Available online at www.sciencedirect.com
SCIENCE DIRECT
Topology and its Applications 150 (2005) 255–265
Topology and its Applications
www.elsevier.com/locate/topol

Some cardinal invariants on the space $C_\alpha(X, Y)$

Süleyman Önal [a,*], Çetin Vural [b]

[a] *Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey*
[b] *Gazi University, Department of Mathematics, 06500 Ankara, Turkey*

Received 15 September 2004; received in revised form 21 November 2004; accepted 27 November 2004

Abstract

Let $C_\alpha(X, Y)$ be the set of all continuous functions from X to Y endowed with the set-open topology where α is a hereditarily closed, compact network on X such that closed under finite unions. We define two properties $(E1)$ and $(E2)$ on the triple (α, X, Y) which yield new equalities and inequalities between some cardinal invariants on $C_\alpha(X, Y)$ and some cardinal invariants on the spaces X, Y such as:

Theorem. *If Y is an equiconnected space with a base consisting of φ-convex sets, then for each $f \in C(X, Y)$, $\chi(f, C_\alpha(X, Y)) = \alpha a(X) . w_e(f(X))$.*

Corollary. *Let Y be a noncompact metric space and let the triple (α, X, Y) satisfy $(E1)$. The following are equivalent*:

(i) *$C_\alpha(X, Y)$ is a first-countable space.*
(ii) *π-character of the space $C_\alpha(X, Y)$ is countable.*
(iii) *$C_\alpha(X, Y)$ is of pointwise countable type.*
(iv) *There exists a compact subset K of $C_\alpha(X, Y)$ such that π-character of K in the space $C_\alpha(X, Y)$ is countable.*
(v) *$\alpha a(X) \leqslant \aleph_0$.*
(vi) *$C_\alpha(X, Y)$ is metrizable.*
(vii) *$C_\alpha(X, Y)$ is a q-space.*

* Corresponding author.
E-mail addresses: osul@metu.edu.tr (S. Önal), cvural@gazi.edu.tr (Ç. Vural).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2004.11.013

(viii) *There exists a sequence* $\{O_n\colon\ n \in \omega\}$ *of nonempty open subset of* $C_\alpha(X, Y)$ *such that each sequence* $\{g_n\colon\ n \in \omega\}$ *with* $g_n \in O_n$ *for each* $n \in \omega$*, has a cluster point in* $C_\alpha(X, Y)$.

MSC: 54C35; 54C05; 54C20

Keywords: Function space; Network; Character; Equiconnected; Arens number

1. Preliminaries

In [4], it have been investigated some relations between some cardinal invariants on the space $C_\alpha(X, \mathbb{R})$ and some cardinal invariants on the space X, where $\mathbb{R}$ is the space of real numbers with the usual metric. In this paper, when the range space Y is an arbitrary topological space having some requisite properties, we investigated some relations between some cardinal invariants on the space $C_\alpha(X, Y)$ and some cardinal invariants on the spaces X, Y.

Throughout this paper X and Y are infinite Tychonoff spaces (i.e., completely regular topological spaces in which finite sets are closed), and $C(X, Y)$ denotes the set of all continuous mappings from X into Y, and α is always a hereditarily closed, compact network on the domain space X. (I.e., α is a network on X such that each member is compact and each closed subset of a member of it is a member of it.) Without loss of generality we can assume that α is closed under finite unions. Throughout this paper ω and $\aleph_0$ denote the first infinite ordinal and the first infinite cardinal, respectively.

Let $A \subseteq X$ and $B \subseteq Y$. The notation $[A, B]$ used to denote

$$[A, B] = \bigl\{f \in C(X, Y)\colon\ f(A) \subseteq B\bigr\}.$$

If $x \in X$ and $B \subseteq Y$, then $[\{x\}, B]$ is abbreviated as $[x, B]$.

The topology generated by the family

$$\mathcal{B} = \left\{\bigcap_{i=1}^{n}[A_i, V_i]\colon\ A_i \in \alpha \text{ and } V_i \text{ is open in } Y \text{ for each } 1 \leqslant i \leqslant n\right\}$$

on the set $C(X, Y)$ is called the *set-open topology*, and the function space $C(X, Y)$ having this topology is denoted by $C_\alpha(X, Y)$. The family $\mathcal{B}$ is called the *standard base* of this topology. For any element $B = \bigcap_{i=1}^{n}[A_i, V_i]$ of $\mathcal{B}$, the set $\bigcup_{i=1}^{n} A_i$ is called the *support of* B, and denoted by $\operatorname{supp}(B)$. The function space $C(X, Y)$ having *the topology of pointwise convergence* is denoted by $C_p(X, Y)$.

Let Z be a subspace of X. Then $\alpha_{|Z}$ denotes the set $\{A \cap Z\colon\ A \in \alpha\}$ and the restriction of a mapping $f : X \to Y$ to the set Z is denoted by $f_{|Z}$.

The cardinality and the closure of a set A is denoted by $|A|$ and $\overline{A}$, respectively.

A family $\mathcal{O}$ of nonempty open subsets of a space X is called a π*-base* for the space X at a set $A \subseteq X$, if for any open set U that contains A there exists an $O \in \mathcal{O}$ such that $O \subseteq U$. The π*-character* of a set A in a topological space X is defined as the smallest cardinal number of the form $|\mathcal{O}|$, where $\mathcal{O}$ is a π*-base* for X at the set A; this cardinal number is denoted by $\pi\chi(A, X)$.

Let β be a family of subsets of X. If every member of α is contained in some member of β, then β is called an *α-cover* of X. The smallest infinite cardinality of such a family β with $\beta \subseteq \alpha$ is called *α-Arens number* of X, and it is denoted by $\alpha a(X)$.

An *external base* for a subspace A of a topological space X is a family $\mathcal{D}$ of open subsets of X with the property that for each $a \in A$ and any neighbourhood U of a in the space X, there exists a $D \in \mathcal{D}$ such that $a \in D \subseteq U$. The *external weight* of a subspace A of a space X is the smallest infinite cardinal number of the form $|\mathcal{D}|$, where $\mathcal{D}$ is an external base for A; this cardinal number is denoted by $w_e(A)$.

The *character* and *weight* of a space X are denoted by $\chi(X)$ and $w(X)$, respectively.

For a space X, we denote the smallest cardinal number κ with the property that for each x in X there exists a compact subset C of X such that $x \in C$ and $\chi(C, X) \leqslant \kappa$ by $h(X)$.

$uw(X)$, the *uniform weight* of an uniform space X, is the smallest infinite cardinality of an uniformity base of X.

Notations and terminology not explained above can be found in [1,3,4].

2. Two properties on the triple (α, X, Y)

It is well known that the topological space $\mathbb{R}$ with the usual metric has a lot of strong properties. For example $\mathbb{R}$ is a linear topological space, and hence the space $C_\alpha(X, \mathbb{R})$ is a linear topological space. But, in general, for any topological space Y, the space $C_\alpha(X, Y)$ does not have most of the properties that $C_\alpha(X, \mathbb{R})$ has.

In this part, we will give two properties on the triple (α, X, Y), and we will investigate (α, X, Y) satisfying these properties.

We say that the triple (α, X, Y) satisfies $(E1)$ when it satisfies the fact that if $y \in Y$, $f \in C(X, Y)$, $A \in \alpha$ and $x \in X \backslash A$, then there exists a $g \in C(X, Y)$ such that $g(x) = y$ and $g_{|A} = f_{|A}$.

We say that the triple (α, X, Y) satisfies $(E2)$ when it satisfies the fact that if $y \in Y$, then there exists an open subset W of Y such that $y \in W$, and if B is a nonempty element of the standard base of $C_\alpha(X, Y)$ and F is a finite subset of X with $F \cap \operatorname{supp}(B) = \emptyset$, then $[F, Y \backslash W] \cap B \neq \emptyset$.

It is easy to see that $(E1)$ implies $(E2)$.

We observe that, if the triple (α, X, Y) satisfies $(E1)$ then $C(X, Y)$ is a dense subset of the product space Y^X.

In order to satisfy $(E1)$, we give some sufficient conditions on the triple (α, X, Y).

Proposition 1. *If X is a zero-dimensional topological space, then (α, X, Y) satisfies $(E1)$.*

Proof. Take any $y \in Y$, $f \in C(X, Y)$, $A \in \alpha$ and $x \in X \backslash A$. Since the space X is zero-dimensional, and A is closed subset of X, then there exists a closed–open subset Z of X such that $x \in Z \subseteq X \backslash A$. Define the function $g : X \to Y$ with $g(p) = y$ for each $p \in Z$, and $g(p) = f(p)$ for each $p \in X \backslash Z$. One can easily verify that g is continuous, $g(x) = y$ and $g_{|A} = f_{|A}$. $\square$

Proposition 2. *If* $(Y, *)$ *is a pathwise connected topological group, then* (α, X, Y) *satisfies* $(E1)$.

Proof. Take any $y \in Y$, $f \in C(X, Y)$, $A \in \alpha$ and $x \in X \backslash A$. Since the space X is Tychonoff, there exists a continuous function $\Psi : X \to [0, 1]$ such that $\Psi(x) = 0$ and $\Psi(A) \subseteq \{1\}$. Pathwise connectedness of Y gives us a continuous function $\Phi : [0, 1] \to Y$ such that $\Phi(0) = (f(x))^{-1} * y$ and $\Phi(1) = e$ (e being the identity of Y). Then define the function $g : X \to Y$ with $g(z) = f(z) * (\Phi(\Psi(z)))$. Thus, g is continuous function that is required. □

Proposition 3. *Let* X *and* Y *be topological spaces. If there exists a continuous map* $\varphi : Y \times Y \times [0, 1] \to Y$ *such that* $\varphi(p, q, 0) = p$ *and* $\varphi(p, q, 1) = q$ *for each* $p, q \in Y$ *with* $p \neq q$, *then* (α, X, Y) *satisfies* $(E1)$.

Proof. Take any $y \in Y$, $f \in C(X, Y)$, $A \in \alpha$ and $x \in X \backslash A$. Since the space X is Tychonoff, there exists a continuous map $\Psi : X \to [0, 1]$ such that $\Psi(x) = 0$ and $\Psi(A) \subseteq \{1\}$. Define the function $g : X \to Y$ with $g(z) = \varphi(y, f(z), \Psi(z))$.Then g is continuous function that is required. □

Recall that an *equiconnected topological space* Y is a topological space with the existence of a continuous map $\varphi : Y \times Y \times [0, 1] \to Y$ such that $\varphi(p, p, t) = p$, $\varphi(p, q, 0) = p$ and $\varphi(p, q, 1) = q$ for every $p, q \in Y$ and $t \in [0, 1]$. The map φ is called an *equiconnecting function. A subset* V of an equiconnected space Y is called a *φ-convex subset* of Y provided that $\varphi(V \times V \times [0, 1]) \subseteq V$.

The last proposition leads us to the following corollary.

Corollary 1. *If* Y *is an equiconnected space, then* (α, X, Y) *satisfies* $(E1)$.

If Y is a topological vector space, or a convex subset of any topological vector space, then Y is an equiconnected space. Also, every retract of any equiconnected space is also an equiconnected space. So, we can state the following.

Corollary 2. *If* Y *is a topological vector space, or a convex subset of any topological vector space, or a retract of a convex subset of any topological vector space, then* (α, X, Y) *satisfies* $(E1)$.

Recall that a topological lattice Y is a topological space, $\leqslant$ is a partial order on Y, every two element set $\{p, q\}$ has the supremum $p \vee q$ and the infimum $p \wedge q$, and the lattice operations $\vee$ and $\wedge$ are continuous.

The following is a sufficient condition on the triple (α, X, Y) for it to satisfy $(E2)$.

Proposition 4. *Let* Y *be a pathwise connected topological lattice having no smallest element. If each compact subset of* Y *is bounded above, then* (α, X, Y) *satisfies* $(E2)$.

Proof. Let $y \in Y$. Since Y has no smallest element, there is a $q \in Y$ such that $q \leqslant y$ and $q \neq y$. Continuity of the operation $\vee : Y \times Y \to Y$ leads us to the fact that $K = \{z \in Y: z \leqslant q\}$ is closed in Y.

Let B be any nonempty element of the standard base of $C_\alpha(X, Y)$ and F be a finite subset of X with $F \cap \operatorname{supp}(B) = \emptyset$. Take an $f \in B$. Since $f(\operatorname{supp}(B))$ is a compact subset of Y, the set $f(\operatorname{supp}(B))$ is bounded above. Let p be an upperbound of $f(\operatorname{supp}(B))$. Since Y is a pathwise connected space and X is a Tychonoff space, we can find a continuous function $h : X \to Y$ such that $h(\operatorname{supp}(B)) \subseteq \{p\}$ and $h(F) \subseteq \{q\}$. Define the function $g : X \to Y$ with $g(x) = h(x) \wedge f(x)$ for each $x \in X$. It is clear that g is continuous, and one can easily verify that $g \in [F, K] \cap B$. □

3. Main results

We give a new definition.

Definition 1. Let X be a topological space, $A \subseteq X$, and let $\mathcal{O}$ be a π-base for X at A. A point x in A is called a *π-accumulation point of* $\mathcal{O}$ at A if for each neighbourhood U of x, and for each neighbourhood V of A, there exists an $O \in \mathcal{O}$ such that $O \cap U \neq \emptyset$ and $O \subseteq V$.

The following lemmas are needed in the proof of the next theorem. The proof of the first lemma is trivial.

Lemma 1. *Let X be a topological space, K be a nonempty compact subset of X, and $\mathcal{O}$ be a π-base for X at K. Then there exists a π-accumulation point of $\mathcal{O}$ at K.*

Lemma 2. *Let $\mathcal{T}$ and $\mathcal{S}$ be two topologies on X such that $(X, \mathcal{S})$ is Hausdorff and $\mathcal{S} \subseteq \mathcal{T}$. Let K be a $\mathcal{T}$-compact subset of X, and let $\mathcal{O}$ be a π-base for the space $(X, \mathcal{T})$ at K. If x is a π-accumulation point (respect to $\mathcal{T}$) of $\mathcal{O}$ at K, then the family*

$$\{U \cap O:\ U \in \mathcal{S},\ x \in U,\ O \in \mathcal{O} \text{ and } U \cap O \neq \emptyset\}$$

is a π-base for the space $(X, \mathcal{T})$ at the point x.

Proof. Take any $T \in \mathcal{T}$ with $x \in T$. Since $x \notin K \setminus T$, and the set $K \setminus T$ is $\mathcal{S}$-compact, and the space $(X, \mathcal{S})$ is Hausdorff, there exist $U, V \in \mathcal{S}$ such that $x \in U$, $K \setminus T \subseteq V$ and $U \cap V = \emptyset$. Since $K \subseteq T \cup V$, $T \cup V \in \mathcal{T}$, and x is a π-accumulation point of $\mathcal{O}$ at K, there exists a $O \in \mathcal{O}$ such that $U \cap O \neq \emptyset$ and $O \subseteq T \cup V$. It is clear that $O \cap U \subseteq T$. □

Lemma 3. *Let the triple (α, X, Y) satisfy $(E2)$, and let K be a nonempty compact subset of $C_\alpha(X, Y)$, and let the family $\mathcal{O}$ be a π-base for the space $C_\alpha(X, Y)$ at K where $\mathcal{O}$ is a subfamily of the standard base of the space $C_\alpha(X, Y)$. Let f be a π-accumulation point of $\mathcal{O}$ at K. Then each element A of α can be covered by a finite subfamily of the family*

$$\big\{B \cup \operatorname{supp}(O):\ B \text{ is a finite subset of } X,\ \text{and } O \in \mathcal{O}\big\}.$$

Proof. Take any $A \in \alpha$. Let us choose an open neighbourhood W_y of y for each $y \in f(A)$ by means of the property $(E2)$. Since the set $f(A)$ is compact, then there exists a finite subset F of $f(A)$ such that $f(A) \subseteq \bigcup\{W_y\colon\ y \in F\}$, and there exists a closed subset C_y of Y for each $y \in F$ such that $f(A) \subseteq \bigcup\{C_y\colon\ y \in F\}$ and $C_y \subseteq W_y$ for each $y \in F$. So, we have $f \in [A \cap f^{-1}(C_y), W_y]$ for each $y \in F$. Since f is a π-accumulation point of $\mathcal{O}$ at K, by Lemma 2, there exists a neighbourhood U_y of f in $C_p(X, Y)$, and an $O_y \in \mathcal{O}$ such that $U_y \cap O_y \neq \emptyset$ and $U_y \cap O_y \subseteq [A \cap f^{-1}(C_y), W_y]$ for each $y \in F$. The choice of W_y's gives us $A \cap f^{-1}(C_y) \subseteq \operatorname{supp}(U_y) \cup \operatorname{supp}(O_y)$ for each $y \in F$, and hence we have that $A \subseteq \bigcup\{\operatorname{supp}(U_y) \cup \operatorname{supp}(O_y)\colon\ y \in F\}$. □

Now we are ready to state and prove one of the main theorems in this paper.

Theorem 1. *Let the triple (α, X, Y) satisfy $(E2)$, and let K be a nonempty compact subset of $C_\alpha(X, Y)$. If $\pi\chi(K, C_\alpha(X, Y)) \leqslant \kappa$, then there exists a closed–open subset Z of X, and a subfamily β of α with $|\beta| \leqslant \kappa$ such that for each $A \in \alpha$ the set $A \setminus Z$ is finite, and β is an $\alpha_{|Z}$-cover of Z.*

Proof. Let $\mathcal{O}$ be a π-base for $C_\alpha(X, Y)$ at K such that $|\mathcal{O}| \leqslant \kappa$, and let f be a π-accumulation point of $\mathcal{O}$ at K. There is no loss of generality by assuming that the family $\mathcal{O}$ is a subfamily of the standard base of the space $C_\alpha(X, Y)$. Let

$$Z = \bigcup\bigl\{\operatorname{supp}(O)\colon\ O \in \mathcal{O}\bigr\}$$

and let β be the family of all finite unions of elements of the family $\{\operatorname{supp}(O)\colon\ O \in \mathcal{O}\}$. It is clear that $|\beta| \leqslant \kappa$, and since α is closed under finite unions, $\beta \subseteq \alpha$.

First, we prove that the set Z is a closed–open subset of X by showing $\overline{Z} \cap \overline{X \setminus Z} = \emptyset$. Let $x \in X$. Let $\mathcal{U}$ be the subfamily of the standard base of $C_p(X, Y)$ such that $\operatorname{supp}(U) \subseteq Z$ and $f \in U$ for each $U \in \mathcal{U}$, and let $\mathcal{F}$ be the family of finite subsets of X which are not meeting with Z. Define

$$A(U) = \bigcup\{O \in \mathcal{O}\colon\ O \cap U \neq \emptyset\}$$

for each $U \in \mathcal{U}$. Since f is a π-accumulation point of $\mathcal{O}$ at K, we have $A(U) \neq \emptyset$ for each $U \in \mathcal{U}$. The property $(E2)$ gives us an open subset W of Y such that $f(x) \in W$ and $[F, Y \setminus W] \cap O \cap U \neq \emptyset$ for each $F \in \mathcal{F}$, $U \in \mathcal{U}$, and $O \in \mathcal{O}$ with $O \cap U \neq \emptyset$. Then define

$$S(F, U) = \overline{[F, Y \setminus W] \cap A(U) \cap U}$$

for each $F \in \mathcal{F}$ and $U \in \mathcal{U}$. Since $[F, Y \setminus W] \cap O \cap U \neq \emptyset$ for each $F \in \mathcal{F}$, $U \in \mathcal{U}$, and $O \in \mathcal{O}$ with $O \cap U \neq \emptyset$, and since f is a π-accumulation point of $\mathcal{O}$ at K, we have that $K \cap S(F, U) \neq \emptyset$ for each $F \in \mathcal{F}$ and $U \in \mathcal{U}$. We observe that the family $\{K \cap S(F, U)\colon\ F \in \mathcal{F},\ U \in \mathcal{U}\}$ of closed subset of K has the finite intersection property. So, there exists a $g \in C(X, Y)$ such that $g \in S(F, U)$ for each $F \in \mathcal{F}$ and $U \in \mathcal{U}$. One can easily prove that $g(X \setminus Z) \cap W = \emptyset$ and $g_{|Z} = f_{|Z}$. Continuity of g, Hausdorffness of Y and $f(x) \in W$ lead us to the fact that $x \notin \overline{Z} \cap \overline{X \setminus Z}$, hence the result.

Now, we prove that the family β is an $\alpha_{|Z}$-cover of Z. Take any $A \in \alpha$. From Lemma 3, there exist $F_1, F_2, \ldots, F_n$ finite subsets of X, and $O_1, O_2, \ldots, O_n \in \mathcal{O}$ such that

$$A \subseteq \bigcup\bigl\{F_i \cup \operatorname{supp}(O_i)\colon\ 1 \leqslant i \leqslant n\bigr\}.$$

Since the set F_i is finite for each i, the set $F_i \cap Z$ can be covered by the supports of finitely many elements of $\mathcal{O}$ for each i with $1 \leqslant i \leqslant n$. Hence the set $A \cap Z$ is contained in some member of β, and so, the family β is an $\alpha_{|Z}$-cover of Z.

Since $\mathrm{supp}(O_i)$ is a subset of Z for each i, we have

$$A \setminus Z \subseteq \bigcup \{F_i \setminus Z\colon\ 1 \leqslant i \leqslant n\}.$$

Therefore $A \setminus Z$ is finite. □

Let Z and K be as in the above theorem. We note that since the set Z is a closed–open subset of X, the mapping

$$\pi : C_\alpha(X, Y) \to C_{\alpha_{|Z}}(Z, Y) \times C_{\alpha_{|X\setminus Z}}(X \setminus Z, Y)$$

defined by $\pi(f) = (f_{|Z}, f_{|X\setminus Z})$ is a homeomorphism. In the above theorem, we have seen that if $A \in \alpha$ with $A \subseteq X \setminus Z$, then A is finite. Hence the space $C_\alpha(X, Y)$ and the product space $C_{\alpha_{|Z}}(Z, Y) \times C_p(X \setminus Z, Y)$ are homeomorphic with the map π. Let $\widetilde{\pi} : C_\alpha(X, Y) \to C_p(X \setminus Z, Y)$ be the projection map with $\widetilde{\pi}(f) = f_{|X\setminus Z}$ for each $f \in C(X, Y)$. One can easily show that $\widetilde{\pi}(K) = C(X \setminus Z, Y)$. Since K is compact and the map $\widetilde{\pi}$ is continuous, the space $C_p(X \setminus Z, Y)$ is compact. If $X \setminus Z \neq \emptyset$, then the space Y closedly embeddable in $C_p(X \setminus Z, Y)$ [4]. So, if Y is not a compact space, then $X \setminus Z = \emptyset$. Thus we can say

Corollary 3. *Let the triple (α, X, Y) satisfy $(E2)$, and let K be a nonempty compact subset of $C_\alpha(X, Y)$. If Y is not a compact space, then $\alpha a(X) \leqslant \pi\chi(K, C_\alpha(X, Y))$.*

If X is a connected space, then we have either $Z = X$ or $Z = \emptyset$ for the set Z. If $Z = \emptyset$, then we have $K = C_\alpha(X, Y)$ for the compact subset K of $C_\alpha(X, Y)$. Therefore we have

Corollary 4. *Let the triple (α, X, Y) satisfy $(E2)$, and let K be a nonempty compact proper subset of $C_\alpha(X, Y)$. If X is a connected space, then*
$\alpha a(X) \leqslant \pi\chi(K, C_\alpha(X, Y))$.

For the set K in Theorem 1, if $K = \{f\}$, where $f \in C(X, Y)$, then we have $Z = X$, and hence we can state the following.

Corollary 5. *For each (α, X, Y) satisfying $(E2)$, and for each $f \in C(X, Y)$, we have that $\alpha a(X) \leqslant \pi\chi(f, C_\alpha(X, Y))$.*

We note that, when Y in the above corollary is $\mathbb{R}$, the α-Arens number of X, the character of $C_\alpha(X, \mathbb{R})$ and the π-character of $C_\alpha(X, \mathbb{R})$ are the same [4, Theorem 4.4.1].

It is known that $\alpha a(X) = |X|$, if α consists of all finite subsets of X [4]. So, by Corollary 5, if the triple (α, X, Y) satisfies $(E2)$ where α consists of all finite subsets of X, then $|X| \leqslant \pi\chi(f, C_p(X, Y))$ for each f in $C(X, Y)$.

Lemma 4. *Let Y be an equiconnected space with equiconnecting function φ. If Y has a base $\mathcal{B}$ consisting of φ-convex subsets of Y, then the family*

$$\sigma = \left\{ \bigcap_{i=1}^{n} [A_i, V_i] \colon n \in \omega,\ A_i \in \alpha,\ V_i \in \mathcal{B} \right\}$$

is a base for the space $C_\alpha(X, Y)$, and σ satisfies the following property:

> *"if $\bigcap_{i=1}^{n}[A_i, V_i] \in \sigma$, $f \in \bigcap_{i=1}^{n}[A_i, V_i]$, $A \in \alpha$, V is an open proper subset of Y and $\bigcap_{i=1}^{n}[A_i, V_i] \subseteq [A, V]$, then there exist some unions of some intersections of elements of the family $\{V_i\colon i = 1, 2, \dots, n\}$ such that the set is a subset of V and contains $f(A)$."*

Proof. Let $\varphi : Y \times Y \times [0, 1] \to Y$ be the equiconnecting function. It is clear that the family σ is a base for the space $C_\alpha(X, Y)$. Let $\bigcap_{i=1}^{n}[A_i, V_i] \in \sigma$, $f \in \bigcap_{i=1}^{n}[A_i, V_i]$, $A \in \alpha$, and let V be an open proper subset of Y such that $\bigcap_{i=1}^{n}[A_i, V_i] \subseteq [A, V]$. By Corollary 1, (α, X, Y) satisfies $(E1)$. The property $(E1)$ leads us to the fact that $A \subseteq \bigcup_{i=1}^{n} A_i$.

The family of all subsets having k elements of the set $\{1, 2, \dots, n\}$ be denoted by $\{1, 2, \dots, n\}^k$ where k is an integer with $1 \leqslant k \leqslant n$. For each k with $1 \leqslant k \leqslant n$, and for each $a \in \{1, 2, \dots, n\}^k$ define $A(a) = \bigcap_{i\in a} A_i$, and define a subset L_k of $\{1, 2, \dots, n\}^k$ by the rule:

$$a \in L_k \iff \big(A \cap A(a)\big) \setminus \bigcup \{A_j\colon 1 \leqslant j \leqslant n \text{ and } j \notin a\} \neq \emptyset.$$

We claim that $V(a) \subseteq V$ for each $a \in L_k$ with $1 \leqslant k \leqslant n$. To prove this, assume the contrary. Let $y \in V(a)\setminus V$ for some $a \in L_k$ with $1 \leqslant k \leqslant n$. Since $a \in L_k$, there exists an $x \in X$ such that $x \in A \cap A(a)$ and $x \notin A_j$ for each $j \in \{1, 2, \dots, n\}\setminus a$. Since f is continuous and the space X is regular, there exists an open neighbourhood W of x such that

$$f(W) \subseteq V(a) \quad \text{and} \quad W \cap \bigcup \{A_j\colon 1 \leqslant j \leqslant n \text{ and } j \notin a\} = \emptyset.$$

Since X is a Tychonoff space, we have a continuous function $\eta : X \to [0, 1]$ such that $\eta(x) = 0$ and $\eta(X\setminus W) \subseteq \{1\}$. Then the function $h : X \to Y$ defined by $h(z) = \varphi(y, f(z), \eta(z))$ for each $z \in X$, is continuous. Since $\eta(x) = 0$, we have $h(x) = y$. We shall now show that $h \in \bigcap_{i=1}^{n}[A_i, V_i]$. Take any i with $1 \leqslant i \leqslant n$, and take a $z \in A_i$. We have that either $i \in a$ or $i \notin a$. If $i \in a$; φ-convexity of V_i leads us to the fact that $h(z) \in V_i$. If $i \notin a$, then we have $A_i \cap W = \emptyset$. So $h(z) = f(z)$, and hence $h(z) \in V_i$. Therefore $h \in \bigcap_{i=1}^{n}[A_i, V_i]$. But, since $x \in A$, $h(x) = y$ and $y \notin V$, we have $h \notin [A, V]$. This gives a contradiction, so we obtain that

$$\bigcup_{k=1}^{n} \bigcup_{a\in L_k} V(a) \subseteq V.$$

Since $A \subseteq \bigcup_{i=1}^{n} A_i$, we have

$$A \subseteq \bigcup_{k=1}^{n} \bigcup_{a\in L_k} A(a).$$

We also have $f(A(a)) \subseteq V(a)$ for each k with $1 \leqslant k \leqslant n$ and $a \in L_k$. Hence we obtain that

$$f(A) \subseteq \bigcup_{k=1}^{n} \bigcup_{a\in L_k} V(a) \subseteq V. \qquad \square$$

Theorem 2. *If Y is an equiconnected space having a base consisting of φ-convex sets, then $w_e(f(X)) \leqslant \chi(f, C_\alpha(X,Y))$ for each $f \in C(X,Y)$.*

Proof. Let f be any element of $C(X,Y)$, and let σ be a base for $C_\alpha(X,Y)$ as in Lemma 4. Let $\mathcal{O} = \{\bigcap_{i=1}^{n_\lambda}[A_i^\lambda, V_i^\lambda]\colon\ \lambda \in I\}$ be a local base at the point f such that $|I| \leqslant \chi(f, C_\alpha(X,Y))$ and $\mathcal{O} \subseteq \sigma$. Let the family $\mathcal{V}$ be the finite unions of the finite intersections of elements of the family $\{V_i^\lambda\colon\ \lambda \in I,\ i = 1,2,\ldots,n_\lambda\}$. We claim that the family $\mathcal{V}$ is an external base for the subspace $f(X)$ of the space Y. To justify our claim, let $y \in f(X)$, and let U be an open neighbourhood of y in Y. Without loss of generality, we can assume that $U \neq Y$. Let $y = f(x)$ for some $x \in X$. Since $f \in [x,U]$ and $[x,U]$ is open in $C_\alpha(X,Y)$, there exists a $\lambda \in I$, such that $f \in \bigcap_{i=1}^{n_\lambda}[A_i^\lambda, V_i^\lambda] \subseteq [x,U]$. From Lemma 4, there exists a $V \in \mathcal{V}$ such that $f(x) \in V \subseteq U$. Hence $\mathcal{V}$ is an external base for $f(X)$. It follows that $w_e(f(X)) \leqslant |\mathcal{V}| \leqslant |I| \leqslant \chi(f, C_\alpha(X,Y))$. $\square$

Theorem 3. *For each X, Y, α and $f \in C(X,Y)$, $\chi(f, C_\alpha(X,Y)) \leqslant \alpha a(X) \,.\, w_e(f(X))$.*

Proof. Let $\alpha a(X) \,.\, w_e(f(X)) = \kappa$. Let $\beta \subseteq \alpha$ be an α-cover of X such that $|\beta| \leqslant \kappa$, and let $\mathcal{V}$ be an external base for the subspace $f(X)$ of the space Y with $|\mathcal{V}| \leqslant \kappa$. Let $\widetilde{\mathcal{V}}$ be the family of all finite unions of the family $\mathcal{V}$, and let $\sigma = \{(V,U) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}}\colon\ \overline{U} \subseteq V\}$, and let $\mathcal{F}$ be the family of all finite intersections of elements of the family $\{[f^{-1}(\overline{U}) \cap A, V]\colon\ (V,U) \in \sigma,\ A \in \beta\}$. It is easily seen that the family $\mathcal{F}$ is a local base at f in $C_\alpha(X,Y)$ and $|\mathcal{F}| \leqslant \kappa$. $\square$

For a dense subset D of a regular space, it is known that $w_e(D) = w(D)$ [2]. By this equality and the above theorem, the following is immediate;

Corollary 6. *For each X, Y, α, and for each almost onto f in $C_\alpha(X,Y)$, we have $\chi(f, C_\alpha(X,Y)) \leqslant \alpha a(X) \,.\, w(f(X))$.*

The following theorem gives a characterization of the character of the space $C_\alpha(X,Y)$ at a point f. Corollaries 1, 5 and Theorems 2, 3 give us the following.

Theorem 4. *If Y is an equiconnected space having a base consisting of φ-convex sets, then for each $f \in C(X,Y)$, $\chi(f, C_\alpha(X,Y)) = \alpha a(X) \,.\, w_e(f(X))$.*

The following theorem is a generalization of [4, Theorem 4.4.2]. Let μ be a compatible uniformity on Y. The function space $C(X,Y)$ having the topology of uniform convergence on α with respect to μ is denoted by $C_{\alpha,\mu}(X,Y)$ as in [4]. $uw(Y)$, the *uniform weight* of an uniform space Y, is the smallest infinite cardinality of an uniformity base of Y.

Theorem 5. *Let Y be an uniform space, and let the triple (α, X, Y) satisfy $(E2)$. If $uw(Y) \leqslant \kappa$, then the following are equivalent*:

(i) $\chi(C_\alpha(X,Y)) \leqslant \kappa$.
(ii) $\pi\chi(C_\alpha(X,Y)) \leqslant \kappa$.
(iii) *There exists a $f \in C(X,Y)$ such that $\pi\chi(f, C_\alpha(X,Y)) \leqslant \kappa$.*
(iv) $\alpha a(X) \leqslant \kappa$.
(v) $uw(C_{\alpha,\mu}(X,Y)) \leqslant \kappa$.

In addition, if Y is not a compact space,

(vi) $h(C_\alpha(X,Y)) \leqslant \kappa$.
(vii) *There exists a compact subset K of $C_\alpha(X,Y)$ with $\pi\chi(K, C_\alpha(X,Y)) \leqslant \kappa$.*

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (vii). The implications (iii) $\Rightarrow$ (iv) and (vii) $\Rightarrow$ (iv) follow from Corollary 5 and Corollary 3, respectively. It is easy to see that $uw(C_{\alpha,\mu}(X,Y)) \leqslant \alpha a(X) \,.\, uw(Y)$. This inequality gives us (iv) $\Rightarrow$ (v). Since α is a hereditarily closed, compact network on X, we have $C_\alpha(X,Y) = C_{\alpha,\mu}(X,Y)$ [4]. Moreover, the character of a topological space whose topology is induced by a uniformity is less than or equal to the uniform weight of it. These give us the implication (v) $\Rightarrow$ (i). □

We recall that a space X is called a *q-space*, if for each point x of X, there exists a sequence $\{U_n\colon n \in \omega\}$ of neighbourhoods of x such that each sequence $\{x_n\colon n \in \omega\}$ with $x_n \in U_n$ for each $n \in \omega$ has a cluster point in X.

It is well known that, for a topological space X, $uw(X) \leqslant \aleph_0$ if and only if X is metrizable space, and $h(X) \leqslant \aleph_0$ if and only if X is of pointwise countable type [3]. So, we have the following.

Corollary 7. *Let Y be a noncompact metric space, and let the triple (α, X, Y) satisfy $(E1)$. The following are equivalent*:

(i) *$C_\alpha(X,Y)$ is a first-countable space.*
(ii) *π-character of the space $C_\alpha(X,Y)$ is countable.*
(iii) *$C_\alpha(X,Y)$ is of pointwise countable type.*
(iv) *There exists a compact subset K of $C_\alpha(X,Y)$ such that π-character of K in the space $C_\alpha(X,Y)$ is countable.*
(v) $\alpha a(X) \leqslant \aleph_0$.
(vi) *$C_\alpha(X,Y)$ is metrizable.*
(vii) *$C_\alpha(X,Y)$ is a q-space.*
(viii) *There exists a sequence $\{O_n\colon n \in \omega\}$ of nonempty open subset of $C_\alpha(X,Y)$ such that each sequence $\{g_n\colon n \in \omega\}$ with $g_n \in O_n$ for each $n \in \omega$ has a cluster point in $C_\alpha(X,Y)$.*

Proof. From Theorem 5, statements from (i) to (vi) are equivalent. It is clear that (vii) $\Rightarrow$ (viii) and (i) $\Rightarrow$ (vii). There is need to prove the implication (viii) $\Rightarrow$ (v). Let $\{O_n\colon n \in \omega\}$

be a sequence as in (viii). Without loss of generality we can assume that, for each $n \in \omega$, O_n is of the form $\bigcap_{i=1}^{k(n)}[A_i^{(n)}, V_i^{(n)}]$ where $k(n) \in \omega$, and $A_i^{(n)} \in \alpha$ and $V_i^{(n)}$ is an open subset of Y for each i with $1 \leqslant i \leqslant k(n)$.

Let

$$\beta = \{\text{supp}(O_n)\colon n \in \omega\}.$$

Since α is closed under finite unions, β is a subfamily of α. We claim that β is an α-cover of X. Assume contrary. Let $A \in \alpha$ such that A is not a subset of $\text{supp}(O_n)$ for each $n \in \omega$. Choose a $x_n \in A \setminus \text{supp}(O_n)$ for each $n \in \omega$. Since Y is not a compact space, there exists a sequence $\{y_n\colon n \in \omega\}$ in Y such that $\{y_n\colon n \in \omega\}$ has no cluster point in Y. Since the triple (α, X, Y) satisfy $(E1)$, there exists an $h_n \in C(X, Y)$ such that $h_n(x_n) = y_n$ and $h_n \in O_n$ for each $n \in \omega$. From the hypothesis, the sequence $\{h_n\colon n \in \omega\}$ has a cluster point h in $C_\alpha(X, Y)$. Let $L = \overline{\{x_n\colon n \in \omega\}}$. Since $\{x_n\colon n \in \omega\} \subseteq A$, $A \in \alpha$ and h is a continuous map, $h(L)$ is a compact subset of Y. So the set $\{y_n\colon n \in \omega\} \cap h(L)$ has to be finite. Then there exists a $k \in \omega$ such that $\{y_n\colon n > k\} \cap h(L) = \emptyset$. Let $V = Y \setminus \{y_n\colon n > k\}$. Since the set $[L, V]$ is an open neighbourhood of h in $C_\alpha(X, Y)$, there exists a $m \in \omega$ with $m > k$ and $h_m \in [L, V]$. Since $h_m(x_m) = y_m$, then we have $y_m \in V$. But this contradicts with the fact that $m > k$. This contradiction shows us that β is an α-cover of X, and hence $a\alpha(X) \leqslant \aleph_0$. □

Acknowledgement

The authors acknowledge Mr. Hasan Gül for the first draft of manuscript.

References

[1] A.V. Arkhangel'skii, Topological Function Spaces, Kluwer Academic, Dordrecth, 1992.
[2] A.V. Arkhangel'skii, V.I. Ponomarev, Fundamentals of General Topology, Reidel, Dordrecth, 1983.
[3] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[4] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Math., vol. 1315, Springer, Berlin, 1988.