An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 38 (1999) 121–131
www.elsevier.nl/locate/camwa

A Hydrodynamic Model Arising in the Context of Granular Media

D. BENEDETTO AND E. CAGLIOTI
Dipartimento di Mathematica
Università di Roma "La Sapienza"
Rome, Italy

F. GOLSE
Université Paris VII and Ecole Normale Supérieure, D.M.I.
45 rue d'Ulm, 75005 Paris, France

M. PULVIRENTI
Dipartimento di Mathematica
Università di Roma "La Sapienza"
Rome, Italy

(Received December 1998; accepted February 1999)

Abstract—In this note, we propose a formal argument identifying the hydrodynamic limit of a Fokker-Planck model for granular media appearing in [1]. More precisely, in the limit of large background temperature and vanishing friction, this hydrodynamic limit is described by the classical system of isentropic gas dynamics with a nonstandard pressure law (specifically, the pressure is proportional to the cube root of the density). Finally, some qualitative properties of the hydrodynamic model are studied. © 1999 Elsevier Science Ltd. All rights reserved.

1. THE FOKKER-PLANCK MODEL

A simple model for granular media, proposed in recent publications (see, for instance, [2]) consists of a one-dimensional system of like N particles subject to inelastic binary collisions. Specifically, each particle moves freely between two consecutive collisions. The change in velocities due to those collisions is given by

$$v_1' = v_1 + \varepsilon\,(v_2 - v_1)\,, \qquad v_2' = v_2 - \varepsilon\,(v_2 - v_1)\,, \tag{1.1}$$

where the parameter $\varepsilon \in]0, 1/2[$ controls the dissipation of kinetic energy while v_1 and v_2 (respectively, v_1' and v_2') are the velocities of the two colliding particles immediately before (respectively, after) the collision occurs. More precisely, assuming the mass of the particles equals 1, the loss of kinetic energy per collision is $(\varepsilon - \varepsilon^2)\times$ relative velocity before collision.

In [2] (see also [3,4]), a Vlasov type kinetic model is derived formally from this particle model in the limit as $N \to +\infty$ and $\varepsilon \to 0$ in such a way that, for some $\lambda > 0$, $N\varepsilon \to \lambda$. This model reads

$$\partial_t f + v\,\partial_x f + \lambda\,\partial_v(Ff) = 0, \tag{1.2}$$

This work was partially supported by MURST and CNR-GNFM (Italy).

PII:S0898-1221(99)00243-6

with

$$F \equiv F(t,x,v) = \int_{\mathbf{R}} |v' - v|\,(v' - v)\,f(t,x,v')\,dv', \tag{1.3}$$

$f \equiv f(t,x,v)$ denoting the density of particles which, at time t, are in position x with velocity v.

The present note is devoted to an inelastic particle system as above, modeled as in (1.2) but immersed in a thermal bath at a constant temperature. In [5] this problem has been considered from a numerical point of view. The simulations show, in certain regimes, nontrivial clustering phenomena. In [1] the effect of the thermal bath is modeled by adding to the Vlasov equation (1.2) a Fokker-Planck term: instead of (1.2), the phase space density f must satisfy

$$\partial_t f + v\,\partial_x f + \lambda\,\partial_v(Ff) = \beta\,\partial_v(vf) + \sigma\,\partial_{vv} f, \tag{1.4}$$

where β is the friction coefficient and σ/β the temperature of the thermal bath.

In the case where the phase space density vanishes at infinity (with sufficiently high order) in the v-variable, one can integrate (1.4) with respect to v to obtain the continuity equation (local conservation of mass)

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1.5}$$

denoting as usual the macroscopic density by ρ and the bulk velocity by u, or in other words

$$\rho(t,x) = \int_{\mathbf{R}} f(t,x,v)\,dv, \qquad \rho(t,x)u(t,x) = \int_{\mathbf{R}} vf(t,x,v)\,dv. \tag{1.6}$$

Multiplying (1.4) by v and integrating with respect to v gives the momentum equation

$$\partial_t(\rho u) + \partial_x \int_{\mathbf{R}} v^2 f\,dv = -\beta\rho u. \tag{1.7}$$

Indeed the contribution of the force is

$$\lambda \int_{\mathbf{R}} v\,\partial_v(Ff)\,dv = -\lambda \int_{\mathbf{R}} Ff\,dv = -\lambda \iint_{\mathbf{R}\times\mathbf{R}} |v' - v|\,(v' - v)\,f(t,x,v)f(t,x,v')\,dv\,dv' = 0,$$

as can be seen by changing variables according to $(v,v') \mapsto (v',v)$ in the double integral. Observe that (1.7) is not in closed form because it involves the second-order moment

$$\int_{\mathbf{R}} v^2 f\,dv,$$

which is not a function of ρ and u unless some closure assumption is made. In addition, if the friction coefficient $\beta = 0$, observe that (1.7) takes the form of a local conservation law, that of momentum.

Instead of looking at the limit as $t \to +\infty$ of only the spatially homogeneous solutions of (1.4) as was done in [1], it is also natural to look for the hydrodynamic limit (i.e., the infinite volume and long time limit) of (1.4). More precisely, one defines a velocity scale $V > 0$, a macroscopic length scale $L > 0$ and consider the dimensionless variables

$$\overline{t} = \frac{tV}{L}, \qquad \overline{x} = \frac{x}{L}, \qquad \overline{v} = \frac{v}{V}. \tag{1.8}$$

Defining

$$\overline{f}\,(\overline{t},\overline{x},\overline{v}) = f(t,x,v) \tag{1.9}$$

leads to the dimensionless form of (1.4)

$$\partial_{\overline{t}}\overline{f} + \overline{v}\,\partial_{\overline{x}}\overline{f} + \lambda LV\,\partial_{\overline{v}}\left(\overline{F}\overline{f}\right) = \frac{\beta L}{V}\,\partial_{\overline{v}}\left(\overline{v}\overline{f}\right) + \frac{\sigma L}{V^3}\,\partial_{\overline{v}\overline{v}}\overline{f}, \tag{1.10}$$

with

$$\overline{F}\left(\bar{t},\overline{x},\overline{v}\right) = \int_{\mathbf{R}} |\overline{v}' - \overline{v}| \left(\overline{v}' - \overline{v}\right) \overline{f}\left(\bar{t},\overline{x},\overline{v}'\right) \, d\overline{v}'. \tag{1.11}$$

The hydrodynamic limit of (1.4) consists in letting $V > 0$ as well as the parameters λ, β, and σ fixed and taking $L \to +\infty$.

Notice that, multiplying (1.10) by $\overline{v}$ and integrating with respect to $\overline{x}$ and $\overline{v}$ under the assumption that the number density $\overline{f}$ vanishes (of sufficiently high order) at infinity in both $\overline{x}$ and $\overline{v}$ leads to

$$\frac{d}{d\bar{t}} \iint_{\mathbf{R}} \overline{v}\overline{f} \, d\overline{x} \, d\overline{v} = -\frac{\beta L}{V} \iint_{\mathbf{R}} \overline{v}\overline{f} \, d\overline{x} \, d\overline{v}. \tag{1.12}$$

Hence, if $\beta > 0$ and as $L \to +\infty$ and unless $\beta = 0$, the total momentum is instantaneously dissipated under the effect of the large dimensionless friction coefficient, thereby leading to the trivial dynamics

$$\partial_{\bar{t}} \int_{\mathbf{R}} \overline{f}\left(\bar{t},\overline{x},\overline{v}\right) \, d\overline{v} = 0, \qquad \int_{\mathbf{R}} \overline{v}\overline{f}\left(\bar{t},\overline{x},\overline{v}\right) \, d\overline{v} = 0. \tag{1.13}$$

Therefore, in the sequel, we set once and for all $\beta = 0$. To simplify notations, we define

$$\epsilon = \frac{V^3}{\sigma L} \quad \text{and} \quad \overline{\lambda} = \frac{\lambda V^4}{\sigma}, \tag{1.14}$$

and substitute this in (1.10); after dropping all bars, we get

$$\partial_t f_\epsilon + v \, \partial_x f_\epsilon + \frac{1}{\epsilon} \left[\lambda \, \partial_v \left(F_\epsilon f_\epsilon\right) - \partial_{vv} f_\epsilon\right] = 0, \tag{1.15}$$

with

$$F_\epsilon(t,x,v) = \int_{\mathbf{R}} |v' - v| \left(v' - v\right) f_\epsilon\left(t,x,v'\right) \, dv'. \tag{1.16}$$

This problem is posed for all $(x,v) \in \mathbf{R} \times \mathbf{R}$ with prescribed initial data

$$f_\epsilon(0,x,v) = f^{\text{in}}(x,v), \qquad (x,v) \in \mathbf{R} \times \mathbf{R}. \tag{1.17}$$

The question of global existence and uniqueness for (1.15)–(1.17) with $\epsilon > 0$ is open. In the pure Vlasov case ($\sigma = \beta = 0$), global existence and uniqueness of classical solutions of the Cauchy problem for (1.2) on $\mathbf{R}_x \times \mathbf{R}_v$ is established in [4] under the condition that λ be small enough (in terms of suitable L_∞ norms of the initial data).

In the new dimensionless variables, the hydrodynamic limit of (1.4) consists in the limit as $\epsilon \to 0$ of the singular perturbation problem (1.15)–(1.17).

A last object naturally associated with the Fokker-Planck model (1.4) is the free energy, which for the dimensionless form (1.15) for $\beta = 0$ is defined by

$$\eta_\lambda(\phi) = \int_{\mathbf{R}} \phi(v) \log \phi(v) \, dv + \frac{\lambda}{6} \iint_{\mathbf{R}\times\mathbf{R}} |v' - v|^3 \, \phi(v)\phi\left(v'\right) \, dv \, dv', \tag{1.18}$$

defined on nonnegative measurable functions ϕ of the velocity variable v.

A natural question is that of the existence and uniqueness of stationary states with respect to the free energy η_λ above; these states minimize the free energy under the constraints of given total mass and momentum. Thus, we consider, for all $\rho \geq 0$ and $u \in \mathbf{R}$,

$$\mathbf{K}_{\rho,u} = \left\{\phi \in L^1\left(\mathbf{R}, \left(1 + v^2\right) \, dv\right) \mid \phi \geq 0 \text{ a.e.}, \int_{\mathbf{R}} \phi \, dv = \rho, \int_{\mathbf{R}} v\phi \, dv = \rho u\right\}. \tag{1.19}$$

LEMMA 1. *For all $\lambda > 0$, η_λ is strictly convex with values in $[0, +\infty]$ on $\mathbf{K}_{1,0}$ and reaches its minimum there at a single point denoted by G_λ. This function G_λ is even and belongs $C^\infty(\mathbf{R})$.*

This is precisely Theorem 2.1 of [1] in the case of $\beta = 0$.

COROLLARY 2. *For all $\lambda > 0$ and all $\rho > 0$, $u \in \mathbf{R}$*

$$\inf_{\mathbf{K}_{\rho,u}} \eta_\lambda \text{ is attained by the unique function } v \mapsto \rho G_{\lambda\rho}(v-u).$$

PROOF. Let $\phi \in \mathbf{K}_{\rho,u}$; then $\psi : v \mapsto (1/\rho)\phi(v+u)$ belongs to $\mathbf{K}_{1,0}$ and a trivial change of variables shows that

$$\eta_{\lambda\rho(\psi)} = \frac{1}{\rho}\eta_\lambda(\phi) - \log\rho. \tag{1.20}$$

One then concludes by a direct application of Lemma 1.

As explained in [1], η_λ is a Lyapunov function for the spatially homogeneous version of (1.4), and the functions G_λ above are the spatially homogeneous steady states for (1.4). It also satisfies the self-consistent equation

$$G_\lambda = \frac{e^{-(\lambda/3)\int dv'|v'-v|^3 G_\lambda(v')}}{\int dv\, e^{-(\lambda/3)\int dv'|v'-v|^3 G_\lambda(v')}}, \tag{1.21}$$

which is consequence of the stationary condition $\lambda\,\partial_v(fF) - \partial_{vv} f = 0$ for the homogeneous equation associated to equation (1.15).

2. THE STATIONARY STATES AND THE HYDRODYNAMIC LIMIT

Defining as in (1.6) the macroscopic density ρ_ϵ and bulk velocity u_ϵ associated to the microscopic density f_ϵ solving (1.15)–(1.17), we see that $(\rho_\epsilon, u_\epsilon)$ satisfy the system of conservation laws (1.5) and (1.7) with $\beta = 0$. One formulation of the hydrodynamic limit is to find a closure for (1.7), i.e., a function Φ such that

$$\lim_{\epsilon\to 0}\int_{\mathbf{R}} f_\epsilon v^2\, dv = \Phi\left(\lim_{\epsilon\to 0}\rho_\epsilon, \lim_{\epsilon\to 0}\rho_\epsilon u_\epsilon\right) \tag{2.1}$$

(where the limit as $\epsilon \to 0$ is taken in some weak topology). One way of finding Φ is by postulating the local equilibrium condition, that is, to leading order as $\epsilon \to 0$:

$$f_\epsilon(t,x,v) \simeq \rho_\epsilon(t,x) G_{\lambda\rho_\epsilon(t,x)}\left(v - u_\epsilon(t,x)\right). \tag{2.2}$$

Indeed, since the spatially homogeneous steady states for (1.4) (with $\beta = 0$) are microscopic densities of the form (2.2) with ρ and u constant, it is natural to look for approximate solutions of (1.15),(1.16) on finite time intervals by slowly modulating the parameters of the steady state (that is, by letting them depend on the slow time and space variables defined by (1.8)).

Another, more mathematical method is to apply the Hilbert expansion method to (1.15),(1.16), i.e., to look for solutions f_ϵ of (1.15),(1.16) as formal series

$$f_\epsilon(t,x,v) = \sum_{k\ge 0}\epsilon^k f_k(t,x,v) \in C^\infty(\mathbf{R}\times\mathbf{R}\times\mathbf{R})[[\epsilon]]. \tag{2.3}$$

Observe then that model (1.15) has a nice gradient structure with respect to the free energy η_λ. Indeed, the differential of $\eta_{\lambda,0}$ is

$$D\eta_\lambda(f) \equiv \log f(t,x,v) + \frac{\lambda}{3}\int_{\mathbf{R}} |v'-v|^3 f(t,x,v')\, dv'. \tag{2.4}$$

Since

$$\lambda\left(F_\epsilon f_\epsilon\right) - \partial_v f_\epsilon = -f_\epsilon\,\partial_v D\eta_\lambda\left(f_\epsilon\right), \tag{2.5}$$

equation (1.15) can be recast in the form

$$\partial_t f_\epsilon + v\, \partial_x f_\epsilon = \frac{1}{\epsilon}\, \partial_v \left[f_\epsilon\, \partial_v D\eta_\lambda \left(f_\epsilon \right) \right]. \tag{2.6}$$

The Hilbert expansion method applied to (2.6) gives, to the leading order,

$$\text{at order } \epsilon^{-1}: \quad \partial_v \left[f_0\, \partial_v D\eta_\lambda \left(f_0 \right) \right] = 0. \tag{2.7}$$

Equation (2.7) is the equilibrium equation for the homogeneous version of equation (1.15). The regular, positive, summable, and stationary solutions of the homogeneous equation must coincide with a minimum of the functional η with the constraints for the density and the momentum, being η strictly convex. Then

$$f_0(t,x,v) = \rho(t,x) G_{\lambda\rho(t,x)} \left(v - u(t,x) \right). \tag{2.8}$$

Hence, (2.8) is a justification to the formal approximation

$$f_\epsilon(t,x,v) \simeq \rho(t,x) G_{\lambda\rho(t,x)} \left(v - u(t,x) \right), \qquad \text{as } \epsilon \to 0. \tag{2.9}$$

As a consequence, one is led to define Φ by the relation

$$\Phi(\rho, \rho u) = \int_{\mathbf{R}} \rho G_{\lambda\rho}(v-u) v^2\, dv. \tag{2.10}$$

LEMMA 3. *For all $\lambda > 0$, one has*

$$G_\lambda(v) = \lambda^{1/3} G_1 \left(\lambda^{1/3} v \right), \qquad v \in \mathbf{R}.$$

PROOF. Let $\phi \in \mathbf{K}_{1,0}$; for all $\alpha > 0$ the function $\psi : v \mapsto \alpha\phi(\alpha v)$ also belongs to $\mathbf{K}_{1,0}$ and one has

$$\eta_\lambda(\psi) = \eta_{\lambda/\alpha^3}(\phi) + \log \alpha. \tag{2.11}$$

Putting $\psi = G_\lambda$ and $\alpha = \lambda^{1/3}$ shows that $\phi : v \mapsto \lambda^{-1/3} G_\lambda(\lambda^{-1/3} v)$ minimizes $\eta_{1,0}$ on $\mathbf{K}_{1,0}$ and must by the uniqueness part of Lemma 1 coincide with G_1, thereby proving our claim.

COROLLARY 4. *For all $\rho > 0$ and $m \in \mathbf{R}$, the function Φ defined by (2.10) is given by the formula*

$$\Phi(\rho, m) = \frac{m^2}{\rho} + k\rho^{1/3}, \tag{2.12}$$

with the constant $k > 0$ defined by

$$k = \lambda^{1/3} \int_{\mathbf{R}} G_1(w) w^2\, dw. \tag{2.13}$$

PROOF. Indeed

$$\Phi(\rho, \rho u) = \int_R \rho^{4/3} \lambda^{1/3} G_1 \left(\lambda^{1/3} \rho^{1/3} (v-u) \right) v^2\, dv; \tag{2.14}$$

in the integrand of the right-hand side of (2.14), split

$$v^2 = u^2 + 2u(v-u) + (v-u)^2. \tag{2.15}$$

Since G_1 belongs to $\mathbf{K}_{1,0}$ by definition, changing variables by $v \mapsto w = \lambda^{1/3}\rho^{1/3}(v-u)$ gives

$$\int_R \rho^{4/3} \lambda^{1/3} G_1 \left(\lambda^{1/3} \rho^{1/3} (v-u) \right) (v-u)^2\, dv = k\rho^{1/3}, \tag{2.16}$$

with k given by (2.13), while

$$\int_R \rho^{4/3}\lambda^{1/3}G_1\left(\lambda^{1/3}\rho^{1/3}(v-u)\right)dv=\rho,$$
$$\int_R \rho^{4/3}\lambda^{1/3}G_1\left(\lambda^{1/3}\rho^{1/3}(v-u)\right)(v-u)\,dv=0. \tag{2.17}$$

The announced formula (2.12) directly follows from the decomposition (2.15) in the integral (2.14) and from (2.16),(2.17).

Thus, the hydrodynamic limit of (1.15),(1.16) is formally given by the approximation (2.9), the parameters ρ and u of the stationary state being governed by the system of conservation laws

$$\partial_t\rho+\partial_x(m)=0,$$
$$\partial_t m+\partial_x\left(\frac{m^2}{\rho}+k\rho^{1/3}\right)=0. \tag{2.18}$$

The system (2.18) coincides with the Euler system of isentropic gases with pressure law

$$P(\rho)=k\rho^{\gamma},\qquad \gamma=\frac{1}{3}. \tag{2.19}$$

However, the exponent $\gamma=1/3$ is extremely nonstandard; in classical gas dynamics, $\gamma=1+2/N$ where N is the number of degrees of freedom for the gas molecule. The mathematical theory of the P-system is by now well understood for $\gamma>1$; see [6, p. 275, Remark 8.6] and the references therein. When $0<\gamma<1$, the qualitative properties of the P-system change radically (see the few remarks below); somehow, system (2.18) is intermediate between the standard P-systems with $\gamma>1$ and the system of pressureless gases, the mathematical theory of which is completely different from that of P-systems with $\gamma>1$; see [7,8].

3. SOME QUALITATIVE PROPERTIES OF THE HYDRODYNAMIC MODEL

In this section, we develop a few considerations about the hydrodynamical model (2.18),(2.19) which we also recast in the nonconservative form

$$\partial_t\rho+\partial_x(\rho u)=0,$$
$$\partial_t u+u\,\partial_x u=\frac{1}{2}\,\partial_x\left(\rho^{-2/3}\right). \tag{3.1}$$

Obviously the system (3.1) is equivalent to equations (2.18),(2.19) at least in the case of smooth solutions.

3.A. The Cloud of Gas in the Vacuum: The Local Existence Problem

The first natural problem arising for system (3.1) is the construction of a local solution. We analyze the problem in the case of finite total mass, describing an expanding cloud of a granular gas (maintained at an infinite temperature).

The difference with the usual equations of the dynamics of an isentropic gas with a pressure law $p=k\rho^{\gamma}$, $\gamma>1$ is clear. In the latter case, if $\rho\to0$ as $|x|\to\infty$, the force term given by the pressure vanishes as $|x|\to+\infty$. On the contrary, in the former case the force, given by $(1/2)\,\partial_x\rho^{-2/3}$, becomes infinite, so that the particles are accelerated towards $\pm\infty$. This is a consequence of our scaling. Indeed, both the thermal fluctuations as well as the inelastic interactions are very strong. Thus, a test particle in the vicinity of $+\infty$ has almost all the mass to the left. The thermal fluctuations towards the left are damped by the inelastic collision mechanism while on the right almost no barrier prevents large accelerations. All these facts make it difficult to establish a local existence theorem for the hydrodynamic model (2.18) or its equivalent form (3.1).

To present a more convincing argument consider a smooth solution to system (3.1). In the case of smooth solutions, such a system is equivalent to the following Hamiltonian system:

$$\begin{aligned} &\ddot{\chi}(t,x) = -\partial_x V\left(t, \chi(t,x)\right), \\ &\chi(0,x) = x, \qquad \dot{\chi}(0,x) = u_0(x), \end{aligned} \tag{3.2}$$

where

$$V(t,x) = -\frac{1}{2}\rho^{-2/3}(t,x). \tag{3.3}$$

The density $\rho = \rho(t,x)$ is transported along characteristics $\chi = \chi(t,x)$ according to the formula

$$\int_{\mathbf{R}} \rho(t,x)\phi(x)\,dx = \int_{\mathbf{R}} \rho_0(x)\phi\left(\chi(t,x)\right)\,dx, \tag{3.4}$$

where ϕ is any test function and (ρ_0, u_0) is the initial data for the system (3.1). Assume that for $x > \overline{x}$, ρ_0 is decreasing and that

$$\rho_0 = O\left(|x|^{-\alpha}\right), \tag{3.5}$$

and, for the sake of simplicity, that $u(0,x) = u_0(x) = 0$ for all x. Suppose also that there exists a smooth local solution $(\rho(t,x), u(t,x))$ and a positive time t_0, small enough so that, for $t < t_0$, ρ and $\partial_x \rho$ can be considered for all practical purposes as constant in time. Therefore, we are led to consider the Hamiltonian system

$$\ddot{\chi}(t,x) = -\partial_x V_0\left(\chi(t,x)\right), \tag{3.6}$$

where

$$V_0(x) = -\frac{1}{2}\rho_0^{-2/3}(x). \tag{3.7}$$

For this system the energy

$$E(x) = \frac{1}{2}\left|\dot{\chi}(t,x)\right|^2 + V_0\left(\chi(t,x)\right) \tag{3.8}$$

is conserved. Then the time that a particle initially at position $x_0 > \overline{x}$ needs to reach $+\infty$ is given by

$$\begin{aligned} t\left(x_0\right) &= \int_{x_0}^{\infty} \frac{dx}{\sqrt{2\left(E\left(x_0\right) - V_0(x)\right)}} \simeq \int_{x_0}^{\infty} \frac{dx}{\sqrt{\rho_0^{-2/3}(x) - \rho_0^{-2/3}\left(x_0\right)}} \\ &= \int_{x_0}^{\infty} \frac{dx}{\sqrt{x^{2\alpha/3} - x_0^{2\alpha/3}}} = x_0^{1-\alpha/3} \int_1^{\infty} \frac{dy}{\sqrt{y^{2\alpha/3} - 1}}. \end{aligned} \tag{3.9}$$

Thus, if $\alpha > 3$, $t(x_0) \to 0$ as $x_0 \to +\infty$.

In particular this implies that there exist particles which reach an infinite speed in an arbitrary small time. As a consequence we do not expect existence of local smooth solutions if ρ decays faster than $1/x^3$.

If $\alpha \leq 3$, the energy

$$E = \frac{1}{2}\int \rho u^2\,dx - \frac{3}{2}\int \rho^{1/3}\,dx, \tag{3.10}$$

and the second term in the right-hand side of (3.10)—the potential energy—is unbounded. Therefore, the only possibility for the energy to be finite is that both the kinetic energy and the potential energy be infinite and that their divergences be compensated. In particular, an upper bound on the energy of a solution does not guaranty that the solution remains bounded for all times. In the next section, we exhibit a family of solutions all the mass of which concentrates at a single point in finite time, but which has finite energy, in the case where (3.5) holds with $\alpha = 3$.

3.B. The Cloud of Gas in the Vacuum: Concentrations

The physical nature of the system under consideration and its similarity with the pressureless gas (see Section 3.D below) suggests to look for solutions concentrating in a finite time. We seek such solutions in the class of self-similar solutions of (3.1).

Let $t_0 > 0$, set $\tau = t_0 - t$ and

$$\rho(t,x) = \tau^{-\alpha}\rho_0\left(\tau^{-\alpha}x\right), \tag{3.11}$$

for some critical exponent α and some universal function $\rho_0 > 0$, $\rho_0 \in L^1(\mathbf{R})$. Obviously, as $t \to t_0$, $\rho \to M\delta(x)$ in the sense of weak convergence of bounded measures on $\mathbf{R}$, where $M = \int_{\mathbf{R}} \rho_0(x)\,dx$ is the total mass of the density ρ_0.

Next assume that ρ satisfy the continuity equation (i.e., the first equation in system (3.1)): it is found that

$$u(t,x) = -\alpha\frac{x}{\tau}. \tag{3.12}$$

On the other hand, the momentum conservation equation (i.e., the second equation in (3.1)) gives

$$\alpha(\alpha-1)\frac{x}{\tau^2} = -\frac{1}{3}\tau^{-\alpha/3}\rho_0\left(\frac{x}{\tau^\alpha}\right)^{-5/3}\rho_0'\left(\frac{x}{\tau^\alpha}\right), \tag{3.13}$$

which entails

$$\alpha = \frac{3}{2}, \tag{3.14}$$

as well as

$$\left(\rho_0^{-5/3}\rho_0'\right)(x) = -\frac{9}{4}x. \tag{3.15}$$

Solving (3.15) we finally obtain

$$\rho(t,x) = \frac{1}{(3/4)^{3/2}}\frac{1}{(t_0-t)^{3/2}}\frac{1}{\left(\left(x/(t_0-t)^{3/2}\right)^2 + c\right)^{3/2}}, \tag{3.16}$$

where the constant $c = 16/3\sqrt{3}\,M$ explicitly depends upon the total mass $M = \int_{\mathbf{R}} \rho(t,x)\,dx$, which is left arbitrary.

It is easy to check that, while both ρu^2 and $\rho^{1/3}$ behave like $1/|x|$ as $x \to \pm\infty$ and therefore, are not integrable, $(1/2)\rho u^2 - (3/2)\rho^{1/3}$ behaves like $1/|x|^3$ and therefore, is integrable. Thus, the self-similar solution defined by (3.12)–(3.16) concentrates in finite time but has finite energy.

3.C. Concentration, Shocks, and Entropy

If the granular gas is confined in a slab or if it is assumed that its density does not vanish at infinity, one can prove the existence and uniqueness of smooth solutions to (3.1) on some short time interval (i.e., before such singularities as shocks, blowing-up or vanishing of the density occur). A natural problem is to investigate which type of the singularity occurs first. The following example shows that this first singularity can be both a shock and a concentration that occur at the same time.

Consider the following initial data for system (3.1):

$$\rho(0,x) = \frac{\mu}{t_0}, \qquad u(0,x) = \begin{cases} \dfrac{a_0}{t_0}, & \text{for } x < -a_0, \\ -\dfrac{x}{t_0}, & \text{for } -a_0 \le x \le a_0, \\ -\dfrac{a_0}{t_0}, & \text{for } x > a_0, \end{cases} \tag{3.17}$$

where a_0, t_0, μ are positive constants. With this initial data the system exhibits a singularity in the density at time $t = t_0$, if $w = a_0/t_0 - \sqrt{3}(t_0/\mu)^{1/3} > 0$.

The solutions can be constructed as follows. Let $0 \le t \le t_0$; define

$$a(t) = (t_0 - t)\left(w + \frac{\sqrt{3}}{\mu^{1/3}}(t_0 - t)^{1/3}\right). \tag{3.18}$$

For $-a(t) \le x \le a(t)$, the pair $\rho(t,x) = \mu/(t_0 - t)$, $u(t,x) = -x/(t_0 - t)$ obviously solves the system (the force term being equal to zero). Notice that $(\rho(t,-a(t)), u(t,-a(t)))$ is on the first rarefaction curve (see [9] for this notion)

$$u = w + \sqrt{3}\frac{1}{\rho^{1/3}}. \tag{3.19}$$

We construct the solution for $0 < t < t_0$, $x < -a(t)$ by transporting the values of (ρ, u) at $(t,x) = (\bar{t}, -a(\bar{t}))$ by the first characteristic field, the propagation speed of which is given by

$$\lambda_1(\bar{t}) = u(\bar{t}, -a(\bar{t})) - \frac{1}{\sqrt{3}}\frac{1}{\rho(\bar{t}, -a(\bar{t}))^{1/3}} = w + \frac{2}{\sqrt{3}}\frac{1}{\mu^{1/3}}(t_0 - \bar{t})^{1/3}. \tag{3.20}$$

In other words, we have constructed a rarefaction wave matching the solution for $-a(t) \le x \le a(t)$ with left constant state given by $(\rho, u) = (\mu/t_0, -a_0/t_0)$.

Let $0 \le \bar{t} < t < t_0$ and

$$x(t,\bar{t}) = -a(\bar{t}) + (t - \bar{t})\lambda_1(\bar{t}). \tag{3.21}$$

It is easy to verify that $-a_0 + t\lambda_1(0) < x(t,\bar{t}) < -a(t)$ and that $\frac{\partial x}{\partial \bar{t}} > 0$. Then we can define its inverse function $\bar{t}(t,x)$. By construction, $\bar{t}$ solves

$$\partial_t \bar{t} + \lambda_1(\bar{t})\,\partial_x \bar{t} = 0, \tag{3.22}$$

and

$$(\rho(t,x), u(t,x)) = \left(\frac{\mu}{(t_0 - \bar{t}(x,t))}, w + \frac{\sqrt{3}}{(\mu/(t_0 - \bar{t}(x,t)))\mu^{1/3}}(t_0 - \bar{t}(x,t))^{1/3}\right) \tag{3.23}$$

is a rarefaction wave, which solves the system for $-a_0 + t\lambda_1(0) < x < -a(t)$. For $x \le -a_0 + t\lambda_1(0)$ the solution is given by the left state $(\rho, u) = (\mu/t_0, -a_0/t_0)$. A similar construction gives the solution for $x > a(t)$.

The solution so constructed is a continuous solution with velocity field bounded for all time. As $t \to t_0$, the density blows up, and develops a Dirac component at $x = 0$ with mass

$$\lim_{t \to t_0} \int_{-a(t)}^{a(t)} \rho(t,x)\,dx = \lim_{t \to t_0} 2a(t)\frac{\mu}{t_0 - t} = 2\mu w. \tag{3.24}$$

More precisely, for $|x| \le t_0^{4/3}/\sqrt{3}\mu^{1/3}$,

$$\begin{aligned} &\lim_{t \to t_0} \rho(t,x) = 2\mu w \delta(x) + \frac{\mu}{(\sqrt{3}\mu^{1/3}|x|)^{3/4}}, \\ &\lim_{t \to t_0} u(t,x) = \operatorname{sgn}(x)\left(w + \frac{\sqrt{3}}{\mu^{1/3}}\left(\sqrt{3}\mu^{1/3}|x|\right)^{1/4}\right). \end{aligned} \tag{3.25}$$

Let us finally consider the problem of shock wave solutions to the hydrodynamic model (3.1) or (2.18),(2.19). A first question is to select what are the physically relevant weak solutions. In the usual isentropic gas dynamics case ($p = k\rho^\gamma$, $\gamma > 1$) the usual entropic solutions predicted by the global theory of 2×2 systems of conservation laws due to DiPerna and completed by Lions, Perthame and Souganidis (see [6] for a survey of these results) are *not* physically relevant. As it is well known (see for instance [10]) one must add the energy or temperature equation to the mass and momentum conservation, that is, the full compressible Euler system. As long as the solution remains smooth the two descriptions are equivalent for isentropic states (i.e., with temperature proportional to $\rho^{\gamma-1}$). In the case of solutions involving only weak shocks (as in Glimm's theory; see [9]), the isentropic Euler system is a good approximation of the full compressible Euler, as was recently proved by Saint-Raymond [11]. In presence of shocks of arbitrary amplitude however, the state equation $p = k\rho^\gamma$ is not valid anymore because the solution of the full compressible Euler system ceases to be isentropic. Entropic solutions of the full compressible Euler system are the natural hydrodynamic limit of the classical kinetic theory (see for instance [12]).

The classical kinetic theory of gases suggests to add to (2.18),(2.19) the limiting entropy inequality

$$\partial_t \int dv\, \eta(f_\epsilon) + \int dv\, v\, D\eta(f_\epsilon)\, \partial_x f_\epsilon = -\frac{1}{\epsilon} \int f_\epsilon \left(\partial_v D\eta(f_\epsilon)\right)^2 \leq 0, \tag{3.26}$$

verified by the solution of the kinetic model (2.6) before considering its hydrodynamic limit (2.18), (2.19). However, at variance with the classical result pertaining to the compressible Euler limit of the Boltzmann equation, the term $\int v\, D\eta(f_\epsilon)\, \partial_x f_\epsilon$ is not the derivative with respect to x of an entropy flux. This is in accordance to the fact that the free energy functional evaluated on the local equilibrium distribution is of the form $(4/3)\rho \log \rho + \text{const}\, \rho$, which is not an entropy for system (2.18),(2.19).

3.D. The Large Mass Limit

We return to the kinetic model (1.15),(1.17) *again with Fokker-Planck constant* σ/ϵ (instead of simply $1/\epsilon$ as after the scalings (1.14)). Consider an initial data f^{in} of the form

$$f_M^{\text{in}}(x,v) = M g^{\text{in}}(x,v), \qquad \iint g^{\text{in}}(x,v)\, dx\, dv = 1. \tag{3.27}$$

Let f_M^ϵ be the solution of (1.15),(1.16); we normalize it consistently with (3.1), by considering $g_M^\epsilon = f_M^\epsilon / M$. Clearly, g_M^ϵ satisfies

$$\partial_t g_M^\epsilon + v\, \partial_x g_M^\epsilon + \frac{\lambda M}{\epsilon}\, \partial_v \left(g_M^\epsilon \int_{\mathbf{R}} |v' - v|\, (v' - v)\, g_M^\epsilon(v')\, dv' \right) = \frac{\sigma}{\epsilon}\, \partial_v^2 g_M^\epsilon. \tag{3.28}$$

PROPOSITION 5. *Assume* $\sigma \geq 0$, $g^{in} \geq 0$ *a.e. with* $\iint g^{\text{in}} v^2\, dx\, dv < +\infty$. *Then, any limit point* g *of* g_M^ϵ *in* $w - L^\infty(\mathbf{R}_+; \mathcal{M}(\mathbf{R}_x \times \mathbf{R}_v))$ *when* $M \to +\infty$, *is of the form*

$$g(t,x,v) = \rho(t,x) \delta\left(v - u(t,x)\right). \tag{3.29}$$

If moreover, $\iint g^{\text{in}} v^4\, dx\, dv < +\infty$, *then the functions* ρ *and* u *in (3.3) satisfy the system of pressureless gasses*

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) &= 0. \end{aligned} \tag{3.30}$$

PROOF. Multiplying (3.28) by $(\epsilon/M)\chi(v)$ and integrating on $[0,T] \times \mathbf{R}_x \times \mathbf{R}_v$ (assuming enough decay for g_M^ϵ as $(x,v) \to +\infty$ for fixed ϵ and M) leads to

$$\begin{aligned} &\frac{\epsilon}{M} \iint g_M^\epsilon(T,x,v) \chi(v)\, dx\, dv \\ &\quad + \frac{\lambda}{2} \int_0^T \iiint |v - v'|\, (v' - v) \cdot \left(\chi'(v') - \chi'(v)\right) g_M^\epsilon(t,x,v) g_M^\epsilon(t,x,v')\, dt\, dx\, dv\, dv' \\ &= \frac{\sigma}{M} \int_0^T \iint g_M^\epsilon(t,x,v) \chi''(v)\, dx\, dv + \frac{\epsilon}{M} \iint g^{\text{in}}(x,v) \chi(v)\, dx\, dv. \end{aligned} \tag{3.31}$$

Choose first $\chi(v) \equiv v^2$; equation (3.31) implies that

$$\int_0^T \iiint |v - v'|^3 \, g_M^\epsilon(t,x,v) g_M^\epsilon(t,x,v') \, dt \, dx \, dv \, dv' \le \frac{Cst\epsilon + 2\sigma T}{M}, \tag{3.32}$$

and that

$$\iint g_M^\epsilon(T,x,v) \, dx \, dv \le \frac{Cst\epsilon + 2\sigma T}{\epsilon M}. \tag{3.33}$$

Inequality (3.32) obviously establishes (3.29).

Applying again (3.31) with $\chi(v) = v^4$, and using (3.33)

$$\iint g_M^\epsilon(T,x,v)\chi(v) \, dx \, dv \le 12\sigma T \frac{Cst\,\epsilon + 2\sigma T}{\epsilon^2} + \iint g^{\text{in}}(x,v)\chi(v) \, dx \, dv, \tag{3.34}$$

which allows to take limits in the system (1.5),(1.7) with $\beta = 0$ and g_M^ϵ in the place of f (this simply amounts to divide (1.5) and (1.7) by M). The limiting system obviously reduces to (3.30) as announced.

REFERENCES

1. D. Benedetto, E. Caglioti, J. Carrillo and M. Pulvirenti, A non-Maxwellian steady distribution for one-dimensional granular media, *J. of Stat. Phys.* **91** (5/6), 979–990, (1998).
2. S. McNamara and W.R. Young, Kinetic of a 1-dimensional medium in the quasi-elastic limit, *Phys. of Fluids A* **7** (3), 34–45, (1995).
3. Y. Du, H. Li and L. Kadanoff, Breakdown of hydrodynamics in a 1-dimensional system of inelastic particles, *Phys. Rev. Lett.* **74** (8), 1268–1271, (1995).
4. D. Benedetto, E. Caglioti and M. Pulvirenti, A kinetic equation for granular media, *RAIRO Modél. Math. Anal. Numér.* **31** (5), 615–641, (1997).
5. A. Puglisi, V. Loreto, U. Marini Bettolo Marconi, A. Petri and A. Vulpiani, Clustering and non-Gaussian behavior in granular matter, *Phys. Rev. Lett.* **81**, 3848, (1998).
6. P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. 2: Compressible Models*, Clarendon Press, Oxford, (1998).
7. E. Grenier, Existence globale pour el système de gaz sans pression, *C. R. Acad. Sci. Paris Sér. I Math.* **431** (2), 171–174, (1995).
8. F. Bouchut and F. James, Duality solutions for pressureless gases, monotone scalar conservation laws and uniqueness, Preprint, (1998).
9. J. Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer-Verlag, Berlin, (1983).
10. A.J. Chorin and J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, New York, (1979).
11. L. Saint-Raymond, Approximation isentropique du système d'Euler compressible en dimenion 1, *C. R. Acad. Sci.* **327** (Série I), 613–616, (1998).
12. C. Cercignani, R. Illner and M. Pulvirenti, The mathematical theory of dilute gases, In *Applied Mathematical Sciences*, Vol. 106, Springer-Verlag, New York, (1994).
13. P. Constantin, E. Grossman and M. Mungan, Inelastic collisions of 3 particles on the line as a 2-dimensional billiards, *Physica D* **83**, 409–420, (1995).