Zygmund’s inequality for entire functions of exponential type

Milutin R. Dostanić

Matematički Fakultet, Studentski trg 16, 11000 Belgrade, Serbia

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Abstract

Let \( f \in L^p(\mathbb{R}) \) be an entire function of exponential type \( \sigma \) whose conjugate indicator diagram lies on interval \([0, \sigma i]\). We prove that

\[
\int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq c_p \sigma^p \int_{-\infty}^{\infty} |\Re f(x)|^p \, dx \quad (p \geq 1)
\]

where \( c_p := \sqrt{\pi} \Gamma(1+\frac{p}{2}) \Gamma(\frac{1}{2}) \) is a best possible constant.

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1. Introduction

In paper [9] A. Zygmund proved the following result:

**Theorem 1.** If \( P \) is an algebraic polynomial of degree \( n \) and \( p \geq 1 \) then

\[
\int_{-\pi}^{\pi} \left| P'(e^{i\theta}) \right|^p d\theta \leq c_p n^p \int_{-\pi}^{\pi} \left| \Re P(e^{i\theta}) \right|^p d\theta
\]  

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E-mail address: domi@matf.bg.ac.yu.

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where \( c_p = \sqrt{\pi} \frac{\Gamma(1 + \frac{p}{2})}{\Gamma\left(\frac{1+p}{2}\right)} \) (\( \Gamma(\cdot) \) is Euler’s Gamma function) and the constant \( c_p \) is the best possible. Here \( \Re z = \frac{1}{2} (z + \overline{z}) \).

Let \( f \) be an entire function of exponential type.

The indicator function of \( f \) is defined as

\[
 h_f(\theta) = \lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r}.
\]

In [4], T. Genčev proved the following result.

**Theorem 2.** Let \( f \) be an entire function of exponential type \( \sigma \) which is bounded on the real line.

If \( h_f \left( \frac{\pi}{2} \right) \leq 0 \) and \( \sup_{\mathbb{R}} |\Re f(x)| = 1 \) then

\[
 |f'(x)| \leq \sigma \quad \text{for every } x \in \mathbb{R}.
\]

The previous inequality can be written in the following way:

\[
 \sup_{\mathbb{R}} |f'(x)| \leq \sigma \sup_{\mathbb{R}} |\Re f(x)| \quad \text{(i.e. } \|f'\|_{\infty} \leq \sigma \|\Re f\|_{\infty}) .
\]

(2)

(The equality in (2) is attained for \( f(z) = e^{i\sigma z} \).

In the proof of Theorem 2, Genčev used the Levitan–Hörmander Theorem [1,5] on the approximation of entire functions by trigonometric polynomials. This Theorem was also used in [3,8] to establish some results regarding entire functions of exponential type.

Having on mind (1) and (2) and the fact that \( \lim_{p \to \infty} \frac{1}{p} = 1 \) it is natural to ask whether

\[
 \int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq c_p \sigma^p \int_{-\infty}^{\infty} |\Re f(x)|^p \, dx \quad (1 \leq p < \infty)
\]

for a suitable class of entire functions.

In this paper we extend Zygmund’s result to a class of entire functions of exponential type and establish optimality of the constant \( c_p \) for this class.

For more on terminology regarding entire functions (indicator and conjugate indicator diagram) see [2] (p. 70–75) or [6] (p. 100–114).

**2. Main result**

**Theorem 3.** Let \( f \) be an entire function of exponential type \( \sigma \) such that

\[
 \int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty \quad (p \geq 1)
\]

and let its conjugate indicator diagram belong to interval \([0, \sigma i] \). Then

\[
 \int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq c_p \sigma^p \int_{-\infty}^{\infty} |\Re f(x)|^p \, dx \quad (p \geq 1)
\]

where \( c_p = \sqrt{\pi} \frac{\Gamma(1 + \frac{p}{2})}{\Gamma\left(\frac{1+p}{2}\right)} \) and the constant \( c_p \) is the best possible.

**Remark 1.** The condition that the conjugate indicator diagram of the function \( f \) belongs to the interval \([0, \sigma i] \) can be changed (under the assumptions of Theorem 3) to the equivalent condition \( h_f \left( \frac{\pi}{2} \right) \leq 0 \). (This follows from [2] Th. 5.3.7, p. 74.)
To prove Theorem 3 we need several Lemmas.

**Lemma 1** ([7, p. 21, Theorem 1.9]). Let \((f_n)_{n=1}^\infty\) be a sequence of measurable functions on \(\mathbb{R}\) such that \(f_n \to f\) (\(n \to \infty\)) almost everywhere, and let there exist \(C < \infty\) and \(p > 0\) such that \(\int_{\mathbb{R}} |f_n(x)|^p \, d\mu \leq C\) for any \(n \in \mathbb{N}\). Then

\[
\lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p \, dx = 0.
\]

(A corollary of this Lemma is the following equality:

\[
\int_{\mathbb{R}} |f_n(x)|^p \, dx = \int_{\mathbb{R}} |f(x)|^p \, dx + \int_{\mathbb{R}} |f_n(x) - f(x)|^p \, dx + o(1), \ n \to \infty).
\]

For the remainder of this paper, we deal with entire functions of exponential type \(\tau\) which belong to \(L^p\) space on the real axis for some \(p \geq 1\).

So, let \(g\) be such a function; so we know that \(\int_{-\infty}^{\infty} |g(x)|^p \, dx < \infty \ (p \geq 1)\).

It is well known [2] that such a function is bounded on \(\mathbb{R}\).

Let

\[
\varphi(x) = \left( \frac{\sin \pi x}{\pi x} \right)^2
\]

and

\[
g_h(x) = \sum_{\nu=-\infty}^{\infty} \varphi(v + hx) \ g \left( x + \frac{\nu}{h} \right), \ h > 0.
\]

The properties of \(g_h\) that we need are summed up in the following Lemma.

**Lemma 2.**

1. \(g_h\) may be represented as

\[
g_h(z) = \sum_{\nu=-N}^{N} a_\nu e^{2\pi i \nu h z}, \quad z \in \mathbb{C}
\]

with \(N = \left[ \frac{\pi}{2\pi h} \right] + 1\) and complex coefficients \(a_\nu\) \((\nu = -N, -N + 1, \ldots, N - 1, N)\).

2. \(|g_h(x)| \leq \max_{x \in \mathbb{R}} |g(x)|\).

3. \(\lim_{h \to 0^+} g_h(z) = g(z)\) uniformly on all compact subsets of \(\mathbb{C}\).

4. \(\int_{\frac{\pi}{2\pi h}}^{\frac{\pi}{4\pi h}} |g_h(x)|^p \, dx \leq \int_{-\infty}^{\infty} |g(x)|^p \, dx\).

The trigonometric polynomials \(g_h\) are called Levitan polynomials. Proofs of properties 1, 2, and 3 can be found in [1] or in [5] and the proof of 4 in [8].

Let \(h = h_n = \frac{\tau}{2\pi n} \ (n = 1, 2, 3, \ldots)\). Then properties 1, 3, and 4 from Lemma 2 can be written as:

(A) \(g_{h_n}(z) = \sum_{\nu=-N}^{N} a_\nu e^{\frac{i\nu z}{h_n}}, \ N = n + 1\).

(B) \(\lim_{h \to 0^+} g_{h_n}(z) = g(z)\) uniformly on all compact subsets of \(\mathbb{R}\).

(C) \(\int_{I_n} |g_{h_n}(x)|^p \, dx \leq \int_{-\infty}^{\infty} |g(x)|^p \, dx\) where \(I_n = \left[ \frac{-n\pi}{\tau}, \frac{n\pi}{\tau} \right]\).

**Lemma 3.** Let \(g\) be an entire function of exponential type \(\tau\) such that

\[
\int_{-\infty}^{\infty} |g(x)|^p \, dx < \infty \quad (p \geq 1)
\]
Then
\[
\lim_{n \to \infty} \int_{I_n} |\text{Re} \left( e^{i(n+1)2\pi \frac{h}{n} t} \cdot g_h(t) \right) |^p \, dt = \int_{-\infty}^{\infty} |\text{Re} \left( e^{i\tau t} \cdot g(t) \right) |^p \, dt. \tag{4}
\]

**Proof.** Let \( \mathcal{X}_{I_n} \) be the characteristic function of interval \( I_n \). From (B) and (C), according to Fatou’s Lemma, it follows that
\[
\int_{-\infty}^{\infty} |g(x)|^p \, dx \geq \lim_{n \to \infty} \int_{-\infty}^{\infty} \mathcal{X}_{I_n}(t) |g_h(t)|^p \, dt \geq \int_{-\infty}^{\infty} \lim_{n \to \infty} \mathcal{X}_{I_n}(t) |g_h(t)|^p \, dt
\]
\[
= \int_{-\infty}^{\infty} |g(t)|^p \, dt.
\]
Therefore,
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \mathcal{X}_{I_n}(t) |g_h(t)|^p \, dt = \int_{-\infty}^{\infty} |g(t)|^p \, dt. \tag{5}
\]
From (C) it follows that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \mathcal{X}_{I_n}(t) |g_h(t)|^p \, dt \leq \int_{-\infty}^{\infty} |g(t)|^p \, dt \tag{6}
\]
and so from (5) and (6) we get
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \mathcal{X}_{I_n}(t) |g_h(t)|^p \, dt = \int_{-\infty}^{\infty} |g(t)|^p \, dt,
\]
i.e.
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |\mathcal{X}_{I_n}(t) g_h(t)|^p \, dt = \int_{-\infty}^{\infty} |g(t)|^p \, dt. \tag{7}
\]
From (7), by applying Lemma 1 to the sequence of functions \( f_n(t) = \mathcal{X}_{I_n}(t) g_h(t) \) and the function \( f(t) = g(t) \) we get
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |\mathcal{X}_{I_n}(t) g_h(t) - g(t)|^p \, dt = 0.
\]
Since \( \int_{-\infty}^{\infty} |g(t)|^p \, dt < \infty \), from the previous inequality we obtain
\[
\lim_{n \to \infty} \int_{I_n} |g_h(t) - g(t)|^p \, dt = 0. \tag{8}
\]
Let us now demonstrate that
\[
\lim_{n \to \infty} \int_{I_n} \left| e^{i(n+1)2\pi \frac{h}{n} t} g_h(t) - e^{i\tau t} g(t) \right|^p \, dt = 0. \tag{9}
\]
By integrating inequalities
\[
\left| e^{i(n+1)2\pi \frac{h}{n} t} g_h(t) - e^{i\tau t} g(t) \right|^p \leq 2^{p-1} \left| g_h(t) - g(t) \right|^p
\]
\[
+ 2^{p-1} \left| e^{i(n+1)2\pi \frac{h}{n} t} - e^{i\tau t} \right|^p |g(t)|^p
\]
Lemma 1, we obtain
\[
\int_{I_n} \left| e^{i(n+1)2\pi h_n t} g_{h_n}(t) - e^{i\tau t} g(t) \right|^p dt \leq 2^{p-1} \int_{I_n} \left| g_{h_n}(t) - g(t) \right|^p dt + 2^{p-1} \int_{-\infty}^{\infty} X_{I_n}(t) \left| e^{i(n+1)2\pi h_n t} - e^{i\tau t} \right|^p |g(t)|^p dt.
\]

The first integral on the right-hand side of the previous inequality tends to 0 according to (8), and the second one tends to 0 according to the Lebesgue dominated convergence theorem. This proves (9).

From this, it follows directly that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left| X_{I_n}(t) \Re \left( e^{i(n+1)2\pi h_n t} g_{h_n}(t) \right) \right|^p dt = 0.
\]

Since \( \int_{-\infty}^{\infty} \left| \Re \left( e^{i\tau t} g(t) \right) \right|^p dt < +\infty \), from the previous inequality we obtain
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left| X_{I_n}(t) \Re \left( e^{i(n+1)2\pi h_n t} g_{h_n}(t) \right) \right|^p dt = \int_{-\infty}^{\infty} \left| \Re \left( e^{i\tau t} g(t) \right) \right|^p dt = 0.
\]

The sequence of functions \( f_n(t) = X_{I_n}(t) \Re \left( e^{i(n+1)2\pi h_n t} g_{h_n}(t) \right) \) and the function \( f(t) = \Re \left( e^{i\tau t} g(t) \right) \) satisfy the conditions of Lemma 1, and so applying it to (10) we obtain
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left| X_{I_n}(t) \Re \left( e^{i(n+1)2\pi h_n t} g_{h_n}(t) \right) \right|^p dt = \int_{-\infty}^{\infty} \left| \Re \left( e^{i\tau t} g(t) \right) \right|^p dt
\]
from which (4) follows directly. □

Lemma 4. If \( p \geq 1 \) then
\[
\lim_{\lambda \to +\infty} \frac{\int_{0}^{+\infty} \left| \sin x \right|^r dx \left| \cos \lambda x \right|^p dx}{\int_{0}^{+\infty} \left| \sin x \right|^p dx} = \frac{\Gamma \left( \frac{1+p}{2} \right)}{\sqrt{\pi} \Gamma \left( 1 + \frac{p}{2} \right)}.
\]

Proof. Let \( I \overset{\text{def}}{=} \frac{1}{\pi} \int_{0}^{+\infty} \left| \sin x \right|^r dx, r > 1 \). Then
\[
\lim_{\lambda \to +\infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \sin \frac{n\pi}{\lambda} \right|^r = \frac{1}{\pi} \lim_{h \to 0^+} h \sum_{n=1}^{\infty} \left| \sin h \theta \right|^r = I
\]
and hence
\[
\lim_{\lambda \to +\infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \sin \frac{(2n-1)\pi}{2\lambda} \right|^r = I.
\]
(The previous equality follows from the identity
\[
\frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \sin \frac{2n-1}{2\lambda} \right|^r = \frac{2}{\pi} \sum_{n=1}^{\infty} \left| \sin \frac{n\pi}{2\lambda} \right|^r - \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \sin \frac{n\pi}{\lambda} \right|^r
\]
and the fact that
\[
\frac{\pi}{2\lambda} \sum_{n=1}^{\infty} \left| \sin \frac{n\pi}{2\lambda} \right|^r \to \infty \int_{0}^{+\infty} \left| \sin x \right|^r dx
\]
and
\[ \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi}{\lambda}}{\frac{n\pi}{\lambda}} \right|^r \rightarrow_{\lambda \to +\infty} I. \]

Let us now demonstrate that for \(|x| \leq \frac{\pi}{2}\) the following holds:
\[ \lim_{\lambda \to +\infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r = I. \] (12)

For \(|x| \leq \frac{\pi}{2}\) we have
\[ \frac{1}{\left( \frac{(2n-1)\pi}{2\lambda} \right)^r} \geq \frac{1}{\left( \frac{n\pi + x}{\lambda} \right)^r} \geq \frac{1}{\left( \frac{(2n+1)\pi}{2\lambda} \right)^r} \]
and so we get
\[ \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r \leq \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r \leq \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r. \] (13)

If \(p \geq 1\), it is easy to see that the function
\[ g(z) = \frac{|1 + z|^p - |z|^p}{1 + |z|^{p-1}} \]
is continuous and bounded on \(C\). Replacing \(z\) by \(\frac{z}{w}\), we see that
\[ |z + w|^p - |z|^p \leq c \left( |z|^{p-1} |w| + |w|^p \right) \]
for all \(z\) and \(w\), where \(c\) is a positive constant that only depends on \(p\).

From the previous inequality we obtain
\[ |z|^p - c \left( |z|^{p-1} |w| + |w|^p \right) \leq |z + w|^p \leq |z|^p + c \left( |z|^{p-1} |w| + |w|^p \right). \] (14)

First, we prove that
\[ \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r \leq I \]
holds (for every \(x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]\)).

Let
\[ a_n(\lambda) = \sin \left( 2n - 1 \right) \frac{\pi}{2\lambda}, \quad b_n(\lambda) = \cos \left( 2n - 1 \right) \frac{\pi}{2\lambda}, \]
\[ \varphi(\lambda, x) = \cos \frac{\pi + 2x}{2\lambda}, \quad \psi(\lambda, x) = \sin \frac{\pi + 2x}{2\lambda}. \]

Then we have
\[ \sin \frac{n\pi + x}{\lambda} = a_n(\lambda) \varphi(\lambda, x) + b_n(\lambda) \psi(\lambda, x) \] (15)
and also
\[ \sin \frac{n\pi + x}{\lambda} = a_{n+1}(\lambda) \varphi(\lambda, -x) - b_{n+1}(\lambda) \psi(\lambda, -x). \] (16)
From (13) and (15), using the right side of inequality (14) we get
\[ \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi + x}{\lambda}}{\frac{n\pi + x}{\lambda}} \right|^r \leq I_1(\lambda) + I_2(\lambda) + I_3(\lambda) \] (17)
where
\[ I_1(\lambda) = |\varphi(\lambda, x)|^r \cdot \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin (2n - 1) \frac{\pi}{2\lambda}}{\frac{2n - 1}{2\lambda}} \right|^r, \]
\[ I_2(\lambda) = c \sum_{n=1}^{\infty} \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|^{r-1}}{(2n - 1) \frac{\pi}{2\lambda}} \cdot |b_n(\lambda)| \cdot |\varphi(\lambda, x)|^{r-1} |\psi(\lambda, x)|, \]
and
\[ I_3(\lambda) = c \sum_{n=1}^{\infty} \frac{|b_n(\lambda)|^r |\psi(\lambda, x)|^r}{((2n - 1) \frac{\pi}{2\lambda})^r}. \]

Having in mind that \( \varphi(\lambda, x) \to 1 \) when \( \lambda \to +\infty \), we obtain from (11) that
\[ \lim_{\lambda \to \infty} I_1(\lambda) = I. \] (18)
Since \( |b_n(\lambda)| \leq 1, |\varphi(\lambda, x)|^{r-1} \leq 1 \) and \( |\psi(\lambda, x)| \leq \frac{2\pi}{\lambda} \) we get
\[ I_2(\lambda) \leq \frac{2\pi c}{\lambda^2} \sum_{n=1}^{\infty} \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|^{r-1}}{(2n - 1) \frac{\pi}{2\lambda}} \] (19)
and
\[ I_3(\lambda) \leq c \left( \frac{2\pi}{\lambda} \right)^r \sum_{n=1}^{\infty} \frac{1}{((2n - 1) \frac{\pi}{2\lambda})^r}. \] (20)
From (20) (because \( r > 1 \)) it directly follows that
\[ \lim_{\lambda \to +\infty} I_3(\lambda) = 0. \] (21)
Now, we prove that if \( r > 1 \) then
\[ \lim_{\lambda \to +\infty} \frac{2\pi c}{\lambda^2} \sum_{n=1}^{\infty} \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|^{r-1}}{(2n - 1) \frac{\pi}{2\lambda}} = 0 \] (22)
holds.
Indeed, if \( 1 < r < 2 \) then we have
\[ \frac{2\pi c}{\lambda^2} \sum_{n=1}^{\infty} \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|^{r-1}}{(2n - 1) \frac{\pi}{2\lambda}} \leq 2\pi c \left( \frac{2}{\pi} \right)^r \lambda^{r-2} \sum_{n=1}^{\infty} \frac{1}{((2n - 1))^r} \]
from which (22) follows directly.
If \( r \geq 2 \) then (because \( \left| \frac{\sin x}{x} \right|^\alpha \leq 1 \) for \( \alpha \geq 0 \))
\[ \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|^{r-1}}{(2n - 1) \frac{\pi}{2\lambda}} \leq \frac{4\lambda^2}{\pi^2} \frac{|\sin (2n - 1) \frac{\pi}{2\lambda}|}{((2n - 1))^2} \]
and so, we have
\[
\frac{2\pi c}{\lambda^2} \sum_{n=1}^{\infty} \left| \frac{\sin (2n-1) \frac{\pi}{2\lambda}}{(2n-1) \frac{\pi}{2\lambda}} \right|^{r-1} \leq \frac{8c}{\pi} \sum_{n=1}^{\infty} \left| \frac{\sin (2n-1) \frac{\pi}{2\lambda}}{(2n-1)^2} \right|
\]
from which (22) follows.

From (17)–(22) it follows that (for \(x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\))
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}} \right| \leq I.
\]
In a similar way, using the left side of (13) and (14), and equality (16), one obtains
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}} \right|^r \geq I
\]
which proves (12). ■

Note that along the way it was demonstrated that there exists \(K < \infty\) such that for any \(x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) and any \(\lambda \geq 1\) the following holds:
\[
\frac{1}{\lambda} \sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi+x}{\lambda}}{\frac{n\pi+x}{\lambda}} \right| \leq K.
\]

Let now \(p \geq 1\).

Since
\[
\lim_{\lambda \to +\infty} \int_0^\infty \left| \frac{\sin x}{x} \right|^2 |\cos \lambda x|^p \, dx = \lim_{\lambda \to +\infty} \sum_{s=1}^{\infty} \int_{(2s-1)\frac{\pi}{2\lambda}}^{(2s+1)\frac{\pi}{2\lambda}} \left| \frac{\sin x}{x} \right|^2 |\cos \lambda x|^p \, dx
\]
\[
= \lim_{\lambda \to +\infty} \sum_{s=1}^{\infty} \frac{1}{\lambda} \int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \left| \frac{\sin \frac{s\pi+x}{\lambda}}{\frac{s\pi+x}{\lambda}} \right|^2 |\cos x|^p \, dx,
\]
by applying the Beppo Levi theorem, (23) and the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{\lambda \to +\infty} \int_0^\infty \left| \frac{\sin x}{x} \right|^2 |\cos \lambda x|^p \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos x|^p \left( \lim_{\lambda \to +\infty} \frac{1}{\lambda} \sum_{s=1}^{\infty} \left| \frac{\sin \frac{s\pi+x}{\lambda}}{\frac{s\pi+x}{\lambda}} \right|^2 \right) \, dx.
\]
(24)

From (12) (putting \(r = 2p\), \(p \geq 1\)) and (24) it follows that
\[
\lim_{\lambda \to +\infty} \frac{\int_0^\infty \left| \frac{\sin x}{x} \right|^2 |\cos \lambda x|^p \, dx}{\int_0^\infty \left| \frac{\sin x}{x} \right|^2 \, dx} = \frac{I \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos x|^p \, dx}{\pi \cdot I} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\cos x)^p \, dx
\]
\[
= \frac{\Gamma \left( \frac{1+p}{2} \right)}{\sqrt{\pi} \Gamma \left( 1 + \frac{p}{2} \right)}
\]
which proves Lemma 4.
3. Proof of Theorem 2

Let \( \tau = \frac{n}{\pi} \) and \( g (z) = e^{-i\tau z} f (z) \). The function \( g \) is an entire function of exponential type whose conjugate indicator diagram lies on the interval \([-\tau, \tau]\) and for which

\[
\int_{-\infty}^{\infty} |g (x)|^p dx < \infty.
\]

Let \( P \) be an algebraic polynomial of degree \( 2n + 2 \) such that

\[
P (e^{ix}) = e^{i(n+1)x} \cdot g_{hn} \left( \frac{x}{2\pi h_n} \right)
\]

(the notation is the same as in Properties (A), (B) and (C) after Lemma 2).

Since

\[
P' (e^{ix}) = e^{i(n+1)x} g_{hn} \left( \frac{x}{2\pi h_n} \right) + e^{i(n+1)x} g'_{hn} \left( \frac{x}{2\pi h_n} \right) \cdot \frac{1}{2\pi h_n},
\]

then, applying Theorem 1, we obtain

\[
\int_{-\pi}^{\pi} \left| \frac{n}{2\pi h_n} \right| g_{hn} \left( \frac{x}{2\pi h_n} \right) + g'_{hn} \left( \frac{x}{2\pi h_n} \right) \right|^p dx
\]

\[
\leq c_p (2n + 2)^p (2\pi h_n)^p \int_{-\pi}^{\pi} \left| \Re \left( e^{i(n+1)x} \cdot g_{hn} \left( \frac{x}{2\pi h_n} \right) \right) \right|^p dx
\]

from which, after substituting \( \frac{x}{2\pi h_n} = t, \quad t \in I_n = \left[ -\frac{n\pi}{\pi}, \frac{n\pi}{\pi} \right] \), it follows that

\[
\int_{I_n} \left| \frac{n}{2\pi h_n} g_{hn} (t) + g'_{hn} (t) \right|^p dt
\]

\[
\leq c_p \left( \frac{2n + 2}{n} \right)^p \pi^p \int_{I_n} \left| \Re \left( e^{i(n+1)\tau t} \cdot g_{hn} (t) \right) \right|^p dt.
\]  

(25)

Let \( a > 0 \) be such that \( a < \frac{n\pi}{\tau} \). Then from (25) it follows that

\[
\int_{-a}^{a} \left| \frac{n}{2\pi h_n} g_{hn} (t) + g'_{hn} (t) \right|^p dt
\]

\[
\leq c_p \left( \frac{2n + 2}{n} \right)^p \pi^p \int_{I_n} \left| \Re \left( e^{i(n+1)\tau t} \cdot g_{hn} (t) \right) \right|^p dt.
\]  

(26)

Since according to Lemma 2 \( \lim_{n \to \infty} g_{hn} (t) = g (t) \) and \( \lim_{n \to \infty} g'_{hn} (t) = g' (t) \) where convergence is uniform on \([-a, a]\), by applying Lemma 3, we get (from (26), letting \( n \to \infty \))

\[
\int_{-a}^{a} |i\tau g (t) + g' (t)|^p dt \leq c_p \sigma^p \int_{-\infty}^{\infty} \left| \Re \left( e^{i\tau t} \cdot g (t) \right) \right|^p dt.
\]

Since \( a > 0 \) is arbitrary, from the previous inequality we obtain

\[
\int_{-\infty}^{\infty} |i\tau g (t) + g' (t)|^p dt \leq c_p \sigma^p \int_{-\infty}^{\infty} \left| \Re \left( e^{i\tau t} \cdot g (t) \right) \right|^p dt.
\]  

(27)

Since \( f (x) = e^{ix} \cdot g (x) \), we have \( |f' (x)| = |g' (x) + i\tau g (x)| \) and from (27) we obtain (3).
Let $B_p$ be the smallest constant such that the following inequality holds:

$$
\int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq B_p \int_{-\infty}^{\infty} |\Re (f(x))|^p \, dx
$$

for all entire functions of exponential type whose conjugate indicator diagram lies on interval $[0, \sigma i]$ and for which

$$
\int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty.
$$

Having in mind the definition of $B_p$, it follows that

$$
B_p \leq c_p \sigma_p.
$$

Let us now show that $B_p \geq c_p \sigma_p$ ($p \geq 1$).

Let $0 < \varepsilon < \frac{\sigma}{2}$. Starting from entire function $\lambda \mapsto \frac{\sin \lambda}{\lambda}$ (a “blue blood” function) we form the function

$$
f_\varepsilon(z) = e^{iz(\sigma - \varepsilon)} \frac{\sin^2 \frac{\varepsilon x}{x^2}}{\varepsilon^2}.
$$

The function $f_\varepsilon$ is an entire function of exponential type $\sigma$ whose conjugate indicator diagram belongs to the interval $[0, \sigma i]$ and $\int_{-\infty}^{\infty} |f_\varepsilon(x)|^p \, dx < \infty$ ($p \geq 1$).

Therefore

$$
\|f_\varepsilon\|_p \leq B_p^\frac{1}{p} \|\Re f_\varepsilon\|_p.
$$

(Here $\|\varphi\|_p = \left(\int_{-\infty}^{\infty} |\varphi(x)|^p \, dx\right)^{\frac{1}{p}}$.)

Note that

$$
\Re f_\varepsilon(x) = \cos(\sigma - \varepsilon) x \cdot \frac{\sin^2 \frac{\varepsilon x}{x^2}}{\varepsilon^2}
$$

and

$$
|f'_\varepsilon(x)| = |f_1^\varepsilon(x) + f_2^\varepsilon(x)|
$$

where

$$
f_1^\varepsilon(x) = i(\sigma - \varepsilon) \frac{\sin^2 \frac{\varepsilon x}{x^2}}{\varepsilon^2}
$$

and

$$
f_2^\varepsilon(x) = \frac{d}{dx} \left( \frac{\sin^2 \frac{\varepsilon x}{x^2}}{\varepsilon^2} \right).
$$

Since $\|f'_\varepsilon\|_p \geq \|f_1^\varepsilon\|_p - \|f_2^\varepsilon\|_p$, then from (28) it follows that

$$
B_p^\frac{1}{p} \geq \|f'_\varepsilon\|_p \geq \|f_1^\varepsilon\|_p - \|f_2^\varepsilon\|_p.
$$
Let us demonstrate that

$$\lim_{\epsilon \to 0^+} \frac{\| f_\epsilon^x \|_p}{\| \mathcal{R}e f_\epsilon \|_p} = 0.$$  \hfill (30)$$

We can easily obtain

$$\frac{\| f_\epsilon^x \|_p}{\| \mathcal{R}e f_\epsilon \|_p} = \left( \frac{\epsilon}{2} \right)^p \frac{\int_0^\infty \left\vert \sin \frac{2x - 2\sin^2 x}{x^3} \right\vert^p dx}{\int_0^\infty \left\vert \cos \frac{2(\sigma - \epsilon)x}{\epsilon} \right\vert^p \cdot \frac{\sin x}{x} \cdot \left\vert \sin \frac{x}{\sigma - \epsilon} \right\vert \cdot \left( \sigma - \epsilon \right)^p}$$

where

$$\lambda = \frac{2}{\epsilon} (\sigma - \epsilon) \to +\infty \text{ when } \epsilon \to 0^+.$$  

So, for the proof (30) it is enough to prove that

$$\lim_{\lambda \to +\infty} \lambda^p \int_0^{\infty} \left\vert \cos \lambda x \right\vert^p \cdot \left\vert \frac{\sin x}{x} \right\vert^2 dx = +\infty$$  \hfill (31)

holds.

Since $\sin x \geq \frac{2}{\pi} x$ for $x \in \left[ 0, \frac{\pi}{2} \right]$, we have

$$\lambda^p \int_0^{\infty} \left\vert \cos \lambda x \right\vert^p \cdot \left\vert \frac{\sin x}{x} \right\vert^2 dx \geq \lambda^p \int_0^{\frac{\pi}{2}} \left\vert \cos \lambda x \right\vert^p \cdot \left\vert \frac{\sin x}{x} \right\vert^2 dx$$

$$\geq \lambda^p \left( \frac{2}{\pi} \right)^{2p} \int_0^{\frac{\pi}{2}} \left\vert \cos \lambda x \right\vert^p \cdot dx$$

$$= \left( \frac{2}{\pi} \right)^{2p} \lambda^{p-1} \int_0^{\frac{5\pi}{2}} \left\vert \cos x \right\vert^p \cdot dx.$$  

If $p \geq 1$ then

$$\lim_{\lambda \to +\infty} \lambda^{p-1} \int_0^{\frac{5\pi}{2}} \left\vert \cos x \right\vert^p \cdot dx = +\infty;$$

so, (31) holds.

Let us now determine

$$\lim_{\epsilon \to 0^+} \frac{\| f_\epsilon^x (x) \|_p}{\| \mathcal{R}e f_\epsilon \|_p}.$$  

Since

$$\frac{\| f_\epsilon^x \|_p}{\| \mathcal{R}e f_\epsilon \|_p} = \left( \sigma - \epsilon \right)^p \frac{\int_0^\infty \left\vert \frac{\sin x}{x} \right\vert^2 dx}{\int_0^\infty \left\vert \cos \frac{2(\sigma - \epsilon)x}{\epsilon} \right\vert^p \cdot dx}$$
then, as $\varepsilon \to 0^+$, applying Lemma 4 (here $\lambda = 2\pi / \varepsilon \to +\infty$ when $\varepsilon \to 0^+$) we get
\[
\lim_{\varepsilon \to 0^+} \frac{\|f_{\varepsilon}^p\|_p}{\|\Re f_{\varepsilon}\|_p} = \sigma^p \frac{\sqrt{\pi} \Gamma \left(1 + \frac{p}{2}\right)}{\Gamma \left(\frac{1+p}{2}\right)}.
\]
(32)

Then, from (29), (30) and (32) it follows that
\[
B_{\frac{p}{2}} \geq \sigma^p \frac{\sqrt{\pi} \Gamma \left(1 + \frac{p}{2}\right)}{\Gamma \left(\frac{1+p}{2}\right)}.
\]
Therefore, $B_p \geq \sigma^p c_p$, which proves Theorem 3. ■

**Remark 2.** Theorem 3 also holds when function $f$ is an entire function of exponential type whose conjugate indicator diagram lies on the interval $[-\sigma i, 0]$ and which satisfies the condition $\int_{-\infty}^{\infty} |f(x)|^p dx < \infty$ ($p \geq 1$). It is sufficient just to apply Theorem 2 to the function $h(z) = f(-z)$.

**Remark 3.** It also follows from Theorem 3 that
\[
\int_{-\infty}^{\infty} \left|f^{(s)}(x)\right|^p dx \leq \left(c_p \cdot \sigma^p\right)^s \int_{-\infty}^{\infty} |\Re f(x)|^p dx
\]
for every $s = 1, 2, 3, \ldots$. This inequality is obtained by induction over $s$ using $|\Re z| \leq |z|$.

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**References**