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Spin Hecke algebras of finite and affine types

Weiqiang Wang

Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

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Abstract

We introduce the spin Hecke algebra, which is a q-deformation of the spin symmetric group algebra, and its affine generalization. We establish an algebra isomorphism which relates our spin (affine) Hecke algebras to the (affine) Hecke–Clifford algebras of Olshanski and Jones–Nazarov. Relation between the spin (affine) Hecke algebra and a nonstandard presentation of the usual (affine) Hecke algebra is displayed, and the notion of covering (affine) Hecke algebra is introduced to provide a link between these algebras. Various algebraic structures for the spin (affine) Hecke algebra are established.

MSC: primary 20C08

Keywords: Hecke algebra; Spin symmetric group; Clifford algebra

1. Introduction

1.1. A basic question

The spin (or projective) representations of the symmetric group were first developed by I. Schur [8] in 1911. We refer to Józefiak [3] for an excellent modern exposition of Schur's work by a systematic use of superalgebras. The symmetric group S_n admits a double cover $S_n^{\sim}$, nontrivial for $n \geqslant 4$:

$$1 \to \mathbb{Z}_2 \to S_n^{\sim} \to S_n \to 1. \tag{1.1}$$

E-mail address: ww9c@virginia.edu.

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A spin representation of S_n is equivalent to a representation of the algebra $\mathbb{C}S_n^- := \mathbb{C}S_n^\sim/\langle z+1\rangle$, the quotient of the group algebra $\mathbb{C}S_n^\sim$ by the ideal $\langle z+1\rangle$, where z denotes the central generator of order 2 coming from $\mathbb{Z}_2$. The algebra $\mathbb{C}S_n^-$ has a presentation with generators t_i $(1 \leqslant i \leqslant n-1)$ subject to the relations:

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \tag{1.2}$$

$$t_i t_j = -t_j t_i \quad \big(|i-j| > 1\big). \tag{1.3}$$

As is well known, Hecke algebras have played an important role in various aspects of representation theory (for some recent developments see the books of Ariki [1] and Kleshchev [5] and the references therein). We ask the following basic question: Is there a natural q-deformation (i.e. Hecke algebra) for $\mathbb{C}S_n^-$ and $\mathbb{C}S_n^\sim$? It is conceivable that a canonical solution to this question, if it exists, might open the door to further new developments in representation theory.

However, there is no standard procedure to define Hecke algebras except for Coxeter groups and perhaps for complex reflection groups. The group $S_n^\sim$ is neither a Coxeter group nor a complex reflection group.

1.2. An affirmative answer

In this paper we introduce the spin Hecke algebra $\mathcal{H}_n^-$ and the covering Hecke algebra $\mathcal{H}_n^\sim$ as q-deformations of $\mathbb{C}S_n^-$ and $\mathbb{C}S_n^\sim$, respectively. We also introduce the spin and the covering affine Hecke algebras, denoted by $\widehat{\mathcal{H}}_n^-$ and $\widehat{\mathcal{H}}_n^\sim$. The spin (affine) Hecke algebras arise from different setups and they enjoy various favorable properties.

Set

$$\varepsilon = q - q^{-1}.$$

The spin affine Hecke algebra $\widehat{\mathcal{H}}_n^-$ is the $\mathbb{C}(q)$-algebra generated by R_i, $1 \leqslant i \leqslant n-1$, and p_i, q_i, $1 \leqslant i \leqslant n$, subject to the following relations:

$$\begin{gathered}
R_i^2 = -\varepsilon^2 - 2,\\
R_i R_j = -R_j R_i \quad \big(|i-j| > 1\big),\\
R_i R_{i+1} R_i - R_{i+1} R_i R_{i+1} = \varepsilon^2 (R_{i+1} - R_i),\\
R_i p_i = p_{i+1} R_i + \varepsilon(q_i - q_{i+1}),\\
R_i q_i = -q_{i+1} R_i - \varepsilon(p_i + p_{i+1}),\\
R_i p_j = p_j R_i, \quad R_i q_j = -q_j R_i \quad (j \neq i, i+1),\\
p_i p_j = p_j p_i, \quad q_i q_j = -q_j q_i \quad (i \neq j),\\
p_i^2 + q_i^2 = 1,\\
p_i q_j = q_j p_i \quad (\forall i, j).
\end{gathered}$$

The algebra $\widehat{\mathcal{H}}_n^-$ can be viewed as a quantum version of the degenerate spin affine Hecke algebra introduced in [10] (see Section 6.4). The subalgebra generated by R_i $(1 \leqslant i \leqslant n-1)$ is the spin Hecke algebra $\mathcal{H}_n^-$ of finite type. Among the noteworthy features of $\mathcal{H}_n^-$ and $\widehat{\mathcal{H}}_n^-$ are the deformed braid relations and the two dependent sets of loop generators. Both $\widehat{\mathcal{H}}_n^-$ and $\mathcal{H}_n^-$ admit superalgebra structures with each p_i being even and q_i, R_i being odd.

1.3. Several related algebras

To formulate a certain Schur–Jimbo type duality, Olshanski [7] introduced a Hecke–Clifford algebra $\mathcal{H}c_n$, which is a q-deformation of the semidirect product $\mathcal{C}_n \rtimes \mathbb{C}S_n$ and is generated by the usual Hecke algebra $\mathcal{H}_n$ for S_n and the Clifford algebra $\mathcal{C}_n$ in n variables. The affine Hecke–Clifford algebra $\widehat{\mathcal{H}c}_n$ was introduced by Jones–Nazarov [4] to study the q-Young symmetrizer for $\mathcal{H}c_n$, and the modular representation theory of $\widehat{\mathcal{H}c}_n$ has been developed by Brundan–Kleshchev [2]. A degenerate version of $\widehat{\mathcal{H}c}_n$ was introduced earlier by Nazarov [6] (called affine Sergeev algebra) to study representations of $\mathbb{C}S_n^-$.

It is known from the works of Sergeev, Józefiak and Stembridge that the representation theory of $\mathbb{C}S_n^-$ is essentially equivalent to that of $\mathcal{C}_n \rtimes \mathbb{C}S_n$. This phenomenon has subsequently been clarified by the construction of a superalgebra isomorphism between $\mathcal{C}_n \rtimes \mathbb{C}S_n$ and $\mathcal{C}_n \otimes \mathbb{C}S_n^-$, due to Sergeev [9] and Yamaguchi [11] independently. (We will say that $\mathcal{C}_n \rtimes \mathbb{C}S_n$ and $\mathbb{C}S_n^-$ are *Morita super-equivalent*; for a justification of the terminology, cf. [2, Lemma 9.9] or [5, 13.2], or our Section 3.1). Such a super-equivalence has been extended by the author [10] to one between the degenerate spin Hecke algebra introduced in [10] and Nazarov's degenerate affine Hecke–Clifford algebra.

1.4. Properties of the spin (affine) Hecke algebra

We establish a Morita super-equivalence between $\mathcal{H}c_n$ and $\mathcal{H}_n^-$ (respectively, between $\widehat{\mathcal{H}c}_n$ and $\widehat{\mathcal{H}}_n^-$) by constructing explicitly a q-deformed version of the above Morita super-equivalences [9–11] in both finite and affine setups:

$$\Phi : \mathcal{H}c_n \xrightarrow{\simeq} \mathcal{C}_n \otimes \mathcal{H}_n^-, \qquad \Phi : \widehat{\mathcal{H}c}_n \xrightarrow{\simeq} \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-.$$

Our key observation on the existence of natural subalgebras of $\mathcal{H}c_n$ and $\widehat{\mathcal{H}c}_n$ which super-commute with $\mathcal{C}_n$ paves the way for the presentations of $\mathcal{H}_n^-$ and $\widehat{\mathcal{H}}_n^-$.

A fundamental construction in the classical theory of the spin symmetric group is an algebra homomorphism from $\mathbb{C}S_n^-$ to $\mathcal{C}_{n-1}$ which gives rise to the basic spin $\mathbb{C}S_n^-$-supermodule [3,8]. We obtain a natural q-deformation of this construction in which the spin Hecke algebra $\mathcal{H}_n^-$ fits nicely.

We construct standard bases for $\mathcal{H}_n^-$ and for $\widehat{\mathcal{H}}_n^-$, describe the center of $\widehat{\mathcal{H}}_n^-$, and further introduce the intertwiners for $\widehat{\mathcal{H}}_n^-$, which have their counterparts in [4]. We introduce the cyclotomic spin Hecke algebras and show that they are Morita super-equivalent to the cyclotomic Hecke–Clifford algebras introduced in [2]. We remark that all of the definitions and constructions in this paper can make sense over a field of characteristic different from 2 (which is occasionally assumed to contain $\sqrt{2}$), and often even over the ring $\mathbb{Z}[\frac{1}{2}]$. It is possible to develop the representation theory of $\mathcal{H}_n^-$ and $\widehat{\mathcal{H}}_n^-$ parallel to the principal results for $\mathcal{H}c_n$ and $\widehat{\mathcal{H}c}_n$ in [2,4]. The new perspective of spin (affine) Hecke algebras can in turn help to clarify the work on the (affine) Hecke–Clifford algebras.

1.5. Relation to the (affine) Hecke algebra

There is a different setup where the spin (affine) Hecke algebra appears to be relevant. One easily writes down a nonstandard presentation for the usual Hecke algebra $\mathcal{H}_n$ with new generators $\mathcal{T}_i := T_i + T_i^{-1}$ instead of the familiar ones T_i. The definition of $\mathcal{H}_n^-$ and the nonstandard presentation of $\mathcal{H}_n$ are surprisingly compatible and this leads to a notion of a covering Hecke algebra $\mathcal{H}_n^\sim$ which is a q-deformation of $\mathbb{C}S_n^\sim$. The quotient of the algebra $\mathcal{H}_n^\sim$ by the ideal $\langle z+1\rangle$ (respectively, $\langle z-1\rangle$) is isomorphic to $\mathcal{H}_n^-$ (respectively, $\mathcal{H}_n$).

It is remarkable that such a compatibility extends to the spin affine Hecke algebra $\widehat{\mathcal{H}}_n^-$ and the usual affine Hecke algebra $\widehat{\mathcal{H}}_n$ of type GL, where we have to adopt a nonstandard presentation of $\widehat{\mathcal{H}}_n$ via the generators $\mathcal{T}_i$ and $\frac{1}{2}(X_i \pm X_i^{-1})$ instead of the Bernstein–Lusztig presentation via the generators T_i and X_i. This leads to the definition of the covering affine Hecke algebra $\widehat{\mathcal{H}}_n^\sim$, whose quotient by the ideal $\langle z+1\rangle$ (respectively, $\langle z-1\rangle$) is isomorphic to $\widehat{\mathcal{H}}_n^-$ (respectively, $\widehat{\mathcal{H}}_n$).

1.6. The organization

The paper is organized as follows. In Section 2, we recall the Hecke–Clifford algebra and introduce the spin and covering Hecke algebras of finite type. In Section 3, we establish the Morita super-equivalence between $\mathcal{H}c_n$ and $\mathcal{H}_n^-$. We provide standard bases for $\mathcal{H}_n^-$, $\mathcal{H}_n^\sim$ and their even subalgebras (which are q-deformations of a double cover of the alternating group and its spin quotient), and also construct the basic spin $\mathcal{H}_n^-$-supermodule. In Section 4, we present the affine Hecke algebra counterpart of Section 2. In Section 5, we establish the Morita super-equivalence between $\widehat{\mathcal{H}c}_n$ and $\widehat{\mathcal{H}}_n^-$. We describe the intertwiners, a standard basis, and the center for $\widehat{\mathcal{H}}_n^-$. In Section 6, we introduce the cyclotomic spin Hecke algebras and the Jucys–Murphy elements for $\mathcal{H}_n^-$. We explain the degeneration of $\widehat{\mathcal{H}}_n^-$ and its cyclotomic version. It is our view that a general notion of spin Hecke algebras exists naturally beyond the setup in this paper. We will return to this elsewhere.

2. Spin and covering Hecke algebras (of finite type)

2.1. The Hecke–Clifford algebra

Let q be a formal parameter.

Definition 2.1. [7] The *Hecke–Clifford algebra* $\mathcal{H}c_n$ is the $\mathbb{C}(q)$-algebra generated by T_i $(1 \leqslant i \leqslant n-1)$ and c_i $(1 \leqslant i \leqslant n)$, subject to the following relations:

$$(T_i - q)\big(T_i + q^{-1}\big) = 0, \tag{2.1}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iT_j = T_jT_i \quad \big(|i-j| > 1\big), \tag{2.2}$$

$$T_ic_i = c_{i+1}T_i, \qquad T_ic_j = c_jT_i \quad (j \neq i, i+1), \tag{2.3}$$

$$c_i^2 = 1, \qquad c_ic_j = -c_jc_i \quad (i \neq j). \tag{2.4}$$

The algebra $\mathcal{H}c_n$ was introduced by Olshanski [7]. It is naturally a super- (i.e. $\mathbb{Z}_2$-graded) algebra with c_i $(1 \leqslant i \leqslant n)$ being odd and T_i $(1 \leqslant i \leqslant n-1)$ being even. The subalgebra generated by T_i subject to the relations (2.1)–(2.2) is the usual Hecke algebra $\mathcal{H}_n$ associated to the symmetric group S_n. We define $T_\sigma := T_{i_1} \cdots T_{i_r}$ as usual for any reduced expression $\sigma = s_{i_1} \cdots s_{i_r} \in S_n$. The $\mathbb{C}$-algebra generated by $c_1, \dots, c_n$ is a Clifford (super)algebra and will be denoted by $\mathcal{C}_n$. It is known that $T_\sigma c_1^{\epsilon_1} \cdots c_n^{\epsilon_n}$, where $\sigma \in S_n$ and $\epsilon_1, \dots, \epsilon_n \in \{0, 1\}$, is a basis for $\mathcal{H}c_n$. Here are some useful identities derived from (2.1)–(2.4):

$$T_i c_{i+1} = c_i T_i - \varepsilon(c_i - c_{i+1}), \tag{2.5}$$

$$T_i(c_i - c_{i+1})T_i = c_{i+1} - c_i, \tag{2.6}$$

$$(c_i - c_j)(c_j - c_k)(c_i - c_j) = 2(c_k - c_i) \quad \text{for distinct } i, j, k. \tag{2.7}$$

2.2. The spin Hecke algebra

Recall $\varepsilon = q - q^{-1}$. We now introduce the first new concept of the paper.

Definition 2.2. The *spin Hecke algebra* $\mathcal{H}_n^-$ is a $\mathbb{C}(q)$-algebra generated by R_i, $1 \leqslant i \leqslant n-1$, subject to the following relations:

$$R_i^2 = -\varepsilon^2 - 2 \equiv -\left(q^2 + q^{-2}\right), \tag{2.8}$$

$$R_i R_j = -R_j R_i \quad \left(|i - j| > 1\right), \tag{2.9}$$

$$R_i R_{i+1} R_i - R_{i+1} R_i R_{i+1} = \varepsilon^2 (R_{i+1} - R_i). \tag{2.10}$$

The algebra $\mathcal{H}_n^-$ is naturally a superalgebra by requiring each R_i to be odd, since the defining relations for $\mathcal{H}_n^-$ are $\mathbb{Z}_2$-homogeneous with respect to such a grading.

2.3. (Anti-)involutions of $\mathcal{H}_n^-$

There are several involutions (i.e. algebra automorphisms of order 2) of the algebra $\mathcal{H}_n^-$. Define

$$\begin{aligned} \sigma&: \quad R_i \mapsto R_{n-i}, \quad q \mapsto q, \\ s&: \quad R_i \mapsto -R_i, \quad q \mapsto q, \\ -&: \quad R_i \mapsto R_i, \quad q \mapsto q^{-1}, \end{aligned}$$

where $1 \leqslant i \leqslant n-1$. By inspection of the defining relations for $\mathcal{H}_n^-$, σ, s and $-$ can be extended to homomorphisms of $\mathcal{H}_n^-$ and they are indeed involutions of $\mathcal{H}_n^-$ (regarded as an algebra over $\mathbb{C}$), Furthermore, $\sigma, s, -$ commute with each other, and their products give rise to several more involutions.

We also define an anti-involution τ of $\mathcal{H}_n^-$ by letting $\tau(R_i) = -R_i$ for each i. One obtains more anti-involutions by composing τ with the involutions above (which commute with τ).

2.4. A nonstandard presentation of Hecke algebra

Denote

$$\mathcal{T}_i := T_i + T_i^{-1} \equiv 2T_i - \varepsilon.$$

Proposition 2.3. *The algebra $\mathcal{H}_n$ has a presentation with generators $\mathcal{T}_i$ $(1 \leqslant i \leqslant n-1)$ subject to the following relations*:

$$\mathcal{T}_i^2 = q^2 + q^{-2} + 2, \tag{2.11}$$

$$\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i \quad \big(|i-j|>1\big), \tag{2.12}$$

$$\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i - \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1} = \varepsilon^2(\mathcal{T}_{i+1} - \mathcal{T}_i). \tag{2.13}$$

Proof. Follows by a direct computation. □

2.5. The covering Hecke algebra

Definition 2.4. The *covering Hecke algebra $\mathcal{H}_n^\sim$* is a $\mathbb{C}(q)$-superalgebra generated by the even generator z and the odd generators $\widetilde{T}_i$ $(1 \leqslant i \leqslant n-1)$, subject to the following relations:

$$z^2 = 1, \quad z \text{ is central},$$

$$\widetilde{T}_i^2 = z\big(q^2 + q^{-2} + 1\big) + 1, \tag{2.14}$$

$$\widetilde{T}_i\widetilde{T}_j = z\widetilde{T}_j\widetilde{T}_i \quad \big(|i-j|>1\big), \tag{2.15}$$

$$\widetilde{T}_i\widetilde{T}_{i+1}\widetilde{T}_i - \widetilde{T}_{i+1}\widetilde{T}_i\widetilde{T}_{i+1} = \varepsilon^2(\widetilde{T}_{i+1} - \widetilde{T}_i). \tag{2.16}$$

We shall denote by $\langle a, b, \ldots\rangle$ a two-sided ideal generated by $a, b, \ldots$. The quotient of the covering Hecke algebra $\mathcal{H}_n^\sim$ by the ideal $\langle z-1\rangle$ is isomorphic to the usual Hecke algebra $\mathcal{H}_n$ with nonstandard presentation (where the canonical image of $\widetilde{T}_i$ matches $\mathcal{T}_i$) and the quotient by $\langle z+1\rangle$ is isomorphic to the spin Hecke algebra $\mathcal{H}_n^-$ (where the canonical image of $\widetilde{T}_i$ matches R_i).

3. Algebraic structures of the spin Hecke algebra

3.1. A Morita super-equivalence

Note that the multiplication in a tensor product $\mathcal{C} \otimes \mathcal{B}$ of two superalgebras $\mathcal{C}$ and $\mathcal{B}$ has a suitable sign convention:

$$(c' \otimes b')(c \otimes b) = (-1)^{|b'||c|}(c'c \otimes b'b).$$

We shall write a typical element in $\mathcal{C} \otimes \mathcal{B}$ as cb rather than $c \otimes b$, and use short-hand notations $c = c \otimes 1$, $b = 1 \otimes b$.

Theorem 3.1. *There exists a superalgebra isomorphism*

$$\Phi : \mathcal{H}c_n \xrightarrow{\simeq} \mathcal{C}_n \otimes \mathcal{H}_n^-$$

which extends the identity map on $\mathcal{C}_n$ and sends

$$T_i \mapsto T_i^\Phi := -\frac{1}{2}R_i(c_i - c_{i+1}) + \frac{\varepsilon}{2}(1 - c_ic_{i+1}), \quad 1 \leqslant i \leqslant n-1. \tag{3.1}$$

Its inverse map Ψ extends the identity map on $\mathcal{C}_n$ and sends

$$R_i \mapsto R_i^{\Psi} := (c_i - c_{i+1})T_i + \varepsilon c_{i+1}, \quad 1 \leqslant i \leqslant n-1. \tag{3.2}$$

Remark 3.2. The isomorphism in Theorem 3.1 in the $q \mapsto 1$ limit reduces to the superalgebra isomorphism $\mathcal{C}_n \rtimes \mathbb{C}S_n \cong \mathcal{C}_n \otimes \mathbb{C}S_n^-$ found in [9,11].

By (2.1)–(2.4), we have the following equivalent expressions for R_i^{Ψ}:

$$R_i^{\Psi} = -T_i(c_i - c_{i+1}) + \varepsilon c_i = c_i T_i - c_{i+1}T_i^{-1} = T_i c_{i+1} - T_i^{-1} c_i. \tag{3.3}$$

Thanks to the above isomorphisms Φ, Ψ, we can define exact functors

$$\mathfrak{F} : \mathcal{H}_n^-\text{-smod} \to \mathcal{H}c_n\text{-smod}, \qquad \mathfrak{F} := \Phi^*(U_n \otimes ?),$$
$$\mathfrak{G} : \mathcal{H}c_n\text{-smod} \to \mathcal{H}_n^-\text{-smod}, \qquad \mathfrak{G} := \operatorname{Hom}_{\mathcal{C}_n}\big(U_n, \Psi^*(?)\big),$$

where U_n denotes the basic spin $\mathcal{C}_n$-supermodule and $\mathcal{H}_n^-$-smod (respectively, $\mathcal{H}c_n$-smod) denotes the category of finite-dimensional supermodules of $\mathcal{H}_n^-$ (respectively, of $\mathcal{H}c_n$). For n even, $\mathfrak{F}$ and $\mathfrak{G}$ establish the equivalence of categories, and indeed $\mathcal{H}_n^-$ and $\mathcal{H}c_n$ are Morita equivalent in the usual sense since $\mathcal{C}_n$ is a simple algebra. For n odd, $\mathcal{C}_n$ is a simple *super*algebra of type Q, $\mathfrak{F}$ and $\mathfrak{G}$ establish an *almost* Morita equivalence of categories which involves some $\mathbb{Z}_2$-parity change functor (see [2, Lemma 9.9] or [5, Proposition 13.2.2] for a precise statement in a similar setup).

Let us call two superalgebras $\mathcal{A}$ and $\mathcal{B}$ *Morita super-equivalent* if there is a superalgebra isomorphism $\mathcal{A} \cong \mathcal{C}_n \otimes \mathcal{B}$ or $\mathcal{B} \cong \mathcal{C}_n \otimes \mathcal{A}$ for some Clifford algebra $\mathcal{C}_n$. In particular, $\mathcal{H}c_n$ and $\mathcal{C}_n \otimes \mathcal{H}_n^-$ are Morita super-equivalent. (This restrictive definition of the Morita super-equivalence is all we need in this paper, though it is possible and potentially useful in other contexts to give a more general definition which incorporates the usual Morita equivalence.)

3.2. Proof of the isomorphism Theorem 3.1

We start with several lemmas.

Lemma 3.3. *The R_i^{Ψ} super-commute with $\mathcal{C}_n$ for every* $1 \leqslant i \leqslant n-1$.

Proof. Clearly, $R_i^{\Psi} c_j = -c_j R_i^{\Psi}$ for $j \neq i, i+1$. By (3.2), (2.3) and (2.4),

$$R_i^{\Psi} c_i = (c_i - c_{i+1})c_{i+1}T_i + \varepsilon c_{i+1}c_i = -c_i(c_i - c_{i+1})T_i - \varepsilon c_i c_{i+1} = -c_i R_i^{\Psi}.$$

We leave to the reader the similar verification that $R_i^{\Psi} c_{i+1} = -c_{i+1} R_i^{\Psi}$. $\square$

Lemma 3.4. *The R_i^{Ψ}, $1 \leqslant i \leqslant n-1$, satisfy the relations* (2.8) *and* (2.9).

Proof. Clearly, $R_i^\Psi R_j^\Psi = -R_j^\Psi R_i^\Psi$ for $|i-j|>1$. By (2.1)–(2.4) and (3.3), we calculate that

$$\begin{aligned}\big(R_i^\Psi\big)^2 &= \big(c_iT_i - c_{i+1}T_i^{-1}\big)\big(T_ic_{i+1} - T_i^{-1}c_i\big)\\ &= c_i(\varepsilon T_i + 1)c_{i+1} + c_{i+1}\big(1+\varepsilon^2 - \varepsilon T_i\big)c_i - 2\\ &= c_i\big(\varepsilon T_i^{-1} + 1 + \varepsilon^2\big)c_{i+1} + c_{i+1}\big(1+\varepsilon^2-\varepsilon T_i\big)c_i - 2\\ &= c_i\varepsilon T_i^{-1}c_{i+1} - c_{i+1}\varepsilon T_ic_i - 2 = -\varepsilon^2 - 2.\end{aligned}$$

This verifies (2.8) and (2.9) for R_i^Ψ. □

Lemma 3.5. *The relation* (2.10) *holds for* R_i^Ψ*, that is, for every* $1 \leqslant i \leqslant n-2$,

$$\big(R_i^\Psi R_{i+1}^\Psi + \varepsilon^2\big)R_i^\Psi = R_{i+1}^\Psi\big(R_i^\Psi R_{i+1}^\Psi + \varepsilon^2\big).$$

Proof. Formulas (2.1) through (2.7) are frequently used in this proof. We have

$$\begin{aligned}R_i^\Psi R_{i+1}^\Psi &= \big((c_i - c_{i+1})T_i^{-1} + \varepsilon c_i\big)\big((c_{i+1}-c_{i+2})T_{i+1} + \varepsilon c_{i+2}\big)\\ &= (c_i-c_{i+1})(c_i-c_{i+2})(T_i-\varepsilon)T_{i+1} + \varepsilon c_i(c_{i+1}-c_{i+2})T_{i+1}\\ &\quad + \varepsilon(c_i-c_{i+1})c_{i+2}(T_i-\varepsilon) + \varepsilon^2c_ic_{i+2}\\ &= (c_i-c_{i+1})(c_i-c_{i+2})T_iT_{i+1} + \varepsilon c_{i+2}(c_{i+1}-c_{i+2})T_{i+1}\\ &\quad - \varepsilon c_{i+2}(c_i-c_{i+1})T_i + \varepsilon^2c_{i+1}c_{i+2}.\end{aligned}$$

Recalling $R_i^\Psi = (c_i - c_{i+1})T_i + \varepsilon c_{i+1}$ from (3.2), we have that

$$\begin{aligned}&\big(R_i^\Psi R_{i+1}^\Psi + \varepsilon^2\big)R_i^\Psi\\ &\quad= (c_i-c_{i+1})(c_i-c_{i+2})(c_{i+1}-c_{i+2})T_iT_{i+1}T_i\\ &\qquad + \varepsilon c_{i+2}(c_{i+1}-c_{i+2})(c_i-c_{i+2})T_{i+1}T_i\\ &\qquad - \varepsilon c_{i+2}(c_i-c_{i+1})(c_{i+1}-c_i) + \varepsilon^2(c_i - c_{i+1} + c_{i+2} + c_ic_{i+1}c_{i+2})T_i\\ &\qquad + \varepsilon(c_i-c_{i+1})(c_i-c_{i+2})c_{i+2}T_iT_{i+1} + \varepsilon^2c_{i+2}(c_{i+1}-c_{i+2})c_{i+2}T_{i+1}\\ &\qquad - \varepsilon^2c_{i+2}(c_i-c_{i+1})c_iT_i + \varepsilon^3c_{i+2}(c_i-c_{i+1})(c_i-c_{i+1}) + \varepsilon^3(c_{i+1}-c_{i+2})\\ &\quad= 2(c_i-c_{i+2})T_iT_{i+1}T_i + \varepsilon(c_{i+1}+c_{i+2}-c_i-c_ic_{i+1}c_{i+2})T_{i+1}T_i\\ &\qquad + \varepsilon(c_{i+1}+c_{i+2}-c_i+c_ic_{i+1}c_{i+2})T_iT_{i+1} - \varepsilon^2(c_{i+1}+c_{i+2})T_{i+1}\\ &\qquad + \varepsilon^2(c_i-c_{i+1})T_i + 2\varepsilon c_{i+2} + \varepsilon^3(c_{i+1}+c_{i+2}).\end{aligned}$$

On the other hand, recalling $R_{i+1}^\Psi = T_{i+1}(c_{i+2}-c_{i+1}) + \varepsilon c_{i+1}$, we have

$$
\begin{aligned}
&R^{\Psi}_{i+1}\big(R^{\Psi}_i R^{\Psi}_{i+1}+\varepsilon^2\big)\\
&\quad=2(c_i-c_{i+2})T_{i+1}T_iT_{i+1}+\varepsilon T_{i+1}(c_{i+2}-c_{i+1})c_{i+2}(c_{i+1}-c_{i+2})T_{i+1}\\
&\qquad-\varepsilon T_{i+1}(c_{i+2}-c_{i+1})c_{i+2}(c_i-c_{i+1})T_i-2\varepsilon^2c_{i+2}T_{i+1}\\
&\qquad+\varepsilon c_{i+1}(c_i-c_{i+1})(c_i-c_{i+2})T_iT_{i+1}+\varepsilon^2c_{i+1}c_{i+2}(c_{i+1}-c_{i+2})T_{i+1}\\
&\qquad-\varepsilon^2c_{i+1}c_{i+2}(c_i-c_{i+1})T_i+\varepsilon^3(c_{i+1}+c_{i+2})\\
&\quad=2(c_i-c_{i+2})T_{i+1}T_iT_{i+1}+2\varepsilon c_{i+2}(\varepsilon T_{i+1}+1)\\
&\qquad-\varepsilon(c_{i+1}-c_{i+2})c_{i+1}(c_i-c_{i+2})T_{i+1}T_i+\varepsilon^2(c_{i+1}-c_{i+2})c_{i+1}(c_i-c_{i+1})T_i\\
&\qquad-2\varepsilon^2c_{i+2}T_{i+1}+\varepsilon c_{i+1}(c_i-c_{i+1})(c_i-c_{i+2})T_iT_{i+1}+\varepsilon^2c_{i+1}c_{i+2}(c_{i+1}-c_{i+2})T_{i+1}\\
&\qquad-\varepsilon^2c_{i+1}c_{i+2}(c_i-c_{i+1})T_i+\varepsilon^3(c_{i+1}+c_{i+2}),
\end{aligned}
$$

which can be shown by a simple rewriting to coincide with the right-hand side of the previous formula for $(R^{\Psi}_i R^{\Psi}_{i+1}+\varepsilon^2)R^{\Psi}_i$. □

Lemma 3.6. *For every $1\leqslant i\leqslant n-1$, we have $\Psi(T^{\Phi}_i)=T_i$, and $\Phi(R^{\Psi}_i)=R_i$.*

Proof. By (3.3), we have

$$
\begin{aligned}
\Psi\big(T^{\Phi}_i\big)&=\Psi\left(\frac{1}{2}R_i(c_{i+1}-c_i)+\frac{\varepsilon}{2}(1-c_ic_{i+1})\right)\\
&=\frac{1}{2}\big(-T_i(c_i-c_{i+1})+\varepsilon c_i\big)(c_{i+1}-c_i)+\frac{1}{2}\varepsilon(1-c_ic_{i+1})=T_i,\\
\Phi\big(R^{\Psi}_i\big)&=\Phi\big(T_i(c_{i+1}-c_i)+\varepsilon c_i\big)\\
&=-\left(\frac{1}{2}R_i(c_{i+1}-c_i)+\frac{\varepsilon}{2}(1-c_ic_{i+1})\right)(c_i-c_{i+1})+\varepsilon c_i=R_i. \qquad \square
\end{aligned}
$$

Proof of Theorem 3.1. By Lemmas 3.3–3.5, Ψ is a (super)algebra homomorphism. By Lemma 3.6 and $\Psi(c_i)=c_i$, Ψ is surjective.

Denote by $\mathcal{H}^-_{n,\Psi}$ the subalgebra of $\mathcal{H}c_n$ generated by R^{Ψ}_i, $1\leqslant i\leqslant n-1$. By Lemma 3.3, we have $\mathcal{H}c_n\supseteq\mathcal{C}_n\otimes\mathcal{H}^-_{n,\Psi}$. By Lemma 3.6, we have

$$
T_i=\frac{1}{2}(c_i-c_{i+1})R^{\Psi}_i+\frac{\varepsilon}{2}(1-c_ic_{i+1})\in\mathcal{C}_n\otimes\mathcal{H}^-_{n,\Psi},
$$

and thus all generators T_i, c_i of $\mathcal{H}c_n$ lie in $\mathcal{C}_n\otimes\mathcal{H}^-_{n,\Psi}$. Therefore, $\mathcal{H}c_n=\mathcal{C}_n\otimes\mathcal{H}^-_{n,\Psi}$ and $\dim\mathcal{H}^-_{n,\Psi}=n!$. By Proposition 3.8 below (whose proof is elementary and in particular does not use this theorem), we have $\dim\mathcal{H}^-_n\leqslant n!$. Thus for dimension reason the surjective homomorphism $\Psi|_{\mathcal{H}^-_n}:\mathcal{H}^-_n\to\mathcal{H}^-_{n,\Psi}$ is indeed an isomorphism and $\dim\mathcal{H}^-_n=n!$. Since both $\mathcal{H}c_n$ and $\mathcal{C}_n\otimes\mathcal{H}^-_n$ have dimensions equal to $2^nn!$, the surjective homomorphism Ψ is an algebra isomorphism.

By Lemma 3.6, Ψ and Φ are inverse isomorphisms. □

Remark 3.7. A somewhat different argument for Theorem 3.1 goes as follows. We can verify directly that Φ is an algebra homomorphism in a way similar to Ψ, which involves a tedious verification of the braid relations for T_i^{Φ}. Then Lemma 3.6 implies that Φ and Ψ are inverse isomorphisms. This argument does not use Proposition 3.8 below.

3.3. Bases for $\mathcal{H}_n^-$ and $\mathcal{H}_n^\sim$

Introduce the following monomials in $\mathcal{H}_n^-$:

$$R_{i,a} := R_i R_{i-1} \cdots R_{i-a+1}, \quad 0 \leqslant a \leqslant i,\ 1 \leqslant i \leqslant n-1, \tag{3.4}$$

where it is understood that $R_{i,0} \equiv 1$ for all i. We refer to a product $R_{i_1} R_{i_2} \cdots R_{i_s}$ of generators in $\mathcal{H}_n^-$ as a *monomial* in $\mathcal{H}_n^-$, and call a monomial *standard* if it is of the form $R_{1,a_1} R_{2,a_2} \cdots R_{n-1,a_{n-1}}$, where $0 \leqslant a_i \leqslant i$ for each $1 \leqslant i \leqslant n-1$.

Proposition 3.8. *The standard monomials $R_{1,a_1} R_{2,a_2} \cdots R_{n-1,a_{n-1}}$, where $0 \leqslant a_i \leqslant i$ and $1 \leqslant i \leqslant n-1$, linearly span $\mathcal{H}_n^-$. In particular,* $\dim \mathcal{H}_n^- \leqslant n!$.

Proof. Since the number of standard monomials in $\mathcal{H}_n^-$ is $n!$, it suffices to prove the first statement on linear span.

Claim 1. *Any monomial of $\mathcal{H}_n^-$ is spanned by the monomials in which R_{n-1} appears at most once.*

We prove Claim 1 by induction on n. The claim trivially holds for $n = 1$. For any monomial in which R_{n-1} appears more than twice, we can apply the argument below to a portion of the monomial which contains exactly two R_{n-1}'s to reduce the number of R_{n-1}'s. So, let us assume that a given monomial is of the form $R_{n-1} \cdot R_{i_1} R_{i_2} \cdots R_{i_s} \cdot R_{n-1}$, where $1 \leqslant i_1, \ldots, i_s \leqslant n-2$ (for some s). By applying (2.9) to move the R_i's outbound the two R_{n-1}'s whenever possible, we are reduced to the case $s = 0$ or $i_1 = i_s = n-2$, where $s \geqslant 1$. The reduction of the number of R_{n-1}'s in the case $s = 0$ is done by (2.8), while in the case $s = 1$ is by (2.10) (note that here we got a linear combination of monomials because (2.10) is not the usual braid relation). In the case $s \geqslant 2$, we reduce to the previous cases by applying the claim for $n-1$, which is the induction step.

Claim 2. *Any monomial of $\mathcal{H}_n^-$ in which R_{n-1} appears exactly once can be written as a linear combination of monomials of the form $R_{i_1} R_{i_2} \cdots R_{i_s} \cdot R_{n-1,a_{n-1}}$, where $1 \leqslant a_{n-1} \leqslant n-1$ and $1 \leqslant i_1, \ldots, i_s \leqslant n-2$ for some s.*

We again argue by induction on n. The claim is trivial for $n = 1, 2$. By permuting R_{n-1} via (2.9) to the right as much as possible, we rewrite the monomial (up to a sign) such that R_{n-1} appears at the very end or it is followed by R_{n-2}. We continue with the second possibility, otherwise we are done. Since the part of the monomial starting from R_{n-2} to the right lies in H_{n-1}^-, we may apply Claim 1 to reduce to the case when R_{n-2} appears to the right of R_{n-1} exactly once. Now the induction step for $n-1$ of Claim 2 is applicable to complete the proof of Claim 2.

We now proceed by induction on n. The proposition holds trivially for $n = 1$. If a monomial in $\mathcal{H}_n^-$ does not contain R_{n-1} and thus is a monomial in $\mathcal{H}_{n-1}^-$, then it is a linear combination of the standard monomials as the induction step applies. Otherwise, the proposition follows by the induction step, Claims 1 and 2. $\square$

Theorem 3.9. *The standard monomials* $R_{1,a_1}R_{2,a_2}\cdots R_{n-1,a_{n-1}}$, *where* $0\leqslant a_i\leqslant i$ *and* $1\leqslant i\leqslant n-1$, *form a basis for* $\mathcal{H}_n^-$. *Also,* $\dim\mathcal{H}_n^-=n!$.

Proof. The statement that $\dim\mathcal{H}_n^-=n!$ is a consequence of (the proof of) Theorem 3.1. The number of standard monomials in $\mathcal{H}_n^-$ is $n!$, and thus the theorem follows from Proposition 3.8. $\square$

We define the monomial $\widetilde{T}_{i,a}\in\mathcal{H}_n^\sim$ (respectively, $\mathcal{T}_{i,a}\in\mathcal{H}_n$), with $\widetilde{T}$ (respectively, $\mathcal{T}$) replacing R in the definition (3.4) of $R_{i,a}$.

Proposition 3.10. *The elements* $\widetilde{T}_{1,a_1}\widetilde{T}_{2,a_2}\cdots\widetilde{T}_{n-1,a_{n-1}}$, $z\widetilde{T}_{1,a_1}\widetilde{T}_{2,a_2}\cdots\widetilde{T}_{n-1,a_{n-1}}$, *where* $0\leqslant a_i\leqslant i$ *and* $1\leqslant i\leqslant n-1$, *form a basis for* $\mathcal{H}_n^\sim$. *Also,* $\dim\mathcal{H}_n^\sim=2n!$.

Proof. The same argument for Proposition 3.8 shows that the elements in the proposition form a spanning set for $\mathcal{H}_n^\sim$. It remains to prove the linear independence.

By definitions, $\mathcal{H}_n^\sim/\langle z-1\rangle\cong\mathcal{H}_n$ and $\mathcal{H}_n^\sim/\langle z+1\rangle\cong\mathcal{H}_n^-$. Denote the corresponding canonical maps by $p_+:\mathcal{H}_n^\sim\to\mathcal{H}_n$ and $p_-:\mathcal{H}_n^\sim\to\mathcal{H}_n^-$. Clearly, $p_+(\widetilde{T}_i)=\mathcal{T}_i$, $p_-(\widetilde{T}_i)=R_i$, and $p_\pm(z)=\pm1$. We shall use the short-hand notation $\widetilde{T}_{\mathbf{a}}=\widetilde{T}_{1,a_1}\widetilde{T}_{2,a_2}\cdots\widetilde{T}_{n-1,a_{n-1}}$. Assume there is a relation

$$(\star)\qquad \sum_{\mathbf{a}}(\alpha_{\mathbf{a}}\widetilde{T}_{\mathbf{a}}+\beta_{\mathbf{a}}z\widetilde{T}_{\mathbf{a}})=0\quad\text{for some constants }\alpha_{\mathbf{a}},\beta_{\mathbf{a}}.$$

By applying the canonical map p_- to $(\star)$ and Theorem 3.9, we conclude that $\alpha_{\mathbf{a}}-\beta_{\mathbf{a}}=0$ for each $\mathbf{a}$. On the other hand, it is (a variant of) a classical fact that $\mathcal{T}_{1,a_1}\mathcal{T}_{2,a_2}\cdots\mathcal{T}_{n-1,a_{n-1}}$, where $0\leqslant a_i\leqslant i$ and $1\leqslant i\leqslant n-1$, form a linear basis for $\mathcal{H}_n^-$. By applying p_+ to $(\star)$, we conclude that $\alpha_{\mathbf{a}}+\beta_{\mathbf{a}}=0$ for each $\mathbf{a}$. So $\alpha_{\mathbf{a}}=\beta_{\mathbf{a}}\equiv0$. $\square$

Remark 3.11. By (2.8)–(2.10) and Theorem 3.9, $\mathcal{H}_n^-$ reduces to $\mathbb{C}S_n^-$ as q goes to 1. By a standard deformation argument, the algebra $\mathcal{H}_n^-$ is semisimple. Similarly, $\mathcal{H}_n^\sim$ is a flat deformation of $\mathbb{C}S_n^\sim$ by Proposition 3.10, which justifies the terminology of "covering Hecke algebra" for $\mathcal{H}_n^\sim$.

3.4. Spin Hecke algebra for the alternating group

Note that the number of generators R_i's appearing in the monomial $R_{1,a_1}R_{2,a_2}\cdots R_{n-1,a_{n-1}}$ is $a_1+\cdots+a_{n-1}$ (called the *length* of the monomial). Denote by $\mathcal{H}_{n,\bar0}^-$ (respectively, $H_{n,\bar0}^\sim$) the even subalgebra of the superalgebra $\mathcal{H}_n^-$ (respectively, $\mathcal{H}_n^\sim$).

Proposition 3.12. *Let* $n\geqslant2$.

(1) *The standard monomials of even length, that is,* $R_{1,a_1}R_{2,a_2}\cdots R_{n-1,a_{n-1}}$, *where* $0\leqslant a_i\leqslant i$, $1\leqslant i\leqslant n-1$ *such that* $a_1+\cdots+a_{n-1}$ *is even, form a basis for the algebra* $\mathcal{H}_{n,\bar0}^-$. *In particular,* $\dim\mathcal{H}_{n,\bar0}^-=\frac12 n!$.

(2) *The elements* $\widetilde{T}_{1,a_1}\widetilde{T}_{2,a_2}\cdots\widetilde{T}_{n-1,a_{n-1}}, z\widetilde{T}_{1,a_1}\widetilde{T}_{2,a_2}\cdots\widetilde{T}_{n-1,a_{n-1}}$, *where* $0 \leqslant a_i \leqslant i$, $1 \leqslant i \leqslant n-1$ *such that* $a_1 + \cdots + a_{n-1}$ *is even, form a basis for the algebra* $\mathcal{H}^{\sim}_{n,\bar{0}}$. *In particular,* $\dim \mathcal{H}^{\sim}_{n,\bar{0}} = n!$.

Proof. Follows from Theorem 3.9 and Proposition 3.10. □

Recall that the alternating group A_n is a subgroup of S_n of index 2. The short exact sequence (1.1) gives rise to a subgroup $A^{\sim}_n$ of $S^{\sim}_n$ of index 2 which is a double cover of A_n. It follows that $\mathcal{H}^{\sim}_{n,\bar{0}}$ is a q-deformation of the algebra $\mathbb{C}A^{\sim}_n$, while $\mathcal{H}^{-}_{n,\bar{0}}$ is a q-deformation of the algebra $\mathbb{C}A^{\sim}_n/\langle z+1\rangle$.

Definition 3.13. The algebra $\mathcal{H}^{-}_{n,\bar{0}}$ is called the *spin Hecke algebra for the alternating group* A_n. The algebra $\mathcal{H}^{\sim}_{n,\bar{0}}$ is called the *covering Hecke algebra for the alternating group* A_n.

We leave it to the reader to write down a presentation for the algebra $\mathcal{H}^{-}_{n,\bar{0}}$ using the generators R_1R_{i+1} $(1 \leqslant i \leqslant n-2)$ and a similar presentation for $\mathcal{H}^{\sim}_{n,\bar{0}}$.

3.5. The basic spin supermodule

Theorem 3.14. *There exists a homomorphism of superalgebras*

$$\pi_q : \mathcal{H}^{-}_n \to \mathcal{C}_n \otimes \mathbb{C}(q)$$

which sends

$$R_i \mapsto \sqrt{-1}\big(qc_i - q^{-1}c_{i+1}\big), \quad 1 \leqslant i \leqslant n-1.$$

The image is isomorphic to $\mathcal{C}_{n-1} \otimes \mathbb{C}(q)$.

Proof. We need to check the relations (2.8)–(2.10) with $\gamma_i := \sqrt{-1}(qc_i - q^{-1}c_{i+1})$ replacing R_i therein. Clearly we have

$$\gamma_i^2 = -2 - \varepsilon^2, \qquad \gamma_i\gamma_j = -\gamma_j\gamma_i \quad \big(|i-j| > 1\big).$$

By a straightforward computation, we have

$$\begin{aligned}
\gamma_i\gamma_{i+1}\gamma_i &= \sqrt{-1}\big(2qc_i - \big(q^{-1} - q^3\big)c_{i+1} - \big(q^{-3} + q\big)c_{i+2}\big) \\
&= 2\gamma_i + \big(q^2 + q^{-2}\big)\gamma_{i+1}, \\
\gamma_{i+1}\gamma_i\gamma_{i+1} &= \sqrt{-1}\big(\big(q^3 + q^{-1}\big)c_i - \big(q^{-3} - q\big)c_{i+1} - 2q^{-1}c_{i+2}\big) \\
&= \big(q^2 + q^{-2}\big)\gamma_i + 2\gamma_{i+1}.
\end{aligned}$$

Thus, $\gamma_i\gamma_{i+1}\gamma_i - \gamma_{i+1}\gamma_i\gamma_{i+1} = \varepsilon^2(\gamma_{i+1} - \gamma_i)$.

Since the image of the linear span of R_i $(1 \leqslant i \leqslant n-1)$ is by definition a subspace of dimension $n-1$ of the linear span of c_i's, the image of $\mathcal{H}^{-}_n$ under π is a Clifford algebra in $n-1$ generators. □

It is well known that $\mathcal{C}_{n-1}$ has a unique simple supermodule, which is of dimension $2^{[n/2]}$. Here $[n/2]$ denotes the largest integer no greater than $n/2$. The pullback via π_q gives rise to a simple $\mathcal{H}_n^-$-supermodule of dimension $2^{[n/2]}$, which we will refer to as the *basic spin $\mathcal{H}_n^-$-supermodule*. Indeed, this module is a q-deformation of the basic spin $\mathbb{C}S_n^-$-supermodule and the homomorphism π_q is the q-deformation of a classical fundamental construction [3,8].

4. Spin and covering affine Hecke algebras

4.1. The affine Hecke–Clifford algebra

Definition 4.1. [4] The *affine Hecke–Clifford algebra* $\widehat{\mathcal{H}c}_n$ is the $\mathbb{C}(q)$-algebra generated by T_i $(1 \leqslant i \leqslant n-1)$ and $c_i, X_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$, subject to the relations (2.1)–(2.4) of T_i, c_i in $\mathcal{H}c_n$ and the following additional relations:

$$(T_i + \varepsilon c_i c_{i+1}) X_i T_i = X_{i+1}, \tag{4.1}$$

$$T_i X_j = X_j T_i \quad (j \neq i, i+1), \tag{4.2}$$

$$X_i X_j = X_j X_i, \tag{4.3}$$

$$X_i c_i = c_i X_i^{-1}, \qquad X_i c_j = c_j X_i \quad (i \neq j). \tag{4.4}$$

The affine Hecke–Clifford algebra $\widehat{\mathcal{H}c}_n$ was introduced by Jones–Nazarov [4]. The algebra $\widehat{\mathcal{H}c}_n$ admits a canonical superalgebra structure with T_i, X_i being even and c_i being odd. It is known that $X_1^{\alpha_1} \cdots X_n^{\alpha_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} T_\sigma$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$, $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ and $\sigma \in S_n$, form a *standard* basis for $\widehat{\mathcal{H}c}_n$ [4] (also cf. [2]). By definition, $\widehat{\mathcal{H}c}_n$ contains $\mathcal{H}c_n$ as a subalgebra.

The convention that $c_i^2 = -1$ was used in [4], and so our c_i matches with their $\sqrt{-1}c_i$. Our convention that $c_i^2 = 1$ is consistent with [2,5]. The different convention leads to a different sign whenever a quadratic term $c_i c_j$ appears. The following useful identities follow from the definition:

$$(T_i + \varepsilon c_i c_{i+1})^{-1} = T_i + \varepsilon c_i c_{i+1} - \varepsilon,$$

$$T_i X_i = X_{i+1} T_i - \varepsilon (X_{i+1} + c_i c_{i+1} X_i),$$

$$T_i X_{i+1} = X_i T_i + \varepsilon (1 - c_i c_{i+1}) X_{i+1}.$$

4.2. The spin affine Hecke algebra

Now we introduce the main new concept of the paper.

Definition 4.2. The *spin affine Hecke algebra*, denoted by $\widehat{\mathcal{H}}_n^-$, is the $\mathbb{C}(q)$-algebra generated by R_i $(1 \leqslant i \leqslant n-1)$ and p_i, q_i $(1 \leqslant i \leqslant n)$, subject to the relations (2.8)–(2.10) for R_i's in $\mathcal{H}_n^-$ and the following additional relations:

$$p_i p_j = p_j p_i, \quad q_i q_j = -q_j q_i \quad (i \neq j), \tag{4.5}$$

$$p_i^2 + q_i^2 = 1, \quad p_i q_j = q_j p_i \quad (\forall i, j), \tag{4.6}$$

$$R_i p_j = p_j R_i, \quad R_i q_j = -q_j R_i \quad (j \neq i, i+1), \tag{4.7}$$

$$R_i p_i = p_{i+1} R_i + \varepsilon(q_i - q_{i+1}), \tag{4.8}$$

$$R_i q_i = -q_{i+1} R_i - \varepsilon(p_i + p_{i+1}). \tag{4.9}$$

The algebra $\widehat{\mathcal{H}}_n^-$ has a canonical superalgebra structure with each p_i being even and each q_i, R_i being odd.

Proposition 4.3. *Assume only the relation* (2.8). *The three pairs of relations* (4.8)–(4.9)*,* (4.10)–(4.11) *and (*4.12)–(4.13) *are equivalent*:

$$R_i p_{i+1} = p_i R_i - \varepsilon(q_i - q_{i+1}), \tag{4.10}$$

$$R_i q_{i+1} = -q_i R_i - \varepsilon(p_i + p_{i+1}); \tag{4.11}$$

$$p_{i+1} = -\frac{1}{2} R_i p_i R_i + \frac{\varepsilon}{2}(q_i R_i + R_i q_i) + \frac{\varepsilon^2}{2} p_i, \tag{4.12}$$

$$q_{i+1} = \frac{1}{2} R_i q_i R_i + \frac{\varepsilon}{2}(p_i R_i + R_i p_i) - \frac{\varepsilon^2}{2} q_i. \tag{4.13}$$

In particular, the algebra $\widehat{\mathcal{H}}_n^-$ is generated by p_1, q_1 and R_i $(1 \leqslant i \leqslant n-1)$.

Proof. The last statement follows readily from (4.12)–(4.13). Recall from (2.8) that $R_i^2 = -2 - \varepsilon^2$.

(i) (4.8)–(4.9) $\Rightarrow$ (4.12)– (4.13): The right multiplication of (4.8) by R_i gives us

$$R_i q_i R_i = \left(-2 - \varepsilon^2\right) p_{i+1} + \varepsilon q_i R_i - \varepsilon q_{i+1} R_i. \tag{4.14}$$

Rewrite (4.9) as $-q_{i+1} R_i = R_i q_i + \varepsilon(p_i + p_{i+1})$. Plugging this equation into (4.14) and reorganizing the terms, we obtain (4.12). The proof of (4.13) is almost identical.

(ii) (4.12)–(4.13) $\Rightarrow$ (4.10)–(4.11): Multiplying (4.12) by R_i on the left gives us

$$R_i p_{i+1} = -\frac{1}{2}\left(-2 - \varepsilon^2\right) p_i R_i + \frac{1}{2}\varepsilon R_i q_i R_i + \frac{1}{2}\varepsilon\left(-2 - \varepsilon^2\right) q_i + \frac{1}{2}\varepsilon^2 R_i p_i. \tag{4.15}$$

Rewrite (4.13) as $R_i q_i R_i = 2q_{i+1} - \varepsilon(p_i R_i + R_i p_i) + \varepsilon^2 q_i$. Plugging this into (4.15) and reorganizing the terms, we obtain (4.10). The proof of (4.11) is almost identical.

We will skip the analogous proofs for (4.10)–(4.11) $\Rightarrow$ (4.12)–(4.13) as well as for (4.12)–(4.13) $\Rightarrow$ (4.8)–(4.9). $\square$

4.3. (Anti-)involutions of $\widehat{\mathcal{H}}_n^-$

There are several involutions of the algebra $\widehat{\mathcal{H}}_n^-$ which are extensions of the involutions σ, s and $-$ of $\mathcal{H}_n^-$ in Section 2.3. We can extend σ in two ways to involutions $\sigma_\pm : \widehat{\mathcal{H}}_n^- \to \widehat{\mathcal{H}}_n^-$ (where q is fixed):

$$\begin{aligned} \sigma_+ &: \quad p_i \to p_{n+1-i}, \quad q_i \to q_{n+1-i}, \quad R_i \mapsto R_{n-i}, \\ \sigma_- &: \quad p_i \to -p_{n+1-i}, \quad q_i \to -q_{n+1-i}, \quad R_i \mapsto R_{n-i}, \end{aligned}$$

for all possible i. We extend s to two involutions s_p, s_q of $\widehat{\mathcal{H}}_n^-$ (where q is fixed):

$$s_p\colon\quad p_i \to -p_i,\quad q_i \to q_i,\quad R_i \mapsto -R_i,$$
$$s_q\colon\quad p_i \to p_i,\quad q_i \to -q_i,\quad R_i \mapsto -R_i,$$

for all possible i. We also extend $-$ to involutions $-_p, -_q$ of $\widehat{\mathcal{H}}_n^-$ (fixing each R_i):

$$-_p\colon\quad p_i \to -p_i,\quad q_i \to q_i,\quad R_i \mapsto R_i,\quad q \mapsto q^{-1},$$
$$-_q\colon\quad p_i \to p_i,\quad q_i \to -q_i,\quad R_i \mapsto R_i,\quad q \mapsto q^{-1},$$

for all possible i. By inspection, all these involutions commute with each other, and their products give rise to many more involutions of $\widehat{\mathcal{H}}_n^-$.

Extending the anti-involution τ on $\mathcal{H}_n^-$, we also have an anti-involution τ on $\widehat{\mathcal{H}}_n^-$ by letting, for all possible i,

$$\tau(p_i) = p_i,\qquad \tau(q_i) = -q_i,\qquad \tau(R_i) = -R_i.$$

One obtains more anti-involutions on $\widehat{\mathcal{H}}_n^-$ by composing τ with the various involutions above (which commute with τ).

4.4. A nonstandard presentation of the affine Hecke algebra

The affine Hecke algebra $\widehat{\mathcal{H}}_n$ is generated by T_i $(1 \leqslant i \leqslant n-1)$, X_j $(1 \leqslant j \leqslant n)$ subject to the relations (2.1)–(2.2) of T_i's in $\mathcal{H}_n$ and the following additional relations:

$$X_iX_j = X_jX_i \quad (\forall i, j),$$
$$T_iX_iT_i = X_{i+1},$$
$$T_iX_j = X_jT_i \quad (j \neq i, i+1).$$

Recall $\mathcal{T}_i := T_i + T_i^{-1} \equiv 2T_i - \varepsilon$, and further introduce

$$P_i := \frac{1}{2}\big(X_i + X_i^{-1}\big),\qquad Q_i := \frac{1}{2}\big(X_i - X_i^{-1}\big).$$

It follows that

$$X_i = P_i + Q_i,\qquad X_i^{-1} = P_i - Q_i.$$

Proposition 4.4. *The algebra $\widehat{\mathcal{H}}_n$ is generated by $\mathcal{T}_i$ $(1 \leqslant i \leqslant n-1)$, P_j and Q_j $(1 \leqslant j \leqslant n)$, subject to the relations* (2.11)–(2.13) *for $\mathcal{T}_i$'s and the following additional relations*:

$$P_iP_j = P_jP_i,\quad Q_iQ_j = Q_jQ_i \quad (i \neq j), \tag{4.16}$$
$$P_i^2 - Q_i^2 = 1,\quad P_iQ_j = Q_jP_i \quad (\forall i, j), \tag{4.17}$$
$$\mathcal{T}_iP_i = P_{i+1}\mathcal{T}_i - \varepsilon(Q_{i+1} + Q_i), \tag{4.18}$$
$$\mathcal{T}_iQ_i = Q_{i+1}\mathcal{T}_i - \varepsilon(P_i + P_{i+1}). \tag{4.19}$$

Proof. This follows by a direct computation. Let us illustrate by the derivation of (4.18). Indeed, recalling $T_i^{-1} = T_i - \varepsilon$, we have

$$\begin{aligned}
\mathcal{T}_i P_i - P_{i+1}\mathcal{T}_i &= \frac{1}{2}(2T_i - \varepsilon)\big(X_i + X_i^{-1}\big) - \frac{1}{2}\big(X_{i+1} + X_{i+1}^{-1}\big)(2T_i - \varepsilon) \\
&= T_i X_i - \frac{1}{2}\varepsilon X_i + (T_i - \varepsilon)X_i^{-1} + \frac{1}{2}\varepsilon X_i^{-1} \\
&\quad - X_{i+1}(T_i - \varepsilon) - \frac{1}{2}\varepsilon X_{i+1} - X_{i+1}^{-1}T_i + \frac{1}{2}\varepsilon X_{i+1}^{-1} \\
&= \big(T_i X_i - X_{i+1}T_i^{-1}\big) + \big(T_i^{-1}X_i^{-1} - X_{i+1}^{-1}T_i\big) - \varepsilon(Q_{i+1} + Q_i) \\
&= -\varepsilon(Q_{i+1} + Q_i).
\end{aligned}$$

In the last equation, we have used $T_i X_i = X_{i+1}T_i^{-1}$ and $T_i^{-1}X_i^{-1} = X_{i+1}^{-1}T_i$. $\square$

One further checks that in the presence of (2.11), the relations (4.18)–(4.19) are equivalent to the two equations below:

$$\begin{aligned}
\mathcal{T}_i P_{i+1} &= P_i \mathcal{T}_i + \varepsilon(Q_{i+1} + Q_i), \\
\mathcal{T}_i Q_{i+1} &= Q_i \mathcal{T}_i + \varepsilon(P_i + P_{i+1}).
\end{aligned}$$

4.5. The covering affine Hecke algebra

Definition 4.5. The *covering affine Hecke algebra* $\widehat{\mathcal{H}}_n^{\sim}$ is generated by z, $\widetilde{T}_i$ $(1 \leqslant i \leqslant n-1)$, $\tilde{P}_j$ and $\tilde{Q}_j$ $(1 \leqslant j \leqslant n)$, subject to the relations (2.14)–(2.16) for $\widetilde{T}_i$'s and the following additional relations:

$$\begin{gathered}
z^2 = 1, \quad z \text{ is central} \\
\tilde{P}_i \tilde{P}_j = \tilde{P}_j \tilde{P}_i, \quad \tilde{Q}_i \tilde{Q}_j = z\tilde{Q}_j \tilde{Q}_i \quad (i \neq j), \\
\tilde{P}_i^2 - z\tilde{Q}_i^2 = 1, \quad \tilde{P}_i \tilde{Q}_j = \tilde{Q}_j \tilde{P}_i \quad (\forall i, j),
\end{gathered}$$

$$\widetilde{T}_i \tilde{P}_i = \tilde{P}_{i+1}\widetilde{T}_i - \varepsilon(\tilde{Q}_{i+1} + z\tilde{Q}_i), \tag{4.20}$$

$$\widetilde{T}_i \tilde{Q}_i = z\tilde{Q}_{i+1}\widetilde{T}_i - \varepsilon(\tilde{P}_i + \tilde{P}_{i+1}). \tag{4.21}$$

Similar to Proposition 4.3, we can show, by assuming only (2.14), that the three pairs of relations (4.20)–(4.21), (4.22)–(4.23), and (4.24)–(4.25) below are equivalent to each other:

$$\widetilde{T}_i \tilde{P}_{i+1} = \tilde{P}_i \widetilde{T}_i + \varepsilon(\tilde{Q}_{i+1} + z\tilde{Q}_i), \tag{4.22}$$

$$\widetilde{T}_i \tilde{Q}_{i+1} = z\tilde{Q}_i \widetilde{T}_i + z\varepsilon(\tilde{P}_i + \tilde{P}_{i+1}); \tag{4.23}$$

$$\tilde{P}_{i+1} = \frac{1}{8}(3-z)\big(z\widetilde{T}_i \tilde{P}_i \widetilde{T}_i + \varepsilon\widetilde{T}_i \tilde{Q}_i + \varepsilon\tilde{Q}_i \widetilde{T}_i + \varepsilon^2 \tilde{P}_i\big), \tag{4.24}$$

$$\tilde{Q}_{i+1} = \frac{1}{8}(3-z)\big(\widetilde{T}_i \tilde{Q}_i \widetilde{T}_i + \varepsilon\widetilde{T}_i \tilde{P}_i + \varepsilon\tilde{P}_i \widetilde{T}_i + \varepsilon^2 z\tilde{Q}_i\big). \tag{4.25}$$

By definition, the quotient of the covering affine Hecke algebra $\widehat{\mathcal{H}}_n^{\sim}$ by the ideal $\langle z-1\rangle$ is isomorphic to the usual affine Hecke algebra in the nonstandard presentation above, where the canonical images of $\tilde{P}_i$, $\tilde{Q}_i$ are identified with P_i, Q_i, respectively. Also, the quotient of $\widehat{\mathcal{H}}_n^{\sim}$ by the ideal $\langle z+1\rangle$ is isomorphic to the spin affine Hecke algebra $\widehat{\mathcal{H}}_n^{-}$, where the canonical images of $\tilde{P}_i$, $\tilde{Q}_i$ are identified with p_i, q_i, respectively.

5. Structures of the spin affine Hecke algebra

5.1. Morita super-equivalence for $\widehat{\mathcal{H}}_n^{-}$

Theorem 5.1. *There exists an isomorphism of superalgebras*

$$\Phi : \widehat{\mathcal{H}c}_n \xrightarrow{\simeq} \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^{-}$$

which extends the isomorphism $\Phi : \mathcal{H}c_n \to \mathcal{C}_n \otimes \mathcal{H}_n^{-}$ and is such that

$$\Phi(X_i) = p_i - c_i q_i, \qquad \Phi\big(X_i^{-1}\big) = p_i + c_i q_i.$$

The inverse Ψ is an extension of $\Psi : \mathcal{C}_n \otimes \mathcal{H}_n^{-} \to \mathcal{H}c_n$ such that

$$\Psi(p_i) = \frac{1}{2}\big(X_i + X_i^{-1}\big), \qquad \Psi(q_i) = \frac{1}{2}\big(X_i - X_i^{-1}\big)c_i.$$

5.2. Proof of the isomorphism Theorem 5.1

We will adopt the convention $\Psi(a) = a^{\Psi}$, $\Phi(b) = b^{\Phi}$. We start with several lemmas.

Lemma 5.2. *In $\widehat{\mathcal{H}c}_n$, we have*

$$\begin{aligned} R_i^{\Psi} p_i^{\Psi} &= p_{i+1}^{\Psi} R_i^{\Psi} + \varepsilon\big(q_i^{\Psi} - q_{i+1}^{\Psi}\big), \\ R_i^{\Psi} q_i^{\Psi} &= -q_{i+1}^{\Psi} R_i^{\Psi} - \varepsilon\big(p_i^{\Psi} + p_{i+1}^{\Psi}\big). \end{aligned}$$

Proof. By (3.3), we have

$$\begin{aligned} R_i^{\Psi} X_i &= \big(T_i(c_{i+1} - c_i) + \varepsilon c_i\big) X_i \\ &= T_i X_i c_{i+1} - T_i X_i^{-1} c_i + \varepsilon X_i^{-1} c_i \\ &= X_{i+1} T_i c_{i+1} - \varepsilon(X_{i+1} + c_i c_{i+1} X_i) c_{i+1} \\ &\quad - X_{i+1}^{-1} T_i c_i - \varepsilon\big(X_i^{-1} c_i - X_{i+1}^{-1} c_{i+1}\big) + \varepsilon X_i^{-1} c_i \\ &= X_{i+1} T_i c_{i+1} - X_{i+1}^{-1} T_i c_i - \varepsilon\big(X_{i+1} c_{i+1} - X_{i+1}^{-1} c_{i+1} + X_i^{-1} c_i\big). \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
R_i^{\Psi} X_i^{-1} &= \big(T_i(c_{i+1}-c_i)+\varepsilon c_i\big)X_i^{-1} \\
&= T_i X_i^{-1} c_{i+1} - T_i X_i c_i + \varepsilon X_i c_i \\
&= X_{i+1}^{-1} T_i c_{i+1} + \varepsilon\big(X_i^{-1} + c_i c_{i+1} X_{i+1}\big)c_{i+1} \\
&\quad - \big(X_{i+1}T_i - \varepsilon(X_{i+1} + c_i c_{i+1} X_i)\big)c_i + \varepsilon X_i c_i \\
&= X_{i+1}^{-1} T_i c_{i+1} - X_{i+1} T_i c_i + \varepsilon\big(X_{i+1}c_i + X_{i+1}^{-1} c_i + X_i c_i\big).
\end{aligned}
$$

Now the lemma follows by adding and subtracting these two identities for $R_i^{\Psi} X_i$ and $R_i^{\Psi} X_i^{-1}$ (as well as multiplying with c_i). $\square$

Lemma 5.3. *Let $1 \leqslant i \leqslant n$. In $\widehat{\mathcal{H}}c_n$, the element p_i^{Ψ} commutes with $\mathcal{C}_n$ while q_i^{Ψ} super-commutes with $\mathcal{C}_n$.*

Proof. Follows directly from (2.4) and (4.4). $\square$

Lemma 5.4. *The $T_i^{\Phi}, X_i^{\Phi}, (X_i^{-})^{\Phi}, c_i$ satisfy the relations* (4.1)–(4.4).

Proof. The relations (4.2)–(4.4) for $X_i^{\Phi}, (X_i^{-})^{\Phi}, c_i$ are easy to verify from the definitions. It remains to check (4.1). We shall use repeatedly (4.8) and (4.9) below. Recalling T_i^{Φ} from (3.1), we calculate that

$$
\begin{aligned}
2X_i^{\Phi} T_i^{\Phi} &= (p_i - c_i q_i)\big(R_i(c_{i+1}-c_i) + \varepsilon(1-c_i c_{i+1})\big) \\
&= \big(R_i p_{i+1} + \varepsilon(q_i - q_{i+1})\big)(c_{i+1}-c_i) - \big(-R_i q_{i+1} - \varepsilon(p_i + p_{i+1})\big)c_i(c_{i+1}-c_i) \\
&\quad + \varepsilon p_i(1-c_i c_{i+1}) + \varepsilon q_i c_i(1-c_i c_{i+1}) \\
&= (R_i p_{i+1} - \varepsilon q_{i+1})(c_{i+1}-c_i) - (R_i q_{i+1} + \varepsilon p_{i+1})(1-c_i c_{i+1}).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
4\big(T_i^{\Phi} + \varepsilon c_i c_{i+1}\big)X_i^{\Phi} T_i^{\Phi} &= \big[R_i(c_{i+1}-c_i) + \varepsilon(1-c_i c_{i+1})\big] \\
&\qquad \times \big[(R_i p_{i+1} - \varepsilon q_{i+1})(c_{i+1}-c_i) - (R_i q_{i+1} + \varepsilon p_{i+1})(1-c_i c_{i+1})\big] \\
&= -R_i(R_i p_{i+1} - \varepsilon q_{i+1})(c_{i+1}-c_i)^2 \\
&\quad - R_i(R_i q_{i+1} + \varepsilon p_{i+1})(c_{i+1}-c_i)(1-c_i c_{i+1}) \\
&\quad + \varepsilon(R_i p_{i+1} - \varepsilon q_{i+1})(1+c_i c_{i+1})(c_{i+1}-c_i) \\
&\quad - \varepsilon(R_i q_{i+1} + \varepsilon p_{i+1})(1+c_i c_{i+1})(1-c_i c_{i+1}) \\
&= 4(p_{i+1} - c_{i+1} q_{i+1}) = 4X_{i+1}^{\Phi}.
\end{aligned}
$$

This completes the proof of the lemma. $\square$

Proof of Theorem 5.1. It is straightforward to check that $p_i^{\Psi}, q_i^{\Psi}, R_i^{\Psi}$ satisfy (4.5)–(4.7). Together with Lemmas 3.3–3.5, 5.2, 5.3, this implies that $\Psi : \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^- \to \widehat{\mathcal{H}c}_n$ is an algebra homomorphism.

Clearly $X_i^{\Phi} \cdot (X_i^{-1})^{\Phi} = 1$. Recalling that $\Phi|_{\mathcal{H}c_n} : \mathcal{H}c_n \to \mathcal{C}_n \otimes \mathcal{H}_n^-$ is an algebra isomorphism by Theorem 3.1, we have by Lemma 5.4 that $\Phi : \widehat{\mathcal{H}c}_n \to \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ is an algebra homomorphism.

By a direct computation, the homomorphisms Ψ and Φ are inverses on the generators, and thus they are inverse algebra isomorphisms. □

5.3. A basis for $\widehat{\mathcal{H}}_n^-$

We recall the definition of R_{i,a_i} from (3.4).

Theorem 5.5. *The algebra $\widehat{\mathcal{H}}_n^-$ has a basis*

$$p_1^{k_1} \cdots p_n^{k_n} q_1^{\epsilon_1} \cdots q_n^{\epsilon_n} \cdot R_{1,a_1} R_{2,a_2} \cdots R_{n-1,a_{n-1}}$$

where $k_1, \ldots, k_n \in \mathbb{Z}_+, \epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$, $0 \leqslant a_i \leqslant i$ and $1 \leqslant i \leqslant n-1$.

Proof. The subalgebra A_i generated by $c_i, p_i^{\Psi}, q_i^{\Psi}$ (for a fixed i) is identical to the subalgebra generated by c_i, X_i, X_i^{-1}, and it has a linear basis given by $c_i^{\alpha} X_i^{a}$ ($\alpha \in \{0, 1\}$, $a \in \mathbb{Z}$). By the standard basis for $\widehat{\mathcal{H}c}_n$, Theorems 3.1 and 5.1, we have the following isomorphisms of vector spaces:

$$\widehat{\mathcal{H}c}_n \cong A_1 \otimes \cdots \otimes A_n \otimes \mathcal{H}_n^- \cong \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-. \tag{5.1}$$

Claim. *The algebra A_i has another basis $\{c_i^{\alpha}(p_i^{\Psi})^k (q_i^{\Psi})^{\beta} \mid \alpha, \beta \in \{0, 1\},\ k \in \mathbb{Z}_+\}$. Equivalently, the subalgebra B_i generated by $(X_i + X_i^{-1})$ and $(X_i - X_i^{-1})$ has a basis $\{(X_i + X_i^{-1})^k (X_i - X_i^{-1})^{\beta} \mid \beta \in \{0, 1\},\ k \in \mathbb{Z}_+\}$.*

We prove the second (equivalent) part of the claim. Since any even power of $(X_i - X_i^{-1})$ can be written as a polynomial in $(X_i + X_i^{-1})$, the algebra B_i is spanned by the elements $(X_i + X_i^{-1})^k (X_i - X_i^{-1})^{\beta}$, with the constraint $\beta \in \{0, 1\}$. It remains to prove the linear independence of these elements. Assume otherwise

$$f(X) := \sum_k a_k \big(X_i + X_i^{-1}\big)^k \big(X_i - X_i^{-1}\big) + \sum_k b_k \big(X_i + X_i^{-1}\big)^k = 0,$$

for some $a_k, b_k \in \mathbb{C}(q)$, all but finitely many being zero. Then $f(X^{-1}) = 0$, and thus

$$2 \sum_k b_k \big(X_i + X_i^{-1}\big)^k = f(X) + f\big(X^{-1}\big) = 0.$$

By looking at the highest degree term in X_i of this equation, we see that all $b_k = 0$, and similarly all $a_k = 0$. This proves the claim.

The theorem now follows from (5.1) and the claim. □

Accordingly, we obtain a similar basis for the covering affine Hecke algebra $\widehat{\mathcal{H}}_n^{\sim}$ (compare Proposition 3.10).

5.4. The center of $\widehat{\mathcal{H}}_n^-$

Theorem 5.6. *The even center of the algebra $\widehat{\mathcal{H}}_n^-$ is the algebra of symmetric polynomials $\mathbb{C}[p_1, p_2, \dots, p_n]^{S_n}$.*

Proof. The center of $\widehat{\mathcal{H}c}_n$ is equal to $\mathbb{C}[X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}]^{S_n}$, the algebra of symmetric polynomials in $X_k + X_k^{-1}$ $(1 \leqslant k \leqslant n)$, according to Jones–Nazarov [4] (cf. [2]). The map $\Phi : \widehat{\mathcal{H}c}_n \xrightarrow{\cong} \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ sends $\mathbb{C}[X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}]^{S_n}$ onto $\mathbb{C}[p_1, p_2, \dots, p_n]^{S_n}$, and so $\mathbb{C}[p_1, p_2, \dots, p_n]^{S_n}$ is the center of $\mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$. It follows that $\mathbb{C}[p_1, p_2, \dots, p_n]^{S_n}$ is contained in the (even) center of $\widehat{\mathcal{H}}_n^-$. On the other hand, any given even central element e of $\widehat{\mathcal{H}}_n^-$ commutes with $\mathcal{C}_n$ thanks to the evenness of e, and thus lies in the center of $\mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$, which is $\mathbb{C}[p_1, p_2, \dots, p_n]^{S_n}$. $\square$

Proposition 5.7. *Let $n > 0$ be odd. Then $\mathfrak{q} := q_1 q_2 \cdots q_n$ is an odd central element of $\widehat{\mathcal{H}}_n^-$. However, $\mathfrak{q}^{\Psi}$ does not lie in the center of $\widehat{\mathcal{H}c}_n$.*

Proof. $\mathfrak{q}^{\Psi}$ does not lie in the center of $\widehat{\mathcal{H}c}_n$, since $\mathfrak{q}^{\Psi} c_i = -c_i \mathfrak{q}^{\Psi}$.

By definition, $\mathfrak{q}$ commutes with each p_i. Since n is odd, $\mathfrak{q}$ commutes with each q_i by a direct computation. It remains to show that $R_i \mathfrak{q} = \mathfrak{q} R_i$ for each i. Indeed, by (4.9) and (4.11), we have

$$\begin{aligned} R_i q_i q_{i+1} &= \big(-q_{i+1} R_i - \varepsilon(p_i + p_{i+1})\big) q_{i+1} \\ &= -q_{i+1}\big(-q_i R_i - \varepsilon(p_i + p_{i+1})\big) - \varepsilon(p_i + p_{i+1}) q_{i+1} = -q_i q_{i+1} R_i. \end{aligned}$$

This, together with (4.7) and the oddness of n, implies that $R_i \mathfrak{q} = \mathfrak{q} R_i$. $\square$

5.5. The intertwiners

By Theorem 5.6, $\delta := \prod_{1 \leqslant i < j \leqslant n} (p_i - p_j)^2$ is an even central element in $\widehat{\mathcal{H}}_n^-$. Denote by $(\widehat{\mathcal{H}}_n^-)_\delta$ the localization of $\widehat{\mathcal{H}}_n^-$ at δ. In particular, $(p_i - p_{i+1})^{-1} \in (\widehat{\mathcal{H}}_n^-)_\delta$.

Define

$$\daleth_i = R_i - \varepsilon \frac{q_i - q_{i+1}}{p_i - p_{i+1}} \in \big(\widehat{\mathcal{H}}_n^-\big)_\delta.$$

It is understood here and below that $\frac{A}{B} = B^{-1} A$.

Proposition 5.8. *The elements $\daleth_i$ $(1 \leqslant i \leqslant n-1)$ satisfy the following relations*:

$$\daleth_i^2 = -2 + 2\varepsilon^2 \frac{p_i p_{i+1} - 1}{(p_i - p_{i+1})^2}, \tag{5.2}$$

$$\daleth_i \daleth_{i+1} \daleth_i = \daleth_{i+1} \daleth_i \daleth_{i+1}, \tag{5.3}$$

$$\daleth_i \daleth_j = -\daleth_j \daleth_i \quad \big(|i - j| > 1\big),$$

$$\beth_i p_i = p_{i+1}\beth_i, \qquad \beth_i p_{i+1} = p_i\beth_i,$$
$$\beth_i p_j = p_j\beth_i, \quad \beth_i q_j = -q_j\beth_i \quad (j \neq i, i+1),$$
$$\beth_i q_i = -q_{i+1}\beth_i, \qquad \beth_i q_{i+1} = -q_i\beth_i.$$

Proof. All of these relations can be verified by direct computation. Below we describe an alternative way by making connections with the intertwiners ϕ_i for $\widehat{\mathcal{H}c}_n$ introduced in [4, (3.6)]. Recall that

$$\phi_i := T_i + \frac{\varepsilon}{X_i X_{i+1}^{-1} - 1} - \frac{\varepsilon}{X_i X_{i+1} - 1} \cdot c_i c_{i+1}$$

in a suitable localization of $\widehat{\mathcal{H}c}_n$ isomorphic to $\mathcal{C}_n \otimes (\widehat{\mathcal{H}}_n^-)_\delta$. One can show that

$$\phi_i c_i = c_{i+1}\phi_i, \qquad \phi_i c_{i+1} = c_i\phi_i.$$

Claim. *The isomorphism $\Phi : \widehat{\mathcal{H}c}_n \to \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ sends ϕ_i to $\frac{1}{2}(c_i - c_{i+1}) \otimes \beth_i$. Indeed, we have*

$$\begin{aligned}
\Phi(\phi_i) &= \frac{1}{2}R_i(c_{i+1} - c_i) + \frac{\varepsilon}{2}(1 - c_i c_{i+1}) \\
&\quad + \varepsilon\Phi\left(\frac{X_{i+1} - X_i^{-1} - (X_{i+1}^{-1} - X_i^{-1})c_i c_{i+1}}{X_i + X_i^{-1} - (X_{i+1} + X_{i+1}^{-1})}\right) \\
&= \frac{1}{2}R_i(c_{i+1} - c_i) + \frac{\varepsilon}{2} \cdot \frac{q_i c_i + q_{i+1}c_{i+1} - q_{i+1}c_i - q_i c_{i+1}}{p_i - p_{i+1}} \\
&= \frac{1}{2}\left(R_i - \varepsilon\frac{q_i - q_{i+1}}{p_i - p_{i+1}}\right)(c_{i+1} - c_i) = \frac{1}{2}(c_i - c_{i+1}) \otimes \beth_i.
\end{aligned}$$

With the help of the claim, all of the identities in the proposition follow from the corresponding statements for ϕ_i in [4, (3.7)] and [4, Proposition 3.1]. Let us illustrate by proving (5.2) in detail below. Recall from [4, Proposition 3.1] that

$$\phi_i^2 = 1 - \varepsilon^2\left(\frac{X_i X_{i+1}^{-1}}{(X_i X_{i+1}^{-1} - 1)^2} + \frac{X_i^{-1} X_{i+1}^{-1}}{(X_i^{-1} X_{i+1}^{-1} - 1)^2}\right).$$

By the above claim, we have

$$\begin{aligned}
\beth_i^2 &= \big((c_i - c_{i+1})\Phi(\phi_i)\big)^2 = -2\Phi\big(\phi_i^2\big) \\
&= -2 + 2\varepsilon^2\frac{X_i^{-1}X_{i+1}^{-1} - 2 + X_i X_{i+1} + X_i X_{i+1}^{-1} - 2 + X_i^{-1}X_{i+1}}{(X_{i+1} + X_{i+1}^{-1} - X_i - X_i^{-1})^2} \\
&= -2 + 2\varepsilon^2\frac{(X_{i+1} + X_{i+1}^{-1})(X_i + X_i^{-1}) - 4}{(X_{i+1} + X_{i+1}^{-1} - X_i - X_i^{-1})^2} \\
&= -2 + 2\varepsilon^2\frac{p_i p_{i+1} - 1}{(p_i - p_{i+1})^2}.
\end{aligned}$$

For the braid relation (5.3), the following identity can be useful:

$$(c_i - c_{i+1})(c_{i+1} - c_{i+2})(c_i - c_{i+1}) = (c_{i+1} - c_{i+2})(c_i - c_{i+1})(c_{i+1} - c_{i+2}). \qquad \square$$

6. Cyclotomic spin Hecke algebras

6.1. The definition

Recall $p_1 q_1 = q_1 p_1$. Consider the subalgebra

$$\mathcal{A}_1 := \mathbb{C}[p_1, q_1]/\langle p_1^2 + q_1^2 - 1\rangle$$

of $\widehat{\mathcal{H}}_n^-$ which is commutative and $\mathbb{Z}_2$-graded with p_1 being even and q_1 odd.

Proposition 6.1. *A nonzero $\mathbb{Z}_2$-homogeneous ideal I_1 of $\mathcal{A}_1$ is one of the following*:

(1) $\langle f(p_1)\rangle$, *for some nonzero polynomial f in one variable*;
(2) $\langle g(p_1)q_1\rangle$, *for some nonzero polynomial g in one variable*;
(3) $\langle (p_1+1)g(p_1), g(p_1)q_1\rangle$, *for some nonzero polynomial g*;
(4) $\langle (p_1-1)g(p_1), g(p_1)q_1\rangle$, *for some nonzero polynomial g*.

Proof. Let I_1 be a nonzero $\mathbb{Z}_2$-homogeneous ideal of $\mathcal{A}_1$. Let f and g be the unique monic polynomials of minimal degree such that $f(p_1) \in I_1$, $g(p_1)q_1 \in I_1$. By the $\mathbb{Z}_2$-homogeneity, $I_1 = \langle f(p_1), g(p_1)q_1\rangle$.

Note that $f(p_1)q_1 \in I_1$ and $(p_1^2 - 1)\cdot g(p_1) = -g(p_1)q_1^2 \in I_1$. By assumption of minimal degrees on f, g, we have

$$f(p_1) \mid (p_1 - 1)(p_1 + 1)\cdot g(p_1), \tag{6.1}$$

and thus $\deg f \leqslant \deg g + 2$. Also $\deg g \leqslant \deg f$, and $g = f$ if $\deg g = \deg f$.

In the case when $\deg f = \deg g$ and thus $g = f$, the ideal I_1 is of the form (1).

In the case when $\deg f = \deg g + 2$, we have $f(p_1) = (p_1^2 - 1)\cdot g(p_1)$ by (6.1), and thus I_1 is of the form (2).

Finally assume that $\deg f = \deg g + 1$ and consider two subcases: (i) $g(p_1) \mid f(p_1)$; (ii) $g(p_1) \nmid f(p_1)$. Thanks to (6.1), in case (i), I_1 is of the form (3) or (4). We now claim the subcase (ii) is empty. Indeed, by (6.1), (ii) and $\deg f = \deg g + 1$, we have $f(p_1) = (p_1^2 - 1)h(p_1) = -q_1^2 h(p_1)$ and $g(p_1) = (p_1 - a)h(p_1)$ for some constant $a \neq \pm 1$ and some polynomial h of degree equal to $(\deg g - 1)$. Therefore,

$$h(p_1)q_1 = \frac{1}{1-a^2}\big((p_1 + a)\cdot g(p_1)q_1 - q_1 \cdot f(p_1)\big) \in I_1.$$

This contradicts with the choice of $g(p)$ of minimal degree. $\square$

Definition 6.2. The *cyclotomic spin Hecke algebra* $\mathcal{H}_n^{I,-}$ is the quotient algebra of $\widehat{\mathcal{H}}_n^-$ by the two-sided ideal $I = \langle I_1\rangle$ generated by a nonzero $\mathbb{Z}_2$-homogeneous ideal $I_1 \subset \mathcal{A}_1$. (Note that $\mathcal{H}_n^{I,-}$ inherits a superalgebra structure from $\widehat{\mathcal{H}}_n^-$.)

Remark 6.3. As a byproduct of the above proof of Proposition 6.1, the ideal I in Definition 6.2 is generated by $f(p_1)$ and $g(p_1)q_1$, where f and g are the unique monic polynomials of minimal degree such that $f(p_1) \in I_1$, $g(p_1)q_1 \in I_1$. We will sometimes write $f = f_I$ and $g = g_I$ to indicate its dependence on I. More specifically, I is generated by one or two elements given in Proposition 6.1.

6.2. Relation to cyclotomic Hecke–Clifford algebras

We refer to Ariki [1] for more on the classical cyclotomic Hecke algebras.

Let F be a polynomial of the form

$$F(X_1) = a_d X_1^d + a_{d_1} X_1^{d-1} + \cdots + a_1 X_1 + a_0$$

which satisfies the condition

$$a_d = 1, \qquad a_i = a_0 a_{d-i} \quad (\forall 0 \leqslant i \leqslant d). \tag{6.2}$$

Associated to such an F, Brundan–Kleshchev [2] introduced the cyclotomic Hecke–Clifford algebra, which will be denoted by $\mathcal{H}c_n^F$ in this paper, as the quotient algebra $\widehat{\mathcal{H}c}_n/\langle F(X_1)\rangle$. The technical condition (6.2) was imposed so that the resulting cyclotomic algebra $\mathcal{H}_n^F$ has an expected basis and dimension. The next proposition shows that the condition (6.2) is natural from the perspective of cyclotomic spin Hecke algebras.

Theorem 6.4. *There is a bijection between the set of cyclotomic spin Hecke algebras and the set of cyclotomic Hecke–Clifford algebras. More explicitly, every cyclotomic Hecke–Clifford algebra $\mathcal{H}c_n^F$ is isomorphic to $\mathcal{C}_n \otimes \mathcal{H}_n^{I,-}$ for some cyclotomic spin Hecke algebra $\mathcal{H}_n^{I,-}$ via Φ. Conversely, for each $\mathcal{H}_n^{I,-}$, the algebra $\mathcal{C}_n \otimes \mathcal{H}_n^{I,-}$ is isomorphic to some cyclotomic Hecke–Clifford algebra via Ψ.*

Proof. Note that $a_0 = \pm 1$ by (6.2). Divide the degree d polynomials F which satisfy the condition (6.2) into the following four cases:

(1) $d = 2k$ is even and $a_0 = 1$;
(2) $d = 2k$ is even and $a_0 = -1$;
(3) $d = 2k + 1$ is odd and $a_0 = 1$;
(4) $d = 2k + 1$ is odd and $a_0 = -1$.

Then it follows by a case-by-case elementary verification that the isomorphism $\Phi : \widehat{\mathcal{H}c}_n \to \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ sends $X_1^{-k} F(X_1)$ for F in each case bijectively onto the corresponding set below:

(1) $\{f(p_1) \mid f$ is a polynomial of degree $k\}$;
(2) $\{g(p_1)q_1 \mid g$ is a polynomial of degree $(k-1)\}$;
(3) $\{(p_1 + 1 - c_1 q_1) \cdot g(p_1) \mid g$ is a polynomial of degree $k\}$;
(4) $\{(p_1 - 1 - c_1 q_1) \cdot g(p_1) \mid g$ is a polynomial of degree $k\}$.

Clearly, the ideal $\mathfrak{I}$ in $\mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ generated by an element in (1) or (2) above coincides with $\mathcal{C}_n \otimes \langle I_1\rangle$ where I_1 is given by Proposition 6.1(1) or (2), respectively. Now the proposition follows by the following claim.

Claim. *The ideal* $\mathfrak{I}$ *in* $\mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^-$ *generated by the element* $(p_1 \pm 1 - c_1 q_1) \cdot g(p_1)$ *in* (3) *or* (4) *coincides with* $\mathcal{C}_n \otimes \langle I_1 \rangle$*, where* $\langle I_1 \rangle$ *is the ideal in* $\widehat{\mathcal{H}}_n^-$ *generated by* I_1 *in Proposition* 6.1(3) *or* (4)*, respectively.*

Let us prove the claim for (3) and skip a similar proof for (4). Indeed, it is clear that $\mathfrak{I} \subseteq \mathcal{C}_n \otimes \langle I_1 \rangle$. On the other hand, we have

$$(p_1 + 1)g(p_1) = \frac{1}{2}(p_1 + 1 - c_1 q_1)(p_1 + 1 + c_1 q_1)g(p_1) \in \mathfrak{I},$$

and thus also $g(p_1)q_1 = c_1(p_1 + 1 + c_1 q_1)g(p_1) - c_1(p_1 + 1)g(p_1) \in \mathfrak{I}$. Therefore, $\mathfrak{I} \supseteq \mathcal{C}_n \otimes \langle I_1 \rangle$. □

It is known [2] that $\dim \mathcal{H}c_n^F = (\deg F)^n 2^n n!$. From the explicit relations between (the generators of) the corresponding ideals in $\widehat{\mathcal{H}c}_n$ and $\widehat{\mathcal{H}}_n^-$ presented in the above proof, we have the following.

Corollary 6.5. *Let* f_I *and* g_I *be the unique monic polynomials of minimal degree such that* $f_I(p_1)$ *and* $g_I(p_1)q_1$ *generate the ideal* I *in* $\widehat{\mathcal{H}}_n^-$*. Then,* $\dim \mathcal{H}_n^{I,-} = (\deg f_I + \deg g_I)^n n!$.

Conjecturally, a basis for $\mathcal{H}_n^{I,-}$ consists of $p_1^{\alpha_1} q_1^{\epsilon_1} \cdots p_n^{\alpha_n} q_n^{\epsilon_n} R$, where $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$, $0 \leqslant \alpha_i < \deg f_I$ if $\epsilon_i = 0$ and $0 \leqslant \alpha_i < \deg g_I$ if $\epsilon_i = 1$, and R runs over all standard monomials in $\mathcal{H}_n^-$.

6.3. Jucys–Murphy elements for $\mathcal{H}_n^-$

We observe that the spin Hecke algebra $\mathcal{H}_n^-$ coincides with the (smallest) cyclotomic spin Hecke algebra $\mathcal{H}_n^{I,-}$, where $I = \langle p_1 - 1, q_1 \rangle$. Similarly, the Hecke–Clifford algebra $\mathcal{H}c_n$ is a special case of the cyclotomic Hecke–Clifford algebras $\mathcal{H}c_n^F$ with $F(X_1) = X_1 - 1$.

Proposition 6.6. *There exists a unique algebra homomorphism*

$$\mathcal{JM} : \widehat{\mathcal{H}}_n^- \to \mathcal{H}_n^-$$

which extends the identity map on $\mathcal{H}_n^-$ *and is such that* $\mathcal{JM}(p_1) = 1$*,* $\mathcal{JM}(q_1) = 0$*.*

Proof. There exists a unique algebra homomorphism $\mathrm{JM} : \widehat{\mathcal{H}c}_n \to \mathcal{H}c_n$, which extends the identity map on $\mathcal{H}c_n$ and is such that $\mathrm{JM}(X_1) = 1$, according to Jones–Nazarov [4, Proposition 3.5]. By (4.1), the images J_i of X_i $(1 \leqslant i \leqslant n)$ under JM, called the Jucys–Murphy elements for $\widehat{\mathcal{H}c}_n$, are given recursively by $J_{i+1} = (T_i + \varepsilon c_i c_{i+1}) J_i T_i$. By Theorems 3.1 and 5.1, there exists a homomorphism $\mathrm{JM}' : \widehat{\mathcal{H}}_n^- \to \mathcal{H}_n^-$ to make the following diagram commutative:

$$\begin{array}{ccc} \widehat{\mathcal{H}c}_n & \xrightarrow{\mathrm{JM}} & \mathcal{H}c_n \\ \Phi \big\downarrow \cong & & \Phi \big\downarrow \cong \\ \mathcal{C}_n \otimes \widehat{\mathcal{H}}_n^- & \xrightarrow{\mathrm{JM}'} & \mathcal{C}_n \otimes \mathcal{H}_n^- . \end{array}$$

Since $\mathrm{JM}(X_1) = 1$, it follows by definition of Φ that $\mathrm{JM}'(p_1) = 1$, $\mathrm{JM}'(q_1) = 0$. Moreover, since $\mathrm{JM}'|_{\mathcal{C}_n \otimes \mathcal{H}_n^-}$ is the identity and the images of p_i, q_i are given recursively by Proposition 4.3, we conclude that JM' is of the form $\mathrm{I} \otimes \mathcal{JM}$ for a unique homomorphism $\mathcal{JM} : \widehat{\mathcal{H}}_n^- \to \mathcal{H}_n^-$ with given images of p_1 and q_1. Note that $\mathcal{JM}(p_1) = 1$ and $\mathcal{JM}(q_1) = 0$. □

We will call the images $\mathfrak{p}_i, \mathfrak{q}_i \in \mathcal{H}_n^-$ $(1 \leqslant i \leqslant n)$ of the elements p_i, q_i's under the homomorphism $\mathcal{JM}$ the *Jucys–Murphy elements* for $\mathcal{H}_n^-$, following the convention for the symmetric group and the usual Hecke algebras. The relations (4.5)–(4.11), with $\mathfrak{p}_i$ and $\mathfrak{q}_i$ replacing p_i and q_i, are satisfied. Alternatively, it follows from the proof of Proposition 6.6 that

$$\mathfrak{p}_i = \frac{1}{2}\Phi\big(J_i + J_i^{-1}\big), \qquad \mathfrak{q}_i = \frac{1}{2}\Phi\big(\big(J_i - J_i^{-1}\big)c_i\big).$$

Note the nontrivial implication that $\Phi(J_i + J_i^{-1})$ and $\frac{1}{2}\Phi((J_i - J_i^{-1})c_i)$ lie in $\mathcal{H}_n^-$. A direct computation using the recursive formula in Proposition 4.3 gives us the first few cases of the Jucys–Murphy elements:

$$\begin{aligned} 1 &= \mathfrak{p}_1, & \mathfrak{q}_1 &= 0, \\ 1 + \varepsilon^2 &= \mathfrak{p}_2, & \mathfrak{q}_2 &= \varepsilon R_1, \\ \frac{\varepsilon^2}{2}(R_1R_2 + R_2R_1) + \big(1 + \varepsilon^2\big)^2 &= \mathfrak{p}_3, & \mathfrak{q}_3 &= \frac{\varepsilon}{2}\big(R_1R_2R_1 + \big(2 + \varepsilon^2\big)R_2\big). \end{aligned}$$

These elements will play important roles in analyzing further the structures and the representation theory of $\mathcal{H}_n^-$ as in the usual (non-spin) setup.

6.4. A degeneration of $\widehat{\mathcal{H}}_n^-$ and $\mathcal{H}_n^{I,-}$

Recall that the spin symmetric group algebra $\mathbb{C}S_n^-$ is generated by t_i $(1 \leqslant i \leqslant n-1)$ subject to the relations (1.2)–(1.3). The degenerate spin affine Hecke algebra $\widehat{\mathcal{B}}$, introduced in [10], is the superalgebra with odd generators b_i $(1 \leqslant i \leqslant n)$ and t_i $(1 \leqslant i \leqslant n-1)$, subject to the relations (1.2)–(1.3) for t_i's and the following additional relations:

$$b_ib_j = -b_jb_i \quad (i \neq j),$$

$$t_ib_i = -b_{i+1}t_i + 1,$$

$$t_ib_j = -b_jt_i \quad (j \neq i, i+1).$$

Remark 6.7. The algebra $\widehat{\mathcal{B}}$ can be obtained from $\widehat{\mathcal{H}}_n^-$ by a suitable degeneration. Set $q = e^{\hbar/2}$. As q goes to 1, keeping in mind $p_i^2 + q_i^2 = 1$, we set

$$p_i \approx 1 + \hbar^2 b_i^2 + o\big(\hbar^2\big), \qquad q_i \approx \hbar\sqrt{-2} \cdot b_i + o(\hbar), \qquad R_i \approx \sqrt{-2} \cdot t_i + o(\hbar).$$

Then, as q goes to 1, the defining relations (2.8)–(2.10), (4.5)–(4.7), (4.9) for $\widehat{\mathcal{H}}_n^-$ reduce to the defining relations for $\widehat{\mathcal{B}}$. The remaining relation (4.8) for $\widehat{\mathcal{H}}_n^-$ reduces to $t_ib_i^2 = b_{i+1}^2t_i + (b_i - b_{i+1})$, which follows from the defining relations for $\widehat{\mathcal{B}}$.

Remark 6.8. The isomorphism in Theorem 5.1 degenerates in the sense of Remark 6.7 to the superalgebra isomorphism between the degenerate affine Hecke–Clifford algebra and $\mathcal{C}_n \otimes \widehat{\mathcal{B}}$ established in [10].

We define the *degenerate cyclotomic spin Hecke algebras* as the quotient algebras $\mathcal{B}^f := \widehat{\mathcal{B}}/\langle f(b_1)\rangle$, where f is an even or an odd polynomial in one variable. The condition on f is precisely such that $\mathcal{B}^f$ inherits a canonical superalgebra structure from $\widehat{\mathcal{B}}$. Using the Morita super-equivalence [10] between $\widehat{\mathcal{B}}$ and Nazarov's degenerate affine Hecke–Clifford algebra, it is straightforward to see that the degenerate cyclotomic spin Hecke algebras correspond bijectively to the degenerate cyclotomic Hecke–Clifford algebras [2,5] (also known as the cyclotomic Sergeev algebras) via a Morita super-equivalence (compare Theorem 6.4).

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