F-algebras in which order ideals are ring ideals

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It is the main purpose of this paper to find conditions in an Archimedean semiprime f-algebra A which are equivalent to the statement that every order ideal in A is a ring ideal (i.e., an algebra ideal).

1. PRELIMINARIES

For unexplained terminology and the basic results on vector lattices (Riesz spaces) and f-algebras we refer to [4] and [5]. Recall that a vector lattice A is called a lattice ordered algebra if there exists an associative multiplication in A with respect to which A is an algebra and with the additional property that

$$x, y \in A^+ \Rightarrow xy \in A^+$$

(A^+ denotes the positive cone of A). A lattice ordered algebra A is called an f-algebra whenever

$$x \wedge y = 0 \Rightarrow (xz) \wedge y = (zx) \wedge y = 0 \text{ for all } z \in A^+.$$

We shall assume throughout this paper that A is an Archimedean f-algebra (and hence A is commutative [5, 140.10]).

A positive operator T on A is called a positive orthomorphism if it follows from $x \wedge y = 0$ in A that $x \wedge Ty = 0$. The difference of two positive orthomorphisms is called an orthomorphism. The collection Orth(A) of all orthomorphisms on A is an Archimedean f-algebra with unit element I, the identity mapping on A ([5, 140.9]). The principal order ideal in Orth(A) generated by I is called the center of A and is denoted by $Z(A)$.

For every $x \in A$, the operator $T_x : A \to A$, defined by $T_x(y) = xy$ for all $y \in A$, satisfies $T_x \in \text{Orth}(A)$. The mapping $\phi : A \to \text{Orth}(A)$ which assigns T_x to x is a lattice homomorphism and a ring homomorphism. Furthermore, $\phi(A)$ is a ring ideal and a vector sublattice of Orth(A). Moreover, ϕ is injective if and only if A is semiprime ([3, 12.1]). A norm on A is called a Riesz norm (or an absolutely monotone norm) if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Such a norm is called an M-norm if $\|x \vee y\| = \max(\|x\|, \|y\|)$, for all $x, y \in A^+$.

We can introduce in $A^{\sim\sim}$, the order bidual of A, a multiplication, the so-called Arens multiplication ([2, 4.1]) as follows: given $a, b \in A$, $f \in A^{\sim}$ and $F, G \in A^{\sim\sim}$, we define $f \cdot a \in A^{\sim}$, $F \cdot f \in A^{\sim}$, and $F \cdot G \in A^{\sim\sim}$ by the equations

(1) $\quad (f \cdot a)(b) = f(ab)$

(2) $\quad (F \cdot f)(a) = F(f \cdot a)$

(3) $\quad (F \cdot G)(f) = F(G \cdot f)$

Then $A^{\sim\sim}$ is an Archimedean lattice ordered algebra with respect to the Arens multiplication. If $A^{\sim}$ separates the points of A we can embed A in $A^{\sim\sim}$. The Arens multiplication in $A^{\sim\sim}$ extends the original multiplication in A.

2. MAIN RESULTS

In this paper we consider the following question: What conditions can be imposed on an Archimedean f-algebra A to ensure that every order ideal in A is a ring ideal? Partial solutions of this problem were obtained in ([1, 1.10]), where it is shown that in an FF-Banach lattice algebra A every order ideal is a ring ideal if and only if A is an Banach f-algebra and in ([3, 17.8]), where it is noticed that in an Archimedean f-algebra A with unit element e every order ideal is a ring ideal if and only if e is a strong order unit (equivalently, A has a strong order unit).

Let us first show that the problem in question only makes sense in the class of f-algebras.

PROPOSITION 1. *Let A be an Archimedean lattice ordered algebra. If every order ideal in A is a ring ideal, then A is an f-algebra.*

PROOF. Take $x, y \in A^+$ such that $x \wedge y = 0$ and $z \in A^+$ arbitrary. The principal order ideal I_x generated by x is, by hypothesis, a ring ideal, so $x \cdot z \in I_x$. In other words, $0 \leq x \cdot z \leq \lambda x$ for appropriate real $\lambda > 0$. It follows from $\lambda x \wedge y = 0$ that $x \cdot z \wedge y = 0$. Similarly, $z \cdot x \wedge y = 0$ and we are done.

REMARK 2. The converse of proposition 1 does not hold in general. Indeed, $A = C(\mathbb{R})$ is an Archimedean f-algebra, but the order ideal generated by i (with $i(x) = x$ for all $x \in \mathbb{R}$) is not a ring ideal.

THEOREM 3. *Let A be an Archimedean f-algebra. The following statements are equivalent:*

(i) *every order ideal in A is a ring ideal of A,*

(ii) *every order ideal in A is a subalgebra of A,*

(iii) *$\phi(A)$ is a subset of $Z(A)$.*

PROOF. (i)→(ii) Obvious.

(ii)→(iii) For any $x \in A^+$, the order ideal I_x is a subalgebra of A, so $x^2 \in I_x$. This implies that $x^2 \leq \lambda x$ for some $\lambda > 0$. Hence, $\phi(x)^2 = \phi(x^2) \leq \lambda \phi(x)$. But Orth($A$) is semiprime, so $0 \leq \phi(x) \leq \lambda I$, as by [3, 12.3] $0 \leq u^2 \leq uv$ implies $u \leq v$. This shows that $\phi(A) \subset Z(A)$.

(iii)→(i) Evidently, it is sufficient to prove that, for any $x \in A^+$, I_x is a ring ideal. To this end, take $y \in A^+$ and observe that $\phi(y) = T_y \in Z(A)^+$, i.e., $0 \leq T_y \leq \lambda I$ for some $\lambda > 0$. This yields $0 \leq T_y(x) = xy \leq \lambda x$ and thus $xy \in I_x$. This holds for all $y \in A^+$, so I_x is a ring ideal.

The next corollary generalizes ([1, 1.10]).

COROLLARY 4. *For a Banach lattice algebra A the following conditions are equivalent:*

(i) *A is an f-algebra,*

(ii) *every order ideal in A is a ring ideal.*

PROOF. This follows immediately from proposition 1, theorem 3 and the fact that $Z(A) = \text{Orth}(A)$ for any Banach lattice A ([5, 144.3]).

We are now in a position to prove the main theorem of this paper.

THEOREM 5. *Let A be an Archimedean semiprime f-algebra. Then the following statements are equivalent.*

(i) *every order ideal in A is a ring ideal of A,*

(ii) *every order ideal in A is a subalgebra of A,*

(iii) *A is lattice and algebra isomorphic to a sub-f-algebra of $Z(A)$,*

(iv) *there exists an M-norm in A,*

(v) *there exists a Riesz norm in A.*

PROOF. The equivalence of (i), (ii) and (iii) follows from theorem 3 and the fact that ϕ is injective. We shall prove (iii)⇒(iv)⇒(v)⇒(i).

(iii)→(iv) The gauge $j(T) = \inf\{\lambda > 0 : -\lambda I \leq T \leq \lambda I\}$ defines an M-norm on $Z(A)$, the restriction of which is an M-norm in A.

(iv)→(v) Trivial.

(v)→(i) The norm dual A^* of A is a Banach lattice, which separates the points of A. Moreover, A^* is an order ideal in $A^\sim$ (the order dual of A) ([5, 102.3]). Observe now that $(A^*)_n^\sim$ is an Archimedean f-algebra with respect to the Arens multiplication in which A can be embedded as a sub-f-algebra (see

[2,4.4 and the remarks following corollary 4.5]). But $(A^*)_n^\sim$ is a band in $(A^*)^\sim = A^{**}$ (note that A^* is a Banach lattice [5, 102.3]) and hence $(A^*)_n^\sim$ is closed. It follows that $(A^*)_n^\sim$ is a Banach lattice on its own. By corollary 4, every order ideal in $(A^*)_n^\sim$ is a ring ideal. Now, let I be an order ideal in A and denote by J the order ideal in $(A^*)_n^\sim$ generated by I. Then J is a ring ideal in $(A^*)_n^\sim$ by the above. Hence $x \in I^+$, $y \in A^+$ implies $xy \in J \cap A$. It follows from $xy \in J$ that $0 \le xy \le z$ for some $z \in I^+$, so the fact that I is an order ideal in A together with $xy \in A$ yields $xy \in I$. Consequently, I is a ring ideal in A.

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