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Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: Crossings and extremes

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Abstract

We propose a new method to get the Hermite polynomial expansion of crossings of any level by a stationary Gaussian process, as well as the one of the number of maxima in an interval, under some assumptions on the spectral moments of the process.

Keywords: Gaussian processes; Crossings; Extremes; Hermite polynomial expansion

1. Introduction

In this work we consider the Hermite expansion (or equivalently expansion into the Itô–Wiener Chaos) for non-smooth functionals of a stationary Gaussian process. This type of expansion has been studied by Slud (1991) for the zero crossings by the process, then has been generalized to different levels in a recent paper by the same author (Slud, 1994). We propose here a new method. We define an analytical formula involving the Dirac function to approximate the number of crossings; it makes then explicit formulas for MWI expansions much easier to obtain than was true in the papers of Slud (1991, 1994), although the expressions obtained there were more general. In particular expanding $|\dot{X}_s|$ in Hermite polynomials in $\dot{X}_s$ rather than in X_s as was implicitly done by Slud (1994), quite simplify the calculations. Moreover we get also in a direct way the Hermite expansion of other functionals of Gaussian processes, like for instance the number of maxima in an interval.

Let us give an outline of our procedure. First we generalize the following formula proved by Kac (1943). Let $N_t(0)$ be the number of zero crossings by the stationary

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Gaussian process X_s, with mean zero and variance one. Suppose also that the variance of the derivative of the process is one. Then Kac formula is given by

$$N_t(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t \cos(\zeta X_s)|\dot{X}_s|\,\mathrm{d}s\,\mathrm{d}\zeta \quad \text{with probability one.}$$

Now consider the following approximation defined for all $x \in \mathbb{R}$ and with $\sigma > 0$ by

$$N_t^\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\mathrm{i}x\zeta - \frac{\sigma^2\zeta^2}{2}\right)\int_0^t \exp(\mathrm{i}\zeta X_s)|\dot{X}_s|\,\mathrm{d}s\,\mathrm{d}\zeta.$$

First we prove that $N_t^\sigma(x) \to N_t(x)$ a.e. as $\sigma \to 0$ (the convergence holds in $L^2(\Omega)$ too), where $N_t(x)$ is the number of crossings at the level $x \in \mathbb{R}$ by the process X_s. Then we show that there exists an Hermite expansion for $N_t^\sigma(x)$. Finally, by using the convergence in $L^2(\Omega)$ and the diagram formula, we get the expansion for $N_t(x)$.

The Hermite expansion for the number of crossings is given in Section 2, under the condition that the Gaussian process has a finite fourth spectral moment. In Section 3, we get a more general result by approximating the number of crossings of a process having a non-smooth derivative, by the number of crossings of a process with a smooth one. Section 4 provides the Chaos expansion for the number of maxima in an interval, under the condition that the process has a sixth spectral moment finite. Note that in this problem the functional depends on the first and the second derivative of the process as on the process itself. It makes then the development into the Itó–Wiener chaos more difficult, since the random variables are not all independent for a fixed t. To overcome this difficulty, we transform these random variables in orthogonal ones. Finally, we give a new approximation for moments of the number of crossings in the Appendix. The authors want to emphasize that the ideas through this paper are inspired in part by the work of Berman, in particular in his recent book on the Sojourns and Extremes of Stochastic Processes (1992), and are strongly motivated by the results of Slud (1991, 1994a, b).

2. Expansion for crossings

We suppose in this section that X_t is a mean zero stationary Gaussian process with variance one, that the function of covariance r has two derivatives and that

$$-r''(0) = 1 \quad \text{and} \quad r^{(\mathrm{iv})}(0) < \infty. \tag{1}$$

Note that the conditions $\mathrm{Var}(X_t) = 1$ and $-r''(0) = 1$ are made with no loss of generality since X_t can be replaced respectively by $X_t/\sqrt{\mathrm{Var}(X_t)}$ and by $X_{t/\sqrt{-r''(0)}}$. Note also that the condition (1) implies the Geman condition

$$\frac{r'(t) - r''(0)}{t} = L(t) \in L^1([0,\delta], \mathrm{d}x). \tag{2}$$

The nth Hermite polynomial H_n can be defined as

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < \alpha_k = t$ be the points where the change of sign of the derivative of X_s occurs. They are in a finite number because the process has a finite fourth spectral moment (condition (1)).

Note that $N_t(x) = \sum_{i=0}^{k-1} |Y_x(X_{\alpha_{i+1}}) - Y_x(X_{\alpha_i})|$, where $Y_x(u) = 1_{[x,\infty)}(u)$ is the Heaviside function whose generalized derivative is the "Dirac function" $\delta_x(u) = \infty$ if $u = x$ and 0 if not.

So we can write formally

$$N_t(x) = \int_0^t \delta_x(X_s)|\dot{X}_s|\,\mathrm{d}s. \tag{3}$$

To make it precise, it is sufficient to approximate δ_x by $\Phi'_{\sigma,x}$, where $\Phi_{\sigma,x}$ is the Gaussian distribution function with mean x and variance σ^2. Hence we introduce

$$N_t^\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_0^t \exp\left(-\frac{(X_s - x)^2}{2\sigma^2}\right)|\dot{X}_s|\,\mathrm{d}s. \tag{4}$$

Note that

$$N_t^\sigma(x) = \int_{-\infty}^{+\infty} N_t(u)\,\mathrm{d}\Phi_{\sigma,x}(u).$$

We have

$$N_t^\sigma(x) = \int_0^t \Phi'_{\sigma,x}(X_s)|\dot{X}_s|\,\mathrm{d}s = \sum_{j=1}^{k-1} |\Phi_{\sigma,x}(X_{\alpha_j}) - \Phi_{\sigma,x}(X_{\alpha_{j-1}})|$$

and since $\Phi_{\sigma,x}(u) \to Y_x(u)$ for each $u \neq x$ as $\sigma \to 0$, it follows that $N_t^\sigma(x) \to N_t(x)$ a.s. as $\sigma \to 0$. Moreover via Rice formula, we get

$$E[N_t^\sigma(x)] = \frac{t}{\pi}\frac{\mathrm{e}^{-x^2/2(1+\sigma^2)}}{(1+\sigma^2)^{1/2}} \to \frac{1}{\pi}t\mathrm{e}^{-x^2/2} = E(N_t(x)) \quad \text{as } \sigma \to 0.$$

The almost sure convergence and the last result imply the $L^1(\Omega)$ convergence. Let us prove that the convergence holds in $L^2(\Omega)$ too.

Lemma 1. *Under* (1) *we have*

$$N_t^\sigma \to N_t \quad \textit{as } \sigma \to 0 \textit{ in } L^2.$$

Proof. Since the Geman condition (2) is satisfied, we have that $E[N_t(x)^2] < \infty$, and by Fatou's lemma, $EN_t^2(x) \leqslant \liminf_{\sigma\to 0} E[N_t^\sigma(x)^2]$. Moreover by Jensen inequality,

$$E[N_t^\sigma(x)^2] = E\left(\left(\int_{\mathbb{R}} N_t(u)\,\mathrm{d}\Phi_{\sigma,x}(u)\right)^2\right) \leqslant \int_{\mathbb{R}} E[N_t^2(u)]\,\mathrm{d}\Phi_{\sigma,x}(u).$$

Further $E[N_t^2(u)]$ is a bounded and continuous function of u (cf. Cramér and Leadbetter, 1967) and $\mathrm{d}\Phi_{\sigma,x}(u)$ converges in distribution to the Dirac measure in x. We conclude that $\limsup_{\sigma\to 0} E[N_t^\sigma(x)^2] \leqslant E[N_t^2(x)]$ and that the convergence is in $L^2(\Omega)$, since the r.v. $N_t^\sigma(x)$ converge a.s. □

Proposition 1. *Let X_t be a mean zero stationary Gaussian process, with variance one and satisfying hypothesis* (1). *Then the following expansion holds in* $L^2(\Omega)$:

$$N_t(x) = \sum_{q=0}^{\infty} \sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}(\dot{X}_s)\, \mathrm{d}s,$$

with

$$b_k(x) = \frac{1}{k!\sqrt{2\pi}} \mathrm{e}^{-x^2/2} H_k(x).$$

Remarks. (i) This result is obtained by Slud (1994b) under a weaker condition, the Geman condition, but by using the MWI approach. Here we give a direct approach which can be applied to many other functionals and which may be adapted to the multidimensional case. This approach is natural in the sense that formally the Dirac function δ_x has the generalized Hermite expansion $\delta_x(u) = \sum_{k=0}^{\infty} b_k(x) H_k(u)$ with

$$b_k(x) = \int_{-\infty}^{\infty} \delta_x(y) \frac{1}{k!} H_k(y) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}\, \mathrm{d}y = \frac{1}{k!} H_k(x) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}};$$

then

$$N_t(x) = \int_0^t \delta_x(X_s) |\dot{X}_s|\, \mathrm{d}s$$

has the corresponding development given by Lemma 2 below, made precise by approximating δ_x by $\Phi'_{\sigma,x}$.

Note also that we get by the same way the Rice formula:

$$E(N_t(x)) = \int_0^t E[\delta_x(X_s)] E|\dot{X}_s|\, \mathrm{d}s = \frac{t}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \sqrt{-r''(0)};$$

more generally if g is a positive function on $\mathbb{R}$, if G is a primitive of g, then (cf. Cabaña, 1985)

$$\int_{\mathbb{R}} N_t(x) g(x)\, \mathrm{d}x = \int_0^t \left[\int_{\mathbb{R}} \delta_{X_s}(x) g(x)\, \mathrm{d}x \right] |\dot{X}_s|\, \mathrm{d}s = \int_0^t g(X_s) |\dot{X}_s|\, \mathrm{d}s$$
$$= \sum_{i=0}^{k-1} |G(X_{\alpha_{i+1}}) - G(X_{\alpha_i})|.$$

We get the Kac formula too: since $\hat{\delta}_0(t) = 1$, by applying formally the Fourier inversion formula, we have

$$\delta_0(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \cos(tu)\, \mathrm{d}t,$$

$$N_t(0) = \int_0^t \delta_0(X_s) |\dot{X}_s|\, \mathrm{d}s = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t \cos(\xi X_s) |\dot{X}_s|\, \mathrm{d}\xi\, \mathrm{d}s.$$

(ii) The heuristic formula (3) presents some analogy with the formula for the local time of the brownian motion. Indeed, let (X_t) be a standard brownian motion and L_t^x its local time. We know that

$$L_t^x = \int_0^t \delta_x(X_s)\,\mathrm{d}s \quad \text{and} \quad \int_0^t f(X_s)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x) L_t^x\,\mathrm{d}x.$$

We can compare those formulae respectively to the ones involving the number of crossings for a Gaussian process, namely

$$N_t(x) = \int_0^t \delta_x(X_s)|\dot{X}_s|\,\mathrm{d}s \quad \text{and} \quad \int_0^t f(X_s)|\dot{X}_s|\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x) N_t(x)\,\mathrm{d}x.$$

For the proof of Proposition 1, to get the Hermite polynomial expansion for $N_t^\sigma(x)$ defined in (4), we need the following result:

Lemma 2. *Let $f \in L^2(\phi(x)\,\mathrm{d}x)$ and let $(c_k, k \geq 0)$ be its Hermite coefficients. One has the following expansion*

$$\begin{aligned}\int_0^t f(X_s)|\dot{X}_s|\,\mathrm{d}s &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k a_{2l} \int_0^t H_k(X_s) H_{2l}(\dot{X}_s)\,\mathrm{d}s \\ &= \sum_{q=0}^{\infty}\sum_{l=0}^{[q/2]} c_{q-2l} a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}(\dot{X}_s)\,\mathrm{d}s,\end{aligned}$$

where $(a_k, k \geq 0)$ are the Hermite coefficients of the function $|x|$, defined by $a_0 = (2/\pi)^{1/2}$ and

$$a_{2l} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{l+1}}{2^l l!(2l-1)} \quad \textit{if } l \geq 1.$$

Proof. Recall that

$$H(X) = \bigoplus_{k=0}^{\infty} H_k$$

(called Wiener chaos), where $H(X)$ denotes the space of real square integrable functionals of the process $X(=X_t)$. (For more details, we refer the reader for instance to Chambers and Slud (1989)). Let us define

$$\zeta^{K,L} = \sum_{k=0}^{K}\sum_{l=0}^{L} c_k a_{2l} \int_0^t H_k(X_s) H_{2l}(\dot{X}_s)\,\mathrm{d}s.$$

Then

$$E[\zeta^{K',L'} - \zeta^{K,L}]^2 = E\left[\sum_{k=K+1}^{K'}\sum_{l=L+1}^{L'} c_k a_{2l} \int_0^t H_k(X_s) H_{2l}(\dot{X}_s)\,\mathrm{d}s\right]^2.$$

Cauchy–Schwarz inequality, the independence between X_s and $\dot{X}_s$ and the orthogonality of the chaos give

$$E[\zeta^{K',L'} - \zeta^{K,L}]^2 \leq t^2 \sum_{k=K+1}^{K'} c_k^2 k! \sum_{l=L+1}^{L'} a_{2l}^2 (2l)!.$$

Thus

$$\zeta^{K,L} \to \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k a_{2l} \int_0^t H_k(X_s) H_{2l}(\dot{X}_s)\,ds \quad \text{in } L^2(\Omega) \text{ as } K, L \to \infty.$$

And we deduce from the Hermite expansions of $|x|$ and $f(x)$ that

$$E\left[\int_0^t f(X_s)|\dot{X}_s|\,ds - \zeta^{K,L}\right]^2 \to 0 \quad \text{as } K, L \to \infty.$$

The second expansion that appears in the statement of the lemma is a consequence of the orthogonality. □

Proof of Proposition 1. Let us denote by $b_k^\sigma(x)$ the kth Hermite coefficient of the function

$$\Phi'_{\sigma,x}(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right).$$

Note that

$$b_k^\sigma(x) = \frac{1}{2\pi\sigma k!}\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right)\exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) H_k(y)\,dy$$

$$\to \frac{1}{k!\sqrt{2\pi}} e^{-x^2/2} H_k(x) = b_k(x) \quad \text{as } \sigma \to 0. \tag{5}$$

Then by using Lemma 2 and (4), we have that for each σ,

$$N_t^\sigma(x) = \sum_{q=0}^{\infty}\sum_{l=0}^{[q/2]} b_{q-2l}^\sigma(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}(\dot{X}_s)\,ds$$

and then

$$\sum_{q=0}^{Q} E\left[\sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}(\dot{X}_s)\,ds\right]^2 \leqslant \lim_{\sigma\to 0} E[N_t^\sigma(x)]^2 = E[N_t(x)]^2 \tag{6}$$

by using Fatou's lemma in the first inequality and the L^2 convergence of $N_t^\sigma(x)$ to $N_t(x)$ as $\sigma \to 0$ in the equality. Thus the following development defines a random variable in $L^2(\Omega)$:

$$\mathcal{N}_t(x) = \sum_{q=0}^{\infty}\sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}(\dot{X}_s)\,ds.$$

Now we can write

$$E[N_t(x) - \mathcal{N}_t(x)]^2 \leqslant 2(E[N_t(x) - N_t^\sigma(x)]^2 + E[N_t^\sigma(x) - \mathcal{N}_t(x)]^2).$$

The first term on the right-hand side tends to zero via Lemma 1. Let us prove that the second one tends to zero too. We have

$$E[N_t^\sigma(x) - \mathcal{N}_t(x)]^2 \leqslant 3\Bigg(\sum_{q=Q+1}^{\infty} E\left[\sum_{l=0}^{[q/2]} b_{q-2l}(x)a_{2l}\int_0^t H_{q-2l}(X_s)H_{2l}(\dot{X}_s)\,ds\right]^2 + \sum_{q=0}^{Q} E\left[\sum_{l=0}^{[q/2]} (b_{q-2l}^\sigma(x) - b_{q-2l}(x))a_{2l}\int_0^t H_{q-2l}(X_s)H_{2l}(\dot{X}_s)\,ds\right]^2 + \sum_{q=Q+1}^{\infty} E\left[\sum_{l=0}^{[q/2]} b_{q-2l}^\sigma(x)a_{2l}\int_0^t H_{q-2l}(X_s)H_{2l}(\dot{X}_s)\,ds\right]^2\Bigg)$$

by using that $\mathcal{N}_t(x)$ belongs to $L^2(\Omega)$ and the orthogonality of the components for different q. The first two terms on the right-hand side tend to 0 as respectively $Q \to \infty$ and $\sigma \to 0$. Since $N_t^\sigma(x) \in H(X)$, we have $\lim_{\sigma\to 0}\|P_Q(N_t^\sigma(x))\|^2 = \|P_Q(N_t(x))\|^2$, where P_Q denotes the orthogonal projection over the Chaos subspace $\oplus_{Q+1}^{\infty} H_q$. But

$$\sum_{q=Q+1}^{\infty} E\left[\sum_{l=0}^{[q/2]} b_{q-2l}^\sigma(x)a_{2l}\int_0^t H_{q-2l}(X_s)H_{2l}(\dot{X}_s)\,ds\right]^2 = \|P_Q(N_t^\sigma(x))\|^2,$$

so the limit as $Q \to \infty$ and $\sigma \to 0$ is 0, which concludes the proof. □

3. Generalization to other processes

Our approach does not allow us to get a result as general as in Slud (1991, 1994), i.e. under only the Geman condition. Nevertheless it can be applied to a large class of processes by using the method of regularization of Wschebor (1985); we establish in particular a lemma of approximation which is of interest on its own (cf. Appendix).

Let X_t be a mean zero stationary Gaussian with variance one satisfying the Geman condition (2). In addition we will assume that $\theta''(t) = tL(t)$ satisfies

$$|\theta''(t+h) - \theta''(t)| \leqslant |h| L_1(h), \tag{7}$$

where $L_1(h)$ is an even function belonging to $L^1([0,\delta], dx)$. Condition (2) with $L(\cdot)$ satisfying this last assumption is denoted by condition (7).

Proposition 2. *Let X_t be a mean zero stationary Gaussian process with variance one satisfying hypothesis* (7). *Then the following expansion holds*:

$$\frac{N_t(x)}{\sqrt{-r''(0)}} = \sum_{q=0}^{\infty}\sum_{l=0}^{[q/2]} b_{q-2l}(x)a_{2l}\int_0^t H_{q-2l}(X_s)H_{2l}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds.$$

Proof. As in Wschebor (1985), we define the regularized process

$$X_t^\varepsilon = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi\left(\frac{t-s}{\varepsilon}\right)X_s\,ds$$

where φ is a compact support even probability density having two continuous derivatives. Let $Y_t^\varepsilon = X_t^\varepsilon/\sigma_\varepsilon$ where $\sigma_\varepsilon^2 = \operatorname{var} X_t^\varepsilon$. Then Y_t^ε is a mean zero stationary

Gaussian process with variance one. Let us consider the following formal development:

$$\frac{\tilde{\mathcal{N}}_t(x)}{\sqrt{-r''(0)}} = \sum_{q=0}^{\infty} \sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) \mathrm{d}s \tag{8}$$

and let us prove that this last expansion belongs to L^2. Indeed by using the diagram formula given in Major (1981), we can show that the partial finite developments of $N_t^\varepsilon(x)/\sqrt{-r_\varepsilon''(0)}$ converge to the same developments of $\tilde{\mathcal{N}}_t(x)/\sqrt{-r_\varepsilon''(0)}$ defined respectively in (8) and (9). We deduce that for each fixed q

$$\sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}\left(\frac{X_s^\varepsilon}{\sigma_\varepsilon}\right) H_{2l}\left(\frac{\dot{X}_s^\varepsilon}{\sigma_\varepsilon\sqrt{-r''(0)}}\right) \mathrm{d}s$$

$$\to \sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) \mathrm{d}s, \quad \text{in probability as } \varepsilon \to 0.$$

By applying Fatou's lemma, we get for each Q positive

$$\sum_{q=0}^{Q} E\left[\sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}(X_s) H_{2l}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) \mathrm{d}s\right]^2$$

$$\leqslant \lim_{\varepsilon\to 0} E\left[\frac{N_t^\varepsilon(x)}{\sqrt{-r_\varepsilon''(0)}}\right]^2 = E\left[\frac{N_t(x)}{\sqrt{-r''(0)}}\right]^2,$$

by applying Lemma 6 of the Appendix in the last equality, which gives the result. We want to show now that in $H(X)$

$$N_t(X) = \tilde{\mathcal{N}}_t(x).$$

We denote by $N_t^\varepsilon(x)$ the number of crossings of Y_t^ε of the level x and by $r_\varepsilon(t)$ the covariance of Y_t^ε; then by using the remark after Proposition 1, we have that

$$\frac{N_t^\varepsilon(x)}{\sqrt{-r_\varepsilon''(0)}} = \sum_{q=0}^{\infty} \sum_{l=0}^{[q/2]} b_{q-2l}(x) a_{2l} \int_0^t H_{q-2l}\left(\frac{X_s^\varepsilon}{\sigma_\varepsilon}\right) H_{2l}\left(\frac{\dot{X}_s^\varepsilon}{\sigma_\varepsilon\sqrt{-r''(0)}}\right) \mathrm{d}s \tag{9}$$

and

$$E\left(\frac{N_t(x)}{\sqrt{-r''(0)}} - \frac{\tilde{\mathcal{N}}_t(x)}{\sqrt{-r''(0)}}\right)^2 \leqslant 2\left(E\left(\frac{N_t(x)}{\sqrt{-r''(0)}} - \frac{N_t^\varepsilon(x)}{\sqrt{-r_\varepsilon''(0)}}\right)^2 + E\left(\frac{N_t^\varepsilon(x)}{\sqrt{-r_\varepsilon''(0)}} - \frac{\tilde{\mathcal{N}}_t(x)}{\sqrt{-r''(0)}}\right)^2\right).$$

Step 1: $E(N_t(x)/\sqrt{-r''(0)} - N_t^\varepsilon(x)/\sqrt{-r_\varepsilon''(0)})^2 \to 0$ as $\varepsilon \to 0$.

Proof. We have

$$E(N_t(x) - N_t^\varepsilon(x))^2 = E(N_t(x))^2 + E(N_t^\varepsilon(x))^2 - 2E(N_t(x)N_t^\varepsilon(x)).$$

According to Lemma 6, $E(N_t(x))^2 = \lim_{\varepsilon\to 0} E(N_t^\varepsilon(x))^2$. On the other hand, the uniform convergence of X_s^ε to X_s as ε tends to 0, implies that for all crossing by X_s of the level x, there exists at least one crossing by X_s^ε of the level x. Hence for a.s. ω, $\exists\, \varepsilon'(\omega)$, $\varepsilon < \varepsilon'(\omega)$ s.t. $N_t(x) \leqslant N_t^\varepsilon(x)$. Therefore the following inequality holds

$$N_t(x) \leqslant \liminf_{\varepsilon\to 0} N_t^\varepsilon(x). \tag{10}$$

Now let $\Delta = \{\omega\colon N_t(x) < \liminf_{\varepsilon\to 0} N_t^\varepsilon(x)\}$ and suppose $P(\Delta) \neq 0$. Via Fatou's lemma, we see that on Δ

$$E[N_t(x)]^2 < \liminf_{\varepsilon\to 0} E[N_t^\varepsilon(x)]^2,$$

which is a contradiction with the convergence for the L^2 norms that we get through Lemma 6 (see the Appendix). Thus $P(\Delta) = 0$, and therefore, combined with (10), it follows that $N_t(x) = \liminf_{\varepsilon\to 0} N_t^\varepsilon(x)$ a.s. Then we can write

$$E[N_t(x)]^2 = E\left[N_t(x) \liminf_{\varepsilon\to 0} N_t^\varepsilon(x)\right] \leqslant \liminf_{\varepsilon\to 0} E[N_t(x) N_t^\varepsilon(x)] \leqslant E[N_t(x)]^2$$

by applying Fatou then Schwarz in the last two inequalities, and finally we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} E[N_t(x)N_t^\varepsilon(x)] &\leqslant (E[N_t(x)]^2)^{1/2} \limsup_{\varepsilon\to 0} (E[N_t^\varepsilon(x)]^2)^{1/2}\\ &= E[N_t(x)]^2 = \liminf_{\varepsilon\to 0} E[N_t(x)N_t^\varepsilon(x)].\end{aligned}$$

Therefore we can conclude this first step.

Step 2: $E(N_t^\varepsilon(x)/\sqrt{-r_\varepsilon''(0)} - \tilde{\mathcal{N}}_t(x)/\sqrt{-r''(0)})^2 \to 0$ as $\varepsilon \to 0$.

Proof. We proceed exactly in the same way as in the last part of the proof of Proposition 1, namely we decompose

$$E\left(\frac{N_t^\varepsilon(x)}{\sqrt{-r_\varepsilon''(0)}} - \frac{\tilde{\mathcal{N}}_t(x)}{\sqrt{-r''(0)}}\right)^2$$

in three terms, then prove the convergence to 0 of each of them.

4. Hermite expansion for the number of maxima in an interval

As an application of the heuristic of Section 2, we may look at various functionals related to crossings, for example at the number of maxima of the process in an interval. Let $M_{[\beta_1,\beta_2]}^X$ be the number of local maxima of X_s, $0 \leqslant s \leqslant t$, lying in the real interval $[\beta_1, \beta_2]$ and let $r(0) = 1$. Formally,

$$M_{[\beta_1,\beta_2]}^X = \int_0^t 1_{[\beta_1,\beta_2]}(X_s)\delta_0(\dot{X}_s)|\ddot{X}_s| 1_{[0,\infty)}(\ddot{X}_s)\,\mathrm{d}s.$$

Lemma 3. *If we suppose that $r^{(vi)}(0) < \infty$, then*

$$M^X_{[\beta_1,\beta_2]} = -\lim_{\sigma\to 0}\int_0^t 1_{[\beta_1,\beta_2]}(X_s)\frac{\exp(-\dot X_\varepsilon^2/2\sigma^2)}{\sigma\sqrt{2\pi}}\ddot X_s 1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s, \quad \textit{a.s. and in } L^2.$$

Proof. Let us define

$$M^X_{[\beta_1,\beta_2]}(\sigma) := -\int_0^t 1_{[\beta_1,\beta_2]}(X_s)\frac{\exp(-\dot X_\varepsilon^2/2\sigma^2)}{\sigma\sqrt{2\pi}}\ddot X_s 1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s, \quad \textit{in } L^2$$

$$= -\sum_{i=0}^{N(\omega)}\int_{\gamma_i}^{\xi_i}\frac{\exp(-\dot X_\varepsilon^2/2\sigma^2)}{\sigma\sqrt{2\pi}}\ddot X_s 1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s,$$

where $\bigcup_{i=0}^{N(\omega)}[\gamma_i,\xi_i] = \{s\in[0,t]\colon 1_{[\beta_1,\beta_2]}(X_s) = 1\}$.

But

$$\int_{\gamma_i}^{\xi_i}\frac{\exp(-\dot X_\varepsilon^2/2\sigma^2)}{\sigma\sqrt{2\pi}}|\ddot X_s|1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s \to \dot D_{[\gamma_i,\xi_i]}(0) \quad \text{a.s. and in } L^2 \text{ as } \sigma\to 0.$$

where $\dot D_{[\gamma_i,\xi_i]}(0)$ denotes the number of zero downcrossings of $\dot X_s$ in $[\gamma_i,\xi_i]$ (by an argument similar to that one used to prove that $N_t^\sigma(x)$ converges to $N_t(x)$). Then we adapt the proof of Proposition 1 (cf. Section 2) to prove the L^2 convergence (the main change being that the r.v. considered are not independent for a fixed t, so we will have to transform them into orthogonal ones). Namely

Lemma 4. *Under the hypothesis that* $-r^{(vi)}(0) < \infty$, *we have*

$$\lim_{\sigma\to 0} E[M^X_{[\beta_1,\beta_2]}(\sigma) - M^X_{[\beta_1,\beta_2]}]^2 = 0.$$

Proof. We briefly outline the main steps. Since we have a.s. convergence, the only thing to prove is that $E[M^X_{[\beta_1,\beta_2]}(\sigma)]^2$ converges towards $E[M^X_{[\beta_1,\beta_2]}]^2$ as $\sigma\to 0$. First note that, by the results of Section 2 applied to the process $\dot X_t$,

$$M^X_{[\beta_1,\beta_2]}(\sigma) \leqslant M^X(\sigma) := \int_0^t \frac{\exp(-\dot X_\varepsilon^2/2\sigma^2)}{\sigma\sqrt{2\pi}}|\ddot X_s|1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s \to \dot D_t(0)$$

$$= M^X \quad \text{as } \sigma\to 0,$$

where M^X is the total number of maxima of X_s, $0\leqslant s\leqslant t$. The generalized Lebesgue theorem gives us the result if

$$\lim_{\sigma\to 0} E[M^X(\sigma)]^2 = E\left[\lim_{\sigma\to 0} M^X(\sigma)\right]^2 := E[M^X]^2,$$

which is true by using exactly the same arguments given in the proof of Lemma 1, Section 2, but applied to the downcrossings of $\dot X_s$ instead of the crossings of X_s. Indeed as in Section 2, we can write formally

$$\dot D_t(u) = -\int_0^t \delta_u(\dot X_s)\ddot X_s 1_{(-\infty,0)}(\ddot X_s)\,\mathrm{d}s$$

so

$$M^X(\sigma) = \int_{-\infty}^{\infty} \dot{D}_t(u)\,\mathrm{d}\phi_{\sigma,0}(u). \qquad \square$$

The next step is to provide the expansion for the number of maxima, as we did in Lemma 2, Section 2. For this it is necessary to get the Hermite expansion for a regular functional depending on the Gaussian vector (X_s, $\dot{X}_s$, $\ddot{X}_s$). Consider three functions: $f_2 \in L^2(\phi(x)\,\mathrm{d}x)$ and $f_1, f_3 \in L^4(\phi(x)\,\mathrm{d}x)$. We must study the Hermite expansion for

$$\int_0^t f_1(X_s) f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) f_3\left(-\frac{\ddot{X}_s}{\sqrt{r^{(\mathrm{iv})}(0)}}\right)\mathrm{d}s.$$

By using the independence between $(X_s, \dot{X}_s)$ and also between $(\dot{X}_s, \ddot{X}_s)$, then Schwarz inequality, we have

$$E\left[f_1(X_s) f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) f_3\left(-\frac{\ddot{X}_s}{\sqrt{r^{(\mathrm{iv})}(0)}}\right)\right]^2$$

$$= E\left[f_1(X_s) f_3\left(\frac{\ddot{X}_s}{\sqrt{r^{(\mathrm{iv})}(0)}}\right)\right]^2 E\left[f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)\right]^2 \leqslant \|f_1^2\|^2 \|f_3^2\|^2 \|f_2^2\|^2.$$

Therefore such a functional belongs to $L^2(\Omega)$. Moreover to obtain its development in the Itô–Wiener Chaos we need to do the following change of variables, in order to get orthogonal components of the vector (X_s, $\dot{X}_s$, $\ddot{X}_s$), s being fixed. Let

$$\frac{\ddot{X}_s}{\sqrt{r^{(\mathrm{iv})}(0)}} = \rho_1 X_s + \rho_2 Z_s,$$

where Z_s is a r.v. independent of X_s and of $\dot{X}_s$ (for each s fixed). The constants ρ_1, ρ_2 do not depend on s, and

$$\rho_1 = \frac{r''(0)}{\sqrt{r^{(\mathrm{iv})}(0)}}, \qquad \rho_2 = \sqrt{1-\rho_1^2}.$$

Hence we can write

$$\int_0^t f_1(X_s) f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) f_3\left(\frac{\ddot{X}_s}{\sqrt{r^{(\mathrm{iv})}(0)}}\right)\mathrm{d}s$$

$$= \int_0^t f_1(X_s) f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) f_3(\rho_1 X_s + \rho_2 Z_s)\,\mathrm{d}s.$$

To use the independence, it is necessary to consider the function

$$g(x, z) = f_1(x) f_3(\rho_1 x + \rho_2 z).$$

$g(x, z) \in L^2(\phi(x)\phi(z)\,\mathrm{d}x\,\mathrm{d}z)$ and its coefficients in the Hermite expansion are given by

$$c_{nm} = \frac{1}{n!m!}\int_{\mathbb{R}^2} f_1(x) f_3(\rho_1 x + \rho_2 z) H_n(x) H_m(z) \phi(x)\phi(z)\,\mathrm{d}x\,\mathrm{d}z.$$

Let

$$b_l := \frac{1}{l!}\int_{\mathbb{R}} f_2(y) H_l(y)\phi(y)\,\mathrm{d}y.$$

Then we have the following:

Lemma 5

$$\int_0^t f_1(X_s) f_2\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) f_3(\rho_1 X_s + \rho_2 Z_s)\,\mathrm{d}s$$

$$= \sum_{n,m=0}^{\infty}\sum_{l=0}^{\infty} c_{nm} b_l \int_0^t H_n(X_s) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) H_m(Z_s)\,\mathrm{d}s$$

$$= \sum_{q=0}^{\infty}\sum_{0\leqslant n+m\leqslant q}^{\infty} c_{nm} b_{q-(n+m)} \int_0^t H_n(X_s) H_{q-(n+m)}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) H_m(Z_s)\,\mathrm{d}s, \text{ in } H(X).$$

The proof is similar to the proof of Lemma 2.

Theorem 1. *Under the condition* $-r^{(\mathrm{vi})}(0) < \infty$, *we have*

$$M^X_{[\beta_1,\beta_2]} = -\sqrt{\frac{r^{(\mathrm{iv})}(0)}{-r''(0)}}\sum_{q=0}^{\infty}\sum_{0\leqslant n+m\leqslant q}^{\infty}\delta_{nm}\frac{1}{(q-(m+n))!\sqrt{2\pi}} H_{q-(m+n)}(0)$$

$$\int_0^t H_n(X_s) H_{q-(n+m)}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) H_m(Z_s)\,\mathrm{d}s$$

where δ_{nm} *is defined in* (11) *below.*

Remark. We could weaken the condition of the previous theorem by using the same method as in Section 3 and by taking a condition similar to (7) for the fourth derivative of $r(\cdot)$. It would then provide that $E(M^X_{[\beta_1,\beta_2]})^2 < \infty$.

Proof of Theorem 1. We apply Lemma 5 to the three functions

$$f_1(x) = 1_{[\beta_1,\beta_2]}(x), \qquad f_2(x) = \frac{\exp-(x^2/2\sigma^2)}{\sigma\sqrt{2\pi}} \quad \text{and} \quad f_3(x) = |x|\,1_{(-\infty,0)}(x),$$

with $b_l := b_l^{\sigma}(0)$ (cf. Section 2) and

$$c_{nm} = \delta_{nm} = \int_{\beta_1}^{\beta_2}\int_{\mathbb{R}} (\rho_1 x + \rho_2 z) 1_{(-\infty,0)}(\rho_1 x + \rho_2 z) H_n(x) H_m(z)\phi(x)\phi(z)\,\mathrm{d}x\,\mathrm{d}z. \quad (11)$$

Let us compute those last coefficients. If $m \geqslant 1$, the definition of Hermite polynomials, namely

$$\sqrt{2\pi} H_m(z)\phi(z) = (-1)^m \frac{\mathrm{d}^m(\exp - z^2/2)}{\mathrm{d}z^m},$$

and a certain number of integration by parts provide

$$\delta_{nm}(\beta_1,\beta_2)=\frac{-\rho_2}{n!m!}(-1)^{n+m-2}\int_{\beta_1}^{\beta_2}\phi^{(n)}(x)\phi^{(m-2)}\left(-\frac{\rho_1}{\rho_2}x\right)\mathrm{d}x.$$

For $m=0$, we have

$$\delta_{n0}=\frac{1}{n!}(-1)^n\int_{\beta_1}^{\beta_2}\phi^{(n)}(x)\left(\rho_1 x\phi^{(-1)}\left(-\frac{\rho_1}{\rho_2}x\right)-\rho_2\phi^{(0)}\left(-\frac{\rho_1}{\rho_2}x\right)\right)\mathrm{d}x,$$

here $\phi^{(-1)}$ means the standard Gaussian distribution! Then we get

$$\sqrt{\frac{-r''(0)}{r^{(\mathrm{iv})}(0)}}M^X_{[\beta_1,\beta_2]}(\sigma\sqrt{-r''(0)})$$

$$=-\sum_{q=0}^{\infty}\sum_{0\leqslant n+m\leqslant q}^{\infty}\delta_{nm}b^\sigma_{q-(n+m)}(0)\int_0^t H_n(X_s)H_{q-(n+m)}\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)H_m(Z_s)\,\mathrm{d}s.$$

Recall that

$$b^\sigma_q(0)\to\frac{1}{q!\sqrt{2\pi}}H_q(0)\quad\text{as }\sigma\to 0$$

(cf. Section 2). Hence we can conclude by using the convergence in the chaos. □

5. Remark

As an application, we will use those representations to study the asymptotic behavior for the estimator of the square root of the second spectral moment defined in Cabaña (1985), which generalizes the one of Lindgren (1974).

Let us modify slightly the estimator of Cabaña as

$$\gamma=\frac{\pi}{t}\int_{-\infty}^{\infty}N_t(x)\,\mathrm{d}\alpha(x),$$

where $\alpha(x)$ is a distribution function on $\mathbb{R}$. Two cases can be considered, on one hand when the measure α has a density function $\mu(\cdot)$ and on the other hand when α is a combination of Dirac measures (of course the third case would be when we take a combination of those two types of measures).

We will be interested in the first case, since the second one provides in fact the estimator of Lindgren and has already been studied via the Hermite expansion by Slud (1994a). Under some classical conditions, we will prove the asymptotic normality for γ and will compute the asymptotic variance. Those results will be the object of another paper.

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Appendix

First recall some notations and results. Suppose that X_t is a mean zero stationary Gaussian process with variance one, that the function of covariance r has two derivatives and satisfies the Geman condition (2). $\phi_u(x, \dot{x}, y, \dot{y})$ denotes the density function of $(X_0, \dot{X}_0, X_u, \dot{X}_u)$, and $\boldsymbol{x} = (x, \dot{x}, y, \dot{y})$. If $x \neq y$, we have

$$E(N_t(x)N_t(y)) = 2\int_0^t (t-u)\int_{\mathbb{R}^2} |\dot{x}||\dot{y}|\phi_u(x, \dot{x}, y, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u,$$

cf. Wschebor (1985), and if $x = y$,

$$E[N_t^2(x)] = EN_t(x) + 2\int_0^t (t-u)\int_{\mathbb{R}^2} |\dot{x}||\dot{y}|\phi_u(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u,$$

cf. Cramér et al. (1967). Let us compute

$$I := \int_{\mathbb{R}^2} |\dot{x}||\dot{y}|\phi_u(x, \dot{x}, y, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}.$$

$$I = \phi_u(x, y)E[|\dot{X}_0\dot{X}_u||X_0 = x, X_u = y],$$

where $\phi_u(x, y)$ is the density function of (X_0, X_u). Hence

$$I = \phi_u(x, y)E[|\xi + \alpha x + \beta y||\xi' - \beta x - \alpha y|],$$

with

$$\operatorname{var}(\xi) = \operatorname{var}(\xi') = -r''(u) - \frac{r'^2(u)}{1 - r^2(u)}, \qquad E(\xi\xi') = -r''(u) - \frac{r(u)r'^2(u)}{1 - r^2(u)}$$

(ξ, ξ' normal r.v.) and

$$\beta(u) := \frac{r'(u)}{(1 - r^2(u))}, \qquad \alpha(u) := -r(u)\beta(u).$$

Suppose now that we are under the conditions of Section 3.

Lemma 6. *Under condition* (7), *we have*

$$E[N_t^\varepsilon(x)]^2 \to EN_t^2(x) \quad \text{as } \varepsilon \to 0.$$

Proof. In the following, K will denote a positive constant which can change from one inequality to the other. Since $\sigma_\varepsilon \to 1$ as $\varepsilon \to 0$, we work on the process X_t^ε instead of Y_t^ε. As a consequence of the Rice formula and the Geman condition, we have $EN_t^\varepsilon(x) \to EN_t(x)$ as $\varepsilon \to 0$, so we have to prove that

$$\int_0^t (t-u)\int_{\mathbb{R}^2} |\dot{x}||\dot{y}|\phi_u^\varepsilon(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u$$

$$\to \int_0^t (t-u)\int_{\mathbb{R}^2} |\dot{x}||\dot{y}|\phi_u(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u \quad \text{as } \varepsilon \to 0.$$

Let us divide the interval of integration into two sets $[0, \delta]$ and $[\delta, t]$ such that $r(u) \neq 1$ if $u \in [\delta, t]$. Using the uniform convergence, we get for all $\delta > 0$,

$$\lim_{\varepsilon \to 0} \int_\delta^t (t-u) \int_{\mathbb{R}^2} |\dot{x}|\,|\dot{y}|\,\phi_u^\varepsilon(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u$$

$$= \int_\delta^t (t-u) \int_{\mathbb{R}^2} |\dot{x}|\,|\dot{y}|\,\phi_u(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u.$$

Let us now prove that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_0^\delta (t-u) \int_{\mathbb{R}^2} |\dot{x}|\,|\dot{y}|\,\phi_u^\varepsilon(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u = 0. \tag{A.1}$$

Since

$$r_\varepsilon(t) = 1 + \frac{r_\varepsilon''(0)}{2} t^2 + \theta_\varepsilon(t), \tag{A.2}$$

with $\theta_\varepsilon(t) > 0$, according to Cramér and Leadbetter (1967, p. 210), we have

$$\int_0^\delta (t-u) \int_{\mathbb{R}^2} |\dot{x}|\,|\dot{y}|\,\phi_u^\varepsilon(x, \dot{x}, x, \dot{y})\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}u \leqslant K \int_0^\delta \frac{\theta_\varepsilon'(u)}{u^2}\,\mathrm{d}u. \tag{A.3}$$

We have also

$$r_\varepsilon(t) = \int_{-\infty}^{\infty} \varphi * \varphi(u) r(t - \varepsilon u)\,\mathrm{d}u.$$

Hence

$$\theta_\varepsilon'(t) = \int_{-\infty}^{\infty} \varphi * \varphi(u)[\theta'(t + \varepsilon u) - 2t\theta''(\varepsilon u)]\,\mathrm{d}u,$$

$$\theta'(\varepsilon u) = \int_{-\infty}^{\infty} \varphi * \varphi(v)[\theta'(v + \varepsilon u) - \theta'(\varepsilon v) - u\theta''(\varepsilon v)]\,\mathrm{d}v,$$

$$\theta'(\varepsilon u) = \int_{-\infty}^{\infty} \varphi * \varphi(v)\,\mathrm{d}v \int_0^u [\theta''(\varepsilon v + z) - \theta''(\varepsilon v)]\,\mathrm{d}z \leqslant \int_0^u z L_1(z)\,\mathrm{d}z.$$

Therefore, under (7),

$$\int_0^\delta \frac{\theta_\varepsilon'(u)}{u^2}\,\mathrm{d}u \leqslant K \int_0^\delta \frac{1}{u^2}\,\mathrm{d}u \int_0^u z L_1(z)\,\mathrm{d}z \leqslant K \int_0^\delta L_1(z)\,\mathrm{d}z,$$

which tends to 0 as δ goes to 0. □

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