Green's $\mathcal{J}$-order and the rank of tropical matrices

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ABSTRACT

We study Green's $\mathcal{J}$-order and $\mathcal{J}$-equivalence for the semigroup of all $n \times n$ matrices over the tropical semiring. We give an exact characterisation of the $\mathcal{J}$-order, in terms of morphisms between certain tropical convex sets. We establish connections between the $\mathcal{J}$-order, isometries of tropical convex sets, and various notions of rank for tropical matrices. We also study the relationship between the relations $\mathcal{J}$ and $\mathcal{D}$; Izhakian and Margolis have observed that $\mathcal{D} \neq \mathcal{J}$ for the semigroup of all 3×3 matrices over the tropical semiring with $-\infty$, but, in contrast, we show that $\mathcal{D} = \mathcal{J}$ for all full matrix semigroups over the finitary tropical semiring.

1. Introduction

Tropical algebra (also known as max-plus algebra or max algebra) is the algebra of the real numbers (typically augmented with $-\infty$, and sometimes also with $+\infty$) under the operations of addition and maximum. It has been an active area of study in its own right since the 1970's [9] and has applications in a variety of scientific and mathematical disciplines. Tropical algebra and many of its basic properties have been independently rediscovered many times by researchers in these application areas.

Many problems arising from applications are naturally expressed using (max-plus) linear equations, so much of tropical algebra concerns matrices. From an algebraic perspective, a key object is the semigroup of all square matrices of a given size over the tropical semiring. This semigroup obviously plays a role analogous to that of the full matrix semigroup over a field. Perhaps less obviously, the relative scarcity of invertible matrices over an idempotent semifield means that much less is to be learnt here by studying only invertible matrices. Many classical matrix problems can be reduced to questions about invertible matrices, and hence about the general linear group. In tropical algebra, however, there is typically no such reduction, so problems often entail a detailed analysis of non-invertible matrices. In this respect, the full matrix semigroup takes on the mantle of the general linear group. Its algebraic structure has thus been the subject of considerable study, at first on an *ad hoc* basis, but more recently moving towards a systematic understanding using the tools of semigroup theory (see for example [11,16,18,20,22,26]).

Green's relations [15,6] are five equivalence relations ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$) and three pre-orders ($\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$) which can be defined upon any semigroup, and which encapsulate the structure of its maximal subgroups and principal left, right and two-sided ideals. They are powerful tools for understanding semigroups and monoids, and play a key role in almost every aspect of modern semigroup theory. The relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ can be described in generality for the full matrix semigroup over a semiring with identity, and hence present no particular challenge in the tropical case; see [22] or Section 3 below for details. In [22], we initiated the study of Green's relations in tropical matrix semigroups, by describing the remaining relations in the case of the 2×2 tropical matrix semigroup. In [16], Hollings and the second author gave a complete description of the $\mathcal{D}$-relation in arbitrary finite dimensions, based on some deep connections with the phenomenon of *duality* between the row and column space of a tropical matrix.

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In the present paper, we turn our attention to the equivalence relation $\mathcal{J}$ and pre-order $\leq_{\mathcal{J}}$ in the full tropical matrix semigroup of arbitrary dimension. For the classical case of finite-dimensional matrices over a field, it is well known that the relations $\mathcal{D}$ and $\mathcal{J}$ coincide. Previous work of the authors showed that this correspondence holds for 2×2 tropical matrices [22], but Izhakian and Margolis [19] have shown that it does not extend to higher dimensional tropical matrix semigroups over the tropical semiring with $-\infty$; indeed, they have found an example of a $\mathcal{J}$-class in the 3×3 tropical matrix semigroup which contains infinitely many $\mathcal{D}$-classes.

Our main technical result (Theorem 5.3) gives a precise characterisation of the $\mathcal{J}$-order (and hence also of $\mathcal{J}$-equivalence) in terms of morphisms between certain tropical convex sets: specifically, $A \leq_{\mathcal{J}} B$ exactly if there exists a tropical convex set Y such that the row space of B maps surjectively onto Y, and the row space of A embeds injectively into Y. Using duality theorems, we also show (Theorem 5.4) that if $A \leq_{\mathcal{J}} B$ then the row space of A admits an isometric (with respect to the Hilbert projective metric) embedding into the row space of B, but in general it need *not* admit a linear embedding. From these results, we are able to deduce that the semigroup of $n \times n$ matrices over the *finitary* tropical semiring (without $-\infty$) does satisfy $\mathcal{D} = \mathcal{J}$.

In the classical case of a finite-dimensional full matrix semigroup over a field, it is well known that the $\mathcal{J}$-relation (and the $\mathcal{D}$-relation, with which it coincides) encapsulates the concept of *rank*, with the $\mathcal{J}$-order corresponding to the obvious order on ranks. Thus, for the semigroup of matrices over a more general ring or semiring, the $\mathcal{J}$-class of a matrix may be thought of as a natural analogue of its rank. This idea is rather different to traditional notions of rank since it is non-numerical, taking values in a poset (the $\mathcal{J}$-order), rather than the natural numbers.

For tropical matrices, several different (numerical) notions of rank have been proposed and studied, both separately and in relation to one another (see for example [1,2,5,12,21,25]). Each of these clearly has merit for particular applications, but overall we suggest that the proliferation of incompatible definitions is evidence that the kind of information given by the "rank" of a classical matrix cannot, in the tropical case, be encapsulated in a single natural number. We believe that the $\mathcal{J}$-class of a matrix may serve as a "general purpose" analogue of rank for tropical mathematics, and partly with this in mind, the final section of this paper discusses the relationship between $\mathcal{J}$-class and some existing notions of rank.

2. Preliminaries

In this section we briefly recall the foundational definitions of tropical algebra, and establish some elementary properties which will be required later.

The *finitary tropical semiring* $\mathbb{FT}$ is the semiring (without additive identity) consisting of the real numbers under the operations of addition and maximum. We write $a \oplus b$ to denote the maximum of a and b, and $a \otimes b$ or just ab to denote the sum of a and b. Note that both operations are associative and commutative and that $\otimes$ distributes over $\oplus$.

The *tropical semiring* $\mathbb{T}$ is the finitary tropical semiring augmented with an extra element $-\infty$ which acts as a zero for addition and an identity for maximum. The *completed tropical semiring* $\overline{\mathbb{T}}$ is the tropical semiring augmented with an extra element $+\infty$, which acts as a zero for both maximum and addition, save that

$$(-\infty)(+\infty) = (+\infty)(-\infty) = -\infty.$$

Thus $\mathbb{FT} \subseteq \mathbb{T} \subseteq \overline{\mathbb{T}}$ and we call the elements of $\mathbb{FT}$ finite elements.

For any commutative semiring S, we denote by $M_n(S)$ the set of all $n \times n$ matrices with entries drawn from S. This has the structure of a semigroup, under the multiplication induced from the semiring operations in the usual way.

We extend the usual order $\leq$ on $\mathbb{R}$ to a total order on $\mathbb{T}$ and $\overline{\mathbb{T}}$ by setting $-\infty < x < +\infty$ for all $x \in \mathbb{R}$. Note that $a \oplus b = a$ exactly if $b \leq a$. The semirings $\mathbb{FT}$ and $\overline{\mathbb{T}}$ admit a natural order-reversing involution $x \mapsto -x$, where of course $-(-\infty) = +\infty$ and $-(+\infty) = -\infty$.

For $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$ we shall be interested in the space S^n of *affine tropical vectors*. We write x_i for the ith component of a vector $x \in S^n$. We extend $\oplus$ and $\leq$ to S^n componentwise so that $(x \oplus y)_i = x_i \oplus y_i$ and $x \leq y$ exactly if $x_i \leq y_i$ for all i. We define a scaling action of S on S^n by

$$\lambda \otimes (x_1, \ldots, x_n) = (\lambda \otimes x_1, \ldots, \lambda \otimes x_n)$$

for each $\lambda \in S$ and each $x \in S^n$. Similarly, for $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$ we extend the involution $x \mapsto -x$ on S to S^n by defining $(-x)_i = -(x_i)$. The scaling and $\oplus$ operations give S^n the structure of an *S-module* (sometimes called an *S-semimodule* since the $\oplus$ operation does not admit inverses).

From affine tropical n-space we obtain *projective tropical $(n-1)$-space* by identifying two vectors if one is a tropical multiple of the other by an element of $\mathbb{FT}$. We denote the projective space obtained from $\mathbb{FT}^n$ (respectively, $\mathbb{T}^n$ or $\overline{\mathbb{T}}^n$) by $\mathbb{PFT}^{(n-1)}$ (respectively, $\mathbb{PT}^{(n-1)}$ or $\overline{\mathbb{PT}}^{(n-1)}$).

A *tropical convex set* in S^n is a subset closed under $\oplus$ and scaling by elements of S, that is, an S-submodule of S^n. If $B \subseteq S^n$ then the *tropical convex hull* of B is the smallest tropical convex set containing B, that is, the set of all vectors in S^n which can be written as tropical linear combinations of finitely many vectors from B. Given two tropical convex sets $X \subseteq S^n$ and $Y \subseteq S^m$, we say that $f : X \to Y$ is a *linear map* from X to Y if $f(x \oplus x') = f(x) \oplus f(x')$ and $f(\lambda \otimes x) = \lambda \otimes f(x)$ for all $x, x' \in X$ and all $\lambda \in S$.

Since each tropical convex set $X \subseteq S^n$ is closed under scaling, it induces a subset of the corresponding projective space, termed the *projectivisation* of X. Notice that one tropical convex set contains another exactly if there is a corresponding containment of their projectivisations.

Given a matrix $A \in M_n(S)$ we define the *row space* of A, denoted by $R_S(A)$, to be the tropical convex hull of the rows of A. Thus $R_S(A) \subseteq S^n$. Similarly, we define the *column space* $C_S(A) \subseteq S^n$ to be the tropical convex hull of the columns of A. We shall also be interested in the projectivisation of $C_S(A)$, which we call the projective column space of A and denote by $PC_S(A)$. Dually, the projective row space $PR_S(A)$ is the projectivisation of the row space of A.

We define a scalar product operation $\overline{\mathbb{T}}^n \times \overline{\mathbb{T}}^n \to \overline{\mathbb{T}}$ on affine tropical n-space by setting

$$\langle x \mid y \rangle = \max\{\lambda \in \overline{\mathbb{T}} : \lambda \otimes x \le y\}.$$

This is a *residual* operation in the sense of residuation theory [4], and has been frequently employed in max-plus algebra. Notice that $\langle x \mid y \rangle = +\infty$ if and only if for each i either $x_i = -\infty$ or $y_i = +\infty$. Thus $\langle x \mid x \rangle = +\infty$ if and only if $x_i \in \{-\infty, +\infty\}$ for all i. It also follows that if $x, y \in \mathbb{T}$ with $x \ne (-\infty, \dots, -\infty)$ then $\langle x \mid y \rangle \in \mathbb{T}$. Similarly, we note that $\langle x \mid y \rangle = -\infty$ if and only if there exists j such that either $x_j = +\infty \ne y_j$ or $y_j = -\infty \ne x_j$. Thus if $x, y \in \mathbb{FT}^n$ then $\langle x \mid y \rangle \in \mathbb{FT}$.

Lemma 2.1. *Let $x, y \in \overline{\mathbb{T}}^n$ with $x \ne y$. If $\langle x \mid y \rangle = +\infty$ then $\langle y \mid x \rangle = -\infty$.*

Proof. Since $x \ne y$ there exists j such that $x_j \ne y_j$. Now, $\langle x \mid y \rangle = +\infty$ implies that for each i either $x_i = -\infty$ or $y_i = +\infty$. Thus either $x_j = -\infty \ne y_j$ or $y_j = +\infty \ne x_j$ and hence, by the remarks preceding the lemma, we find that $\langle y \mid x \rangle = -\infty$. □

We define a distance function on $\overline{\mathbb{T}}^n$ by $d_H(x, y) = 0$ if x is a finite scalar multiple of y and

$$d_H(x, y) = -(\langle x \mid y \rangle \otimes \langle y \mid x \rangle)$$

otherwise. By Lemma 2.1, it is easy to see that $d_H(x, y) \ne -\infty$ for all $x, y \in \overline{\mathbb{T}}^n$. Thus $d_H(x, y) = +\infty$ unless both $\langle x \mid y \rangle$ and $\langle y \mid x \rangle$ are finite. Moreover, if $\langle x \mid y \rangle, \langle y \mid x \rangle \in \mathbb{FT}$ then it is easy to check that $d_H(x, y) \ge 0$. It is also easily verified that d_H is invariant under scaling x or y by finite scalars and hence is well-defined on $\mathbb{PT}^{(n-1)}$, $\mathbb{PT}^{(n-1)}$ and $\mathbb{PFT}^{(n-1)}$. For $x, y \in \mathbb{FT}^n$ we see that $d_H(x, y) \in \mathbb{FT}$. In fact, it can be shown that d_H is a metric on $\mathbb{PFT}^{(n-1)}$ and an extended metric on $\mathbb{PT}^{(n-1)}$ and $\mathbb{PT}^{(n-1)}$, called the *(tropical) Hilbert projective metric* (see [16, Proposition 1.6], for example). In particular, d_H induces obvious definitions of *isometry* and *isometric embeddings* between subsets of tropical projective spaces.

Now let $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$ and let $A \in M_n(S)$. Following [8,13] we define a map $\theta_A : R_S(A) \to C_S(A)$ by $\theta_A(x) = A \otimes (-x)^T$ for all $x \in R_S(A)$. Dually, we define $\theta_A' : C_S(A) \to R_S(A)$ by $\theta_A'(x) = (-x)^T \otimes A$ for all $x \in C_S(A)$. We call θ_A and θ_A' the *duality maps* for A. Notice that the duality maps do not make sense over $S = \mathbb{T}$, as the involution $x \mapsto -x$ is not defined for $x = -\infty$. The following lemma recalls some known properties of the duality maps which we shall need.

Lemma 2.2 (*Properties of the Duality Maps [8,13,16]*)**.** *Let $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$ and let $A \in M_n(S)$.*

(i) *θ_A and θ_A' are mutually inverse bijections between $R_S(A)$ and $C_S(A)$.*
(ii) *For all $x, y \in R_S(A)$, $x \le y$ if and only $\theta_A(y) \le \theta_A(x)$.*
    For all $x, y \in C_S(A)$, $x \le y$ if and only $\theta_A'(y) \le \theta_A'(x)$.
    We say that θ_A and θ_A' are order reversing.
(iii) *For all $x \in R_S(A)$ and all $\lambda \in \mathbb{FT}$, $\theta_A(\lambda \otimes x) = -\lambda \otimes \theta_A(x)$.*
    For all $x \in C_S(A)$ and all $\lambda \in \mathbb{FT}$, $\theta_A'(\lambda \otimes x) = -\lambda \otimes \theta_A'(x)$.
    We say that θ_A and θ_A' preserve scaling by finite scalars.

Of the properties in the lemma, part (i) is established for $S = \mathbb{FT}$ in [13, Theorem 23] and for $S = \overline{\mathbb{T}}$ in [8, Section 4.5]. Part (ii) is also shown in [8, Section 4.5]. Part (iii) is proved in [16, Theorem 2.4] and an expository account of all three parts can be found in [16].

We now recall the "metric duality theorem" from [16].

Theorem 2.3 (*Metric Duality Theorem [16, Theorem 2.6]*)**.** *Let $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$ and let $A \in M_n(S)$. Then the duality maps θ_A and θ_A' induce mutually inverse isometries (with respect to the Hilbert projective metric) between $PR_S(A)$ and $PC_S(A)$.*

3. Green's relations

Green's relations are five equivalence relations and three pre-orders, which can be defined on any semigroup, and which together describe the (left, right and two-sided) principal ideal structure of the semigroup. We give here brief definitions; for fuller discussion, proof of claimed properties and equivalent formulations, we refer the reader to an introductory text such as [17].

Let S be any semigroup. If S is a monoid, we set $S^1 = S$, and otherwise we denote by S^1 the monoid obtained by adjoining a new identity element 1 to S. We define a binary relation $\le_{\mathcal{R}}$ on S by $a \le_{\mathcal{R}} b$ if $aS^1 \subseteq bS^1$, that is, if either $a = b$ or there exists q with $a = bq$. We define another relation $\mathcal{R}$ by $a \mathcal{R} b$ if and only if $aS^1 = bS^1$.

The relations $\leq_{\mathcal{L}}$ and $\mathcal{L}$ are the left–right duals of $\leq_{\mathcal{R}}$ and $\mathcal{R}$, so $a \leq_{\mathcal{L}} b$ if $S^1 a \subseteq S^1 b$, and $a \mathcal{L} b$ if $S^1 a = S^1 b$. The relations $\leq_{\mathcal{J}}$ and $\mathcal{J}$ are two-sided analogues, so $a \leq_{\mathcal{J}} b$ if $S^1 a S^1 \subseteq S^1 b S^1$, and $a \mathcal{J} b$ if $S^1 a S^1 = S^1 b S^1$. We also define a relation $\mathcal{H}$ by $a \mathcal{H} b$ if $a \mathcal{L} b$ and $a \mathcal{R} b$. Finally, the relation $\mathcal{D}$ is defined by $a \mathcal{D} b$ if there exists an element $c \in S$ such that $a \mathcal{R} c$ and $c \mathcal{L} a$.

The relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{J}$ and $\mathcal{D}$ are equivalence relations; this is trivial in the first four cases, but requires slightly more work in the case of $\mathcal{D}$. The relations $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ are pre-orders (reflexive, transitive binary relations) each of which induces a partial order on the equivalence classes of the corresponding equivalence relation.

The study of Green's relations for the full tropical matrix semigroups was begun (in the 2×2 case) by the authors [22] and continued in greater generality by Hollings and the second author [16]. Some key results of those papers are summarised in the following two theorems; see [16, Proposition 3.1], [16, Theorem 5.1], [16, Theorem 5.5] and [16, Theorem 3.5] for full details and proofs.

Theorem 3.1 (*Known Characterisations of Green's Relations*). *Let $A, B \in M_n(S)$ for $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$.*

 (i) *$A \leq_{\mathcal{L}} B$ if and only if $R_S(A) \subseteq R_S(B)$;*
 (ii) *$A \mathcal{L} B$ if and only if $R_S(A) = R_S(B)$;*
(iii) *$A \leq_{\mathcal{R}} B$ if and only if $C_S(A) \subseteq C_S(B)$;*
(iv) *$A \mathcal{R} B$ if and only if $C_S(A) = C_S(B)$;*
 (v) *$A \mathcal{H} B$ if and only if $R_S(A) = R_S(B)$ and $C_S(A) = C_S(B)$;*
(vi) *$A \mathcal{D} B$ if and only if $C_S(A)$ and $C_S(B)$ are isomorphic as S-modules;*
(vii) *$A \mathcal{D} B$ if and only if $R_S(A)$ and $R_S(B)$ are isomorphic as S-modules.*

Theorem 3.2 (*Inheritance of $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$*). *Consider $M_n(\mathbb{FT}) \subseteq M_n(\mathbb{T}) \subseteq M_n(\overline{\mathbb{T}})$. Each of Green's pre-orders $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}$ and equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ in $M_n(\mathbb{FT})$ or $M_n(\mathbb{T})$ is the restriction of the corresponding relation in $M_n(\overline{\mathbb{T}})$.*

The $\mathcal{J}$-relation and $\leq_{\mathcal{J}}$ pre-order for the semigroups $M_n(S)$ with $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$ have so far remained rather mysterious. We will give a characterisation of the $\mathcal{J}$-relation in these tropical matrix semigroups.

4. The $\mathcal{J}$-relation is inherited

In many applications one wishes to work with the tropical semiring $\mathbb{T}$, but for theoretical purposes it is often nicer to work over the finitary tropical semiring $\mathbb{FT}$ or the completed tropical semiring $\overline{\mathbb{T}}$. The following result is an analogue for the $\mathcal{J}$-order and $\mathcal{J}$-equivalence of Theorem 3.2 above, saying that these relations in a full matrix semigroup over $\mathbb{T}$ and $\mathbb{FT}$ are inherited from the corresponding semigroup over $\overline{\mathbb{T}}$. Hence, in order to understand these relations in all three cases, it suffices to study them for $\overline{\mathbb{T}}$, and we shall for much of the remainder of the paper work chiefly with $\overline{\mathbb{T}}$.

Proposition 4.1 (*Inheritance of $\mathcal{J}$*). *Consider $M_n(\mathbb{FT}) \subseteq M_n(\mathbb{T}) \subseteq M_n(\overline{\mathbb{T}})$.*

 (i) *Let $A, B \in M_n(\mathbb{FT})$. Then*

$$A \leq_{\mathcal{J}} B \text{ in } M_n(\mathbb{FT}) \quad \text{if and only if} \quad A \leq_{\mathcal{J}} B \text{ in } M_n(\mathbb{T}).$$

(ii) *Let $A, B \in M_n(\mathbb{T})$. Then*

$$A \leq_{\mathcal{J}} B \text{ in } M_n(\mathbb{T}) \quad \text{if and only if} \quad A \leq_{\mathcal{J}} B \text{ in } M_n(\overline{\mathbb{T}}).$$

Proof. (i) It is clear that if $A \leq_{\mathcal{J}} B$ in $M_n(\mathbb{FT})$ then $A \leq_{\mathcal{J}} B$ in $M_n(\mathbb{T})$. Suppose now that $A \leq_{\mathcal{J}} B$ in $M_n(\mathbb{T})$. Thus there exist $P, Q \in M_n(\mathbb{T})$ such that $A = PBQ$ giving

$$A_{i,j} = \bigoplus_{k=1}^{n} \bigoplus_{l=1}^{n} P_{i,k} B_{k,l} Q_{l,j} \tag{4.1}$$

for all $i, j \in \{1, \ldots, n\}$. Since P, B, Q each have finitely many entries we may choose $\delta \in \mathbb{R}$ such that:

(1) $\delta \leq p + b + q - b' - p'$ for every pair of (necessarily finite) entries b, b' in B and all finite entries p, p' in P and q in Q;
(2) $\delta \leq p + b + q - b' - q'$ for every pair of (necessarily finite) entries b, b' in B and all finite entries p in P and q, q' in Q;
(3) $2\delta \leq p + b + q - b'$ for every pair of (necessarily finite) entries b, b' in B and all finite entries p in P and q in Q.

Now let P', Q' be the matrices obtained from P and Q respectively by replacing each $-\infty$ entry by δ. Thus $P', Q' \in M_n(\mathbb{FT})$. We shall show that $A = P'BQ'$.

Let $i, j \in \{1, \ldots, n\}$. By (4.1) we may choose k and l such that $A_{i,j} = P_{i,k} B_{k,l} Q_{l,j} \geq P_{i,h} B_{h,m} Q_{m,j}$ for all h and m. Since $A_{i,j}$ is finite it follows that $P_{i,k}$ and $Q_{l,j}$ are also finite. Thus $P'_{i,k} = P_{i,k}$ and $Q'_{l,j} = Q_{l,j}$, giving $A_{i,j} = P'_{i,k} B_{k,l} Q'_{l,j}$. It then suffices to show that $(P'BQ')_{i,j} = P'_{i,k} B_{k,l} Q'_{l,j}$. Now

$$(P'BQ')_{i,j} = \bigoplus_{h=1}^{n} \bigoplus_{m=1}^{n} P'_{i,h} B_{h,m} Q'_{m,j},$$

so it will suffice to show that $P'_{i,k} B_{k,l} Q'_{l,j} \geq P'_{i,h} B_{h,m} Q'_{m,j}$ for all h and m. There are four cases to consider:

(a) If $P_{i,h}, Q_{m,j} \in \mathbb{FT}$ then

$$P'_{i,h}B_{h,m}Q'_{m,j} = P_{i,h}B_{h,m}Q_{m,j} \leq P_{i,k}B_{k,l}Q_{l,j} = P'_{i,k}B_{k,l}Q'_{l,j}.$$

(b) If $P_{i,h} \in \mathbb{FT}$ and $Q_{m,j} = -\infty$ then

$$\begin{aligned}
P'_{i,h}B_{h,m}Q'_{m,j} &= P_{i,h} + B_{h,m} + \delta \\
&\leq P_{i,h} + B_{h,m} + P_{i,k} + B_{k,l} + Q_{l,j} - B_{h,m} - P_{i,h} \\
&\leq P_{i,k} + B_{k,l} + Q_{l,j} = P_{i,k}B_{k,l}Q_{l,j} = P'_{i,k}B_{k,l}Q'_{l,j},
\end{aligned}$$

by (1).

(c) If $P_{i,h} = -\infty$ and $Q_{m,j} \in \mathbb{FT}$ then we may apply an argument dual to that in case (b), using condition (2) in place of condition (1).

(d) If $P_{i,h} = -\infty$ and $Q_{m,j} = -\infty$ then

$$\begin{aligned}
P'_{i,h}B_{h,m}Q'_{m,j} &= \delta + B_{h,m} + \delta \\
&\leq P_{i,k} + B_{k,l} + Q_{l,j} - B_{h,m} + B_{h,m} \\
&\leq P_{i,k} + B_{k,l} + Q_{l,j} = P_{i,k}B_{k,l}Q_{l,j} = P'_{i,k}B_{k,l}Q'_{l,j},
\end{aligned}$$

by (3).

Thus we see that $(P'BQ')_{i,j} = A_{i,j}$ for all i and j.

(ii) It is clear that if $A \leq_{\mathcal{J}} B$ in $M_n(\mathbb{T})$ then $A \leq_{\mathcal{J}} B$ in $M_n(\overline{\mathbb{T}})$. Suppose now that $A \leq_{\mathcal{J}} B$ in $M_n(\overline{\mathbb{T}})$. Thus there exist $P, Q \in M_n(\overline{\mathbb{T}})$ such that $A = PBQ$. Let P', Q' be the matrices obtained from P and Q respectively by replacing each $+\infty$ entry by 0. We shall show that $A = P'BQ'$.

It is easy to see that $(P' \otimes B \otimes Q')_{i,j} \leq A_{i,j}$ for all i, j. Thus, it will suffice to show that for every pair of indices i, j, there exist h and m such that $P'_{i,h} \otimes B_{h,m} \otimes Q'_{m,j} = A_{i,j}$.

Since $A = P \otimes B \otimes Q$, it is clear that for all i and j there exist h and m such that $P_{i,h} \otimes B_{h,m} \otimes Q_{m,h} = A_{i,j}$. We claim that $P'_{i,h} \otimes B_{h,m} \otimes Q'_{m,h} = A_{i,j}$. If $P_{i,h}, Q_{m,j} \neq +\infty$, then this follows immediately. Thus we may assume that one of $P_{i,h} = +\infty$ and $Q_{h,m} = +\infty$ holds. Suppose that $P_{i,h} = +\infty$. Since $A_{i,j} \in \mathbb{T}$, it follows that we must have $B_{h,m} \otimes Q_{m,h} = -\infty = A_{i,j}$. But then it is easily verified that $B_{h,m} \otimes Q'_{m,h} = -\infty$, giving

$$P'_{i,h} \otimes B_{h,m} \otimes Q'_{m,h} = 0 \otimes B_{h,m} \otimes Q'_{m,h} = 0 \otimes -\infty = -\infty = A_{i,j}.$$

A similar argument holds for the case $Q_{h,m} = +\infty$. Thus, for all i and j we have that $(P' \otimes B \otimes Q')_{i,j} = A_{i,j}$. $\square$

5. Characterising the $\mathcal{J}$-order

In this section we shall give an exact characterisation of the $\mathcal{J}$-order, and hence also of the $\mathcal{J}$-relation, in terms of linear morphisms between column spaces (or dually, row spaces). As discussed in the previous section, we restrict our attention to the semirings $\mathbb{FT}$ and $\overline{\mathbb{T}}$, enabling us to make use of the duality maps and the Metric Duality Theorem (Theorem 2.3 above). Since the duality maps are not defined over $\mathbb{T}$, there is no direct analogue of Theorem 2.3 over $\mathbb{T}$. Thus the methods described in this section apply only to $\mathbb{FT}$ and $\overline{\mathbb{T}}$. However, by Proposition 4.1, we know that the $\mathcal{J}$-order in $M_n(\mathbb{T})$ is the restriction of the $\mathcal{J}$-order in $M_n(\overline{\mathbb{T}})$. Thus the results of this section together with Proposition 4.1 also give a characterisation of the $\mathcal{J}$-order and the $\mathcal{J}$-relation in $M_n(\mathbb{T})$.

We shall make use of [16, Theorem 4.2], which we state here for the reader's convenience:

Theorem 5.1. *[16, Theorem 4.2] Let $A, B \in M_n(S)$ for $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$. Then the following are equivalent:*

(i) $R_S(A) \subseteq R_S(B)$;

(ii) *there is a linear morphism from $C_S(B)$ to $C_S(A)$ taking the ith column of B to the ith column of A for all i;*

(iii) *there is a surjective linear morphism from $C_S(B)$ to $C_S(A)$ taking the ith column of B to the ith column of A for all i.*

We remark that Theorem 5.1 has a left–right dual, obtained by swapping rows with columns and row spaces with column spaces throughout the statement. Theorem 5.1 describes a duality between embeddings of row spaces and surjections of column spaces, which also has an algebraic manifestation:

Theorem 5.2. *Let $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$ and $A, B \in M_n(S)$. Then the following are equivalent:*

(i) $C_S(B)$ *surjects linearly onto* $C_S(A)$;

(ii) $R_S(A)$ *embeds linearly into* $R_S(B)$;

(iii) *there exists* $C \in M_n(S)$ *with* $A \mathrel{\mathcal{R}} C \leq_{\mathcal{L}} B$.

Proof. We prove first that (i) implies (iii). Suppose that $f : C_S(B) \to C_S(A)$ is a linear surjection. Let C be the matrix obtained by applying f to each column of B. Then clearly $C_S(C) = C_S(A)$ so $C \mathcal{R} A$ by Theorem 3.1(iv). Moreover, by Theorem 5.1 and the definition of C we have $R_S(C) \subseteq R_S(B)$, so by Theorem 3.1(i) we have $C \leq_{\mathcal{L}} B$.

Next we show that (iii) implies (ii). Since $A \mathcal{R} C$, in particular $A \mathcal{D} C$, so Theorem 3.1(vii) tells us that there is a linear isomorphism from $R_S(A)$ to $R_S(C)$. Also, since $C \leq_{\mathcal{L}} B$, Theorem 3.1(i) gives that $R_S(C)$ is contained in $R_S(B)$. Thus, the isomorphism gives a linear embedding of $R_S(A)$ into $R_S(B)$.

Finally, suppose (ii) holds, say $f : R_S(A) \to R_S(B)$ is a linear embedding. Let A' be obtained from A by applying f to each row of A. Then $R_S(A)$ is linearly isomorphic to $R_S(A')$, which is contained in $R_S(B)$. By Theorem 5.1, it follows from the latter that there is a linear surjection from $C_S(B)$ onto $C_S(A')$. Moreover, since $R_S(A)$ and $R_S(A')$ are isomorphic as S-modules, Theorem 3.1 parts (vi) and (vii) give that $C_S(A)$ and $C_S(A')$ are isomorphic as S-modules. Composing gives a linear surjection from $C_S(B)$ onto $C_S(A)$. □

We remark that the equivalence of conditions (i) and (ii) in Theorem 5.2 is a manifestation of a more general abstract categorical duality in residuation theory (see for example [7]). Again, the theorem has a left–right dual, obtained by interchanging row spaces with column spaces, $\mathcal{L}$ with $\mathcal{R}$, and $\leq_{\mathcal{L}}$ with $\leq_{\mathcal{R}}$. Theorem 5.2 and its dual lead easily to the main result of this section.

Theorem 5.3 (*Linear Characterisation of the $\mathcal{J}$-Order*)**.** *Let $A, B \in M_n(S)$ for $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$. The following are equivalent:*

(i) $A \leq_{\mathcal{J}} B$;
(ii) *there exists a tropical convex set $Y \subseteq S^n$ such that $R_S(A)$ embeds linearly into Y and $R_S(B)$ surjects linearly onto Y;*
(iii) *there exists a tropical convex set $Y \subseteq S^n$ such that $R_S(A) \subseteq Y$ and $R_S(B)$ surjects linearly onto Y;*
(iv) *there exists a tropical convex set $Y \subseteq S^n$ such that $C_S(A)$ embeds linearly into Y and $C_S(B)$ surjects linearly onto Y;*
(v) *there exists a convex set $Y \subseteq S^n$ such that $C_S(A) \subseteq Y$ and $C_S(B)$ surjects linearly onto Y.*

Proof. We show the equivalence of (i)–(iii), since the equivalence of (i), (iv) and (v) is dual. Suppose that $A \leq_{\mathcal{J}} B$, say $A = PBQ$ for some $P, Q \in M_n(S)$. Thus $A \leq_{\mathcal{L}} BQ$ and $BQ \leq_{\mathcal{R}} B$ by giving $R_S(A) \subseteq R_S(BQ)$ and $C_S(BQ) \subseteq C_S(B)$ by Theorem 3.1. By the dual to Theorem 5.1, there exists a surjective linear map from $R_S(B)$ onto $R_S(BQ)$. Thus, setting $Y = R_S(BQ)$ yields (iii).

That (iii) implies (ii) is trivial, so it remains only to show that (ii) implies (i). Let Y be a tropical convex set with the given properties. Since Y is a morphic image of $R_S(B)$, it is generated by the images of the rows of B. Thus, we may suppose that Y is the row space of some matrix, say $F \in M_n(S)$. Now by the dual to Theorem 5.2, there is a matrix C with $F \mathcal{L} C \leq_{\mathcal{R}} B$, and by Theorem 5.2, there is a matrix D with $A \mathcal{R} D \leq_{\mathcal{L}} F$. Thus, $A \leq_{\mathcal{J}} B$, so (i) holds. □

In [22] we saw that the $\mathcal{J}$-relation on the semigroup of 2×2 tropical matrices was characterised by the notion of mutual isometric embedding of projective column (dually, row) spaces. We show now that isometric embedding of projective column (dually, row) spaces is a necessary condition of the $\mathcal{J}$-order, although we shall see later (Section 7 below) that it is not a sufficient condition. The proof is based on the Metric Duality Theorem from [16] (Theorem 2.3 above).

Theorem 5.4. *Let $A, B \in M_n(S)$ for $S \in \{\mathbb{FT}, \overline{\mathbb{T}}\}$. If $A \leq_{\mathcal{J}} B$ then*

(i) *$PC_S(A)$ embeds isometrically into $PC_S(B)$;*
(ii) *$PR_S(A)$ embeds isometrically into $PR_S(B)$.*

Proof. We show that (i) holds, with (ii) being dual. Since $A \leq_{\mathcal{J}} B$ there exist $P, Q \in M_n(S)$ such that $A = PBQ$. Now $A \leq_{\mathcal{R}} PB \leq_{\mathcal{L}} B$ so by Theorem 3.1 we have $C_S(A) \subseteq C_S(PB)$ and $R_S(PB) \subseteq R_S(B)$. It follows that $PC_S(A) \subseteq PC_S(PB)$ and $PR_S(PB) \subseteq PR_S(B)$. Now by Theorem 2.3, $PC_S(PB)$ is isometric to $PR_S(PB)$, and $PR_S(B)$ is isometric to $PC_S(B)$. By composing inclusions and isometries in the appropriate order, we obtain an isometric embedding of $PC_S(A)$ into $PC_S(B)$. □

We remark that there is no reason to believe that the isometric embedding constructed in the proof of Theorem 5.4 is a linear morphism. The inclusions are of course linear, and the isometries are "anti-isomorphisms" in the sense of [16], but this is not sufficient to ensure that the composition is linear. The distinction here ultimately stems from the distinction between a meet-semilattice morphism (which by definition preserves greatest lower bounds) and an order-morphism of meet-semilattices (which need not).

Theorem 5.2 (and also Theorem 5.3) implies that a *linear* embedding of row spaces is a sufficient condition for two matrices to be related in the $\mathcal{J}$-order, while Theorem 5.4 says that an *isometric* embedding of (projective) row spaces is a necessary condition for the same property. It is very natural to ask, then, whether the former condition is necessary, or the latter condition is sufficient. In Section 7 we shall give examples to show that, in general, neither is the case.

6. $\mathcal{D} = \mathcal{J}$ in the finitary case

One of the most fundamental structural questions about any semigroup is that of whether the relations $\mathcal{D}$ and $\mathcal{J}$ coincide. These relations are always equal in finite semigroups (more generally, in compact topological semigroups), but differ in many important infinite semigroups. Semigroups in which $\mathcal{D} = \mathcal{J}$ have an ideal structure which is considerably easier to analyse.

The full matrix semigroup $M_n(K)$ of matrices over a field K is a well-known example of an (infinite, provided K is infinite) semigroup in which $\mathcal{D} = \mathcal{J}$. In [22, Theorem 3.7] we showed that $M_2(\mathbb{T})$ also has $\mathcal{D} = \mathcal{J}$, but Izhakian and Margolis [19] have recently produced examples to show that $\mathcal{D} \neq \mathcal{J}$ in $M_3(\mathbb{T})$; it follows easily that $\mathcal{D} \neq \mathcal{J}$ in $M_n(\mathbb{T})$ for all $n \geq 3$, and hence using Proposition 4.1 and Theorem 3.2 also in $M_n(\overline{\mathbb{T}})$ for $n \geq 3$. In contrast, in this section we shall show that the finitary tropical matrix semigroup $M_n(\mathbb{FT})$ satisfies $\mathcal{D} = \mathcal{J}$ for all n.

Our proof makes use of some topology. For convenience, we identify $\mathbb{PFT}^{n-1}$ with $\mathbb{R}^{n-1}$ via the correspondence

$$[(x_1, \ldots, x_n)] \mapsto (x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n). \tag{6.1}$$

With this identification, the Hilbert projective metric on $\mathbb{PFT}^{n-1}$ is Lipschitz equivalent to the standard Euclidean metric (which we will denote by d_E) on $\mathbb{R}^{n-1}$. Indeed, it is an easy exercise to verify that for any points $x, y \in \mathbb{R}^{n-1} = \mathbb{PFT}^{n-1}$ we have

$$d_H(x, y) \leq \sqrt{2}\, d_E(x, y) \quad \text{and} \quad d_E(x, y) \leq \sqrt{n-1}\, d_H(x, y).$$

In particular, the two metrics induce the same topology, so we may speak without ambiguity of a sequence of points converging.

Theorem 6.1 ($\mathcal{D} = \mathcal{J}$ for Finitary Tropical Matrices)**.** *Let $A, B \in M_n(\mathbb{FT})$. Then $A\ \mathcal{J}\ B$ if and only if $A\ \mathcal{D}\ B$.*

Proof. Clearly, if A and B are $\mathcal{D}$-related then they are also $\mathcal{J}$-related. Now suppose for a contradiction that $A\ \mathcal{J}\ B$, but A is not $\mathcal{D}$-related to B. We claim first that there is an isometric (with respect to the Hilbert projective metric) self-embedding $f : PR_{\mathbb{FT}}(A) \to PR_{\mathbb{FT}}(A)$ which is not an isometry (that is, which is not surjective).

To construct such a map, we proceed much as in the proof of Theorem 5.4. Since $A \leq_{\mathcal{J}} B$ we may write $A = PBQ$ for some $P, Q \in M_n(\mathbb{FT})$. Letting $X = BQ$, we have $A \leq_{\mathcal{L}} X \leq_{\mathcal{R}} B$, so by Theorem 3.1,

$$PR_{\mathbb{FT}}(A) \subseteq PR_{\mathbb{FT}}(X) \quad \text{and} \quad PC_{\mathbb{FT}}(X) \subseteq PC_{\mathbb{FT}}(B).$$

Now by Theorem 2.3, there is an isometry from $PR_{\mathbb{FT}}(X)$ to $PC_{\mathbb{FT}}(X)$ and an isometry from $PC_{\mathbb{FT}}(B)$ to $PR_{\mathbb{FT}}(B)$. Composing these inclusions and isometries in the appropriate order, we obtain an isometric embedding

$$h : PR_{\mathbb{FT}}(A) \to PR_{\mathbb{FT}}(B).$$

We claim that h is not surjective. Indeed, for h to be surjective we would clearly have to have $PC_{\mathbb{FT}}(X) = PC_{\mathbb{FT}}(B)$ and $PR_{\mathbb{FT}}(A) = PR_{\mathbb{FT}}(X)$, which by Theorem 3.1 would yield $A\ \mathcal{L}\ X\ \mathcal{R}\ B$ and hence $A\ \mathcal{D}\ B$, giving a contradiction.

Dually, since $B \leq_{\mathcal{J}} A$, we may construct a non-surjective isometric embedding $g : PR_{\mathbb{FT}}(B) \to PR_{\mathbb{FT}}(A)$. Now let $f : PR_{\mathbb{FT}}(A) \to PR_{\mathbb{FT}}(A)$ be the non-surjective isometric self-embedding given by the composition $g \circ h$.

By [23, Proposition 2.6], the projective row space $PR_{\mathbb{FT}}(A)$ is a compact subset of $\mathbb{PFT}^{n-1} = \mathbb{R}^{n-1}$. We will use this fact to deduce the desired contradiction. Let $X_0 = PR_{\mathbb{FT}}(A)$. Since f is not a surjection and has closed image, we may choose an $x_0 \in X_0$ and $\varepsilon > 0$ such that $x_0 \notin f(X_0)$ and $d_H(x_0, y) \geq \varepsilon$ for all $y \in f(X_0)$. Now we define a sequence of points by $x_i = f^i(x_0)$, and a decreasing sequence of sets by $X_i = f^i(X_0)$.

Notice that, since f^i is an isometric embedding, it follows from the properties of x_0 and X_0 that each x_i lies in X_i but satisfies $d_H(x_i, y) \geq \varepsilon$ for all $y \in X_{i+1}$. In particular, whenever $j > i$ we have $x_j \in X_j \subseteq X_{i+1}$ so that $d_H(x_i, x_j) \geq \varepsilon$. It follows that the x_i's cannot contain a convergent subsequence, which contradicts the fact that they are contained in the compact set $X_0 = PR_{\mathbb{FT}}(A)$. $\square$

7. Embeddings do not characterise the $\mathcal{J}$-order.

We have already seen that, by Theorem 5.2, a linear embedding of row spaces is a sufficient condition for two matrices to be related in the $\mathcal{J}$-order, while, by Theorem 5.4, an isometric embedding of (projective) row spaces is a necessary condition for the same property. It is very natural to ask, then, whether an exact characterisation of the $\mathcal{J}$-order can be obtained in terms of (linear or isometric) embeddings alone. In this section we answer this question in the negative, by giving examples to show that linear embedding of row spaces is not a necessary condition for two matrices to be related in the $\mathcal{J}$-order, while isometric embedding is not a sufficient condition. By Theorem 5.2 and the fact that the $\mathcal{J}$-order is invariant under taking matrix transposes, this also suffices to exclude linear or isometric embeddings of column spaces, or linear surjections of row or column spaces as exact characterisations of the $\mathcal{J}$-order.

To show that our examples have the claimed properties, we shall need some more concepts and terminology. Let $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$ and let X be a finitely generated tropical convex set in S^n. A set $\{x_1, \ldots, x_k\} \subseteq X$ is called a *weak basis* of X if it is a generating set for X minimal with respect to inclusion. It is known that every finitely generated tropical convex set admits a weak basis, which is unique up to permutation and scaling (see [27, Theorem 1 and Corollary 3.6] for the case $S \in \{\mathbb{T}, \overline{\mathbb{T}}\}$ and [10, Theorem 3.2] for the case $S = \mathbb{FT}$). In particular, any two weak bases have the same cardinality, in view of which we may define the *generator dimension* of a finitely generated tropical convex set X to be the cardinality of a weak basis for X, or equivalently, the minimum cardinality of a generating set for X. Generator dimension is closely related to the notion which was called *linear independence* in [9, Chapter 16].
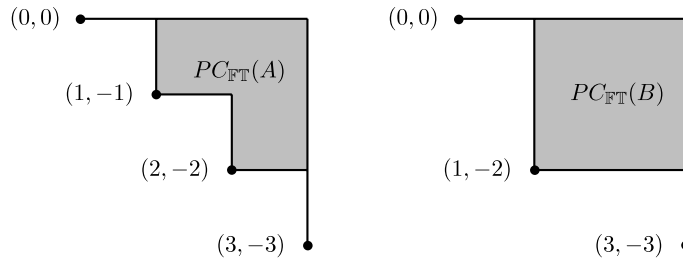
Fig. 1. The projective column spaces of the matrices A and B from Example 7.1.

Note that generator dimension is *not* well-behaved with respect to inclusion: a tropical convex set of generator dimension k may contain a tropical convex set of generator dimension strictly greater than k. In particular, for $n \geq 3$ the generator dimension of a finitely generated tropical convex set in S^n can exceed n.

However, it is easily seen that generator dimension is well-behaved with respect to linear surjections: if $f : X \to Y$ is a linear surjection then the generator dimension of X is at least that of Y. Indeed, if B is a weak basis for X then $f(B)$ is a generating set for Y, and so Y has generator dimension at most $|f(B)| \leq |B|$. In particular, generator dimension is an isomorphism invariant.

Example 7.1 (*Linear Embedding is Not Necessary for the $\mathcal{J}$-Order*). Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in $M_4(\mathbb{FT})$. Then $A \leq_{\mathcal{J}} B$; indeed $A \leq_{\mathcal{R}} B$ since $A = BX$ where

$$X = \begin{pmatrix} 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \\ -1 & 0 & 0 & -1 \\ -3 & -3 & -1 & 0 \end{pmatrix},$$

for example. Thus $C_{\mathbb{FT}}(A) \subseteq C_{\mathbb{FT}}(B)$ by Theorem 3.1(iii). Since every element of $C_{\mathbb{FT}}(A)$ and every element of $C_{\mathbb{FT}}(B)$ has the form $(a, b, c)^T$, the elements of the corresponding projective column spaces all have the form $(a - c, b - c, 0)^T$ (where, as before, we identify $\mathbb{PFT}^3$ with $\mathbb{R}^3$ via the map given in (6.1)). Since the third coordinate is fixed we may therefore draw our projective column spaces in two dimensions, as in Fig. 1.

It is easy to check that $C_{\mathbb{FT}}(A)$ has generator dimension 4, while $C_{\mathbb{FT}}(B)$ has generator dimension 3 (the points labelled in Fig. 1 are the projectivisations of a weak basis for $C_{\mathbb{FT}}(A)$ and $C_{\mathbb{FT}}(B)$ respectively). It follows by the above discussion that there cannot be a surjective linear morphism from $C_{\mathbb{FT}}(B)$ onto $C_{\mathbb{FT}}(A)$, and hence by Theorem 5.2, $R_{\mathbb{FT}}(A)$ does not embed in $R_{\mathbb{FT}}(B)$.

It is an easy exercise to extend the dimension of this example, so as to show that linear embedding of row spaces is not necessary for the $\mathcal{J}$-order in $M_n(\mathbb{FT})$ for all $n \geq 4$ and hence by Proposition 4.1 also in $M_n(\mathbb{T})$ and $M_n(\overline{\mathbb{T}})$ for $n \geq 4$. Using [22, Theorem 3.6] it can be shown that linear embedding of row spaces is an exact characterisation of the $\mathcal{J}$-order in $M_2(\mathbb{T})$ (and hence also in $M_2(\mathbb{FT})$, by Proposition 4.1). The three-dimensional case, that is, whether linear embedding of row spaces is an exact characterisation of the $\mathcal{J}$-order in $M_3(\mathbb{FT})$, $M_3(\mathbb{T})$ and/or $M_3(\overline{\mathbb{T}})$, remains open.

Example 7.1 shows that linear embedding does not (even in the finitary case) characterise the $\mathcal{J}$-order, but it does not rule out the possibility that *mutual* linear embedding characterises the $\mathcal{J}$-equivalence. Indeed, in the finitary case, Theorem 3.1(vi) and Theorem 6.1 together imply that $\mathcal{J}$-related matrices have linearly isomorphic column spaces, and hence in particular, mutually embedding column spaces. The following example shows that this characterisation does not extend to the case of $\mathbb{T}$.

Example 7.2 (*Mutual Linear Embedding is Not Necessary for $\mathcal{J}$*). Consider the matrices

$$A = \begin{pmatrix} -\infty & 0 & 1 & 1 \\ -\infty & -\infty & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, \quad B = \begin{pmatrix} -\infty & 0 & 1 & 1 \\ -\infty & -\infty & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

in $M_4(\mathbb{T})$. We first claim that $A \mathcal{J} B$. Let μ denote the linear embedding $\mu : \mathbb{T}^4 \to \mathbb{T}^4$ given by $\mu : (x, y, z, t) \mapsto (x, y, z+1, t)$. It is easily verified that the restriction of μ to $C_{\mathbb{T}}(A)$ gives an embedding $\mu : C_{\mathbb{T}}(A) \to C_{\mathbb{T}}(B)$ and straightforward to check that the restriction of μ to $C_{\mathbb{T}}(B)$ gives an embedding $\mu : C_{\mathbb{T}}(B) \to C_{\mathbb{T}}(A)$. Thus μ gives mutual linear embeddings of the column spaces. Hence, by the dual to Theorem 5.2, we deduce that $A \mathcal{J} B$. It is also easy to check that $C_{\mathbb{T}}(A)$ has generator
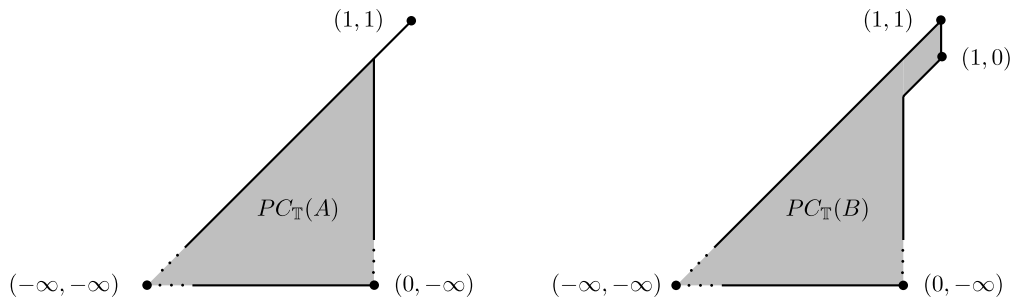
Fig. 2. The projective column spaces of the matrices *A* and *B* from Example 7.2.

dimension 3 while $C_{\mathbb{T}}(B)$ has generator dimension 4. So by the same argument as in Example 7.1, $C_{\mathbb{T}}(A)$ cannot surject onto $C_{\mathbb{T}}(B)$. It can then be shown that $R_{\mathbb{T}}(B)$ cannot embed in $R_{\mathbb{T}}(A)$.

Every non-zero element of $C_{\mathbb{T}}(A)$ and $C_{\mathbb{T}}(B)$ has the form $(a, b, c, -\infty)^T$, where $c \neq -\infty$. Thus, we may identify the elements of the projectivisations with elements of the form $(a - c, b - c, 0, -\infty)^T$. Since the third and fourth coordinates are fixed we may therefore draw our projective column spaces in two dimensions, as in Fig. 2. The projective row space of *A* can be drawn similarly, but that of *B* is harder to illustrate in two dimensions.

Again, it is straightforward to extend this example to higher dimensions, showing that mutual linear embedding of row spaces is not necessary for two matrices to be $\mathcal{J}$-related in $M_n(\mathbb{T})$ for $n \geq 4$, and hence by Proposition 4.1 also in $M_n(\overline{\mathbb{T}})$, for $n \geq 4$. We know from [22, Theorem 3.7] that $\mathcal{D} = \mathcal{J}$ in $M_2(\mathbb{T})$, so by the same argument as in the finitary case we see that mutual linear embedding exactly characterises $\mathcal{J}$. The three-dimensional case, that is, whether mutual linear embedding of row spaces is necessary for $\mathcal{J}$-equivalence in $M_3(\mathbb{T})$ and/or $M_3(\overline{\mathbb{T}})$, remains open.

Example 7.3 (*Isometry is Not Sufficient for $\mathcal{J}$-Equivalence.*) Consider the matrix

$$A = \begin{pmatrix} 1 & 6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $M_3(\mathbb{FT})$. By Theorem 2.3 we have that $PR_{\mathbb{FT}}(A)$ is isometric to $PC_{\mathbb{FT}}(A)$, or equivalently, $PR_{\mathbb{FT}}(A)$ is isometric to $PR_{\mathbb{FT}}(A^T)$. As usual, we identify $\mathbb{PFT}^2$ with $\mathbb{R}^2$ via the map given in (6.1). Fig. 3 illustrates these isometric row spaces. We claim that *A* is not $\mathcal{J}$-related to A^T.

Since *A* has only finite entries, by Theorem 6.1 $A \mathcal{J} A^T$ if and only if $A \mathcal{D} A^T$, which by Theorem 3.1(vii) holds if and only if $R_{\mathbb{FT}}(A)$ and $R_{\mathbb{FT}}(A^T)$ are linearly isomorphic. Thus, it will suffice to show that these spaces are not linearly isomorphic.

Let r_1, r_2, r_3 denote the rows of *A* and c_1, c_2, c_3 denote the columns. It is easy to see that the rows of *A* form a weak basis for $R(A)$ and the columns of *A* form a weak basis for $R(A^T)$. Suppose for a contradiction that $f : R_{\mathbb{FT}}(A) \to R_{\mathbb{FT}}(A^T)$ is an isomorphism of $\mathbb{FT}$-modules. Since f is an isomorphism, it maps weak bases to weak bases. Thus

$$\{f(r_1), f(r_2), f(r_3)\} = \{\alpha_1 \otimes c_1, \alpha_2 \otimes c_2, \alpha_3 \otimes c_3\},$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{FT}$. Moreover, it is easy to see that f must preserve distances, so $d_H(f(r_i), f(r_j)) = d_H(r_i, r_j)$ for all i and j. But a straightforward calculation reveals that $\{d_H(r_i, r_j)\} = \{0, 3, 5, 6\}$, whilst $\{d_H(f(r_i), f(r_j))\} = \{0, 4, 6\}$, thus contradicting the existence of an isomorphism between $R_{\mathbb{FT}}(A)$ and $R_{\mathbb{FT}}(A^T)$.

Again, this example extends easily to $n \geq 3$, showing that an isometry between row spaces is not sufficient to imply that two matrices are $\mathcal{J}$-related in $M_n(\mathbb{FT})$ for all $n \geq 3$, and hence by Proposition 4.1 also in $M_n(\mathbb{T})$ and $M_n(\overline{\mathbb{T}})$. It also follows, in all of these semigroups, that an isometric embedding of the row space of *A* into the row space of *B* does not imply that $A \leq_{\mathcal{J}} B$. In the case $n = 2$, it follows from [22, Theorem 3.7] that isometry of row spaces does give an exact characterisation of $\mathcal{J}$ in $M_2(\mathbb{FT})$ and $M_2(\mathbb{T})$.

8. $\mathcal{J}$ and the rank of a tropical matrix

In the full matrix semigroup $M_n(K)$ of matrices over a field K, two matrices are $\mathcal{J}$-related (and hence also $\mathcal{D}$-related) if and only if they have the same *rank*. In this classical situation there are several equivalent definitions of rank, stemming from the ideas of matrix factorisation, linear independence of rows or columns and singularity. Unfortunately, in the case of matrices over a semiring, these definitions cease to be equivalent. In this section we shall look at how $\mathcal{J}$-classes (and $\mathcal{D}$-classes) relate to various ideas of the rank of a tropical matrix. We shall show that many of the commonly studied ranks are $\mathcal{J}$-class invariants.

We begin by considering an arbitrary commutative semiring *S*. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a *rank function* to be a map
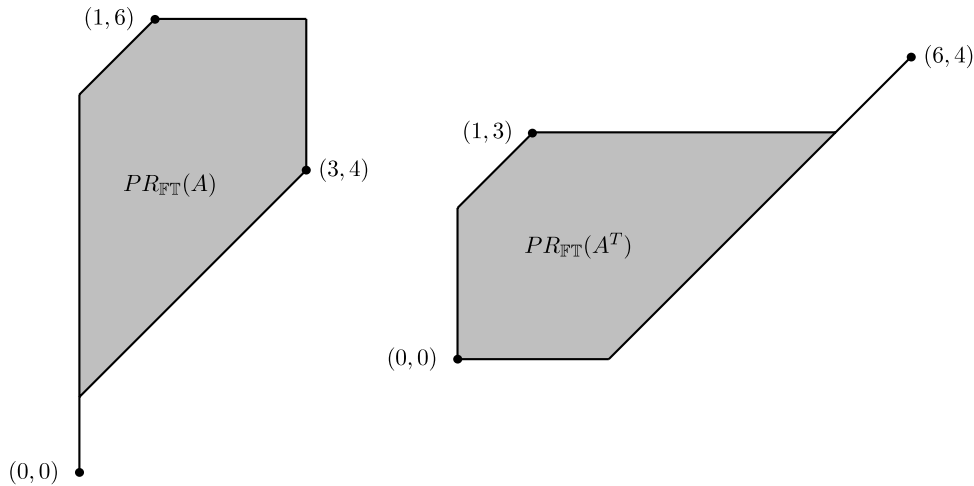
$$\text{rank} : M_n(S) \to \mathbb{N}_0.$$

Fig. 3. The projective row spaces of the matrices A and A^T from Example 7.3.

We say that a given rank function *respects the $\mathcal{J}$-order* if $A \leq_{\mathcal{J}} B$ implies rank$(A) \leq$ rank(B). Clearly, any function which respects the $\mathcal{J}$-order is in particular a $\mathcal{J}$-class invariant, although the converse need not hold. We say that a rank function *satisfies the rank-product inequality* if and only if for all $A, B \in M_n(S)$ we have

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)).$$

The following elementary proposition observes that a rank function respects the $\mathcal{J}$-order exactly if it satisfies the rank-product inequality.

Proposition 8.1. *Let S be a semiring and* rank $: M_n(S) \to \mathbb{N}_0$ *a rank function. Then the function respects the $\mathcal{J}$-order if and only if*

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$$

for all $A, B \in M_n(S)$.

Proof. Suppose first that rank$(AB) \leq \min(\text{rank}(A), \text{rank}(B))$ for all A and B. If $X \leq_{\mathcal{J}} Y$ then we may write $X = PYQ$, where $P, Q \in M_n(S)^1$. If $P, Q \in M_n(S)$ then

$$\text{rank}(X) \leq \min(\text{rank}(P), \text{rank}(YQ))$$
$$\leq \min(\text{rank}(P), \text{rank}(Y), \text{rank}(Q)) \leq \text{rank}(Y).$$

Similar arguments apply when one or both of P and Q is an adjoined identity.

Conversely, suppose rank respects the $\mathcal{J}$-order. Then for $A, B \in M_n(S)$ we have $AB \leq_{\mathcal{J}} A$ and $AB \leq_{\mathcal{J}} B$, so rank$(AB) \leq$ rank(A) and rank$(AB) \leq$ rank(B) as required. □

We shall see below that a number of natural notions of rank (over the tropical semiring, or sometimes over more general semirings) have been shown to satisfy the rank-product inequality. It follows that all of these respect the $\mathcal{J}$-order, and hence in particular are $\mathcal{J}$-class invariants.

Example 8.2 (*Factor Rank*). Let S be a commutative semiring with addition denoted by $\oplus$ and multiplication by juxtaposition. Let A be a non-zero $m \times n$ matrix over S. Recall that the *factor rank* of A is the smallest natural number k such that A can be written as a product $A = BC$ with B an $m \times k$ matrix and C a $k \times n$ matrix. Equivalently, the factor rank is the smallest natural number k such that A can be written as

$$A = c_1 r_1 \oplus \cdots \oplus c_k r_k,$$

for some column m-vectors $c_1, \ldots, c_k$ and row n-vectors $r_1, \ldots, r_k$. Put another way, the factor rank of A is the smallest cardinality of a set of n-vectors whose S-linear span contains the rows of A. By convention, the zero matrix has factor rank 0, and it is clear that no other matrix has factor rank 0.

In the case where S is a field, factor rank coincides with the usual definition of rank. Factor rank has been widely studied over the Boolean semiring (where it is called *Schein rank* [24]) and the tropical semiring (where it is sometimes called *Barvinok rank* [12]). It has been observed in various semirings (see for example [3, Proposition 4.4]) that factor rank satisfies the rank-product inequality, and hence in our terminology respects the $\mathcal{J}$-order. However, this fact does not appear to have been stated for commutative semirings in full generality; for this reason we include a very brief proof.

Corollary 8.3. *Let S be a commutative semiring and $n \in \mathbb{N}$. Then factor rank respects the $\mathcal{J}$-order, and hence is a $\mathcal{J}$-class invariant, in $M_n(S)$.*

Proof. Let $A, B \in M_n(S)$ with $A \leq_{\mathcal{J}} B$. We shall show that the factor rank of A does not exceed that of B. Note that if S has a zero and B is the zero matrix then the result holds trivially, since then we must have that $A = B$. Thus we may assume that B is non-zero. Let $k > 0$ be the factor rank of B. Then k is the smallest natural number such that we may write

$$B = c_1 r_1 \oplus \cdots \oplus c_k r_k,$$

where each c_i is a $n \times 1$ column vector and each r_i is a $1 \times n$ row vector. Since $A \leq_{\mathcal{J}} B$ we have $A = PBQ$ for some matrices $P, Q \in M_n(S)$[1]. Thus, by associativity and distributivity of matrix multiplication,

$$A = (Pc_1)(r_1 Q) \oplus \cdots \oplus (Pc_k)(r_k Q),$$

where each Pc_i is a $n \times 1$ column vector and each $r_i Q$ is a $1 \times n$ row vector. This gives that the factor rank of A is less than or equal to k. In other words, the factor rank of A is less than or equal to the factor rank of B. $\square$

Example 8.4 (*Column and Row Rank*). Let S be a semiring, $n \in \mathbb{N}$ and A be a non-zero matrix in $M_n(S)$. The *column rank* of A is the cardinality of the smallest generating set for the column space of A. In the case where $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$, the column rank is simply the generator dimension of the column space, as discussed in Section 7. The *row rank* of A is defined dually; it is well known that the row rank and column rank of a tropical matrix can differ (see Example 7.1, for instance). The zero matrix, in the case where S has a zero element, has column rank and row rank 0.

Column rank and row rank are closely connected to the notion which was called *weak independence*, which was first introduced in [9, Chapter 16]; see [5] for survey of these ideas.

In Example 7.2 of Section 7 above we exhibited two $\mathcal{J}$-related matrices in $M_4(\mathbb{T})$ whose column spaces have different generator dimension. Hence column rank (and, by symmetry, row rank) is not a $\mathcal{J}$-class invariant, and does not respect the $\mathcal{J}$-order. However, it follows easily from results of Hollings and the second author [16] (quoted as part of Theorem 3.1 above) that they are $\mathcal{D}$-class invariants:

Corollary 8.5. *Let $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$, $n \in \mathbb{N}$ and $A, B \in M_n(S)$. If $A\mathcal{D}B$ then A and B have the same column rank and the same row rank.*

Proof. By Theorem 3.1(vi) and (vii), A and B have isomorphic column spaces and isomorphic row spaces, and it is immediate from the definitions that isomorphic spaces have the same generator dimension. Thus, A and B have the same column rank and the same row rank. $\square$

An interesting observation is the following. If X is a finitely generated tropical convex set (say of row vectors), then by Theorem 3.1 and Corollary 8.5, there exists a $k \in \mathbb{N}$ which is the column rank of *every* matrix whose row space is X. In other words, X, in addition to its own generator dimension as a space of row vectors, admits an invariant which one might call its *dual dimension*. One might ask whether this dimension manifests itself in a "coordinate-free" manner in the space X itself. In fact, over the semirings $\mathbb{FT}$ or $\overline{\mathbb{T}}$, X is *anti-isomorphic* to the column space of any matrix of which it is the row space (see [16, Theorem 4.4]), so this dual dimension is exactly the minimum cardinality of a generating set for X under the operations of scaling and *greatest lower bound* within X. (Note that greatest lower bound within X does *not* necessarily coincide with componentwise minimum, since X need not be closed under the latter operation.)

While column rank and row rank are not $\mathcal{J}$-class invariants in general (as shown for example in Example 7.2 above), we know that $M_2(\mathbb{T})$ (by [22, Theorem 3.7]) and $M_n(\mathbb{FT})$ for all n (by Theorem 6.1) satisfy $\mathcal{D} = \mathcal{J}$, so in these semigroups column rank and row rank *are* $\mathcal{J}$-class invariants.

Example 8.6 (*Gondran–Minoux Rank [14]*). Let S be a commutative semiring with zero element. We say that vectors $x_1, \ldots, x_t$ are *Gondran–Minoux independent* if whenever

$$\sum_{i \in I} \alpha_i x_i = \sum_{j \in J} \alpha_j x_j,$$

with $I, J \subset \{1, \ldots, t\}, I \cap J = \emptyset$ and $\alpha_1, \ldots, \alpha_t \in S$ it follows that $\alpha_1, \ldots, \alpha_t$ are all zero. Now let $A \in M_n(S)$. The *maximal Gondran–Minoux column [row] rank* of A is the maximal number of Gondran–Minoux independent columns [rows] of A.

Note that this notion of rank is explicitly defined in terms of the actual columns of A, rather than just the column space, so there is no immediate reason to suppose that it is a column space invariant (or equivalently by Theorem 3.1(iv), an $\mathcal{R}$-class invariant). Rather surprisingly, however, recent results of Shitov [25] imply that maximal Gondran–Minoux column rank even respects the $\mathcal{J}$-order in full matrix semigroups over $\mathbb{T}$ and $\overline{\mathbb{T}}$. Indeed, in [25, Theorem 2.9] it is shown that for a class of semirings including $\mathbb{T}$ and $\overline{\mathbb{T}}$ (specifically, idempotent *quasi-selective* semirings with zero and without zero divisors), this notion of rank satisfies the rank-product inequality, so Proposition 8.1 yields:

Corollary 8.7. *Let $S \in \{\mathbb{T}, \overline{\mathbb{T}}\}$ and $n \in \mathbb{N}$. Then maximal Gondran–Minoux column rank and maximal Gondran–Minoux row rank respect the $\mathcal{J}$-order, and hence are $\mathcal{J}$-class invariants, in $M_n(S)$.*

It is interesting that, although the Gondran–Minoux ranks give $\mathcal{J}$-class invariants in both $\mathbb{T}$ and $\overline{\mathbb{T}}$, the rank functions themselves are dependent upon which tropical semiring one works in (whilst the $\mathcal{J}$-classes are not, by Proposition 4.1). For instance, the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

has maximal Gondran–Minoux column rank 2 over $\mathbb{T}$, but 1 over $\overline{\mathbb{T}}$.

Example 8.8 (*Determinantal Rank*). Another equivalent way to define rank in classical linear algebra is as the dimension of the largest non-singular submatrix. In a semiring, without negation, it is not entirely clear how to define singularity. However, one reasonable approach is to regard a matrix as singular if the terms which would normally have positive coefficients in the determinant have the same sum as the terms as those which would normally have negative coefficients.

More formally, for a $k \times k$ matrix M over a commutative semiring, we define

$$|M|^+ = \sum_{\substack{\sigma \in S_k, \\ \text{sign}(\sigma)=1}} m_{1,\sigma(1)} \cdots m_{k,\sigma(k)}$$

$$|M|^- = \sum_{\substack{\sigma \in S_k, \\ \text{sign}(\sigma)=-1}} m_{1,\sigma(1)} \cdots m_{k,\sigma(k)},$$

where S_k denotes the symmetric group on $\{1, \ldots, k\}$. The *determinantal rank* of A is the largest integer k such that A has a $k \times k$ minor M with $|M|^+ \neq |M|^-$. In [2, Theorem 9.4] it was shown that determinantal rank over $\mathbb{T}$ (and hence also over $\mathbb{FT}$) satisfies the rank-product inequality, which combined with Proposition 8.1 yields:

Corollary 8.9. *Let $S \in \{\mathbb{FT}, \mathbb{T}\}$ and $n \in \mathbb{N}$. Then determinantal rank respects the $\mathcal{J}$-order, and hence is a $\mathcal{J}$-class invariant, in $M_n(S)$.*

Example 8.10 (*Tropical Rank*). Another natural, and frequently used, notion of singularity for tropical matrices is the following. For $S \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}\}$ we define a matrix $M \in M_k(S)$ to be *strongly regular* if there is no non-empty subset $T \subseteq S_k$ such that

$$\bigoplus_{\sigma \in T} m_{1,\sigma(1)} \cdots m_{k,\sigma(k)} = \bigoplus_{\sigma \in S_k \backslash T} m_{1,\sigma(1)} \cdots m_{k,\sigma(k)}.$$

Now for $A \in M_n(S)$, the *tropical rank* of A is the largest integer k such that A has a strongly regular $k \times k$ minor.

Strongly regular matrices and tropical rank were first studied in [9, Chapters 16 and 17], where they are called just *regular* matrices and *rank* respectively. A number of equivalent formulations have since been discovered (see for example [2,5,12,21]). Perhaps most interestingly, over $\mathbb{FT}$ tropical rank is the maximum topological dimension of the row (or column) space viewed as a subset of $\mathbb{R}^n$ with the usual topology [12, Theorem 4.1]. In [2, Theorem 9.4] it was shown that tropical rank of matrices over $\mathbb{T}$ (and hence also over $\mathbb{FT}$) satisfies the rank-product inequality. Combining this with Proposition 8.1 we have:

Corollary 8.11. *Let $S \in \{\mathbb{FT}, \mathbb{T}\}$ and $n \in \mathbb{N}$. Then tropical rank respects the $\mathcal{J}$-order, and hence is a $\mathcal{J}$-class invariant, in $M_n(S)$.*

Finally, we remark briefly that there are a number of other notions of rank for tropical matrices (see for example [2]), and we do not claim that the study presented in this section is exhaustive. One which has proved to be of interest for applications in algebraic geometry is that of *Kapranov rank* (see for example [12]); its relationship with Green's relations deserves detailed study.

Acknowledgements

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