mathbf{p}\in\operatorname{Min}\operatorname{Ass}(M)} M(\mathbf{p})$. It will follow that $\operatorname{Ass}(M) = \bigcup_{\mathbf{p}\in\operatorname{Min}\operatorname{Ass}(M)} \operatorname{Ass}(M(\mathbf{p})) = \bigcup_{\mathbf{p}\in\operatorname{Min}\operatorname{Ass}(M)} \{\mathbf{p}\} = \operatorname{Min}\operatorname{Ass}(M)$ and the result will follow.

It is easy to prove that, for any family of submodules $(M_i)_{i\in I}$ of a given Γ-module M, one has $\operatorname{Supp}(\sum_{i\in I} M_i) = \bigcup_{i\in I} \operatorname{Supp}(M_i)$. This implies that $\operatorname{Min}\operatorname{Ass}(\sum_{i\in I} M_i) \subseteq \bigcup_{i\in I} \operatorname{Min}\operatorname{Ass}(M_i)$ using Proposition 2.1. We apply this to our particular case. Let us fix $\mathbf{p} \in \operatorname{Min}\operatorname{Ass}(M)$ and take

$$x \in M(\mathbf{p}) \cap \left(\sum_{\mathbf{q}\in\operatorname{Min}\operatorname{Ass}(M),\mathbf{q}\neq\mathbf{p}} M(\mathbf{q}) \right).$$

Then we have inclusions

$$\operatorname{Ass}(\Gamma x) \subseteq \operatorname{Ass}(M(\mathbf{p})) \cap \operatorname{Ass}\left(\sum_{\mathbf{q}\in\operatorname{Min}\operatorname{Ass}(M),\mathbf{q}\neq\mathbf{p}} M(\mathbf{q}) \right) \subseteq \operatorname{Ass}(M(\mathbf{p}))$$
$$\cap \left(\bigcup_{\mathbf{q}\in\operatorname{Min}\operatorname{Ass}(M),\mathbf{q}\neq\mathbf{p}} \operatorname{Ass}(M(\mathbf{q})) \right) \subseteq \{\mathbf{p}\} \cap (\operatorname{Min}\operatorname{Ass}(M) \setminus \{\mathbf{p}\}) = \emptyset.$$

It follows that $x = 0$ and, hence, the sum of the $M(\mathbf{q})$, with $\mathbf{q} \in \operatorname{Min}\operatorname{Ass}(M)$, is direct.

Let us consider now $Z := \operatorname{Supp}(M)$, which is a subset of $\operatorname{Spec}\Gamma$ closed under specialization. Then, by Theorem 2.2, M belongs to $\mathcal{T}_Z$ and hence $t_Z(M) = M$. If now $x \in M$ then Proposition 2.3 guarantees the existence of distinct prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_r \in \operatorname{Min}\operatorname{Supp}(M)$ and positive integer $n_1, \ldots, n_r > 0$ such that $\mathbf{p}_1^{n_1} \cdot \ldots \cdot \mathbf{p}_r^{n_r} x = 0$. The $\mathbf{p}_i$ are pairwise coprime since $\operatorname{Min}\operatorname{Supp}(M) = \operatorname{Min}\operatorname{Ass}(M)$ (see Proposition 2.1). But then it follows easily that the ideals $\mathbf{p}_i^{n_i}$ are also pairwise coprime. Then Γx is a module over the factor ring $\Gamma/\mathbf{p}_1^{n_1} \cdot \ldots \mathbf{p}_r^{n_r}$. But, by Lemma 2.6, we know that $\mathbf{p}_1^{n_1} \cdot \ldots \cdot \mathbf{p}_r^{n_r} = \bigcap_{1\le i\le r} \mathbf{p}_i^{n_i}$, and then the canonical map

$$\Gamma/\mathbf{p}_1^{n_1} \cdot \ldots \cdot \mathbf{p}_r^{n_r} \longrightarrow \prod_{1\le i\le r} \Gamma/\mathbf{p}_i^{n_i}$$

is a ring isomorphism. It follows that in the ring $\Gamma/\mathbf{p}_1^{n_1} \cdot \ldots \cdot \mathbf{p}_r^{n_r}$ we can decompose $\bar{1} = \bar{g}_1 + \cdots + \bar{g}_r$, where $g_i \in \mathbf{p}_1^{n_1} \cdot \ldots \mathbf{p}_{i-1}^{n_{i-1}} \cdot \mathbf{p}_{i+1}^{n_{i+1}} \ldots \cdot \mathbf{p}_r^{n_r}$. Then $x = \sum_{1\le i\le r} g_i x$ and $\mathbf{p_i}^{n_i} g_i x = 0$, for $i = 1, \ldots, r$. It follows that $x \in \oplus_{\mathbf{p}\in\operatorname{Min}\operatorname{Ass}(M)} M(\mathbf{p})$, and we get the desired equality $M = \oplus_{\mathbf{p}\in\operatorname{Min}\operatorname{Ass}(M)} M(\mathbf{p})$. □

Proposition 2.8. *Let M and N be Γ-modules such that $\mathbf{p}$ and $\mathbf{q}$ are coprime whenever $\mathbf{p} \in \operatorname{Ass}(M)$ and $\mathbf{q} \in \operatorname{Ass}(N)$ (resp. $\mathbf{p} \in \operatorname{Min}\operatorname{Ass}(M)$ and $\mathbf{q} \in \operatorname{Min}\operatorname{Ass}(N)$). The equality*

$$\operatorname{Ext}^i_\Gamma(M, N) = 0 = \operatorname{Ext}^i_\Gamma(N, M)$$

holds for all $i \ge 0$.

Proof. Since we have $\operatorname{Min}\operatorname{Ass}(M) = \operatorname{Min}\operatorname{Supp}(M)$ and similarly for N it follows that $\mathbf{p}$ and $\mathbf{q}$ are coprime whenever $\mathbf{p} \in \operatorname{Supp}(M)$ and $\mathbf{q} \in \operatorname{Supp}(N)$. If

$$0 \to M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots$$

is the minimal injective resolution of M in Γ-Mod and $E(\Gamma/\mathbf{p})$ is an injective indecomposable Γ-module appearing as direct summand of some I^i, then $\mathbf{p} \in \operatorname{Supp}(M)$ (cf. [24][Theorem 18.7]). It follows that $\operatorname{Hom}_\Gamma(N, I^i) = 0$, and hence $\operatorname{Ext}^i_\Gamma(N, M) = 0$, for all $i \ge 0$. That $\operatorname{Ext}^i_\Gamma(M, N) = 0$ for all $i \ge 0$ follows by symmetry. □

3. Algebras with a commutative Harish-Chandra subalgebra and lifting of torsion theories

Throughout the rest of the paper U is an algebra and Γ a commutative subalgebra satisfying the conditions of Setup 1.1. We denote by $j : \Gamma \hookrightarrow U$ the canonical inclusion and by $j_* : U\text{-Mod} \longrightarrow \Gamma\text{-Mod}$ the restriction of scalars functor. It is clear that if $\mathcal{T}$ is a (hereditary) torsion class in Γ-Mod, then $\hat{\mathcal{T}} = j_*^{-1}(\mathcal{T}) := \{T \in U\text{-Mod} : j_*(T) \in \mathcal{T}\}$ is a (hereditary) torsion class in U-Mod. However, if M is an U-module, then its torsion Γ-submodule $t(M)$ and its torsion U-submodule $\hat{t}(M)$ satisfy an inclusion $\hat{t}(M) \subseteq t(M)$ that might be strict. Equality happens exactly when $t(M)$ is an U-submodule of M. That justifies the following.

Definition 2. A torsion theory $(\mathcal{T}, \mathcal{F})$ in Γ-Mod is called *liftable* to U-Mod in case $t(M)$ is a U-submodule of M, for every U-module M.

The following is a general criterion for the lifting of a torsion theory.

Proposition 3.1. *Let $Z \subseteq \mathrm{Spec}\Gamma$ be a closed under specialization subset and $(\mathcal{T}_Z, \mathcal{F}_Z)$ be its associated torsion theory in Γ-*Mod*. The following assertions are equivalent:*

(1) *$(\mathcal{T}_Z, \mathcal{F}_Z)$ is liftable to U-*Mod
(2) *For each prime ideal $\mathbf{p}$ (minimal) in Z, the U-module $U/U\mathbf{p}$ belongs to $\mathcal{T}_Z$ when looked at as Γ-module.*

Proof. (1) $\Longrightarrow$ (2) Let us take $\mathbf{p} \in Z$. Then the canonical generator $x = 1 + U\mathbf{p}$ of $U/U\mathbf{p}$ belongs to $t_Z(U/U\mathbf{p})$ (see Proposition 2.3). Since $t_Z(U/U\mathbf{p})$ is a U-submodule of $U/U\mathbf{p}$ we conclude that $U/U\mathbf{p} = t_Z(U/U\mathbf{p})$ and condition (2) holds.

(2) $\Longrightarrow$ (1) Let $M \neq 0$ be an arbitrary nonzero U-module. If $0 \neq x \in t_Z(M)$ then, by Proposition 2.3, there are $\mathbf{p}_1, \dots, \mathbf{p}_r \in \mathrm{Min}\, Z$ and positive integers $n_1, \dots, n_r > 0$ such that $\mathbf{p}_1^{n_1} \cdot \dots \cdot \mathbf{p}_r^{n_r} x = 0$. We shall prove that $Ux \subseteq t_Z(M)$ by induction on $k = n_1 + \dots + n_r$. If $k = 1$ then we have a $\mathbf{p} \in \mathrm{Min}(Z)$ such that $\mathbf{p}x = 0$. Then we get an epimorphism of U-modules $U/U\mathbf{p} \twoheadrightarrow Ux$ ($\bar{u} = u + U\mathbf{p} \rightsquigarrow ux$) whose domain belongs to $\mathcal{T}_Z$ when viewed as a Γ-module. Then Ux belongs to $\mathcal{T}_Z$ when viewed as a Γ-module, so that $Ux \subseteq t_Z(M)$.

Suppose now that $k > 1$. If $\mathbf{p}_r x = 0$ then we are done. So we can assume that $\mathbf{p}_r x \neq 0$. The induction hypothesis says that $U\mathbf{p}_r x \subseteq t_Z(M)$, from which it follows that the assignment $\bar{u} = u + U\mathbf{p}_r \rightsquigarrow \overline{ux} = ux + t_Z(M)$ gives a well-defined map $f : U/U\mathbf{p}_r \longrightarrow M/t_Z(M)$, which is clearly a homomorphism of Γ-modules. Then we have that $\mathrm{Im}(f) = (Ux + t_Z(M))/t_Z(M) \in \mathcal{T}_Z$ since $U/U\mathbf{p}_r$ belongs to $\mathcal{T}_Z$. But we also have that $\mathrm{Im}(f) \in \mathcal{F}_Z$ because $\mathrm{Im}(f)$ is a Γ-submodule of $M/t_Z(M)$. It follows that $\mathrm{Im}(f) = 0$, so that $Ux \subseteq t_Z(M)$. □

Note that in our setting the commutative algebra Γ always has finite Krull dimension, so that the (co)height of any of its prime ideal is a natural number. We are now ready to prove our main result, which implies Theorem 1.2.

Theorem 3.2. (1) *Let i be any natural number. The torsion theory $(\mathcal{T}_i, \mathcal{F}_i)$ is liftable to U-*Mod.
(2) *Let M be a simple U-module. There exists a (unique) natural number i such that $t_i(M) = M$ and $t_{i-1}(M) = 0$. Hence all prime ideals in $\mathrm{Ass}(M)$ have coheight exactly i.*

Proof. We prove the first statement by induction on i. If $i = 0$ we take $\mathbf{m} \in \mathrm{Min}\, Z_0 = Z_0 = \mathrm{Specm}\, \Gamma$. In order to prove that $U/U\mathbf{m} \in \mathcal{T}_0$, thus ending the proof (cf. Proposition 3.1), it is enough to prove that $\frac{\Gamma u \Gamma + U\mathbf{m}}{U\mathbf{m}} \cong \frac{\Gamma u \Gamma}{\Gamma u \Gamma \cap U\mathbf{m}}$ is a 'left' Γ-module in $\mathcal{T}_0$, for all $u \in U$. Indeed we have an epimorphism in Γ-Mod

$$\frac{\Gamma u \Gamma}{\Gamma u \mathbf{m}} \twoheadrightarrow \frac{\Gamma u \Gamma}{\Gamma u \Gamma \cap U\mathbf{m}}.$$

But since $\Gamma u \Gamma$ is finitely generated as right Γ-module it follows that $\frac{\Gamma u \Gamma}{\Gamma u \mathbf{m}}$ is finite dimensional as K-vector space. In particular $\frac{\Gamma u \Gamma}{\Gamma u \Gamma \cap U\mathbf{m}}$ is a 'left' Γ-module of finite length and hence belongs to $\mathcal{T}_0$.

Suppose now that $i > 0$ and $i < d = K\,dim(\Gamma)$ (the case $i \geq d$ is trivial). If $\mathbf{p} \in \mathrm{Min}\, Z_i$ and $\mathrm{cht}(\mathbf{p}) < i$ then the induction hypothesis says that $U/U\mathbf{p} \in \mathcal{T}_{i-1} \subset \mathcal{T}_i$. We assume then that $\mathrm{cht}(\mathbf{p}) = i$. According to Proposition 3.1, it will be enough to prove that $U/U\mathbf{p}$ belongs to $\mathcal{T}_i$ when viewed as a Γ-module. This is in turn equivalent to prove that, for each $u \in U$, all the prime ideals of Γ containing $\mathrm{ann}_\Gamma(u + U\mathbf{p}) = (U\mathbf{p} : u) := \{g \in \Gamma : gu \in U\mathbf{p}\}$ have coheight $\leq i$ (cf. Proposition 2.3). Therefore our goal is to prove that the Krull dimension of the algebra $\Gamma/(U\mathbf{p} : u)$ is $\leq i$, for all $u \in U$. For that we shall use the fact that the Krull dimension of this latter algebra coincides with its Gelfand–Kirillov dimension (cf [22][Proposition 7.9])

We fix an element $u \in U$, a finite set of generators $\{u = u_1, u_2, \dots, u_n\}$ of $\Gamma u \Gamma$ as right Γ-module and a finite set of generators $\{t_1, \dots, t_m\}$ of Γ as a K-algebra. We consider the filtration $(F_k)_{k \geq 0}$ on Γ obtained by taking as F_k the vector subspace of Γ generated by the monomials of degree $\leq k$ on the t_i. The induced filtration on $\Gamma/(U\mathbf{p} : u)$ is given by $(\frac{F_k + (U\mathbf{p}:u)}{(U\mathbf{p}:u)})_{k \geq 0}$. The multiplication map $\bar{g} \rightsquigarrow gu + U\mathbf{p}$ is a K-linear isomorphism $\frac{F_k + (U\mathbf{p}:u)}{(U\mathbf{p}:u)} \xrightarrow{\cong} \frac{F_k u + U\mathbf{p}}{U\mathbf{p}}$, for each $k \geq 0$.

Due to our choices, we have that $t_i u_j = \sum_{1 \leq l \leq n} u_l g_{ij}^l$, with $g_{ij}^l \in \Gamma$, for all $i = 1, \dots, r$ and $j = 1, \dots, n$. There exists a minimal positive integer $s > 0$ such that $\{g_{ij}^l\} \subset F_s$. An easy induction gives that $F_k u_j \subseteq \sum_{1 \leq i \leq n} u_i F_{sk}$, for all $k \geq 0$ and all $j = 1, \dots, n$. In particular we have $F_k u \subseteq \sum_{1 \leq i \leq n} u_i F_{sk}$, and hence $\frac{F_k u + U\mathbf{p}}{U\mathbf{p}} \subseteq \sum_{1 \leq i \leq n} \frac{u_i F_{ks} + U\mathbf{p}}{U\mathbf{p}}$, for all $k \geq 0$. Note that we have a surjective K-linear map

$$\frac{F_{sk} + U\mathbf{p}}{U\mathbf{p}} \twoheadrightarrow \frac{u_i F_{ks} + U\mathbf{p}}{U\mathbf{p}} (g + U\mathbf{p} \rightsquigarrow u_i g + U\mathbf{p}).$$

Then, taking K-dimensions, we obtain

$$dim\left(\frac{F_k u + U\mathbf{p}}{U\mathbf{p}}\right) \leq s \cdot dim\left(\frac{F_{ks} + U\mathbf{p}}{U\mathbf{p}}\right),$$

and hence

$$\frac{\log\left(dim\left(\frac{F_k u + U\mathbf{p}}{U\mathbf{p}}\right)\right)}{\log(k)} \leq \frac{\log\left(s \cdot dim\left(\frac{F_{ks} + U\mathbf{p}}{U\mathbf{p}}\right)\right)}{\log(k)}, \qquad (*)$$

for all $k > 0$. Note that we obtain a filtration $(F'_k)_{k\geq 0}$ of the algebra Γ by putting $F'_k = F_{sk}$, for all $k \geq 0$. Then, by applying limit superior to the inequality $(*)$ and bearing in mind that the Gelfand–Kirillov dimension decreases by passing to factor algebras, we get that

$$GKdim(\Gamma/(U\mathbf{p} : u)) \leq GKdim(\Gamma/(U\mathbf{p} \cap \Gamma)) \leq GKdim(\Gamma/\mathbf{p}) = i.$$

This proves the first statement of the theorem. Let us now put $i = \min\{j \geq 0 : M \in \mathcal{T}_j\}$. Then we have $t_i(M) = M$ and $t_{i-1}(M) \subsetneq M$ (convening that $t_{-1}(M) = 0$). By (1), it follows that $t_{i-1}(M)$ is a proper U-submodule of M. The simplicity of M gives that $t_{i-1}(M) = 0$ and, using Corollary 2.5, the proof is completed. □

Question and Remark 3.3. According to Proposition 2.7, if M is a simple U-module and the prime ideals in $\mathrm{Ass}(M)$ are pairwise coprime (e.g. if $M \in \mathcal{T}_0$) then we have a decomposition $M = \oplus_{\mathbf{p}\in \mathrm{Ass}(M)} M(\mathbf{p})$ as Γ-module. For an arbitrary simple U-module M, using Theorem 3.2, it is not difficult to see that the sum $\sum_{\mathbf{p}\in \mathrm{Ass}(M)} M(\mathbf{p})$ is direct, so that $\oplus_{\mathbf{p}\in \mathrm{Ass}(M)} M(\mathbf{p})$ is a Γ-submodule of M. Then a natural question arises: is this sum a U-submodule? Note that a positive answer would imply that $M = \oplus_{\mathbf{p}\in \mathrm{Ass}(M)} M(\mathbf{p})$.

Given a simple U-module M, one needs ways of calculating the $i \geq 0$ such that $t_i(M) = M$ and $t_{i-1}(M) = 0$. Recall that a subset $\{g_1, \ldots, g_r\} \subset \Gamma$ is called a *regular sequence* in case $\sum_{1\leq i\leq r} \Gamma g_i \neq \Gamma$ and $\bar{g}_k := g_k + \sum_{1\leq i<k} \Gamma g_i$ is not a zero divisor in $\Gamma / \sum_{1\leq i<k} \Gamma g_i$, for all $k = 1, \ldots, n$. In that case r is called the *length* of the regular sequence. We refer the reader to [24][pages 136 and 250] for the definitions of Cohen–Macaulay and equidimensional commutative rings, that we use in the following result.

Proposition 3.4. *Suppose that Γ is Cohen–Macaulay and equidimensional and let $d = K\dim(\Gamma)$ be its Krull dimension. If M is a U-module such that all ideals in $\mathrm{Ass}(M)$ have the same coheight (e.g. a simple U-module), then the following assertions are equivalent:*

(1) $t_i(M) = M$ *and* $t_{i-1}(M) = 0$
(2) *There is a regular sequence in Γ, maximal with the property of annihilating some $x \in M \setminus \{0\}$, which has length $d - i$.*

Proof. The equidimensionality guarantees that $\mathrm{ht}(\mathbf{p}) + \mathrm{cht}(\mathbf{p}) = d$, for all $\mathbf{p} \in \mathrm{Spec}\Gamma$ (cf. [23][Corollary II.3.6]). Note also that if $\{g_1, \ldots, g_k\}$ is a regular sequence contained in $\mathrm{ann}_\Gamma(x)$, for some $x \in M \setminus \{0\}$, then, replacing if necessary x by some $gx \neq 0$ with $g \in G$, it is not restrictive to assume that $\mathrm{ann}_\Gamma(x) = \mathbf{q}$, for some prime ideal $\mathbf{q} \in \mathrm{Ass}(M)$. So assertion (2) is equivalent to the following:

(2′) There is a regular sequence in Γ of length $d - i$ contained in some $\mathbf{q} \in \mathrm{Ass}(M)$ and maximal with that property.

By [23][Theorem VI.3.14] and the fact that all prime ideals in $\mathrm{Ass}(M)$ have the same (co)height, this condition 2′ is in turn equivalent to say that $d - i = \mathrm{ht}(\mathbf{q})$, for every $\mathbf{q} \in \mathrm{Ass}(M)$. Therefore assertion (2) holds if, and only if, $\mathrm{cht}(\mathbf{q}) = i$ for all $\mathbf{q} \in \mathrm{Ass}(M)$. By Corollary 2.5, this is equivalent to assertion (1). □

4. An approximation to the assassin of a *U*-module

The preceding section shows that, given a simple U-module M, its assassin as Γ-module, $\mathrm{Ass}(M)$, is an important invariant. Therefore it is natural to give ways of approximating this subset of $\mathrm{Spec}(\Gamma)$. We will see in this section that, knowing a prime $\mathbf{p} \in \mathrm{Ass}(M)$ and the finite subset $\{u_1, \ldots, u_n\} \subset U$ of Setup 1.1, one can give a precise subset of $\mathrm{Spec}\Gamma$ in which $\mathrm{Ass}(M)$ is contained.

We will follow the terminology used for maximal ideals in [8] and, given $u \in U$, we denote by X_u the set of pairs $(\mathbf{q}, \mathbf{p}) \in \mathrm{Spec}\Gamma \times \mathrm{Spec}\Gamma$ such that $\frac{\Gamma u \Gamma}{\mathbf{q}u\Gamma + \Gamma u \mathbf{p}} \neq 0$ (or equivalently $\frac{\Gamma}{\mathbf{q}} \otimes_\Gamma \Gamma u \Gamma \otimes_\Gamma \frac{\Gamma}{\mathbf{p}} \neq 0$). For simplicity, we shall write $\mathbf{q} \equiv_u \mathbf{p}$ whenever $(\mathbf{q}, \mathbf{p}) \in X_u$.

Note that, due to Nakayama lemma, if H if a finitely generated Γ-module and $\mathbf{q} \in \mathrm{Supp}(H)$ then $\mathbf{q}H \neq H$. We will use this fact in the proof of the following result, which is a crucial tool for our purposes.

Lemma 4.1. *Let M be a U-module. The following assertions hold:*

(1) *If $u \in U$, $x \in M$ and $\mathbf{q} \in \mathrm{Supp}(\Gamma ux)$, then there exists $\mathbf{p} \in \mathrm{Ass}(\Gamma x)$ such that $\mathbf{q} \equiv_u \mathbf{p}$*
(2) *If all prime ideals in $\mathrm{Ass}(M)$ have the same coheight, then there is an inclusion*

$$\mathrm{Ass}(\Gamma(x + y)) \subseteq \mathrm{Ass}(\Gamma x) \cup \mathrm{Ass}(\Gamma y),$$

for all $x, y \in M$.

Proof. (1) We have $\mathbf{q} \in \mathrm{Supp}(\Gamma ux) \subseteq \mathrm{Supp}(\Gamma u \Gamma x)$. It follows that $\frac{\Gamma u \Gamma x}{\mathbf{q}u\Gamma x} \neq 0$ since our Setup 1.1 guarantees that $\Gamma u \Gamma x$ is a finitely generated Γ-module. The assignment $\bar{v} \otimes y \rightsquigarrow \overline{vy}$ gives a surjective K-linear map

$$\frac{\Gamma u \Gamma}{\mathbf{q}u\Gamma} \otimes_\Gamma \Gamma x \twoheadrightarrow \frac{\Gamma u \Gamma x}{\mathbf{q}u\Gamma x} \neq 0.$$

It follows that $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \Gamma x \neq 0$. But Γx admits a finite filtration with successive factors isomorphic to $\Gamma/\mathbf{p}$, with $\mathbf{p} \in \mathrm{Supp}(\Gamma x)$ (see [23][Proposition VI.2.6]). We conclude that there is a $\mathbf{p}' \in \mathrm{Supp}(\Gamma x)$ such that $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \frac{\Gamma}{\mathbf{p}'} \neq 0$. Choosing now $\mathbf{p} \in \mathrm{Ass}(\Gamma x)$ such that $\mathbf{p} \subseteq \mathbf{p}'$, we get that $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \frac{\Gamma}{\mathbf{p}} \neq 0$ and hence $\mathbf{q} \equiv_u \mathbf{p}$.

(3) Since we have an inclusion $\Gamma(x+y) \subseteq \Gamma x + \Gamma y$ it will be enough to check that $\mathrm{Ass}(\Gamma x + \Gamma y) \subseteq \mathrm{Ass}(\Gamma x) \cup \mathrm{Ass}(\Gamma y)$. To do that, we consider the canonical exact sequence in Γ-Mod:

$$0 \to \Gamma x \cap \Gamma y \longrightarrow \Gamma x \oplus \Gamma y \longrightarrow \Gamma x + \Gamma y \to 0,$$

from which we get that $\mathrm{Ass}(\Gamma x + \Gamma y) \subseteq \mathrm{Supp}(\Gamma x \oplus \Gamma y) = \mathrm{Supp}(\Gamma x) \cup \mathrm{Supp}(\Gamma y)$.

By hypothesis, all prime ideals in $\mathrm{Ass}(M)$ have the same coheight, which implies that all of them are minimal in $\mathrm{Supp}(M)$. As a consequence, if $\mathbf{q} \in \mathrm{Ass}(\Gamma x + \Gamma y)$ and we assume that $\mathbf{q} \in \mathrm{Supp}(\Gamma x)$, then $\mathbf{q}$ is minimal in $\mathrm{Supp}(\Gamma x)$. This implies that $\mathbf{q} \in \mathrm{Min\,Supp}(\Gamma x) = \mathrm{Min\,Ass}(\Gamma x) \subseteq \mathrm{Ass}(\Gamma x)$. We replace x by y in case $\mathbf{q} \in \mathrm{Supp}(\Gamma y)$, and the proof is finished. □

4.1. *Proof of Theorem 1.3*

We are now ready to prove Theorem 1.3.

If $\mathbf{q} \in \mathrm{Ass}(M)$ then we have $\mathbf{q} = \mathrm{ann}_\Gamma(ux)$, for some $u \in U$. If $u \in \Gamma$ then $\mathbf{q} = \mathbf{p}$ and there is nothing to prove. So we assume $u \notin \Gamma$, in which case u is a sum of products of the form

$$g_1 u_{k_1} g_2 \dots g_r u_{k_r} g_{r+1},$$

where the g_k belong to Γ and the $k_1, \dots, k_r$ belong to $\{1, \dots, n\}$. Lemma 4.1 allows us to assume, without loss of generality, that

$$u = g_1 u_{k_1} g_2 \dots g_r u_{k_r} g_{r+1},$$

something that we do from now on in this proof.

We then have

$$\mathbf{q} \in \mathrm{Ass}(\Gamma ux) \subseteq \mathrm{Ass}(\Gamma u_{k_1} g_2 \dots g_r u_{k_r} g_{r+1} x).$$

By Lemma 4.1(1), there is a $\mathbf{q}_1 \in \mathrm{Ass}(\Gamma g_2 u_{k_2} \dots g_r u_{k_r} g_{r+1} x)$ such that $\mathbf{q} \equiv_{u_{k_1}} \mathbf{q}_1$. By induction we get a sequence $\mathbf{q} = \mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_r$ of prime ideals in $\mathrm{Ass}(M)$, whence of coheight exactly $\mathrm{cht}(\mathbf{p})$ (see Theorem 3.2, such that $\mathbf{q}_r \in \mathrm{Ass}(\Gamma g_{r+1} x)$ and $\mathbf{q}_{i-1} \equiv_{u_{k_i}} \mathbf{q}_i$ for $i = 1, \dots, r$. But $\mathrm{Ass}(\Gamma g_{r+1} x) = \{\mathbf{p}\}$ since $\mathrm{ann}_\Gamma(x) = \mathbf{p}$ is a prime ideal and $g_{r+1} x \neq 0$. Then $\mathbf{q}_r = \mathbf{p}$ and the proof is finished.

Theorem 1.3 suggests to define, for each $0 \leq i \leq d$, a (not necessarily symmetric) relation $\equiv$ in the set $\mathrm{Min}\, Z_i$ of prime ideals of coheight i by saying that $\mathbf{q} \equiv \mathbf{p}$ if, and only if, there are a sequence $\mathbf{q} = \mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_s = \mathbf{p}$ in $\mathrm{Min}\, Z_i$ and a sequence of indices $k_1, \dots, k_s \in \{1, \dots, n\}$ such that $\mathbf{q}_{i-1} \equiv_{u_{k_i}} \mathbf{q}_i$, for all $i = 1, \dots, s$.

Corollary 4.2. *If M is a simple U-module and $\mathbf{p}, \mathbf{q} \in \mathrm{Ass}(M)$ then $\mathbf{q} \equiv \mathbf{p}$.*

Proof. As U-module, M is generated by any of its nonzero elements. Choose $0 \neq x \in M$ such that $\mathrm{ann}_\Gamma(x) = \mathbf{p}$ and apply Theorem 3.2. □

We obtain immediately the following refinement of Proposition 2.7.

Corollary 4.3. *Let M be a simple U-module and take $\mathbf{p} \in \mathrm{Ass}(M)$, with $\mathrm{cht}(\mathbf{p}) = i$. Suppose that $\mathbf{q}$ and $\mathbf{q}'$ are coprime whenever $\mathbf{q} \neq \mathbf{q}'$ are distinct prime ideals of Γ of coheight i such that $\mathbf{q} \equiv \mathbf{p}$ and $\mathbf{q}' \equiv \mathbf{p}$. Then we have a decomposition $M = \oplus_{\mathbf{q} \in \mathrm{Ass}(M)} M(\mathbf{q})$ as Γ-module.*

Proof. By Theorem 1.3, we have an inclusion $\mathrm{Ass}(M) \subseteq \{\mathbf{q} \in \mathrm{Spec}\Gamma : \mathrm{cht}(\mathbf{q}) = i \text{ and } \mathbf{q} \equiv \mathbf{p}\}$. Therefore the elements of $\mathrm{Ass}(M)$ are pairwise coprime and Proposition 2.7 applies. □

The following example shows that in some circumstances (usually when the coheight is large), Theorem 1.3 is not sufficient to approximate $\mathrm{Ass}(M)$.

Example 4.4. Let K be algebraically closed of characteristic zero and $U = A_n(K)$ be the Weyl algebra given by generators $X_1, \dots, X_n, Y_1, \dots, Y_n$ subject to the relations

$$X_i X_j - X_j X_i = 0 = Y_i Y_j - Y_j Y_i$$
$$X_i Y_j - Y_j X_i = \delta_{ij},$$

for all $i, j \in \{1, \dots, n\}$, where δ_{ij} is the Kronecker symbol. Assume $n > 1$, put $t_i = X_i Y_i$ and put $\Gamma = K[t_1, \dots, t_n]$. Then Γ and U satisfy the conditions of our Setup 1.1 by taking $u_j = Y_{\sigma(j)}$ (resp. $u_j = X_{\sigma(j)}$), for all $j = 1, \dots, n$, where $\sigma \in S_n$ is any permutation. If $\mathbf{p} = \Gamma(t_1 - 1)$ then $\mathbf{q} \equiv \mathbf{p}$, for every prime ideal $\mathbf{q} \in \mathrm{Spec}(\Gamma)$ of height 1.

Proof. For simplicity put $u_i = Y_i$ ($i = 1, \dots, n$), the other choices being treated similarly. Then one readily shows the equalities

$$Y_i t_j = t_j Y_i \quad (i \neq j)$$
$$Y_i t_i = (t_i - 1)Y_i \quad (\text{equivalently } t_i Y_i = Y_i(t_i + 1)),$$

for all $i = 1, \dots, n$. If $f, g \in \Gamma$ are irreducible polynomials we derive from these equalities that $f \equiv_{Y_i} g$ if and only if the polynomials $s_i(f) := f(t_1, \dots, t_{i-1}, t_i + 1, t_{i+1}, \dots, t_n)$ and g are not coprime (i.e. the prime ideals of Γ generated by them are not coprime). Indeed we have that $fY_i\Gamma = Y_i s_i(f)\Gamma$ and $\Gamma Y_i \Gamma = Y_i \Gamma$ using the above equalities. But then the obvious isomorphism of 'right' Γ-modules $\Gamma \cong Y_i\Gamma$ induces an isomorphism

$$\frac{\Gamma Y_i \Gamma}{fY_i\Gamma} = \frac{Y_i\Gamma}{Y_i s_i(f)\Gamma} \overset{\cong}{\longleftrightarrow} \frac{\Gamma}{(s_i(f))}.$$

It follows that $\frac{\Gamma Y_i \Gamma}{fY_i\Gamma + \Gamma Y_i g} \cong \frac{\Gamma Y_i \Gamma}{fY_i \Gamma} \otimes_\Gamma \frac{\Gamma}{(g)}$ is nonzero if an only if $\frac{\Gamma}{(s_i(f))} \otimes_\Gamma \frac{\Gamma}{(g)} \neq 0$. This happens exactly when $s_i(f)$ and g are not coprime.

We pass now to prove the statement. If $s_i(f)$ is not coprime with $t_1 - 1$, for some $i = 1, \dots, n$, then last paragraph applies with $g = t_1 - 1$. So we assume that $s_i(f)$ is coprime with $t_1 - 1$ for all $i = 1, \dots, n$. (*Note that this situation can actually happen. For instance if $f = a + b(t_1 - 2)^m$, with $m > 0$ $a, b \in K$ and $a \neq 0 \neq a + (-1)^m b$.*) We then put $f' := s_1(f)$ and express it as a sum $\sum_{0 \leq k \leq r} g_k(t_2, \dots, t_n)(t_1 - 1)^k$. Then we get

$$\Gamma = f'\Gamma + (t_1 - 1)\Gamma = g_0\Gamma + (t_1 - 1)\Gamma$$

and it is easy to derive from this that g_0 is a constant polynomial, so that we can rewrite

$$f'(t_1, \dots, t_n) = a + (t_1 - 1)^m g(t_1, \dots, t_2),$$

where $g \in \Gamma \setminus \{0\}$ and $a \in K \setminus \{0\}$. Note that, given any index $i = 2, \dots, n$, we cannot have $g(t_1, \dots, t_{i-1}, \alpha, t_{i+1}, \dots, t_n) = 0$, for all $\alpha \in K$. Indeed in that case the polynomial g would be zero. We then choose $\alpha \in K$ such that $g(t_1, \alpha, t_3, \dots, t_n) \neq 0$ and claim that f' and $t_2 - \alpha$ are not coprime. To see that, note that f' and $t_2 - \alpha$ are coprime if, and only if, $\bar{f}' := f' + (t_2 - \alpha)$ is invertible in $\Gamma/(t_2 - \alpha)$. Using the canonical isomorphism

$$K[t_1, \dots, t_n]/(t_2 - \alpha) \overset{\cong}{\longrightarrow} K[t_1, t_3, \dots, t_n]$$
$$(\bar{h} \rightsquigarrow h(t_1, \alpha, t_3, \dots, t_n)),$$

we immediately find a polynomial $u \in K[t_1, t_3, \dots, t_n]$ satisfying the equality

$$f'(t_1, \alpha, t_3, \dots, t_n)u(t_1, t_3, \dots, t_n) = 1$$

in $K[t_1, t_3, \dots, t_n]$. It follows that

$$f'(t_1, \alpha, t_3, \dots, t_n) = 1 + (t_1 - 1)^m g(t_1, \alpha, t_2, \dots, t_n)$$

is a constant polynomial, something which can only happen when $g(t_1, \alpha, t_2, \dots, t_n) = 0$. But this contradicts our choice of α.

Put now $h := t_2 - \alpha$. We then get that $f \equiv_{Y_1} h$ since $f' = s_1(f)$ is not coprime with $h = t_2 - \alpha$. On the other hand, we also have $h \equiv_{Y_2} t_1 - 1$ since $s_2(h) = h(t_2 + 1) = t_2 + 1 - \alpha$ is not coprime with $t_1 - 1$. We then conclude that $f \equiv t_1 - 1$ as desired. □

We end the section with a result on extensions of U-modules.

Proposition 4.5. *Let M and N be nonzero U-modules and suppose that $\frac{\Gamma u \Gamma}{\mathbf{q}u\Gamma + \Gamma u \mathbf{p}} = 0$, for all $u \in U$, $\mathbf{q} \in \mathrm{Ass}(M)$ and $\mathbf{p} \in \mathrm{Ass}(N)$. The following assertions hold:*

(1) $\mathrm{Ext}^i_\Gamma(M, N) = 0 = \mathrm{Ext}^i_\Gamma(N, M)$, *for all $i \geq 0$*
(2) $\mathrm{Ext}^1_U(N, M) = 0$.

Proof. (1) By taking $u = 1$ above, we see that $\mathbf{p}$ and $\mathbf{q}$ are coprime whenever $\mathbf{p} \in \mathrm{Ass}(M)$ and $\mathbf{q} \in \mathrm{Ass}(N)$. The assertion follows from Proposition 2.8.

(2) Let $0 \to M \longrightarrow X \longrightarrow N \to 0$ be an exact sequence in U-Mod. By assertion 1 we know that it split in Γ-Mod. Then we shall identify $X = M \oplus N$, in which case the external multiplication map $U \times X \longrightarrow X$ ($(u, x) \rightsquigarrow u \cdot x$) is entirely determined by the U-module structures on M and N and by a K-bilinear map $\mu : U \times N \longrightarrow M$ satisfying the following three properties for all $u, u' \in U$, $g \in \Gamma$ and $y \in N$:

(1) $\mu(uu', y) = u\mu(u', y) + \mu(u, u'y)$ (this guarantees that $(uu') \cdot y = u \cdot (u' \cdot y)$)
(2) $\mu(g, y) = 0$, for all $g \in \Gamma$ (this guarantees that the structure of Γ-module on $M \oplus N$ given by restriction of scalars via the inclusion $j : \Gamma \hookrightarrow U$ is that of the direct sum)
(3) $u \cdot y = \mu(u, y) + uy$ (this guarantees that the projection $\begin{pmatrix} 0 & 1 \end{pmatrix} : X = M \oplus N \longrightarrow N$ is a U-homomorphism).

It follows that the assignment $u \otimes y \rightsquigarrow \mu(u, y)$ defines a homomorphism of Γ-modules $\mu' : U \otimes_\Gamma N \longrightarrow M$.

We claim that $\mu' = 0$. Suppose not and take $\mathbf{q} \in \mathrm{Ass}(\mathrm{Im}(\mu')) \subseteq \mathrm{Ass}(M)$. The surjective Γ-homomorphism $U \otimes_\Gamma N \twoheadrightarrow \mathrm{Im}(\mu')$ induces another surjective Γ-homomorphism

$$\bigoplus_{u \in U, y \in N} \Gamma u \Gamma \otimes_\Gamma \Gamma y \twoheadrightarrow \mathrm{Im}(\mu').$$

In particular, we get that $\mathbf{q} \in \mathrm{Supp}(\Gamma u \Gamma \otimes_\Gamma \Gamma y)$, for some $u \in U$ and $y \in N$. Since $\Gamma u \Gamma \otimes_\Gamma \Gamma y$ is an epimorphic image in Γ-Mod of $\Gamma u \Gamma$, which is finitely generated as 'left' Γ-modules, it follows that $\Gamma u \Gamma \otimes_\Gamma \Gamma y$ is a finitely generated Γ-module and thereby that $\mathbf{q}(\Gamma u \Gamma \otimes_\Gamma \Gamma y) \neq \Gamma u \Gamma \otimes_\Gamma \Gamma y$. That means that the left arrow in the exact sequence

$$\mathbf{q} u \Gamma \otimes_\Gamma \Gamma y \longrightarrow \Gamma u \Gamma \otimes_\Gamma \Gamma y \longrightarrow \frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \Gamma y \to 0$$

is not surjective, and hence that $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \Gamma y \neq 0$. The argument of Lemma 4.1(1) shows that there exists a $\mathbf{p} \in \mathrm{Ass}(\Gamma y) \subseteq \mathrm{Ass}(N)$ such that $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma} \otimes_\Gamma \frac{\Gamma}{\mathbf{p}} \neq 0$. We then get $\frac{\Gamma u \Gamma}{\mathbf{q} u \Gamma + \Gamma u \mathbf{p}} \neq 0$, which contradicts the hypothesis. □

5. Applications and some open questions

We start with a proposition which will be useful in what follows, for its hypotheses are satisfied by all examples of this final section.

Proposition 5.1. *In the Setup 1.1 suppose in addition that the following conditions hold:*

(1) *If $Z = Z(U)$ is the center of U then $Z \cap \Gamma$ is equidimensional (see [24], p. 250)*
(2) *Γ is flat as a $Z \cap \Gamma$-module*
(3) *For each simple U-module, the endomorphism algebra $End_U(M)$ has dimension equal to 1 as a K-vector space.*

If $U-fl$ denotes the subcategory of U-modules of finite length, then $\mathcal{T}_i \cap U-fl = \mathcal{T}_j \cap U-fl$, for all $i, j \geq K\,dim(\Gamma) - K\,dim(Z \cap \Gamma)$.

Proof. Let M be a simple U-module. Then the structural map $K \longrightarrow End_U(M)$ is an algebra isomorphism, which we view as an identification. On the other hand, every element $z \in Z$ induces by multiplication an endomorphism $\lambda_z \in End_U(M)$. Put $Z' = Z \cap \Gamma$. The assignment $z \rightsquigarrow \lambda_z$ gives then an isomorphism

$$Z'/ann_{Z'}(M) \xrightarrow{\cong} End_U(M) = K,$$

thus showing that $\mathbf{m} := ann_{Z'}(M)$ is a maximal ideal of Z'. Let now $\mathbf{p} \in \mathrm{Spec}(\Gamma)$ be minimal over $\Gamma\mathbf{m}$. We clearly have $\mathbf{m} = Z' \cap \mathbf{p}$ and we have an equality

$$ht(\mathbf{p}) = ht(\mathbf{m}) + K\,dim\left(\frac{\Gamma_\mathbf{p}}{\Gamma_\mathbf{p}\mathbf{m}}\right)$$

(cf. [24][Theorem 15.1]). But the prime spectrum of $\frac{\Gamma_\mathbf{p}}{\Gamma_\mathbf{p}\mathbf{m}}$ is in bijection with the set of $\mathbf{q} \in \mathrm{Spec}(\Gamma)$ such that $\Gamma\mathbf{m} \subseteq \mathbf{q} \subseteq \mathbf{p}$. By our choice of $\mathbf{p}$, this implies that $\mathrm{Spec}(\frac{\Gamma_\mathbf{p}}{\Gamma_\mathbf{p}\mathbf{m}})$ has one element. It follows that $K\,dim(\frac{\Gamma_\mathbf{p}}{\Gamma_\mathbf{p}\mathbf{m}}) = 0$, so that $ht(\mathbf{p}) = ht(\mathbf{m})$, for all $\mathbf{m} \in \mathrm{Specm}\,Z'$ and all $\mathbf{p} \in \mathrm{Spec}\,\Gamma$ minimal over $\Gamma\mathbf{m}$.

Put $d := K\,dim(\Gamma)$ and $e := K\,dim(Z')$. Equidimensionality of Z' gives that $ht(\mathbf{m}) = e$ (cf. [23][Corollary II.3.6]). Then from the last paragraph and the inequality

$$ht(\mathbf{p}) + K\,dim(\Gamma/\mathbf{p}) \leq K\,dim(\Gamma)$$

we readily derive that

$$K\,dim\left(\frac{\Gamma}{\Gamma\mathbf{m}}\right) = \mathrm{Sup}\{K\,dim(\Gamma/\mathbf{p}) : \mathbf{p} \in \mathrm{Spec}\,\Gamma \text{ minimal over } \Gamma\mathbf{m}\} \leq d - e.$$

This says that the coheight of any $\mathbf{p} \in \mathrm{Spec}\,\Gamma$ containing a maximal ideal of Z' is always $\leq d - e$. In particular that happens for all $\mathbf{p} \in \mathrm{Ass}(M)$, for every simple U-module M. It follows that the simple U-modules in $\mathcal{T}_i$ are the same for all $d - e \leq i \leq d$, which implies the statement. □

Remark 5.2. If our field is algebraically closed, condition (3) in Proposition 5.1 is satisfied whenever U admits an exhaustive filtration $U_0 \subset U_1 \subset \cdots$ such that the associated graded algebra $gr(U)$ is a commutative finitely generated algebra (cf. [6][Lemma 2.6.4]). It is the case for all finite W-algebras (cf. [5], Theorem 10.1 or [17], 4.4).

The following problems are of special interest in the case of enveloping algebras of Lie algebras and finite W-algebras.

Problems 5.3. Suppose that Γ and U satisfy the conditions of Setup 1.1 and also the hypotheses of Proposition 5.1. We propose the following problems:

(1) To identify the set $\mathbf{N}_U$ of natural numbers $0 \le j \le d-e$ for which there exists a simple U-module M such that $t_j(M) = M$ and $t_{j-1}(M) = 0$ (convening that $t_{-1}(M) = 0$).
(2) Given $j \in \mathbf{N}_U$, to identify the set of $\mathbf{p} \in \operatorname{Spec}\Gamma$ such that $cht(\mathbf{p}) = j$ and $\mathbf{p} \in \operatorname{Ass}(M)$, for some simple U-module M
(3) (Local version) Given a character $\chi : Z' = Z \cap \Gamma \longrightarrow K$, to identify the set $\mathbf{N}(\chi)$ of natural numbers $0 \le j \le d-e$ for which there exists a simple U-module M annihilated by $Ker(\chi)$ with $t_j(M) = M$ and $t_{j-1}(M) = 0$. For any $j \in \mathbf{N}(\chi)$, to identify all $\mathbf{p} \in \operatorname{Spec}\Gamma$ such that $cht(\mathbf{p}) = j$, $Ker(\chi) \subset \mathbf{p}$ and $\mathbf{p} \in \operatorname{Ass}(M)$ for some simple U-module M.

We move now to the announced classical examples. In the rest of the paper we assume that the field K is algebraically closed of characteristic zero.

5.1. Finite W-algebras

Associated with a nilpotent element and a good grading in the Lie algebra gl_n, there is associated a finite W-algebra (see [9] for the definition and details). Each finite W-algebra of type $\mathbf{A}$ is determined by a sequence of integers $\tau = (p_1, \ldots, p_m)$ such that $1 \le p_1 \le \cdots \le p_m$ and $p_1 + \cdots + p_m = n$. We denote such an algebra by $W(\tau)$. If for each $k = 1, \ldots, m$ we put $\tau_k = (p_1, \ldots, p_k)$, then we obtain a chain of subalgebras

$$W(\tau_1) \subset \cdots \subset W(\tau_m) = W(\tau).$$

The subalgebra Γ of $W(\tau)$ generated by the centers of the $W(\tau_k)$ is a commutative algebra usually called the *Gelfand–Tsetlin subalgebra* of $W(\tau)$.

As shown in [14,13], the algebra $U = W(\tau)$ and the commutative subalgebra Γ satisfy all the conditions of Setup 1.1 and all the hypothesis of Proposition 5.1, actually with $Z \subset \Gamma$ and hence $Z \cap \Gamma = Z$. Moreover, we have $d = mp_1 + (m-1)p_2 + \cdots + 2p_{m-1} + p_m$ and $e = p_1 + \cdots + p_m$ (see [12,5]), where d and e are as in Proposition 5.1. In particular we get:

Corollary 5.4. *Let us consider the natural number $r = (m-1)p_1 + (m-2)p_2 + \cdots + p_{m-1}$. The following assertions hold:*

(1) *The torsion theories $(\mathcal{T}_i, \mathcal{F}_i)$ $(i = 0, 1, \ldots, d)$ are liftable from Γ-Mod to $W(\tau)$-Mod.*
(2) *If M is a simple $W(\tau)$-module then there is a unique natural number $0 \le j \le r$ such that $t_j(M) = M$ and $t_{j-1}(M) = 0$. In this case all prime ideals in $\operatorname{Ass}(M)$ have coheight exactly j.*

Note that in the case $m = n$ and $p_1 = \cdots = p_m = 1$ the corresponding W-algebra is isomorphic to $U(\mathrm{gl}_n)$.

5.2. The Lie algebra gl_n

Given any positive integer n and any basis $\pi = \{\alpha_1, \ldots, \alpha_n\}$ of the root system of the Lie algebra gl_n, we denote by gl_i the Lie subalgebra corresponding to the simple roots $\alpha_1, \ldots, \alpha_i$. We then have inclusions of Lie algebras

$$\mathrm{gl}_1 \subset \mathrm{gl}_2 \subset \cdots \subset \mathrm{gl}_n$$

inducing corresponding inclusions of associative algebras

$$U_1 \subset U_2 \subset \cdots \subset U_n,$$

where $U_k = U(\mathrm{gl}_k)$ is the universal enveloping algebra of gl_k for each $k > 0$. If we put $U = U_n$ then the subalgebra $\Gamma(\pi)$ of U generated by the centers of $U_1, \ldots, U_n$ is a maximal commutative subalgebra, called the *Gelfand–Tsetlin subalgebra* of U associated to the root system π. The inclusion $\Gamma(\pi) \subset U$ satisfies all the requirements of Setup 1.1 and the hypotheses of Proposition 5.1, again with $Z \subseteq \Gamma$. Concretely $\Gamma(\pi)$ is isomorphic to a polynomial algebra on $\frac{n(n+1)}{2}$ variables (cf. [14,15]) while the center $Z = Z(U)$ is a polynomial algebra on n variables. We therefore have:

Corollary 5.5. *The following assertions hold:*

(1) *The torsion theories $(\mathcal{T}_i, \mathcal{F}_i)$ $(i = 0, 1, \ldots, \frac{n(n+1)}{2})$ are liftable from $\Gamma(\pi)$-Mod to $U(\mathrm{gl}_n)$-Mod.*
(2) *If M is a simple gl_n-module then there is a unique natural number $0 \le j \le \frac{n(n-1)}{2}$ such that $t_j(M) = M$ and $t_{j-1}(M) = 0$. In this case all prime ideals in $\operatorname{Ass}(M)$ have coheight exactly j.*

An interesting phenomenon for $U_n = U(\mathrm{gl}_n)$ is that there are several Gelfand–Tsetlin subalgebras to which we can apply our general theory, namely, one per each choice of a basis of the root system. We denote by $\mathcal{T}_i(\pi)$ the class of U_n-modules M such that, viewed as $\Gamma(\pi)$-module, M belongs to $\mathcal{T}_i$. Since different root systems are conjugated by the Weyl group, one immediately gets:

Proposition 5.6. *Let π and π' be two bases of the root systems of gl_n. The categories $\mathcal{T}_i(\pi)$ and $\mathcal{T}_i(\pi')$ are equivalent for any i.*

Concerning Problems 5.3(1), it is well known that $0 \in \mathbf{N}_U$ when U is a finite W-algebra of type $\mathbf{A}$. For the particular case $U = U(\mathrm{gl}_n)$ we have that $1 \in \mathbf{N}_U$, as the following example show.

Example 5.7. There are simple gl_n-modules which are not in $\mathcal{T}_0$ for all $n > 1$.

Proof. Consider any generic simple non-weight (with respect to any Cartan subalgebra) gl_2-module V, such modules exist by [4]. Then $V \in \mathcal{T}_1$ and is not Gelfand–Tsetlin. Let H be a Cartan subalgebra of gl_3. Fix $a \in \mathbb{C}$. Let (c_1, c_2) be the central character of V (c_1 is an eigenvalue of $e_{11} + e_{22}$ and c_2 is an eigenvalue of the quadratic Casimir element). Let $\mathfrak{P}$ be a parabolic subalgebra of gl_3 whose Levi factor is $\mathrm{gl}_2 + H$. Now consider the induced module $M(V, a) = U(\mathrm{gl}_3) \otimes_{U(\mathfrak{P})} V$ where V is naturally viewed as a $\mathfrak{P}$-module with a trivial action of the radical and $e_{11} + e_{22} + e_{33}$ acts by multiplication by a. Then $M(V, a)$ has a unique simple quotient $L(V, a)$ which belongs to the subcategory $\mathcal{T}_1 \subset \mathrm{gl}_3$-Mod and is not Gelfand–Tsetlin. Similarly, one can induce now from $L(V, a)$ to get a gl_4-module with a unique simple quotient in $\mathcal{T}_1 \subset \mathrm{gl}_4$-Mod which is not Gelfand–Tsetlin. One continues inductively. Hence, for each $n \geq 2$ we construct a simple gl_n-module in $\mathcal{T}_1$ which is not Gelfand–Tsetlin. □

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