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Note

A note on path kernels and partitions

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ABSTRACT

The detour order of a graph G, denoted by $\tau(G)$, is the order of a longest path in G. A subset S of $V(G)$ is called a P_n-kernel of G if $\tau(G[S]) \leq n-1$ and every vertex $v \in V(G) - S$ is adjacent to an end-vertex of a path of order $n-1$ in $G[S]$. A partition of the vertex set of G into two sets, A and B, such that $\tau(G[A]) \leq a$ and $\tau(G[B]) \leq b$ is called an (a, b)-partition of G. In this paper we show that any graph with girth g has a P_{n+1}-kernel for every $n < \frac{3g}{2} - 1$. Furthermore, if $\tau(G) = a + b$, $1 \leq a \leq b$, and G has girth greater than $\frac{2}{3}(a+1)$, then G has an (a, b)-partition.

1. Introduction

Let $G = (V, E)$ be a finite simple graph. The *vertex set* and *edge set* of the graph G are denoted by $V(G)$ and $E(G)$, respectively. If H is a subgraph of G and v is a vertex, the *open H-neighbourhood of* v is the set $N_H(v) = \{u \in V(H) | uv \in E(G)\}$. If S is a subset of $V(G)$, we write $N_H(S) = \cup_{v \in S} N_H(v) - S$, and $v \in S$. Also, we let $N(S) = N_G(S)$. The subgraph of G induced by S is denoted by $G[S]$.

Following Kapoor et al. [5], we call a longest path in a graph a *detour* of the graph. The number of vertices in a detour of G is called the *detour order* of G and is denoted by $\tau(G)$. The cycle of order n and the path of order n are denoted by C_n and P_n respectively. The number of vertices in a shortest cycle of G is called the *girth* of G and denoted by $g(G)$. We shall call a vertex $v \in V(G)$ a *P_n-terminal vertex of* G if v is an end-vertex of a P_n but not of a P_{n+1} in G. A class $\mathcal{P}$ of graphs is said to be a hereditary (an induced hereditary) class of graphs if every subgraph (induced subgraph) of a graph in $\mathcal{P}$ is also in $\mathcal{P}$.

The distance between two vertices u and v in a connected graph G is denoted by $d_G(u, v)$. If $u \in S \subseteq V(G)$ and $v \in B = V(G) - S$, then $d_B(u, v)$ denotes the length of a shortest u–v path with all its internal vertices in B, if such a path exists. If not, we put $d_B(u, v) = \infty$.

A partition of the vertex set of G into two sets, A and B, such that $\tau(G[A]) \leq a$ and $\tau(G[B]) \leq b$ is called an *(a, b)-partition* of G. If G has an (a, b)-partition for every pair (a, b) of positive integers such that $a + b = \tau(G)$, then we say that G is *τ-partitionable* [2]. The following conjecture is known as the *Path Partition Conjecture* (or the PPC, for short).

Conjecture 1. *Every graph is τ-partitionable.*

A summary of the PPC status is given in [3].

A set K of vertices of a graph G is called a *P_n-kernel* of G if $\tau(G[K]) \leq n-1$ and every vertex in $G - K$ is adjacent to a P_{n-1}-terminal vertex of $G[K]$.

It was conjectured that every graph has a P_n-kernel for every integer $n \geq 2$ (see [4,9]), but Aldred and Thomassen [1] disproved the conjecture by presenting a graph G with $\tau(G) = 364$ and containing no P_{364}-kernel. Recently, Katrenič and

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Semanišin constructed a graph without a P_{155}-kernel in [6]. They also constructed, for each $l \geq 0$, a graph with no $P_{\tau(G)-l}$-kernel. However, in order to prove the PPC it would suffice to prove the following conjecture, for which no counterexample is known.

Conjecture 2. *Every graph has a P_{n+1}-kernel for every $n \leq \frac{\tau(G)}{2}$.*

It therefore remains interesting to examine what conditions can guarantee the existence of a P_n-kernel in a graph.

It is proved in [4,8,7] that every graph has a P_n-kernel for every $n \leq 9$.

We shall call a subset S of $V(G)$ a P_n-*semikernel* of G if $\tau(G[S]) \leq n-1$ and every vertex in $N(S)$ is adjacent to a P_{n-1}-terminal vertex of $G[S]$.

Obviously, any P_n-kernel of a graph is a P_n-semikernel of the graph, but the converse does not hold. However in [4] the following result is proved.

Lemma 1.1 *([4]). Let $\mathcal{P}$ be a hereditary class of graphs and let $n \geq 2$ be an integer. If every graph in $\mathcal{P}$ has a P_n-semikernel, then every graph in $\mathcal{P}$ has a P_n-kernel.*

The following conjecture is equivalent to Conjecture 2 and hence its truth would imply the truth of Conjecture 1.

Conjecture 3. *Every graph has a P_{n+1}-semikernel for every $n \leq \frac{\tau(G)}{2}$.*

Dunbar and Frick [4] showed that a graph with girth g has a P_{n+1}-kernel for every $n \leq g+1$ and pointed out the importance of the result in connection with the PPC. In this paper, we extend their result by showing that every graph with girth g has a P_{n+1}-kernel for every $n < \frac{3g}{2} - 1$.

2. Main results

In order to prove our main theorem we need the following lemma.

Lemma 2.1. *Suppose G is a graph with girth g. Let C be a g-cycle in G and $B = V(G) - V(C)$. If $x \in V(C)$ and $u \in B$ such that $d_B(u, x) < \frac{g}{2} - 1$, then u is not adjacent to any vertex in $C - x$.*

Proof. Suppose, to the contrary, that v in $C - x$ is adjacent to u. Let P be the shortest path in B from u to x, and Q be the shortest path in C from x to v; the length of Q is no more than $\frac{g}{2}$. Therefore the length of cycle $xQvuPx$ is less than $\frac{g}{2} - 1 + 1 + \frac{g}{2} = g$. This contradiction completes the proof of the lemma. □

Theorem 2.2. *If G is a graph with girth g, then G has a P_{n+1}-semikernel for every $n < \frac{3g}{2} - 1$.*

Proof. We may assume that G is connected. If $n \leq g+1$, then G has a P_{n+1}-kernel by Theorem 4.5 of [4], and if $n \geq \tau(G)$ then $V(G)$ is a P_{n+1}-kernel of G. Hence we may assume that $g < n < \tau(G)$. Let $C := v_1v_2v_3 \cdots v_gv_1$ be a g-cycle in G. Initially, we put $t := 0$, $S := V(C)$, $B = V(G) - S$ and $A = \emptyset$. Then we move vertices from B to S and to A according to the following steps.

STEP 1: Put $t := t + 1$.

Let λ_t be the order of a longest path in S with v_t as end-vertex;

$M_t := \{u \in B | d_B(u, v_t) \leq n - \lambda_t\}$;

$S := S \cup M_t$;

$B := B - M_t$.

We note the following:

Since $n - \lambda_t \leq n - g < \frac{g}{2} - 1$, it is clear that $G[M_t \cup \{v_t\}]$ is a tree and by Lemma 2.1, no vertex in M_t is adjacent to any vertex in $C - v_t$. Thus no new cycle has been created in S and hence $\tau(G[S]) \leq n$.

By the definition of M_t, every vertex in M_t is either a P_n-terminal vertex of $G[S]$ or has no neighbour in B (because any such neighbour would have now been moved from B to S as well).

STEP 2: Move all the B-neighbours of P_n-terminal vertices of $G[S]$ to A.

Now no vertex in M_t has any neighbour in B.

STEP 3: If $t < g$, then return to STEP 1. Otherwise stop.

Finally, $N(S) \cap B = \emptyset$, every vertex in A is adjacent to a P_n-terminal vertex of $G[S]$ and $\tau(G[S]) = n$. Thus S is a P_{n+1}-semikernel of G. □

The class of graphs with girth equal to $\frac{3g}{2} - 1$ is not a hereditary class; however, graphs with less than $\frac{3g}{2} - 1$ do form a hereditary class.

Since having girth greater than $\frac{2n}{3}$ is a hereditary property, Lemma 1.1 together with Theorem 2.2 implies the following.

Theorem 2.3. *If G is a graph with girth $g > \frac{2}{3}(n+1)$, then G has a P_{n+1}-kernel.*

Theorem 2.3 has improved a result given in [4].

Let G be a graph with $\tau(G) = a + b$, where a and b are positive integers. If G has a P_{a+1}-kernel or a P_{b+1}-kernel, then G is (a, b)-partitionable. So we further obtain:

Corollary 2.4. *Let G be a graph with girth g and suppose $\tau(G) = a + b$, with $1 \leq a \leq b$. If $g > \frac{2}{3}(a + 1)$, then G is (a, b)-partitionable.*

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