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Stochastic Differential Games*

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INTRODUCTION

In recent papers [9–11] we have proved existence theorems for deterministic differential games. These are games between two players y and z, with dynamics given by a system of ordinary differential equations

$$\frac{dx}{dt} = f(t, x, y, z),$$

and a payoff depending on x, y, z and the time when the game terminates. The purpose of the present paper is to obtain such existence theorems for games in which the dynamics is given by a system of stochastic differential equations

$$d\xi = f(t, \xi, y, z)\, dt + \sigma(t, \xi)\, dw,$$

where $\sigma\, dw$ represents the "noise". We shall also consider games with any number of players.

In recent years, Fleming (see [1–3], survey in [4], and joint paper with Nisio [5]) has proved existence theorems for stochastic optimal control problems. A basic approach here is the reduction of the problem to a setting in terms of solutions of the second-order parabolic equations and the use of *a priori* estimates. In the present work we shall combine this approach with some ideas developed in [9]–[12] for deterministic games.

In Section 1, we have collected material on second-order parabolic equations and systems in a form suitable for subsequent applications. In Section 2, we deal with games of perfect observation. Here the number of players is arbitrary. The existence of an equilibrium point in pure strategies is proved. In Section 3 we deal with a 2-player game of partial observation and prove the existence of value and of a saddle point.

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The basic underlying assumptions of this paper are: (i) the "noise" coefficient σ is independent of the controls, that is, $\sigma = \sigma(t, x)$, and (ii) the matrix $\sigma\sigma^*$ is positive definite. Some remarks concerning (ii) are given in Section 4.

1. Auxiliary Results on Parabolic Equations

We denote by x a variable point in the euclidean space R^m, and by t a real variable. A function $f(x)$ is said to be uniformly Hölder continuous (with exponent α) on a set $S \subset R^m$ if, for some $C > 0$, $0 < \alpha \leqslant 1$,

$$|f(x) - f(\bar{x})| \leqslant C|x - \bar{x}|^\alpha \qquad \text{for all} \quad x, \bar{x} \text{ in } S.$$

A function $f(t, x)$ is said to be uniformly Hölder continuous (with exponent α) on a set $G \subset \{-\infty < t < \infty\} \times R^m$ if

$$|f(t, x) - f(\bar{t}, \bar{x})| \leqslant C(|t - \bar{t}|^{\alpha/2} + |x - \bar{x}|^\alpha).$$

The constant C is called a Hölder coefficient (with respect to the exponent α).

Let Ω be a bounded domain in R^m, with boundary $\partial\Omega$. Suppose $\partial\Omega$ can be locally represented in the form

$$x_i = \phi(x_1, \ldots, x_{i-1}, x_{i+1}, x_m), \tag{1.1}$$

where ϕ is in C^2. Then we say that $\partial\Omega$ belongs to C^2. If the second derivatives of ϕ are uniformly Hölder continuous (exponent α), then we say that $\partial\Omega$ belongs to $C^{2+\alpha}$.

Throughout this paper, Q_T will denote the cylinder

$$Q_T = \{(t, x); s < t < T, x \in \Omega\},$$

where s, T are fixed real numbers and Ω is a fixed bounded domain in R^m. We also set

$$S_T = \{(t, x); s < t < T, x \in \partial\Omega\}, \qquad \Omega_\sigma = \{(\sigma, x); x \in \Omega\},$$
$$\partial\Omega_\sigma = \{(\sigma, x); x \in \partial\Omega\}, \qquad \Gamma_T = S_T \cup \bar{\Omega}_T.$$

By D_x and D_x^2 we shall mean any partial derivative of the first and second orders, respectively, with respect to the components x_i of $x = (x_1, \ldots, x_m)$. By the gradient vector $\nabla_x f$ we mean the vector

$$\left(\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_m} f\right);$$

here, either $f = f(x)$ or $f = f(t, x)$.

Let $\partial\Omega$ belong to C^2. A function Φ defined on Γ_T is said to belong to $C^{2,1}(\Gamma_T)$ if (a) in terms of the local parameters (as in (1.1)), Φ, $\partial\Phi/\partial t$, $\partial\Phi/\partial x_j$, $\partial\Phi/\partial x_j\,\partial x_k$ for all $(j \neq i, k \neq i)$ are uniformly continuous on S_T; (b) $\Phi(T, x)$ is in $C^2(\bar{\Omega}_T)$, i.e., $D_x\Phi(T, x)$, $D_x{}^2\Phi(T, x)$ are uniformly continuous in Ω, and (c) Φ is continuous on $\partial\Omega_T$.

Let $\partial\Omega$ belong to $C^{2+\alpha}$. If $\Phi \in C^{2,1}(\Gamma_T)$ and if, in addition, the functions Φ, $\partial\Phi/\partial t$, $\partial\Phi/\partial x_j$, $\partial^2\Phi/\partial x_j\,\partial x_k$ occurring in (a) are uniformly Hölder continuous (exponent α) and $\Phi(T, x) \in C^{2+\alpha}(\bar{\Omega}_T)$, then we say that Φ belongs to $C_\alpha^{2,1}(\Gamma_T)$.

If $\Phi \in C^{2,1}(\Gamma_T)$, then we denote by $\| \Phi \|_{2,1}^{\Gamma_T}$ an upper bound on all the derivatives occurring in (a), (b). If $\Phi \in C_\alpha^{2,1}(\Gamma_T)$, then we set

$$\| \Phi \|_{2,1,\alpha}^{\Gamma_T} = \| \Phi \|_{2,1}^{\Gamma_T} + H_\alpha(\Phi),$$

where $H_\alpha(\Phi)$ is an upper bound on Hölder coefficients (with respect to exponent α) of all the derivatives occurring in (a), (b).

A function Ψ defined in $\bar{Q}_T$ such that Ψ, $D_t\Psi$, $D_x\Psi$, $D_x{}^2\Psi$ are uniformly continuous (uniformly Hölder continuous, exponent α) in Q_T is said to belong to $C^{2,1}(Q_T)$ $(C_\alpha^{2,1}(Q_T))$.

It is well known that $\Phi \in C^{2,1}(\Gamma_T)$ $(\Phi \in C_\alpha^{2,1}(\Gamma_T))$ if and only if there exists a function Ψ in $C_\alpha^{2,1}(Q_T)$ $(C_\alpha^{2,1}(Q_T))$ such that $\Psi = \Phi$ on Γ_T.

A function $u(x)$ is said to belong to $W_p{}^k(\Omega)$ $(1 < p < \infty$, k non-negative integer) if all its weak derivatives of order $\leqslant k$ belong to $L^p(\Omega)$. A function $u(t, x)$ is said to belong to $W_p^{2,1}(Q_T)$ if u and its weak derivatives

$$D_x u,\ D_t u,\ D_x{}^2 u$$

belong to $L^p(Q_T)$. We introduce the norm

$$\| u \|_{W_p^{2,1}(Q_T)} = \| u \|_{L^p(Q_T)} + \sum_{j=1}^{m} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(Q_T)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^p(Q_T)} + \sum_{j,k=1}^{m} \left\| \frac{\partial^2 u}{\partial x_j\,\partial x_k} \right\|_{L^p(Q_T)}.$$

Consider a partial differential equation

$$Lu \equiv \frac{\partial u}{\partial t} + \sum_{j,k=1}^{m} a_{jk}(t, x) \frac{\partial^2 u}{\partial x_j\,\partial x_k} + \sum_{j=1}^{m} b_j(t, x) \frac{\partial u}{\partial x_j} + c(t, x)u = f(t, x) \quad \text{in} \quad Q_T, \tag{1.2}$$

with initial and boundary conditions given by

$$u = \Phi \quad \text{on} \quad \Gamma_T. \tag{1.3}$$

We shall need the following assumptions:

(A_1). For all $(t, x) \in \bar{Q}_T$ and all $\xi \in R^m$,

$$\nu_0 |\xi|^2 \leqslant \sum_{j,k=1}^{m} a_{jk}(t, x)\xi_j\xi_k \leqslant \nu_1 |\xi|^2, \tag{1.4}$$

where ν_0, ν_1 are positive constants.

(A_2). The $a_{jk}(t, x)$ are continuous in $\bar{Q}_T$; denote by ν a modulus of continuity for all these functions.

(A_3). The derivatives $\partial a_{jk}(t, x)/\partial x_i$ are uniformly continuous in Q_T; denote by ν_3 a constant for which

$$\left| \frac{\partial a_{jk}(t, x)}{\partial x_i} \right| \leqslant \nu_3 \qquad \text{for all} \quad (t, x) \in Q_T. \tag{1.5}$$

(A_4). The derivatives $\partial a_{jk}(t, x)/\partial t$ are uniformly continuous in Q_T; denote by ν_4 a constant for which

$$\left| \frac{\partial a_{jk}(t, x)}{\partial t} \right| \leqslant \nu_4 \qquad \text{for all} \quad (t, x) \in Q_T. \tag{1.6}$$

(B). The $b_j(t, x)$ and $c(t, x)$ are measurable functions in $\bar{Q}_T$, and

$$| b_j(t, x)| \leqslant \nu_2, \qquad | c(t, x)| \leqslant \nu_2 \qquad \text{for all} \quad (t, x) \in \bar{Q}_T. \tag{1.7}$$

Let u be a function in $W_p^{2,1}(Q_T)$, for some $p > 1$, continuous in $\bar{Q}_T$, such that (1.2) holds almost everywhere, ($\partial u/\partial t$, $\partial u/\partial x_j$, $\partial^2 u/\partial x_j \partial x_k$ are taken as weak derivatives) and (1.3) holds. Then we say that u is a *solution* in $W_p^{2,1}(Q_T)$ of (1.2), (1.3). When $p = 2$, we simply say that u is a solution of (1.2), (1.3).

LEMMA 1. *Let $\partial\Omega \in C^2$, $\Phi \in C^{2,1}(\Gamma_T)$, and let* ($A_1$), ($A_2$) *and* (B) *hold. Then, for any $p > 1$, $f \in L^p(Q_T)$, there exists a unique solution in $W_p^{2,1}(Q_T)$ of* (1.2), (1.3). *Furthermore, if, also, $f \in L^q(Q_T)$ for some $p \leqslant q < \infty$, then u is the unique solution in $W_q^{2,1}(Q_T)$ of* (1.2), (1.3), *and*

$$\| u \|_{W_q^{2,1}(Q_T)} \leqslant C(\| \Phi \|_{2,1}^{\Gamma_T} + \| f \|_{L^q(Q_T)}). \tag{1.8}$$

Here C is a constant depending only on ν_0, ν_1, ν_2, ν, Q_T.

This lemma is due to Gagliardo [13, 14] for $p = q = 2$, and, in the general case to Solonnikov [19, 20].

We write, for any set Q' in the (t, x)-space,

$$|v|_{\alpha,Q'} = \text{l.u.b.} \frac{|v(t,x) - v(\bar{t},\bar{x})|}{|t-\bar{t}|^{\alpha/2} + |x-\bar{x}|^{\alpha}},$$

where the l.u.b. is taken with respect to $(t, x) \in Q'$, $(\bar{t}, \bar{x}) \in Q'$, $(t, x) \neq (\bar{t}, \bar{x})$.

LEMMA 2. *Let* $\partial\Omega \in C^2$, $\Phi \in C^{2,1}(\Gamma_T)$, *and let* ($A_1$)–($A_3$) *and* (B) *hold. Then there exists an* α, $0 < \alpha < 1$, *such that, for any* $f \in L^\infty(Q_T)$, *the solution* u *of* (1.2), (1.3) *satisfies*

$$|D_x u|_{\alpha,Q'} \leqslant C \tag{1.9}$$

for any set Q' *whose closure is contained in* Q_T; *here* C *is a constant depending only on* $\nu_0, \nu_1, \nu_2, \nu_3, \nu, Q_T, Q'$ *and* $\text{l.u.b.}_{\Gamma_T} |\Phi|$.

Proof. The lemma was proved in Ladyzhenskaja–Solonnikov–Uraltseva [17, Chap. 6] in case u, $D_x u$, $D_x{}^2 u$, $D_t u$ are continuous in Q_T and $|u| \leqslant \mu$; C depends, in this case, only on $\nu_0, \nu_1, \nu_2, \nu_3, \nu, Q_T, Q'$ and μ. Note, however, that μ is bounded by a constant depending only on the ν_j and l.u.b. $|\Phi|$.

To prove the lemma in general (i.e., when u is only in $W_p^{2,1}(Q_T)$), we approximate a_{ij}, b_i, c by sequences of smooth functions a_{ij}^n, b_i^n, c^n:

$$\begin{aligned} a_{ij}^n &\to a_{ij} && \text{uniformly in } Q_T, \\ b_i^n &\to b_i && \text{in } L^p(Q_T) \text{ and almost everywhere}, \\ c_i^n &\to c_i && \text{in } L^p(Q_T) \text{ and almost everywhere}. \end{aligned}$$

The quantities ν_k, ν for the a_{ij}^n, b_i^n, c^n can be taken to be independent of n; they will depend only on the ν_k, ν in (A_1)–(A_3) (B). Hence, the corresponding solution u^n satisfies

$$|D_x u^n|_{\alpha,Q'} \leqslant C, \tag{1.10}$$

C as in the assertion of the lemma. Since the u^n satisfy (by [17; Chap. 6]) a uniform Hölder condition (independently of n), there exists a uniformly convergent subsequence $\{u^{n'}\}$. Its limit is easily seen to be the solution u of (1.2), (1.3). Finally, (1.9) follows from (1.10), upon taking $n = n' \to \infty$.

LEMMA 3. *Let* $0 < \alpha < 1$. *Suppose* $\partial\Omega \in C^{2+\alpha}$, $\Phi \in C_\alpha^{2,1}(\Gamma_T)$, *and let* ($A_1$)–($A_4$) *and* (B) *hold. Then, for any* $f \in L^\infty(Q_T)$, *the solution* u *of* (1.2), (1.3) *satisfies*

$$|D_x u|_{\alpha,Q_T} \leqslant C(\|\Phi\|_{2,1,\alpha}^{\Gamma_T} + \underset{Q_T}{\text{l.u.b.}} |f|), \tag{1.11}$$

where C *is a constant depending only on* $\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu, Q_T$.

Proof. If $\Phi = 0$ and b_i, c_i, f are continuous, then the lemma is due to Friedman [6] (see also [7]). By approximation, the assertion follows for b_i, c_i, f measurable and bounded, in case $\Phi \equiv 0$. If $\Phi \not\equiv 0$, then we extend it into a function $\hat{\Phi}$ defined in the whole (t, x)-space, such that $D_x\hat{\Phi}$, $D_x^2\hat{\Phi}$, $D_t\hat{\Phi}$ are Hölder continuous (exponent α) in $\bar{Q}_T$ with coefficients bounded by

$$C' \| \Phi \|_{2,1,\alpha}^{\Gamma_T} \qquad (C' \text{ depending on } Q_T).$$

Applying the special case of $\Phi = 0$ to $\hat{u} = u - \hat{\Phi}$, we obtain the assertion of the lemma.

Remark. Note that instead of assuming that $\Phi \in C_\alpha^{2,1}(\Gamma_T)$, it suffices to assume that $\Phi \in C^{2,1}(\Gamma_T)$ and $D_x\Phi$ is uniformly Hölder continuous (exponent α) on S_T, $D_x\Phi$ being any tangential derivative at $\partial\Omega$.

We shall deal later on with parabolic systems of the form

$$\frac{\partial u^k}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2 u^k}{\partial x_i \, \partial x_j} + f_k(t, x, \nabla_x u^1, \ldots, \nabla_x u^N) = 0 \text{ in } Q_T, \quad (1.12)$$

$$u^k = \Phi^k \qquad \text{on} \quad \Gamma_T, \quad (1.13)$$

where $k = 1, 2, \ldots, N$ and f_k are nonlinear in the variables $\nabla_x u^j$. We shall write $u = (u_1, \ldots, u_n)$ and say that $u \in W_p^{2,1}(Q_T)$ if and only if $u_j \in W_p^{2,1}(Q_T)$ for all j. We also write

$$\| u \|_{W_p^{2,1}(Q_T)} = \sum_{j=1}^{N} \| u_j \|_{W_p^{2,1}(Q_T)}.$$

A similar notation will be applied to other norms.

If $u \in W_p^{2,1}(Q_T)$, $u \in C(\bar{Q}_T)$, $D_x u \in C(Q_T)$ and if (1.12) holds almost everywhere (with $D_x^2 u$, $D_t u$ being weak derivatives) and (1.13) holds, then we call u a *strong solution* in $W_p^{2,1}(Q_T)$ of (1.12), (1.13). When $p = 2$, we simply call u a strong solution of (1.12), (1.13).

2. N-Person Games with Perfect Observations

We assume that the reader is familiar with the basic theory of stochastic differential equations, and its relation to parabolic equations. Expositions of these topics can be found in [5, 15, 16].

Consider a system of m stochastic differential equations

$$d\xi = f(t, \xi, y_1, \ldots, y_N) \, dt + \sigma(t, \xi) \, dw \quad (2.1)$$

for $s \leqslant t \leqslant T$; here s, T are fixed real numbers, $w(t) = (w_1(t),..., w_m(t))$, where the $w_i(t)$ are independent Brownian motions. We also introduce the initial condition

$$\xi(s) = x_0 , \tag{2.2}$$

where x_0 is a random variable independent of the $w_i(t)$. We shall denote by τ the exit time from Q_T.

Let Y_i be compact subsets of some euclidean spaces R^{k_i}; we shall call Y_i the *control set* for the *player* y_i. When each player y_i chooses a pure strategy, i.e., a measurable function $y_i(t, \xi)$ defined on $[s, T] \times R^m$ with values in Y_i, then the system (2.1) takes the form

$$d\xi = f(t, \xi, y_1(t, \xi),..., y_N(t, \xi))\, dt + \sigma(t, \xi)\, dw. \tag{2.3}$$

Under some standard assumptions on f, σ (stated below) the system (2.3), (2.2) has a unique solution $\xi(t)$.

In addition to (2.1), (2.2), we are given *cost functionals*

$$J_i(y_1 ,..., y_N) = E_{sx_0} \left\{ \int_s^\tau h_i(t, \xi, y_1 ,..., y_N)\, dt + g_i(\tau, \xi(\tau)) \right\}, \tag{2.4}$$

where E_{sx_0} stands for the expectation. When the players choose pure strategies $y_j = y_j(t, x)$, if the solution $\xi = \xi(t)$ is then uniquely determined, then one can compute the costs $J_i(y_1 ,..., y_N)$.

The above setting of the players choosing pure strategies represents a model of a game of perfect observation. In this model, the players make use only of the present position of x. In the deterministic games considered by Friedman [9–12], the players make use also of all the past positions of x. This more general setting can also be extended to games with dynamics (2.3); we shall introduce it in Section 3.

Another remark. We assume throughout this work that the "noise" term σ is independent of the control variables $y_1 ,..., y_N$. If σ depends on the y_i, serious mathematical difficulties occur, and very little is known (even when $N = 1$).

We shall often write $y = (y_1 ,..., y_N)$. If all the y_j are pure strategies, we call y a pure strategy.

Suppose now $\sigma(t, x)$ and $f(t, x, y)$ are measurable in t, uniformly Lipschitz continuous in (x, y), and bounded by const. $(1 + | x |)$. Then, for any pure strategy $y(t, x)$, measurable in t and uniformly Lipschitz continuous in x, there exists a unique solution $\xi(t)$ of (2.3), (2.2) (see [15, 16]). Actually, with a suitable definition of a solution, Stroock and Varadhan [21] proved that a unique solution of (2.2), (2.3) exists whenever $f(t, x, y, (t, x),..., y_N(t, x))$ is bounded and measurable, $\sigma(t, x)$ is bounded and continuous, and $\sigma\sigma^*$ is positive definite (σ^* = transpose of σ). This solution is a continuous Markov process of diffusion, with local drift f and local covariance σ. Further, setting

$$(a_{ij}) = \tfrac{1}{2}\,\sigma\sigma^*$$

and assuming that the random variable $x_0 = x_0(\omega)$ is a constant x_0, the cost functional J_k turns out to be (see [5])

$$J_k = \psi_k(s, x_0), \tag{2.5}$$

where ψ_k is the solution of

$$\frac{\partial \psi_k}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x)\,\frac{\partial^2 \psi_k}{\partial x_i\,\partial x_j} + \sum_{i=1}^{m} f_i(t, x, y_1(t, x),..., y_N(t, x))\,\frac{\partial \psi_k}{\partial x_i}$$
$$+ h_k(t, x, y_1(t, x),..., y_N(t, x)) = 0 \text{ in } Q_T, \tag{2.6}$$

$$\psi_k = g_k \qquad \text{on} \quad \Gamma_T. \tag{2.7}$$

Suppose (a_{ij}) is a positive definite matrix. Then, by Lemma 1, the system (2.6), (2.7) has a unique solution ψ_k under some assumptions on σ, f and the $y_j\,(t, x)$. It will be much more convenient to work with (2.5), (2.6), (2.7), instead of (2.1), (2.2), (2.4). We shall therefore define the concepts of a game and equilibrium point with respect to (2.5), (2.6), (2.7); analogous definitions can be given, however, with respect to (2.1), (2.2), (2.4) also.

We shall assume in what follows that σ is such that (a_{ij}) satisfies (A_1), (A_2) and, at times, (A_3), (A_4). Regarding $f(t, x, y)$ and the h_i, g_i, we shall assume:

(C). $f(t, x, y_i,..., y_N)$ and the $h_i(t, x, y_1,..., y_N)$ are continuous in functions in $[s, T] \times R^m \times Y_1 \times \cdots \times Y_N$, $\partial\Omega \in C^2$, and the $g_i(t, x)$ belong to $C^{2,1}(\Gamma_T)$.

Then, to any pure strategies $y_j(t, x)$ $(1 \leqslant j \leqslant N)$ there corresponds a cost vector

$$J = (J_1, J_2,..., J_N), \qquad \text{where} \quad J_k = \psi_k(s, x_0),$$

and each ψ_k is the unique solution of (2.6), (2.7), in accordance with Lemma 1.

DEFINITIONS. The system (2.5), (2.6), (2.7) is called an *N-person differential game with perfect observation*. Consider the following scheme (or function): Each player chooses a pure strategy, and then the costs J_k are computed. We refer to this scheme as a game of *perfect observation played by pure strategies*, or, briefly (following Fleming [1] for $N = 1$) a *Markovian game*.

DEFINITION. A pure strategy

$$y^*(t, x) = (y_1{}^*(t, x), \dots, y_N{}^*(t, x))$$

is called an *equilibrium point in pure strategies* (or an *equilibrium pure strategy*) of the differential game if

$$J_k(y_1{}^*, \dots, y_{k-1}^*, y_k, y_{k+1}^*, \dots, y_N{}^*) \geqslant J_k(y_1{}^*, \dots, y_{k-1}^*, y_k{}^*, y_{k+1}^*, \dots, y_N{}^*) \tag{2.8}$$

for any pure strategy y_k, $1 \leqslant k \leqslant N$.

An equilibrium point is a "reasonable" solution for noncooperative game of N players. If $N = 2$ and $J_1 + J_2 = 0$, we say that we have a *zero sum 2-person game*; the equilibrium point is then called a *saddle point in pure strategies*.

We shall prove in this section that an equilibrium pure strategy $y^*(t, x)$ exists. $y^*(t, x)$ is also an equilibrium pure strategy for the game determined by (2.1), (2.2), (2.3); the concept of a solution of (2.2), (2.3) is taken as in [21].

Let p_k ($k = 1, 2, \dots, N$) be a variable point in R^m, and consider the function

$$H_k(t, x, y_1, \dots, y_N, p_k) = f(t, x, y_1, \dots, y_N) \cdot p_k + h_k(t, x, y_1, \dots, y_N). \tag{2.9}$$

This function is called the k-th *Hamiltonian function* associated with the game (2.5), (2.6), (2.7).

We shall need the following *generalized minimax condition*:

(D). There exist functions $y_1{}^*(t, x, p), \dots, y_N{}^*(t, x, p)$, where $p = (p_1, \dots, p_N)$, such that

(i) the $y_j{}^*(t, x, p)$ are measurable in $(t, x) \in \bar{Q}_T$ for every p, and continuous in p with modulus of continuity independent of $(t, x) \in \bar{Q}_T$.

(ii) for all $(t, x) \in \bar{Q}_T$ and for all p,

$$y_j{}^*(t, x, p) \in Y_j \qquad (1 \leqslant j \leqslant N);$$

(iii) for all $(t, x) \in \bar{Q}_T$ and for all p,

$$\min_{y_k \in Y_k} H_k(t, x, y_1{}^*(t, x, p), \dots, y_{k-1}^*(t, x, p), y_k, y_{k+1}^*(t, x, p), \dots, y_N{}^*(t, x, p), p_k)$$
$$= H_k(t, x, y_1{}^*(t, x, p), \dots, y_N{}^*(t, x, p), p_k) \tag{2.10}$$

for $1 \leqslant k \leqslant N$.

EXAMPLE. Suppose

$$f(t, x, y_1, \ldots, y_N) = \sum_{j=1}^{N} F_j(t, x, y_k)$$
$$h_k(t, x, y_1, \ldots, y_N) = \sum_{j=1}^{N} h_{kj}(t, x, y_k). \tag{2.11}$$

Then (2.10) is equivalent to

$$\min_{y_k \in Y_k} \{F_k(t, x, y_k) \cdot p_k + h_{kk}(t, x, y_k)\} = F_k(t, x, y_k^*) \cdot p_k + h_{kk}(t, x, y_k^*), \tag{2.12}$$

where $y_k^* = y_k^*(t, x, p_k)$. By Lemma 1 of [1], $y_k^*(t, x, p_k)$ can be taken to be measurable in (t, x) for each p_k. If it is also continuous in p_k, for each (t, x), with modulus of continuity independent of (t, x), then the condition (D) holds.

THEOREM 1. *Let* (A_1)–(A_3), (C) *and* (D) *hold. Then there exists a solution* $\phi^* = (\phi_1^*, \ldots, \phi_N^*)$ *of the semilinear parabolic system*

$$\frac{\partial \phi_k}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2 \phi_k}{\partial x_i \, \partial x_j} + f(t, x, y^*(t, x, \nabla_x \phi)) \cdot \nabla_x \phi_k$$
$$+ h_k(t, x, y^*(t, x, \nabla_x \phi)) = 0 \text{ in } Q_T, \qquad 1 \leqslant k \leqslant N, \tag{2.13}$$

$$\phi_k = g_k \qquad \text{on} \quad \Gamma_T. \tag{2.14}$$

More precisely, ϕ *is continuous in* $\bar{Q}_T$ *and satisfies* (2.14), $\nabla_x \phi^*$ *is a bounded function in* Q_T, *uniformly Hölder continuous (with some exponent* α*) in compact subsets of* Q_T, *the weak derivatives* $\partial^2 \phi_k / \partial x_i \, \partial x_j$ *belong to* $L^r(Q_T)$ *for any* $r > 1$, *and* (2.13) *holds almost everywhere.*

We have used here the notation

$$(\nabla_x \phi) = (\nabla_x \phi_1, \ldots, \nabla_x \phi_N).$$

Proof. Fix $\beta \in (0, 1)$. Let $\{\Omega^n\}$ be a sequence of bounded domains such that $\Omega \subset \Omega^n$, $\partial\Omega^n \in C^{2+\beta}$, and $\partial\Omega^n$ converges to $\partial\Omega$ in the norm of $C^{2+\beta}$, i.e., there is a finite number of neighbourhoods containing $\partial\Omega^n$, $\partial\Omega$ such that in each neighbourhood $\partial\Omega^n$ and $\partial\Omega$ can be represented in the form

$$x_i = h^n(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m),$$
$$x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

for the same i, and, as $n \to \infty$,

$$h^n \to h, \qquad D_x h^n \to D_x h, \qquad D_x^2 h^n \to D_x^2 h \text{ uniformly.}$$

Set

$$Q_T^n = (s, T) \times \Omega^n, \qquad \partial\Omega_T^n = \{(T, x); x \in \partial\Omega^n\},$$

$$S_T^n = (s, T) \times \partial\Omega^n, \qquad \Gamma_T^n = S_T^n \cup \overline{\Omega_T^n}.$$

Let $g^n = (g_1^n, ..., g_N^n)$ be functions in $C^3(\overline{Q_T^n})$ such that

$$\| g^n - g \|_{2,1}^{\Gamma_T} \to 0 \qquad \text{if} \quad n \to \infty.$$

Let $a_{ij}^n(t, x), f^n(t, x, y_1, ..., y_N), h^n(t, x, y_1, ..., y_N)$ be continuously differentiable functions in all their variables, satisfying the conditions (A_1)–(A_3), (C) with ν_i, ν independent of n, such that, as $n \to \infty$,

$$a_{ij}^n \to a_{ij} \text{ uniformly in } Q_T,$$

$$f^n(t, x, y_1, ..., y_N) \to f(t, x, y_1, ..., y_N) \text{ uniformly in } \bar{Q}_T \times Y_1 \times \cdots \times Y_N,$$

$$h^n(t, x, y_1, ..., y_N) \to h(t, x, y_1, ..., y_N) \text{ uniformly in } \bar{Q}_T \times Y_1 \times \cdots \times Y_N.$$

Let $\tilde{y}^n(t, x, p) = (\tilde{y}_1^n(t, x, p), ..., \tilde{y}_N^n(t, x, p))$ be continuously differentiable in all their variables such that

$$| \tilde{y}_j^n(t, x, p)| \leqslant C_0,$$

where C_0 is a constant independent of n, and such that (a) for each p,

$$\tilde{y}^n(t, x, p) \to y^*(t, x, p) \qquad \text{for almost all} \quad (t, x) \in Q_T,$$

as $n \to \infty$, and (b) the $\tilde{y}_j^n(t, x, p)$ are continuous in p, with modulus of continuity independent of t, x, n. We can take, for instance, $\tilde{y}^n$ to be a mollifier of y^* with respect to all the variables.

Consider the semilinear parabolic system

$$\frac{\partial \phi_k}{\partial t} + \sum_{i,j=1}^{m} a_{ij}^n(t, x) \frac{\partial^2 \phi_k}{\partial x_i \partial x_j} + f^n(t, x, \tilde{y}^n(t, x, \nabla_x \phi)) \cdot \nabla_x \phi_k$$

$$+ h_k^n(t, x, \tilde{y}^n(t, x, \nabla_x \phi)) + \lambda_k^n(t, x) = 0 \text{ in } Q_T^n, \tag{2.15}$$

$$\phi_k = g_k^n \qquad \text{on} \quad \Gamma_T^n. \tag{2.16}$$

Here, λ_k^n is a continuously differentiable function having the following properties: (i) $\lambda_k^n(t, x) = 0$ if

$$\text{dist}((t, x), \partial\Omega_T^m) > \frac{1}{m};$$

(ii) $|\lambda_k^n(t, x)| \leqslant C_1$, C_1 is independent of n, and (iii) the relation (2.15) holds at $\partial\Omega_T^n$, when the ϕ_k and their derivatives are computed from (2.16).

By Theorem 7.1 of [17, p. 596], the system (2.15), (2.16) has a unique solution $\phi^n = (\phi_1^n, \ldots, \phi_N^n)$. Indeed, notice first that the following a priori bound hold:

$$\operatorname*{l.u.b.}_{Q_T^n} |\phi^n(t, x)| \leqslant M, \tag{2.17}$$

where M is a constant independent of n. This follows by considering ϕ_k^n as a solution of the k-th equation in (2.15) and using the proof of Theorem 2.9 in [17, p. 23].

The unique solution ϕ^n of (2.15), (2.16) satisfies (by Lemma 6.1, p. 589 of [17])

$$\operatorname*{l.u.b.}_{Q_T^n} | D_x\phi^n(t, x)| \leqslant M_1, \tag{2.18}$$

where M_1 is a constant independent of n. We can therefore apply Lemma 2 and conclude that for any set Q' with closure in Q_T,

$$| D_x\phi^n(t, x)|_{\alpha, Q'} \leqslant M_2, \tag{2.19}$$

where M_2 is a constant independent of n.

From Lemma 1 and (2.18) we also get, for any $r > 1$,

$$\| \phi^n \|_{W_r^{2,1}(Q_T)} \leqslant M_3, \tag{2.20}$$

where M_3 is a constant independent of n. We can now extract a subsequence $\{\phi^{n'}\}$ of $\{\phi^n\}$ such that, as $n' \to \infty$,

$$\begin{aligned} \phi^{n'} &\rightharpoonup \phi^* \quad \text{weakly in } W_r^{2,1}(Q_T), \\ \phi^{n'} &\to \phi^* \quad \text{uniformly in } \bar{Q}_T, \\ \nabla_x\phi^{n'} &\to \nabla_x\phi^* \text{ uniformly in compact subsets of } Q_T. \end{aligned}$$

It follows that, for almost all $(t, x) \in Q_T$,

$$\tilde{y}^{n'}(t, x, \nabla_x\phi^{n'}(t, x)) \to y^*(t, x, \nabla_x\phi^*(t, x))$$

as $n' \to \infty$. Hence, for almost all $(t, x) \in Q_T$,

$$f^{n'}(t, x, \tilde{y}^{n'}(t, x, \nabla_x \phi^{n'}(t, x)) \cdot \nabla_x \phi^{n'}(t, x) + h_k^{n'}(t, x, \tilde{y}^{n'}(t, x, \nabla_x \phi^{n'}(t, x))$$
$$\to f(t, x, y^*(t, x, \nabla_x \phi^*(t, x)) \cdot \nabla_x \phi_k^*(t, x) + h_k(t, x, y^*(t, x, \nabla_x \phi^*(t, x)) \tag{2.21}$$

as $n' \to \infty$. From the Lebesgue bounded convergence theorem we then deduce that (2.21) holds with "$\to$" standing for weak convergence in $L^2(Q_T)$. Since also

$$\frac{\partial \phi_k^n}{\partial t} + \sum_{i,j=1}^m a_{ij}^n(t, x) \frac{\partial^2 \phi_k^n}{\partial x_i \, \partial x_j} \to \frac{\partial \phi_k^*}{\partial t} + \sum_{i,j=1}^m a_{ij}(t, x) \frac{\partial^2 \phi_k^*}{\partial x_i \, \partial x_j}$$

weakly in $L^2(Q_T)$, we conclude that ϕ^* is a solution of (2.13). Clearly, (2.14) also hold. The proof of Theorem 1 is thus complete except for the assertion that $\phi^* \in W_r^{2,1}(Q_T)$ for any $r > 1$. Note that we have constructed ϕ^* as a weak limit of functions in $W_r^{2,1}(Q_T)$ for a particular r. Taking a sequence of increasing r's and using the diagonal method, we obtain a solution ϕ^* that belongs to $W_r^{2,1}(Q_T)$ for any r.

We now state the main result of this section.

THEOREM 2. *Let the assumptions* (A_1)–(A_3), (C) *and* (D). *Write*

$$y_j^*(t, x) = y_j^*(t, x, \nabla_x \phi^*(t, x)), \tag{2.22}$$

where ϕ^* *is as in the assertion of Theorem* 1. *Then*

$$u^*(t, x) = (y_1^*(t, x), \ldots, y_N^*(t, x))$$

is an equilibrium point in pure strategies of the differential game associated with (2.5), (2.6), (2.7).

Proof. Let $y_k(t, x)$ be any pure strategy for the player y_k. Denote by $\phi = (\phi_1, \ldots, \phi_N)$ the unique solution of

$$\frac{\partial \phi_r}{\partial t} + \sum_{i,j=1}^m a_{ij}(t, x) \frac{\partial^2 \phi_r}{\partial x_i \, \partial x_j}$$
$$+ f(t, x, y_1^*(t, x), \ldots, y_{k-1}^*(t, x), y_k(t, x), y_{k+1}^*(t, x), \ldots, y_N^*(t, x)) \cdot \nabla_x \phi_r$$
$$+ h_r(t, x, y_1^*(t, x), \ldots, y_{k-1}^*(t, x), y_k(t, x), y_{k+1}^*(t, x), \ldots, y_N^*(t, x)) = 0 \quad \text{in } Q_T, \tag{2.23}$$

$$\phi_r = g_r \quad \text{on} \quad \Gamma_T. \tag{2.24}$$

By (2.10) we find that the function ϕ_k^* satisfies

$$\frac{\partial \phi_k^*}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2 \phi_k^*}{\partial x_i \partial x_j}$$

$$+ f(t, x, y_1^*(t, x), \ldots, y_{k-1}^*(t, x), y_k(t, x), y_{k+1}^*(t, x), \ldots, y_N^*(t, x)) \cdot \nabla_x \phi_k^*$$

$$+ h(t, x, y_1^*(t, x), \ldots, y_{k-1}^*(t, x), y_k(t, x), y_{k+1}^*(t, x), \ldots, y_N^*(t, x)) \geqslant 0$$

almost everywhere in Q_T. Setting

$$b(t, x) = f(t, x, y_1^*(t, x), \ldots, y_{k-1}^*(t, x), y_k(t, x), y_{k+1}^*(t, x), \ldots, y_N^*(t, x)),$$

we see that the function $\chi = \phi_k^* - \phi_k$ satisfies

$$\frac{\partial \chi}{\partial t} + \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 \chi}{\partial x_i \partial x_j} + b(t, x) \cdot \nabla_x \chi \geqslant 0 \text{ almost everywhere in } Q_T .$$

By Lemma 2 of [1], $\chi \leqslant 0$ in $\bar{Q}_T$. This gives (2.8).

For a zero-sum 2-person game, we can prove Theorem 2 under a condition weaker than (D), called the *minimax condition*:

(D′). For any $(t, x) \in Q_T$ and for any p_1 in R^m,

$$\min_{y_1 \in Y_1} \max_{y_2 \in Y_2} H_1(t, x, y_1, y_2, p_1) = \max_{y_2 \in Y_2} \min_{y_1 \in Y_1} H_1(t, x, y_1, y_2, p_1). \quad (2.25)$$

Note, by Lemma 1 of [1] that there exist measurable functions $y_1 = y_1^*(t, x, p_1)$, $y_2 = y_2^*(t, x, p_1)$ with values in Y_1 and Y_2, respectively, such that

$$\max_{y_2 \in Y_2} H_1(t, x, y_1^*(t, x, p_1), y_2, p_1) = \min_{y_1 \in Y_1} \max_{y_2 \in Y_2} H_1(t, x, y_1, y_2, p_1), \quad (2.26)$$

$$\min_{y_1 \in Y_1} H_1(t, x, y_1, y_2^*(t, x, p_1), p_1) = \max_{y_2 \in Y_2} \min_{y_1 \in Y_1} H_1(t, x, y_1, y_2, p_1). \quad (2.27)$$

From this we infer the condition (2.10). However, we cannot infer, in general, that the $y_j^*(t, x, p_1)$ are continuous in p_1.

Set

$$H(t, x, p) = \min_{y \in Y_1} \max_{y_2 \in Y_2} H_1(t, x, y_1, y_2, p). \quad (2.28)$$

THEOREM 3. *Let $N = 2$, $J_2 = -J_1$, and assume that* (A_1)–(A_3), (C) *and* (D′) *hold. Then there exists a solution ϕ^* of the parabolic equation*

$$\frac{\partial \phi}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + H(t, x, \nabla_x \phi) = 0 \textit{ in } Q_T \quad (2.29)$$

with the initial-boundary conditions

$$\phi = g_1 \qquad on \quad \Gamma_T . \tag{2.30}$$

More precisely, ϕ *is continuous in* $\bar{Q}_T$ *and satisfies* (2.30), $\nabla_x \phi^*$ *is a bounded function in* Q_T , *uniformly Hölder continuous (with some exponent* α*) in compact subsets of* Q_T , *the weak derivatives* $\partial\phi^*/\partial t$, $\partial^2\phi^*/\partial x_i\, \partial x_j$ *belong to* $L^r(Q_T)$ *for any* $r > 1$, *and* (2.13) *holds almost everywhere.*

Since $H(t, x, p)$ is a continuous function in (t, x, p), the proof of Theorem 3 is similar to the proof of Theorem 1.

THEOREM 4. *Let the conditions of Theorem* 3 *hold and let* $y_1^*(t, x, p)$, $y_2^*(t, x, p)$ *be any measurable functions with values in* Y_1 , Y_2 *respectively, satisfying* (2.26), (2.27). *Write*

$$y_1^*(t, x) = y_1^*(t, x, \nabla_x\phi^*(t, x)),$$
$$y_2^*(t, x) = y_2^*(t, x, \nabla_x\phi^*(t, x)).$$

Then $(y_1^*(t, x), y_2^*(t, x))$ *is a saddle point in pure strategies of the differential game associated with* (2.6), (2.7) *(where* $k = 1$, $N = 2$*) and the payoff* (2.5) *(with* $k = 1$*).*

Proof. Let y_1 choose the strategy $y_1^*(t, x)$, and let y_2 choose any strategy $y_2(t, x)$. Denote by ψ the solution of

$$\frac{\partial\psi}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2\psi}{\partial x_i\, \partial x_j} + f(t, x, y_1^*(t, x), y_2(t, x)) \cdot \nabla_x\psi$$
$$+ h_1(t, x, y_1^*(t, x), y_2(t, x)) = 0 \text{ in } Q_T , \quad \psi = g_1 \text{ on } \Gamma_T . \tag{2.31}$$

Since $\phi^* = \psi$ on Γ_T and

$$\frac{\partial\phi^*}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2\phi^*}{\partial x_i\, \partial x_j} + f(t, x, y_1^*(t, x), y_2(t, x)) \cdot \nabla_x\phi^*$$
$$+ h_1(t, x, y_1^*(t, x), y_2(t, x)) \leqslant 0$$

almost everywhere in Q_T , we conclude, by Lemma 2 of [1], that $\phi^* \geqslant \psi$ in Q_T . This gives

$$J_1(y_1^*, y_2^*) \geqslant J_1(y_1^*, y_2).$$

Similarly, one proves that

$$J_1(y_1^*, y_2^*) \leqslant J_1(y_1 , y_2^*).$$

Remark. We have dealt so far with the case where the initial condition $\xi(s)$ is fixed. If $\xi(s)$ is not fixed, that is, if the random variable $x_0 = x_0(\omega)$ is not a constant function of ω, then suppose that its distribution is given by a measure μ_0 defined on Ω. The cost $\psi_k(s, x_0)$ is then to be replaced by the cost

$$J_k = \int \psi_k(s, x)\, d\mu_0(x).$$

The results of this section immediately extend to this more general case.

3. Differential Games with Partial Observation

We consider, in this section, a game played by 2 players, y and z. The dynamics is given by a system of stochastic differential equations

$$d\xi = f(t, \xi, y, z)\, dt + \sigma(t, \xi)\, dw \tag{3.1}$$

with initial point

$$\xi(s) = x_0\,. \tag{3.2}$$

As in Section 2, control sets Y and Z are given, and they compact subsets of some euclidean spaces R^p and R^q, respectively. A payoff is given by

$$P(y, z) = E_{sx_0} \left\{ \int_s^\tau h(\tau, \xi, y, z)\, dt + g(\tau, \xi(\tau)) \right\}, \tag{3.3}$$

where τ is the exit time from Q_T. The player y wants to maximize the payoff, while the player z wants to minimize it.

If y and z make perfect observations, and if they use only pure strategies, then the existence of a saddle point follows by Theorem 4. Suppose now that y and z, at time t, can only observe a quantity $\eta(t)$, and suppose, further, that the manner by which $\eta(t)$ is related to $\xi(t)$ is known to have the form

$$d\eta = \tilde{f}(t, \xi, \eta, y, z)\, dt + \sigma(t, \xi, \eta)\, d\tilde{w},$$

where $\tilde{w}$ is a Brownian motion independent of w. We then consider the pair $\zeta = (\eta, \xi)$ as defining a diffusion process, governed by stochastic differential equations. With respect to this system, the players y and z observe a certain number of components of ζ, namely the components of η. The above setting is thus equivalent (with a different notation) to the following one:

The dynamics of the game is given by (3.1), and the players y, z observe just the first l components $\xi_1, \ldots, \xi_l$ of $\xi = (\xi_1, \ldots, \xi_m)$.

Set

$$\hat{\xi} = (\xi_1, ..., \xi_l), \qquad \hat{\hat{\xi}} = (\xi_{l+1}, ..., \xi_m),$$

so that $\xi = (\hat{\xi}, \hat{\hat{\xi}})$. We define a *pure strategy* for y as a measurable function $y = y(t, \hat{\xi})$ from $[s, T] \times R^l$ into Y, and a *pure strategy* for z as a measurable function $z = z(t, \hat{\xi})$ from $[s, T] \times R^l$ into Z.

As in Section 2, under some assumptions on f, σ, the payoff (2.3) corresponding to the solution of (3.1), (3.2) with $y = y(t, \hat{\xi})$, $z = (t, \hat{\xi})$ can be given as follows: If

$$\frac{\partial\psi}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2\psi}{\partial x_i\, \partial x_j} + f(t, x, y(t, \hat{x}), z(t, \hat{x})) \cdot \nabla_x\psi$$

$$+ h(t, x, y(t, \hat{x})) = 0 \text{ in } Q_T, \tag{3.4}$$

$$\psi = g \qquad \text{on} \quad \Gamma_T, \tag{3.5}$$

then

$$P(y, z) = \psi(s, x_0). \tag{3.6}$$

We shall replace the original setting of (3.1)–(3.3) by the setting (3.4)–(3.6). We can define *saddle point in pure strategies* as in Section 2. However, there is no simple connection between such saddle points and solutions of equations of the Hamilton–Jacobi type. This makes it much more difficult to try to prove the existence of a saddle point in pure strategies. There is also an intuitive reason why one should not expect, in general, the existence of a saddle point in pure strategies: In the lack of perfect observation, each player should make use of all the past history of the game, not just the present state.

We shall now develop an existence theory based on the partial observation of the whole past. This method is analogous to that introduced in [9]–[12] for deterministic games.

Let n be any positive integer, and let $\delta = (T - s)/n$. Denote by I_j the interval $t_{j-1} < t \leqslant t_j$, where $t_j = s + j\delta$. Denote by Y_j (Z_j) the set of all measurable functions $y_j(t, \hat{x})$ $(z_j(t, \hat{x}))$ from $I_j \times R^m$ into $Y(Z)$. An upper δ-strategy Γ^δ for y is a vector

$$\Gamma^\delta = (\Gamma^{\delta,1}, ..., \Gamma^{\delta,n}),$$

where $\Gamma^{\delta,j}$ is a map from

$$Z_1 \times Y_1 \times \cdots \times Z_{j-1} \times Y_{j-1} \times Z_j$$

into Y_j. A lower δ-strategy Δ_δ for z is a vector

$$\Delta_\delta = (\Delta_{\delta,1},..., \Delta_{\delta,n}),$$

where $\Delta_{\delta,1}$ is an element of Z_1, and $\Delta_{\delta,j}$ ($j \geqslant 2$) is a map from

$$Z_1 \times Y_1 \times \cdots \times Z_{j-1} \times Y_{j-1}$$

into Z_j.

We shall assume

(C′). $f(t, x, y, z)$ and $h(t, x, y, z)$ are continuous functions in $[s, T] \times R^m \times Y \times Z$, $\partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0, 1)$, and $g \in C_\alpha^{2,1}(\Gamma_T)$.

Any pair $(\Delta_\delta, \Gamma^\delta)$ defines a unique pair of pure strategies $(y^\delta(t, \hat{x}), z_\delta(t, \hat{x}))$, called the *outcome* of $(\Delta_\delta, \Gamma^\delta)$. If ($A_1$), ($A_2$), and (C′) hold, then there is a unique solution ψ^δ of (3.4), (3.5), when $y = y^\delta(t, \hat{x})$, $z = z_\delta(t, \hat{x})$, and a payoff

$$P(y^\delta, z_\delta) = \psi^\delta(s, x_0).$$

We denote this payoff also by $P[\Delta_\delta, \Gamma^\delta]$, or

$$P[\Delta_{\delta,1}, \Gamma^{\delta,1},..., \Delta_{\delta,n}, \Gamma^{\delta,n}].$$

The above scheme of corresponding a payoff $P(\Delta_\delta, \Gamma^\delta)$ to each pair $(\Delta_\delta, \Gamma^\delta)$, is called an upper δ-game, and is denoted by G^δ. The *upper δ-value* V^δ of this upper δ-game, is defined by

$$V^\delta = \inf_{\Delta_{\delta,1}} \sup_{\Gamma^{\delta,1}} \cdots \inf_{\Delta_{\delta,n}} \sup_{\Gamma^{\delta,n}} P[\Delta_{\delta,1}, \Gamma^{\delta,1},..., \Delta_{\delta,n}, \Gamma^{\delta,n}].$$

Similarly, we define lower δ-game G_δ and lower δ-value V_δ. Here, y uses lower δ-strategies Γ_δ and z uses upper δ-strategies Δ^δ. The pair of sequence

$$G = (\{G^\delta\}, \{G_\delta\}) \qquad \left(\delta = \frac{T-s}{n}, n = 1, 2,...\right)$$

is called the *differential game with partial observation* associated with (3.4)–(3.6). If

$$V^+ = \lim_{\delta\to 0} V^\delta, \qquad V^- = \lim_{\delta\to 0} V_\delta$$

exist, we call them the *upper value* and the *lower value* of the game. If $V^+ = V^-$, then we say that the game has *value* V, where $V = V^+ = V^-$.

A sequence $\Gamma = \{\Gamma_\delta\}$ is called a *strategy* for y. Similarly, a sequence $\Delta = \{\Delta_\delta\}$ is called a strategy for z. Each pair $(\Delta_\delta, \Gamma_\delta)$ determines an outcome

(y_δ, z_δ) and the corresponding solution ψ_δ of (3.4), (3.5). Suppose there exists a subsequence $\{\delta'\}$ of $\{\delta\}$ such that, as $\delta' \to 0$,

$$y_{\delta'}(t, \hat{x}) \rightharpoonup \bar{y}(t, \hat{x}) \qquad \text{weakly in} \quad L^1((s, T); R^p), \tag{3.7}$$

$$z_{\delta'}(t, \hat{x}) \rightharpoonup z(\bar{t}, \hat{x}) \qquad \text{weakly in} \quad L^1((s, T); R^q), \tag{3.8}$$

$$\psi_{\delta'}(t, x) \to \bar{\psi}(t, x) \qquad \text{for each} \quad (t, x) \in \bar{Q}_T, \tag{3.9}$$

where $\bar{y}(t, \hat{x}) \in Y$, $\bar{z}(t, \hat{x}) \in Z$ almost everywhere, and $\bar{\psi}$ is the solution of (3.4), (3.5) corresponding to $y = \bar{y}$, $z = \bar{z}$. Then we say that $(\bar{y}, \bar{z})$, or $(\bar{y}, \bar{z}, \bar{\psi})$, is an *outcome* of (Δ, Γ). The set of all numbers $\bar{\psi}(s, x_0)$, when $(\bar{y}, \bar{z}, \bar{\psi})$ varies over the set of all outcomes of (Δ, Γ), is called the *payoff set* of (Δ, Γ), and is denoted by $P[\Delta, \Gamma]$.

Given two sets of real numbers, A and B, we write $A \leqslant B$ if $a \leqslant b$ for all $a \in A$, $b \in B$. We write $A \leqslant B$ also in case A is empty or B is empty. Suppose the value V exists, and let Δ^*, Γ^* be strategies such that

$$P[\Delta^*, \Gamma] \leqslant P[\Delta^*, \Gamma^*] = \{V\} \leqslant P[\Delta, \Gamma^*] \tag{3.10}$$

for all strategies Δ, Γ. Then we call (Δ^*, Γ^*) a *saddle point.*

We shall denote the payoff $\psi(s, x_0)$ also by $P(\psi)$.

We can extend the concept of an outcome of (Δ, Γ) by omitting the conditions (3.7), (3.8). Thus, a solution $\bar{\psi}$ of (3.4), (3.5) (corresponding to some $y = y(t, \hat{x})$, $z = z(t, \hat{x})$) is called a *generalized outcome* of (Δ, Γ) if (3.9) holds for some subsequence $\{\delta'\}$ of $\{\delta\}$. The set of all numbers $P(\bar{\psi})$, where $\bar{\psi}$ varies over the set of generalized outcomes of (Δ, Γ), is called the *generalized payoff set* of (Δ, Γ), and is denoted by $P_0[\Delta, \Gamma]$. A pair (Δ^*, Γ^*) is called a *generalized saddle point* if it satisfies (3.10) with P replaced by P_0.

The concept of strategy as introduced in [12] differs from that introduced here. In [12], the spaces Y_j (Z_j) consist of all the measurable functions $y_i(t)$ $(z_j(t))$ from I_j into $Y(Z)$. Here they consist of all the measurable functions $y_i(t, x)$ $(z_j(t, x))$ from $I_j \times R^m$ into $Y(z)$. However, if we introduce in [12] the latter spaces (but restrict the $y_i(t, x)$, $z_j(t, x)$ to be uniformly Lipschitz continuous in x and request $f(t, x, y, z)$ to be Lipschitz continuous in (x, y, z), so as to have a unique trajectory for each pair of controls), then the resulting game will have the same value as in [12] (or [9]). In fact, the proof of the formula

$$V^\delta = \inf_{z_1 \in Z} \sup_{y_1 \in Y} \cdots \inf_{z_n \in Z} \sup_{y_n \in Y} P(z_1, y_1, ..., z_n, y_n)$$

given in [12] extends also to the present case where control functions have the form $y(t, x)$, $z(t, x)$. This implies that the concept of the upper value does not change when control functions $y(t)$, $z(t)$ are replaced by control functions $y(t, x)$, $z(t, x)$. Similarly, the concept of the lower value does not change.

Next, we can define a concept of strategy (for deterministic games) based on controls of the form $y(t, x)$, $z(t, x)$. As far as the existence of a saddle point, the max-min principle and the computational methods of [12] are concerned, we shall not get anything new. The numerical results in [12] for saddle points of particular games will be the same for the saddle points based on controls $y(t, x)$, $z(t, x)$. We can, therefore, conclude that the approach of the present section to stochastic games reduces, in fact, to the approach of [9]–[12] when the games are deterministic.

Another remark. To every pure strategy $\tilde{y}(t, \hat{x})$ we can correspond a *constant strategy* $\tilde{\Gamma}$ as follows (cf. [12]):

$$\tilde{\Gamma} = \{\tilde{\Gamma}_\delta\}, \qquad \tilde{\Gamma}_\delta = (\tilde{\Gamma}_{\delta,1}, \ldots, \tilde{\Gamma}_{\delta,n}),$$

where $\tilde{\Gamma}_{\delta,j}$ maps the whole space $Z_1 \times Y_1 \times \cdots \times Z_{j-1} \times Y_{j-1}$ into the point $\tilde{y}_j(t, \hat{x})$, the restriction of $\tilde{y}(t, \hat{x})$ to I_j. Using this correspondence, we can show that the equilibrium point in pure strategies established in Section 2 for a 2-person zero sum game, gives a saddle point in constant strategies—in the context of the present section. The concept of strategy as defined in this section and the last assertion extend also to N-person games.

We shall need the following condition:

(E). The controls y, z appear "separately" in f, h, i.e.,

$$\begin{aligned} f(t, x, y, z) &= f^1(t, x, y) + f^2(t, x, z), \\ h(t, x, y, z) &= h^1(t, x, y) + h^2(t, x, z). \end{aligned}$$

THEOREM 5. *Let the conditions* (A_1)–(A_4), (C′) *and* (E) *hold. Then the differential game with partial observation associated with* (3.4)–(3.6) *has value.*

If the condition (E) is not assumed, one can still prove that V^+ and V^- exist. The proof is obtained by combining the methods of [9] or [12, Chap. 2], with estimates derived in the subsequent proof.

Proof. The proof is similar to the proof of Theorem 1 in [9] (or Theorem 2.3.1 in [12]); the components z_j, y_i, instead of being functions of t only, as in [9, 12], are now functions of $(t, \hat{x})$. Thus, all we need to prove is the following lemma.

LEMMA 4. *Let the conditions* (A_1)–(A_4), (C′), (E) *hold. Let* $y_\lambda(t, \hat{x})$, $z_\lambda(t, \hat{x})$ *be pure strategies for* y *and* z, *respectively, for each* λ *from a sequence* $\{\lambda_n\}$, $\lambda_n \searrow 0$ *if* $n \nearrow \infty$. *Let* $\tilde{z}_\lambda(t, \hat{x})$ *be a pure strategy for* z *satisfying* $\tilde{z}_\lambda(t, \hat{x}) =$

$z_\lambda(t - \lambda, \hat{x})$ for $s + \lambda \leqslant t \leqslant T$, $\lambda \in \{\lambda_n\}$. Denote by ψ_λ and $\tilde{\psi}_\lambda$ the solutions of (3.4), (3.5) corresponding to y_λ, z_λ and y_λ, $\tilde{z}_\lambda$, respectively. Then, there exists a function $\sigma(\lambda)$, independent of y_λ, z_λ, such that $\sigma(\lambda_n) \to 0$ if $n \to \infty$ and

$$\max_{(t,x)\in\bar{Q}_T} |\tilde{\psi}_\lambda(t, x) - \psi_\lambda(t, x)| \leqslant \sigma(\lambda). \tag{3.11}$$

Proof. Denote by $U(t, \tau)$ $(s \leqslant t \leqslant \tau \leqslant T)$ the fundamental solution of the parabolic operator

$$\frac{\partial}{\partial t} + \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \, \partial x_j}$$

corresponding to the boundary condition $u = 0$ on $\partial\Omega_T$ (see [8]). Here we consider the elliptic part

$$A(t) = \sum_{i,j=1}^{m} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \, \partial x_j}$$

as a linear (unbounded) operator in $X = L^r(\Omega)$, for some fixed $r > 1$; later, we shall take $r > n$.

We shall denote by $\| \;\|$ the norm in X. We shall denote norms of bounded linear operators in X by $\| \;\|$ also.

We may assume that the resolvent of $A(t)$ exists for all λ with $\operatorname{Re} \lambda \geqslant 0$, for otherwise we first perform a transformation $\psi \to e^{\beta t}\psi$, where β is a suitable constant. But then, by [8], for $s \leqslant t < \sigma \leqslant T$,

$$\| A^\theta(t) U(t, \sigma)\| \leqslant \frac{C}{(\sigma - t)^\theta} \qquad (0 \leqslant \theta < 1). \tag{3.12}$$

Next, using the identity

$$U(t, \sigma + \lambda)x - U(t, \sigma)x = U(t, \sigma + \lambda) \int_0^{\sigma+\lambda} A(\xi) U(\xi, \sigma)x \, d\xi$$

for $x \in D_A$ (see [8, p. 250]) and estimates on U given in [8, Section 4], we find that for $s \leqslant t < \sigma < \sigma + \lambda \leqslant T$

$$\| A^\theta(t)[U(t, \sigma + \lambda) - U(t, \sigma)]\| \leqslant C \frac{\lambda^{\rho-\theta}}{(\sigma - t)^{\rho'}} \qquad (0 < \theta < \rho < \rho' < 1); \tag{3.13}$$

here and in what follows, various different constants are denoted by the same symbol C.

Set $\phi_\lambda = \tilde{\psi}_\lambda - \psi_\lambda$. Then, with $\phi_\lambda(t) = \phi_\lambda(t, \cdot)$,

$$\begin{aligned}\frac{d\phi_\lambda}{dt} + A(t)\phi_\lambda = &- [f^1(t, x, y_\lambda(t, \hat{x})) \cdot \nabla_x \phi_\lambda] \\ &- [f^2(t, x, \tilde{z}_\lambda(t, \hat{x})) \cdot \nabla_x \tilde{\psi}_\lambda - f^2(t, x, z_\lambda(t, \hat{x})) \cdot \nabla_x \psi_\lambda] \\ &- [h^2(t, x, \tilde{z}_\lambda(t, \hat{x})) - h^2(t, x, z_\lambda(t, \hat{x}))] \\ \equiv &-B_1 - B_2 - B_3 . \end{aligned} \tag{3.14}$$

We shall write $B_i(t) = B_i(t, \cdot)$.

Suppose $y_\lambda(t, \hat{x})$, $z_\lambda(t, \hat{x})$, $\tilde{z}_\lambda(t, \hat{x})$, f and h are all continuously differentiable. By Lemma 3, $\nabla_x \psi_\lambda$, $\nabla_x \tilde{\psi}_\lambda$ are then uniformly Hölder continuous in Q_T. Hence, by [8],

$$\begin{aligned}-\phi_\lambda(t) &= \int_t^T U(t, \sigma)B_1(\sigma)\, d\sigma + \int_t^T U(t, \sigma)B_2(\sigma)\, d\sigma + \int_t^T U(t, \sigma)B_3(\sigma)\, d\sigma \\ &\equiv \Phi_1 + \Phi_2 + \Phi_3 . \end{aligned} \tag{3.15}$$

We shall estimate the Φ_i. First, for any $0 \leqslant \theta < 1$,

$$\begin{aligned}\| A^\theta(t)\Phi_1(t)\| &\leqslant C \int_t^T \| A^\theta(t) U(t, \sigma)\| \, \| \nabla_x \phi_\lambda(\sigma, \cdot)\| \, d\sigma \\ &\leqslant C \int_t^T \frac{\| \nabla_x \phi_\lambda(\sigma, \cdot)\|}{(\sigma - y)^\theta}\, d\sigma. \end{aligned} \tag{3.16}$$

Since (see [8])

$$\| \nabla_x \phi_\lambda(\sigma, \cdot)\| \leqslant C \| A^\theta(\sigma)\phi_\lambda(\sigma, \cdot)\| \qquad \text{if } \tfrac{1}{2} < \theta < 1, \tag{3.17}$$

we get

$$\| A^\theta(t)\Phi_1(t)\| \leqslant C \int_t^T \frac{\| A^\theta(\sigma)\phi_\lambda(\sigma, \cdot)\|}{(\sigma - t)^\theta}\, d\sigma \qquad \text{if } \tfrac{1}{2} < \theta < 1. \tag{3.18}$$

By Lemma 3,

$$| \nabla_x \psi_\lambda |_{\alpha, Q_T} \leqslant C, \qquad | \nabla_x \tilde{\psi}_\lambda |_{\alpha, Q_T} \leqslant C, \tag{3.19}$$

$$\operatorname*{l.u.b.}_{Q_T} | \nabla_x \psi_\lambda(t, x)| \leqslant C, \qquad \operatorname*{l.u.b.}_{Q_T} | \nabla_x \tilde{\psi}_\lambda(t, x)| \leqslant C. \tag{3.20}$$

To estimate Φ_3 , write

$$\Phi_3 = \left[\int_{t+\lambda}^{T} U(t, \sigma)h^2(\sigma, \cdot, z_\lambda(\sigma - \lambda, \cdot))\, d\sigma - \int_{t}^{T-\lambda} U(t, \sigma)h^2(\sigma, \cdot, z_\lambda(\sigma, \cdot))\, d\sigma\right]$$
$$+ \int_{t}^{t+\lambda} U(t, \sigma)h^2(\sigma, \cdot, \tilde{z}_\lambda(\sigma, \cdot))\, d\sigma - \int_{T-\lambda}^{T} U(t, \sigma)h^2(\sigma, \cdot, z_\lambda(\sigma, \cdot))\, d\sigma$$
$$\equiv I_1 + I_2 - I_3 . \tag{3.21}$$

We can write

$$I_1 = \int_{t}^{T-\lambda} [U(t, \sigma + \lambda) - U(t, \sigma)]h^2(\sigma + \lambda, \cdot, z_\lambda(\sigma, \cdot))\, d\sigma$$
$$+ \int_{t}^{T-\lambda} U(t, \sigma + \lambda)[h^2(\sigma + \lambda, \cdot, z_\lambda(\sigma, \cdot)) - h^2(\sigma, \cdot, z_\lambda(\sigma, \cdot))]\, d\sigma$$
$$\equiv I_{11} + I_{12} .$$

By (3.13),

$$\| A^\theta(t)I_{11} \| \leqslant C \int_{t}^{T-\lambda} \frac{\lambda^{\rho-\theta}}{(\sigma - t)^{\rho'}} \| h^2(\sigma + \lambda, \cdot, z_\lambda(\sigma, \cdot))\|\, d\sigma \leqslant C\lambda^{\rho-\theta}$$

if $\theta < \rho < \rho' < 1$. We also have,

$$\| A^\theta(t)I_{12} \| \leqslant C\epsilon(\lambda), \qquad \epsilon(\lambda) \to 0 \quad \text{if} \quad \lambda \to 0,$$

where $\epsilon(\lambda)$ depends on the modulus of continuity of $h^2(t, x, z)$ with respect to t. Hence,

$$\| A^\theta(t)I_1 \| \leqslant C\lambda^{\rho-\theta} + C\epsilon(\lambda). \tag{3.22}$$

Next

$$\| A^\theta(t)I_2 \| \leqslant \int_{T-\lambda}^{T} \frac{C}{(\sigma - t)^\theta}\, d\sigma \leqslant C\lambda^{1-\theta} .$$

Similarly $\| A^\theta(t)I_3 \| \leqslant C\lambda^{-1\theta}$. We conclude that

$$\| A^\theta(t)\Phi_3 \| \leqslant C\lambda^{\rho-\theta} + C\epsilon(\lambda) \qquad \text{for any} \quad 0 < \theta < \rho < 1. \tag{3.23}$$

Next,

$$\Phi_2 = \int_{t}^{T} U(t, \sigma)[f^2(\sigma, x, \tilde{z}_\lambda(\sigma, \hat{x})) \cdot \nabla_x\tilde{\psi}_\lambda(\sigma, x) - f^2(t, x, z_\lambda(\sigma, \hat{x}) \cdot \nabla_x\tilde{\psi}_\lambda(\sigma, x)]\, d\sigma$$
$$+ \int_{t}^{T} U(t, \sigma)[f^2(\sigma, x, z_\lambda(\sigma, \hat{x}) \cdot \nabla_x\phi_\lambda(\sigma, x)]\, d\sigma$$
$$\equiv \Phi_{21} + \Phi_{22} . \tag{3.24}$$

As for Φ_{22}, we have, by (3.17),

$$\| A^\theta(t)\Phi_{22} \| \leqslant \int_t^T \frac{C}{(\sigma - t)^\theta} \| \nabla_x \phi_\lambda(\sigma, \cdot)\| \, d\sigma$$
$$\leqslant C \int_t^T \frac{\| A^\theta(\sigma)\phi_\lambda(\sigma)\|}{(\sigma - t)^\theta} \, d\sigma$$

if $\frac{1}{2} < \theta < 1$. As for $A^\theta(t)\Phi_{21}$, it can be estimated in the same manner as $A^\theta(t)\Phi_3$. Here we make use of (3.19), (3.20). The inequality we get is

$$\| A^\theta(t)\Phi_{31} \| \leqslant C\lambda^{\rho-\theta} + C\epsilon(\lambda) + C\lambda^\alpha.$$

We conclude that

$$\| A^\theta(t)\Phi_2 \| \leqslant C\lambda^{\rho-\theta} + C\epsilon(\lambda) + C\lambda^\alpha + C \int_t^T \frac{\| A^\theta(\sigma)\phi_\lambda(\sigma)\|}{(\sigma - t)^\theta} \, d\sigma. \tag{3.25}$$

Combining this with (3.23), (3.18), and (3.15) and setting

$$\gamma_\lambda(t) = \| A^\theta(t)\phi_\lambda(t)\|, \qquad \beta(\lambda) = \min\{\epsilon(\lambda), \lambda^{\rho-\theta}, \lambda^\alpha\},$$

we get

$$\gamma_\lambda(t) \leqslant C\beta(\lambda) + C \int_t^T \frac{\gamma_\lambda(\sigma)}{(\sigma - t)^\theta} \, d\sigma. \tag{3.26}$$

By iteration we find that

$$\gamma_\lambda(t) \leqslant \beta^*(\lambda), \qquad \beta^*(\lambda) \to 0 \qquad \text{if} \quad \lambda \to 0,$$

i.e.,

$$\| A^\theta(t)\phi_\lambda(t)\| \leqslant \beta^*(\lambda). \tag{3.27}$$

In deriving (3.27) we have assumed that y_λ, z_λ, $\tilde{z}_\lambda$, f and h are continuously differentiable. However, the function $\beta^*(\lambda)$ occurring in (3.27) depends only on the constants which enter into the conditions (A_1)–(A_4) and on bounds and moduli of continuity of f, h. Hence, by approximating y_λ, z_λ, $\tilde{z}_\lambda$, f by smooth functions and applying (3.27) to each of the corresponding ϕ_λ, we conclude that (3.27) holds in general.

Since (by [8], for instance)

$$| \phi_\lambda(t, x)| \leqslant C\| A^\theta(t)\phi_\lambda(t, \cdot)\| \qquad \text{if} \quad r > n, \quad \tfrac{1}{2} < \theta < 1,$$

the assertion of the lemma follows from (3.27).

Our next objective is to prove the existence of a saddle point. We shall need the following conditions:

(F). $f(t, x, y, z)$ and $h(t, x, y, z)$ are linear functions of y, z, i.e.,

$$f(t, x, y, z) = f^0(t, x) + F^1(t, x)y + F^2(t, x)z,$$
$$h(t, x, y, z) = h^0(t, x) + h^1(t, x) \cdot y + h^2(t, x) \cdot z,$$

and Y, Z are convex sets.

(F'). For any $(t, x) \in Q_T$ and $p \in R^m$, the set

$$f(t, x, Y, Z) \cdot p + h(t, x, Y, Z)$$
$$\equiv \{f(t, x, y, z) \cdot p + h(t, x, y, z); y \in Y, z \in Z\}$$

is a convex set.

LEMMA 5. *Let* (A_1)–(A_4) *hold. Let* $f(t, x, y, z)$ *be continuous in* $[s, T] \times R^m \times Y \times Z$, $\partial\Omega \in C^{2+\alpha}$ *(for some* $\alpha > 0$*),* $g \in C_\alpha^{2,1}(\Gamma_T)$, *and let* (F) *hold. Then, given any sequence of pure strategies* $(y_n(t, \hat{x}), z_n(t, \hat{x}))$ *and the corresponding solutions* ψ_n *of* (3.4), (3.5), *there exists a subsequence* $\{n'\}$ *of* $\{n\}$ *and pure strategies* $\bar{y}(t, \hat{x})$, $\bar{z}(t, \hat{x})$, *such that, as* $n' \to \infty$,

$$y_{n'} \rightharpoonup \bar{y} \text{ in } L^1((s, T); R^p), \qquad z_{n'} \rightharpoonup \bar{z} \text{ in } L^1((s, T); R^q), \tag{3.28}$$
$$\psi_{n'}(t, x) \to \bar{\psi}(t, x) \text{ uniformly in } Q_T, \tag{3.29}$$

where $\bar{\psi}$ *is the solution of* (3.4), (3.5) *corresponding to* $\bar{y}$, $\bar{z}$.

Proof. We may assume that $g = 0$, for otherwise, we consider $\psi_n - \hat{g}$, where $\hat{g}$ is a $C_\alpha^{2,1}$ extension of g into $\bar{Q}_T$.

The proof of (3.28) follows by a standard argument (cf. [12, proof of Theorem 2.4.1]). Since, by Lemma 3,

$$| \psi_n(t, x) - \psi_n(t', x')| + | \nabla_x\psi_n(t, x) - \nabla_x\psi_n(t', x')|$$
$$\leqslant C(| t - t' |^{\alpha/2} + | x - x' |^{\alpha}),$$

we may also assume, by the Ascoli–Arzela lemma, that

$$\begin{aligned} \psi_{n'}(t, x) &\to \bar{\psi}(t, x), \\ \nabla_x\psi_{n'}(t, x) &\to \nabla_x\bar{\psi}(t, x) \end{aligned} \tag{3.30}$$

uniformly in Q_T, where $\bar{\psi}$ is some function. Writing

$$\psi_n(t, x) = \int_t^T U(t, \sigma)[f^0(\sigma, x) + F^1(\sigma, x) y_n(\sigma, \hat{x}) + F^2(\sigma, x) z_n(\sigma, \hat{x})] \cdot \nabla_x \psi_n(\sigma, x)\, d\sigma + \int_t^T U(t, \sigma)[h^0(\sigma, x) + h^1(\sigma, x) \cdot y_n(\sigma, \hat{x}) + h^2(\sigma, x) \cdot z_n(\sigma, \hat{x})]\, d\sigma,$$

and taking $n = n' \to \infty$, we find that

$$\bar{\psi}(t, x) = \int_t^T U(t, \sigma)[f^0(\sigma, x) + F^1(\sigma, x) \bar{y}(\sigma, \hat{x}) + F^2(\sigma, x) \bar{z}(\sigma, \hat{x})] \cdot \nabla_x \bar{\psi}(\sigma, x)\, d\sigma + \int_t^T U(t, \sigma)[h^0(\sigma, x) + h^1(\sigma, x) \cdot \bar{y}(\sigma, \hat{x}) + h^2(\sigma, x) \cdot \bar{z}(\sigma, \hat{x})]\, d\sigma. \tag{3.31}$$

The solution of (3.4), (3.5) corresponding to $\bar{y}, \bar{z}$ satisfies the integral Eq. (3.31). Further, from the estimates (3.12), (3.17) we can deduce that there is at most one solution of (3.31). It follows that $\bar{\psi}$ is the solution corresponding to $\bar{y}$, $\bar{z}$. This completes the proof of the lemma.

The next lemma is analogous to Filippov's theorem in "Ordinary Differential Equations."

LEMMA 6. *Let* (A_1)–(A_4) *hold. Let* $f(t, x, y, z)$ *be continuous in* $[s, T] \times R^m \times Y \times Z$, $\partial\Omega \in C^{2+\alpha}$ *(for some* $\alpha > 0$*),* $g \in C^{2,1}(\Gamma_T)$*, and let* (F′) *hold. Then, given any sequence of pure strategies* $(y_n(t, x), z_n(t, x))$ *(here* $l = m$*) and the corresponding solutions* ψ_n *of* (3.4), (3.5) *(with* $\hat{x} = x$*), there exists a subsequence* $\{n'\}$ *of* $\{n\}$ *and a solution* $\bar{\psi}$ *of* (3.4), (3.5) *(for some pure strategies* $y(t, x), z(t, x)$*) such that* (3.29) *holds.*

Proof. Write $U = Y \times Z$, $u_n = (y_n, z_n)$. As in the proof of Lemma 5 we may assume that $g = 0$ and that (3.30), (3.31) hold with $\hat{x} = x$. We can write

$$\psi_n(t, \cdot) = \int_t^T U(t, \sigma) k_n(\sigma, \cdot)\, d\sigma, \tag{3.32}$$

where

$$k_n(\sigma, x) = f(\sigma, x, u_n(\sigma, x)) \cdot \nabla_x \psi_n(\sigma, x) + h_n(\sigma, x, u_n(\sigma, x)). \tag{3.33}$$

We may assume that the subsequence $\{n'\}$ is such that

$$k_{n'}(\sigma, x) \rightharpoonup \tilde{k}(\sigma, x) \quad \text{in} \quad L^1(Q_T),$$
$$k_{n'}(\sigma, \cdot) \rightharpoonup \tilde{k}(\sigma, \cdot) \quad \text{in} \quad L^1((s, T); L^1(\Omega)).$$

By arguments used in the proof of Theorem 2.4.2 of [12], we deduce that

$$\tilde{k}(\sigma, x) \in f(\sigma, x, U) \cdot \nabla_x \bar{\psi}(\sigma, x) + h(\sigma, x, U).$$

From the proof of Filippov's lemma (with t replaced by (t, x)) we deduce that there exists a control function $\tilde{u}(t, x) = (\tilde{y}(t, x), \tilde{z}(t, x))$ (with values in $Y \times Z$) such that

$$\tilde{k}(\sigma, x) = f(\sigma, x, \tilde{u}(\sigma, x)) \cdot \nabla_x \bar{\psi}(\sigma, x) + h(\sigma, x, \tilde{u}(\sigma, x)).$$

But then, from (3.32), (3.33) we conclude that

$$\bar{\psi}(t, \cdot) = \int_t^T U(t, \sigma) f(\sigma, x, \tilde{y}(\sigma, x), \tilde{z}(\sigma, x)) \cdot \nabla_x \psi(\sigma, x)\, d\sigma$$
$$+ \int_t^T U(t, \sigma) h(\sigma, x, \tilde{y}(\sigma, x), \tilde{z}(\sigma, x))\, d\sigma.$$

Hence $\bar{\psi}$ is a solution of (3.4), (3.5) corresponding to $y = \tilde{y}$, $z = \tilde{z}$. This completes the proof of the lemma.

Using Lemma 5 we can apply the method of proof of Theorem 2.5.1 in [12], and thus deduce the following result concerning the existence of a saddle point.

THEOREM 6. *Let the conditions of Theorem* 5 *hold, let* $\partial\Omega \in C^{2+\alpha}$ *(for some* $\alpha > 0$), *and let* (F) *hold. Then there exists a saddle point for the game of partial observation associated with* (3.4)–(3.6).

Similarly, using Lemma 6 one can establish the existence of a generalized saddle point for the game of perfect observation associated with (3.4)–(3.6) (with $\hat{x} = x$), replacing the condition (F) by the weaker condition (F′). However, in Section 2 we have already established a stronger result for games of perfect observation, concerning the existence of a saddle point in pure strategies. The usefulness of Lemma 6 lies then only in establishing the fact that the generalized payoff set $P_0[\Delta, \Gamma]$ is never empty.

Theorems 5 and 6 extend to the case where one player observes $x_{i_1}, \ldots, x_{i_l}$ and another player observes $x_{j_1}, \ldots, x_{j_k}$, for some indices $i_1, \ldots, i_l, j_1, \ldots, j_k$.

Remark 1. One can show that the value of the game $V(s, x_0)$ is a continuous function of (s, x_0). The proof is analogous to the proof of the corresponding result for deterministic games (in [9, 12]). One uses here estimates from Sections 1 and 3.

Remark 2. Theorems 5 and 6 and the last remark extend to the case where the initial random variable $x_0 = x_0(\omega)$ is not the constant function.

4. Stochastic Games with Degeneracy

If $\sigma\sigma^*$ is not positive definite, then the parabolic Eq. (3.4) is degenerate. The results of Sections 2 and 3 are based on estimates that have been established in the literature for nondegenerate parabolic equations only. Therefore at present the results of Sections 2 and 3 cannot be extended to the degenerate case. To illustrate the kind of difficulty that is encountered in the degenerate case, and also to suggest a possible line of approach, consider the following special case: The dynamics is given by

$$\dot{\eta} = k(t, \eta, y, z), \qquad \eta(s) = \eta_0 . \tag{4.1}$$

$$d\xi = f(t, \xi, \eta, y, z)\, dt + \sigma(t, \xi)\, dw, \qquad \xi(s) = \xi_0 , \tag{4.2}$$

where $\sigma\sigma^*$ is positive definite, and

$$P(y, z) = \mu(\eta) + E_{s\xi_0} \left\{ \int_s^\tau h(t, \xi, y, z)\, dt + g(\tau, \xi(\tau)) \right\}. \tag{4.3}$$

$\mu(\eta)$ is a continuous functional of η in a suitable topology. The control functions are either of the form $y(t, \xi, \eta)$, $z(t, \xi, \eta)$, where a uniform Lipschitz condition with respect to η is assumed, or of the form $y(t, \xi)$, $z(t, \xi)$. For simplicity, assume the latter form.

Solving $\eta = \eta(y, z)$ from (4.1) and substituting into (4.2), we obtain

$$d\xi = f(t, \xi, \eta(y, z), y, z)\, dt + \sigma(t, \xi)\, dw, \qquad \xi(s) = \xi_0 . \tag{4.4}$$

Note, however, that this system is not a stochastic differential system of the usual type, since $\eta(y, z)$ at time t depends on the controls $y = y(\sigma, \xi(\sigma))$, $z = z(\sigma, \xi(\sigma))$ for all $s \leqslant \sigma \leqslant t$. Thus (4.4) is, in effect, a stochastic integral equation of Volterra type. By the methods of [16] one can establish the existence of a solution for (4.4). However, there is no connection, in general, between such a system and parabolic equations. Thus, in order to extend the results of Sections 2 and 3 to the present degenerate case, one has to study stochastic integral equations and relate them, perhaps, to some "integro-parabolic" equations.

For the one player case, the system (4.1), (4.2) represents a slightly more specialized case of degeneracy than that treated by Fleming [2]. A solution of the optimal problem for the one-player case has been recently obtained by Rishel [18].

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