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Asymptotic expansions for a class of tests for a general covariance structure under a local alternative

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ABSTRACT

Let $\boldsymbol{S}$ be a $p \times p$ random matrix having a Wishart distribution $W_p(n, n^{-1}\boldsymbol{\Sigma})$. For testing a general covariance structure $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi})$, we consider a class of test statistics $T_h = n\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}}))$, where $\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = \sum_{i=1}^{p} h(\lambda_i)$ is a distance measure from $\boldsymbol{\Sigma}_1$ to $\boldsymbol{\Sigma}_2$, λ_i's are the eigenvalues of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$, and h is a given function with certain properties. Wakaki, Eguchi and Fujikoshi (1990) suggested this class and gave an asymptotic expansion of the null distribution of T_h. This paper gives an asymptotic expansion of the non-null distribution of T_h under a sequence of alternatives. By using results, we derive the power, and compare the power asymptotically in the class. In particular, we investigate the power of the sphericity tests.

1. Introduction

Let $\boldsymbol{S}$ be a $p \times p$ random matrix having a Wishart distribution $W_p(n, n^{-1}\boldsymbol{\Sigma})$. It is assumed that $n \geq p$. We consider the problem of testing

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi}) \quad \text{against} \quad H_1 : \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} \in \boldsymbol{\Xi}$. Here, $\boldsymbol{\Xi}$ is an open subset of $\mathbb{R}^q$. We assume the following.

A1. All the elements of $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ are known C^4-class functions on $\boldsymbol{\Xi}$, and the Jacobian matrix of $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ is of full rank.

$\boldsymbol{\Sigma}(\boldsymbol{\Xi})$ is a smooth subsurface in $\mathbb{R}^{p(p+1)/2}$ with coordinates $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^q)'$. The hypothesis H_0 represents various covariance structures as special cases.

We consider a class of test statistics via minimization of the following divergence measure from $\boldsymbol{S}$ to $\boldsymbol{\Sigma}(\boldsymbol{\xi})$. Let h be a C^4-function on $(0, \infty)$ satisfying

A2. $h(1) = 0$, $h_1 = 0$, and $h_2 = 1$,
A3. $h(\lambda) > 0$ for any $\lambda \neq 1$,

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where h_r denotes the rth derivative of h at $\lambda = 1$. For arbitrary two matrices $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ we define a distance measure from $\boldsymbol{\Sigma}_1$ to $\boldsymbol{\Sigma}_2$ by

$$\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = \sum_{i=1}^{p} h(\lambda_i),$$

where λ_i's are the eigenvalues of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$. Note that $\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) \geq 0$ with equality if and only if $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ because of A3. However, in general, ρ_h is not symmetric and does not satisfy the triangle law.

Wakaki et al. [17] suggested a class of test statistics

$$T_h = n \inf_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} \rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\boldsymbol{\xi})) = n\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}})), \tag{1.1}$$

where $\hat{\boldsymbol{\xi}}$ is the minimizing point. For example, using $h(\lambda_i) = -\log\lambda_i + \lambda_i - 1$, ρ_h is the Kullback divergence and the corresponding statistic T_h is just based on the log-likelihood ratio criterion. Another typical example is $h(\lambda_i) = (\lambda_i - 1)^2/2$.

It may be noted that the asymptotic expansions of the null distributions of T_h's in some special cases have been obtained by many authors (e.g., [1,7,11], etc.). Kollo and von Rosen [5], and Magnus and Neudecker [6] are also useful for deriving asymptotic expansion formulas for random matrices. An emphasis in [17] is put on an asymptotic expansion of the null distribution of T_h in a general case. Many authors also gave the asymptotic expansions of the non-null distributions of T_h's in some special cases (e.g., [3,8,12], etc.). This paper gives an asymptotic expansion of the non-null distribution of T_h in a general case under a sequence of alternatives converging to the null hypothesis with the rate of convergence $n^{-1/2}$. Sequences of local alternatives have often been considered in comparisons of tests. One simple question is about the rate of the convergence. We choose $n^{-1/2}$ because the powers of test statistics converge to a constant which is greater than the significant level. An interesting result was given by Sugiura and Nagao [13] who compared a modified likelihood ratio test with the asymptotically UMP invariant test for testing homogeneity of several variances. They investigated the limiting distributions under sequences of alternatives with arbitrary rate of convergence. In Section 2 we give stochastic expansions of $\hat{\boldsymbol{\xi}}$ as well as T_h. In Section 3 we obtain an asymptotic expansion of the non-null distribution of T_h under the local alternatives up to the order $n^{-1/2}$. In Section 4 we derive the power, and compare the power asymptotically in the class. Especially we consider the power of the sphericity tests.

2. Stochastic expansion of T_h

We consider a sequence of alternative hypotheses

$$H_n : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi}_0) + \frac{1}{\sqrt{n}}\boldsymbol{\Sigma}(\boldsymbol{\xi}_0)^{1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}(\boldsymbol{\xi}_0)^{1/2}$$

for $\boldsymbol{\Sigma} \notin \boldsymbol{\Sigma}(\boldsymbol{\Xi})$, where $\boldsymbol{\Delta}$ is a symmetric matrix and $\boldsymbol{\xi}_0 \in \boldsymbol{\Xi}$. For simplicity, let us denote as $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\boldsymbol{\xi}_0)$ and $\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}})$. We shall expand T_h in terms of

$$\boldsymbol{V} = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1/2} \tag{2.1}$$

which is $O_p(1)$.

First we summarize the notations used in this paper. Let

$$\partial_a = \frac{\partial}{\partial\xi^a}, \qquad \boldsymbol{J}_{ab\cdots} = \boldsymbol{\Sigma}_0^{1/2}\left[\partial_a\partial_b\cdots\boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0}\boldsymbol{\Sigma}_0^{1/2},$$
$$\widehat{\boldsymbol{J}}_{ab\cdots} = \boldsymbol{\Sigma}_0^{1/2}\left[\partial_a\partial_b\cdots\boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}\right]_{\boldsymbol{\xi}=\widehat{\boldsymbol{\xi}}}\boldsymbol{\Sigma}_0^{1/2},$$
$$\boldsymbol{V} = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1/2},$$
$$s_a = -\frac{1}{2}\mathrm{tr}(\boldsymbol{J}_a\boldsymbol{V}), \quad (a = 1, \ldots, q),$$

and

$$\boldsymbol{G} = (g_{ab}), \quad g_{ab} = \mathrm{E}[s_a s_b] = \frac{1}{2}\mathrm{tr}(\boldsymbol{J}_a\boldsymbol{J}_b), \ (a, b = 1, \ldots, q).$$

It follows from A1 that $\boldsymbol{G}$ is nonsingular. Let g^{ab} be the (a, b) element of $\boldsymbol{G}^{-1}$. As another version of $\boldsymbol{J}_{ab}$, let

$$\boldsymbol{J}_{[ab]} = \boldsymbol{J}_{ab} - \frac{1}{2}\boldsymbol{J}_c g^{cd}\mathrm{tr}(\boldsymbol{J}_d\boldsymbol{J}_{ab}),$$

with Einstein's summation convention. When an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values. For example, $\boldsymbol{J}_c g^{cd}$

means $\sum_{c=1}^{q} \boldsymbol{J}_c g^{cd}$ since 'c' appears twice as a superscript and a subscript. The summation convention is used throughout this paper.

Considering the Taylor expansion of h around $\lambda_i = 1$, we have

$$\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}) = \mathrm{tr}\left[\frac{1}{2}(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^2 + \frac{1}{3!}h_3(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^3 + \frac{1}{4!}h_4(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^4\right] + o\left(\mathrm{tr}\left\{(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^4\right\}\right) \tag{2.2}$$

when $\boldsymbol{S} \to \boldsymbol{\Sigma}$. Let

$$\boldsymbol{\Lambda} = \sqrt{n}\boldsymbol{\Sigma}_0^{-1/2}(\boldsymbol{S}\widehat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{I}_p)\boldsymbol{\Sigma}_0^{1/2}. \tag{2.3}$$

Then we obtain an expansion of T_h,

$$T_h = \mathrm{tr}\left[\frac{1}{2}\boldsymbol{\Lambda}^2 + \frac{1}{3!\sqrt{n}}h_3\boldsymbol{\Lambda}^3 + \frac{1}{4!n}h_4\boldsymbol{\Lambda}^4\right] + o_p(n^{-1}). \tag{2.4}$$

In order to obtain an explicit expansion of T_h, it is necessary to obtain an expansion of $\boldsymbol{\Lambda}$. It is shown similarly as in [14] that

$$\bar{\xi}^a = \sqrt{n}(\hat{\xi}^a - \xi_0^a)$$

is asymptotically normal and hence $O_p(1)$. The Taylor expansion of $\widehat{\boldsymbol{\Sigma}}^{-1}$ around $\boldsymbol{\xi}_0$ is given by

$$\sqrt{n}\boldsymbol{\Sigma}_0^{1/2}(\hat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{\Sigma}_0^{-1})\boldsymbol{\Sigma}_0^{1/2} = \boldsymbol{J}_b\bar{\xi}^b + \frac{1}{2\sqrt{n}}\boldsymbol{J}_{bc}\bar{\xi}^b\bar{\xi}^c + O_p(n^{-1}). \tag{2.5}$$

Using (2.1),

$$\boldsymbol{S} = \boldsymbol{\Sigma}_0^{1/2}\left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}(\boldsymbol{V} + \boldsymbol{\Delta}) + \frac{1}{n}(\boldsymbol{V}\boldsymbol{\Delta} + \boldsymbol{\Delta}\boldsymbol{V})\right\}\boldsymbol{\Sigma}_0^{1/2} + O_p(n^{-3/2}). \tag{2.6}$$

Then using (2.5) and (2.6), (2.3) is expanded as

$$\boldsymbol{\Lambda} = \boldsymbol{V} + \boldsymbol{\Delta} + \boldsymbol{J}_b\bar{\xi}^b + \frac{1}{\sqrt{n}}\left\{\frac{1}{2}\boldsymbol{J}_{bc}\bar{\xi}^b\bar{\xi}^c + (\boldsymbol{V} + \boldsymbol{\Delta})\boldsymbol{J}_b\bar{\xi}^b + \frac{1}{2}(\boldsymbol{V}\boldsymbol{\Delta} + \boldsymbol{\Delta}\boldsymbol{V})\right\} + O_p(n^{-1}). \tag{2.7}$$

In order to obtain an explicit expansion of $\boldsymbol{\Lambda}$, it is necessary to obtain an expansion of $\bar{\xi}^a$. The estimates $\hat{\xi}^a$, $(a = 1, \ldots, q)$, satisfy the system of equations

$$[\partial_a\rho(\boldsymbol{S}, \boldsymbol{\Sigma})]_{\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}} = 0, \quad (a = 1, \ldots, q).$$

Using (2.2) it can be seen that $\hat{\xi}^a$'s satisfy

$$\mathrm{tr}\left[\boldsymbol{S}[\partial_a\boldsymbol{\Sigma}^{-1}]_{\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}}\left\{\boldsymbol{S}\hat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{I}_p + \frac{1}{2}h_3(\boldsymbol{S}\hat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{I}_p)^2\right\}\right] = O_p(n^{-3/2}),$$

or equivalently

$$\mathrm{tr}\left[\left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}(\boldsymbol{V} + \boldsymbol{\Delta})\right\}\hat{\boldsymbol{J}}_a(\boldsymbol{\Lambda} + \frac{1}{2\sqrt{n}}h_3\boldsymbol{\Lambda}^2)\right] = O_p(n^{-1}). \tag{2.8}$$

Substituting (2.7) and

$$\hat{\boldsymbol{J}}_a = \boldsymbol{J}_a + \frac{1}{\sqrt{n}}\boldsymbol{J}_{ab}\bar{\xi}^b + O_p(n^{-1})$$

into (2.8), it is seen that $\bar{\xi}^a$'s satisfy

$$\mathrm{tr}\left[\boldsymbol{J}_a(\boldsymbol{V} + \boldsymbol{\Delta} + \boldsymbol{J}_b\bar{\xi}^b)\right] + \frac{1}{\sqrt{n}}\mathrm{tr}\Bigg[\widetilde{h}_3\boldsymbol{J}_a(\boldsymbol{V} + \boldsymbol{\Delta} + \boldsymbol{J}_b\bar{\xi}^b)^2 + \boldsymbol{J}_a\left(\frac{1}{2}\boldsymbol{J}_{bc} - \boldsymbol{J}_b\boldsymbol{J}_c\right)\bar{\xi}^b\bar{\xi}^c$$
$$+\boldsymbol{J}_{ab}(\boldsymbol{V} + \boldsymbol{\Delta} + \boldsymbol{J}_c\bar{\xi}^c)\bar{\xi}^b + \boldsymbol{J}_a\boldsymbol{V}\boldsymbol{\Delta}\Bigg] = O_p(n^{-1}), \quad (a = 1, \ldots, q), \tag{2.9}$$

where $\widetilde{h}_3 = 1 + \frac{1}{2}h_3$. The solution of $\bar{\xi}^a$ in (2.9) can be found in an expanded form

$$\bar{\xi}^a = \kappa^a + \frac{1}{\sqrt{n}}\varepsilon^a + O_p(n^{-1}). \tag{2.10}$$

In fact, substituting (2.10) into (2.9), we obtain

$$\kappa^a = e^a + \delta^a, \quad \varepsilon^a = -\frac{1}{2}g^{ab}\mathrm{tr}\left[\boldsymbol{J}_b\boldsymbol{M} + \boldsymbol{J}_{bc}\widetilde{\boldsymbol{W}}(e^c + \delta^c)\right], \tag{2.11}$$

where

$$e^a = g^{ab}s_b, \qquad \delta^a = -\frac{1}{2}g^{ab}\mathrm{tr}(\boldsymbol{J}_b\boldsymbol{\Delta}), \qquad \widetilde{\boldsymbol{W}} = \boldsymbol{W} + \boldsymbol{W}_\delta, \qquad \boldsymbol{W} = \boldsymbol{V} + \boldsymbol{J}_b e^b,$$

$$\boldsymbol{W}_\delta = \boldsymbol{\Delta} + \boldsymbol{J}_b\delta^b, \qquad \boldsymbol{M} = \widetilde{h}_3\widetilde{\boldsymbol{W}}^2 + \left(\frac{1}{2}\boldsymbol{J}_{bc} - \boldsymbol{J}_b\boldsymbol{J}_c\right)(e^b + \delta^b)(e^c + \delta^c) + \boldsymbol{V}\boldsymbol{\Delta}.$$

Hence, from (2.4), (2.7) and (2.11), we obtain an expansion of T_h given by

$$T_h = \frac{1}{2}\mathrm{tr}(\widetilde{\boldsymbol{W}}^2) + \frac{1}{\sqrt{n}}T_1(\boldsymbol{V}) + O_p(n^{-1}), \tag{2.12}$$

where

$$T_1(\boldsymbol{V}) = -\frac{1}{2}g^{ab}\mathrm{tr}\left[\boldsymbol{J}_a\boldsymbol{M} + \boldsymbol{J}_{ab}\widetilde{\boldsymbol{W}}(e^b + \delta^b)\right]\mathrm{tr}(\boldsymbol{J}_b\widetilde{\boldsymbol{W}}) + \mathrm{tr}\left\{\left(\frac{1}{2}\boldsymbol{J}_{bc} - \boldsymbol{J}_b\boldsymbol{J}_c\right)\widetilde{\boldsymbol{W}}\right\}$$
$$\times (e^b + \delta^b)(e^c + \delta^c) + \mathrm{tr}(\boldsymbol{J}_b\widetilde{\boldsymbol{W}}^2)(e^b + \delta^b) + \mathrm{tr}(\boldsymbol{V}\boldsymbol{\Delta}\widetilde{\boldsymbol{W}}) + \frac{1}{6}h_3\mathrm{tr}(\widetilde{\boldsymbol{W}}^3). \tag{2.13}$$

3. Asymptotic expansion of the non-null distribution of T_h under the local alternative

We can write the characteristic function of T_h as

$$\phi(t) = \mathrm{E}[\exp(\mathrm{i}tT_h)] = \mathrm{E}\left[\left\{\mathrm{etr}\left(\frac{1}{2}\theta\widetilde{\boldsymbol{W}}^2\right)\right\}T(\boldsymbol{V})\right] + O(n^{-1}), \tag{3.1}$$

where

$$\theta = \mathrm{i}t, \quad T(\boldsymbol{V}) = 1 + \frac{1}{\sqrt{n}}\theta T_1(\boldsymbol{V}), \tag{3.2}$$

with the expression $T_1(\boldsymbol{V})$ in (2.13). The probability density function (pdf) of $\boldsymbol{V}$ is expressed as (see e.g., [11, p. 160])

$$f(\boldsymbol{V}) = f_0(\boldsymbol{V})Q(\boldsymbol{V}) + O(n^{-3/2}), \tag{3.3}$$

where

$$\begin{aligned} f_0(\boldsymbol{V}) &= a_p\mathrm{etr}\left(-\frac{1}{4}\boldsymbol{V}^2\right), \quad a_p = \pi^{-p(p+1)/4}2^{-p(p+1)/4}, \\ Q(\boldsymbol{V}) &= 1 + \frac{1}{\sqrt{n}}Q_1(\boldsymbol{V}) + \frac{1}{n}Q_2(\boldsymbol{V}), \\ Q_1(\boldsymbol{V}) &= -\frac{1}{2}(p+1)\mathrm{tr}(\boldsymbol{V}) + \frac{1}{6}\mathrm{tr}(\boldsymbol{V}^3), \\ Q_2(\boldsymbol{V}) &= \frac{1}{2}\{Q_1(\boldsymbol{V})\}^2 - \frac{1}{24}p(2p^2+3p-1) + \frac{1}{4}(p+1)\mathrm{tr}(\boldsymbol{V}^2) - \frac{1}{8}\mathrm{tr}(\boldsymbol{V}^4). \end{aligned} \tag{3.4}$$

Therefore, we have

$$\phi(t) = \int a_p\left\{\mathrm{etr}\left(-\frac{1}{4}\boldsymbol{V}^2 + \frac{1}{2}\theta\widetilde{\boldsymbol{W}}^2\right)\right\}Q(\boldsymbol{V})T(\boldsymbol{V})\mathrm{d}\boldsymbol{V} + O(n^{-1}), \tag{3.5}$$

where $\mathrm{d}\boldsymbol{V} = \mathrm{d}v_{11}\mathrm{d}v_{12}\cdots\mathrm{d}v_{p-1,p}d_{p,p}$.

We prepare some lemmas useful for reductions of (3.5). Note that $\boldsymbol{G}^{-1} = (g^{ab})$ exists. Let

$$e^a = -\frac{1}{2}g^{ab}\mathrm{tr}(\boldsymbol{J}_b\boldsymbol{V}), \qquad \boldsymbol{U} = -\boldsymbol{J}_a e^a, \quad \text{and} \quad \boldsymbol{W} = \boldsymbol{V} - \boldsymbol{U}, \tag{3.6}$$

and similarly

$$\delta^a = -\frac{1}{2}g^{ab}\mathrm{tr}(\boldsymbol{J}_b\boldsymbol{\Delta}), \qquad \boldsymbol{U}_\delta = -\boldsymbol{J}_a\delta^a, \quad \text{and} \quad \boldsymbol{W}_\delta = \boldsymbol{\Delta} - \boldsymbol{U}_\delta. \tag{3.7}$$

Further, let

$$\boldsymbol{M} = (\mathrm{vec}^*(\boldsymbol{J}_1), \ldots, \mathrm{vec}^*(\boldsymbol{J}_q)),$$

where for any $p \times p$ symmetric matrix $\boldsymbol{A} = (a_{ij})$,

$$\mathrm{vec}^*(\boldsymbol{A}) = \left(\frac{a_{11}}{\sqrt{2}}, \ldots, \frac{a_{pp}}{\sqrt{2}}, a_{12}, \ldots, a_{p-1,p} \right)'.$$

Note that $\{\mathrm{vec}^*(\boldsymbol{A})\}' \mathrm{vec}^*(\boldsymbol{B}) = \frac{1}{2}\mathrm{tr}(\boldsymbol{AB})$. We obtain the following lemmas.

Lemma 3.1. *Let* $\boldsymbol{P_M} = \boldsymbol{M}(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}'$. *Then,*

$$\begin{aligned}
&\boldsymbol{e} = (e^1, \ldots, e^q)' = -(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}\mathrm{vec}^*(\boldsymbol{V}),\\
&\boldsymbol{\delta} = (\delta^1, \ldots, \delta^q)' = -(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}'\mathrm{vec}^*(\boldsymbol{\Delta}),\\
&\mathrm{vec}^*(\boldsymbol{U}) = \boldsymbol{P_M}\mathrm{vec}^*(\boldsymbol{V}),\\
&\mathrm{vec}^*(\boldsymbol{U}_\delta) = \boldsymbol{P_M}\mathrm{vec}^*(\boldsymbol{\Delta}),\\
&\mathrm{vec}^*(\boldsymbol{W}) = (\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})\mathrm{vec}^*(\boldsymbol{V}),\\
&\mathrm{vec}^*(\boldsymbol{W}_\delta) = (\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})\mathrm{vec}^*(\boldsymbol{\Delta}).
\end{aligned}$$

Lemma 3.2. *Let θ be any complex number whose real part is smaller than $-\frac{1}{2}$. Let $g(\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{W})$ be a function of $\boldsymbol{V}$, $\boldsymbol{U}$, and $\boldsymbol{W}$. Then,*

$$\begin{aligned}
&\int \mathrm{etr}\left\{ -\frac{1}{4}\boldsymbol{V}^2 + \frac{1}{2}\theta(\boldsymbol{W} + \boldsymbol{W}_\delta)^2 \right\} \times g(\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{W})\mathrm{d}\boldsymbol{V}\\
&\quad = (1-2\theta)^{-r/2} \times \exp\left[\theta(1-2\theta)^{-1}\left\{ \frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2) \right\} \right] \times \int \left(-\frac{1}{4}\boldsymbol{V}^2 \right) g(\dot{\boldsymbol{V}}, \boldsymbol{U}, \dot{\boldsymbol{W}})\mathrm{d}\boldsymbol{V},
\end{aligned} \tag{3.8}$$

where $r = p(p+1)/2 - q$, $\dot{\boldsymbol{V}} = \boldsymbol{U} + (1-2\theta)^{-1/2}\boldsymbol{W} + 2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta$,

$$\dot{\boldsymbol{W}} = (1-2\theta)^{-1/2}\boldsymbol{W} + 2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta.$$

Proof. We shall show that (3.8) is obtained by considering the transformation $\boldsymbol{V} \to \breve{\boldsymbol{V}}$, where

$$\breve{\boldsymbol{V}} = \boldsymbol{U} + (1-2\theta)^{1/2}\boldsymbol{W} - 2\theta(1-2\theta)^{-1/2}\boldsymbol{W}_\delta. \tag{3.9}$$

Using Lemma 3.1, we have

$$\mathrm{vec}^*(\breve{\boldsymbol{V}}) = \left\{ \boldsymbol{P_M} + (1-2\theta)^{1/2}(\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M}) \right\} \left\{ \mathrm{vec}^*(\boldsymbol{V}) - 2\theta(1-2\theta)^{-1}\mathrm{vec}^*(\boldsymbol{W}_\delta) \right\}.$$

This implies that the inverse transformation is

$$\mathrm{vec}^*(\boldsymbol{V}) = \left\{ \boldsymbol{P_M} + (1-2\theta)^{-1/2}(\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M}) \right\} \mathrm{vec}^*(\breve{\boldsymbol{V}}) + 2\theta(1-2\theta)^{-1}\mathrm{vec}^*(\boldsymbol{W}_\delta).$$

It is equivalent to

$$\boldsymbol{V} = \breve{\boldsymbol{U}} + (1-2\theta)^{-1/2}\breve{\boldsymbol{W}} + 2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta,$$

where $\breve{\boldsymbol{U}} = \frac{1}{2}\boldsymbol{J}_a g^{ab}\mathrm{tr}(\boldsymbol{J}_b\breve{\boldsymbol{V}})$, and $\breve{\boldsymbol{W}} = \breve{\boldsymbol{V}} - \breve{\boldsymbol{U}}$. Therefore, the Jacobian of the transformation (3.9) is

$$J(\boldsymbol{V} \to \breve{\boldsymbol{V}}) = |\boldsymbol{P_M} + (1-2\theta)^{-1/2}(\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})| = (1-2\theta)^{-r/2},$$

since the characteristic roots of $\boldsymbol{P_M}$ are one or zero and $\mathrm{rank}(\boldsymbol{P_M}) = q$. Further, it holds that $\boldsymbol{U} = \breve{\boldsymbol{U}}$, and $\boldsymbol{W} = (1-2\theta)^{-1/2}\breve{\boldsymbol{W}} + 2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta$, since $\mathrm{vec}^*(\breve{\boldsymbol{U}}) = \boldsymbol{P_M}\mathrm{vec}^*(\breve{\boldsymbol{V}}) = \mathrm{vec}^*(\boldsymbol{U})$, and $\breve{\boldsymbol{W}} = (1-2\theta)^{1/2}\boldsymbol{W} - 2\theta(1-2\theta)^{-1/2}\boldsymbol{W}_\delta$. These complete the proof. □

Lemma 3.3. *Let $\boldsymbol{V}$ be a $p \times p$ symmetric random matrix with pdf $f_0(\boldsymbol{V})$ in (3.3). Let e^a, $\boldsymbol{U}$, and $\boldsymbol{W}$ be the random variables defined by (3.5). Then*

(1) *$\boldsymbol{e} = (e^1, \ldots, e^q)'$ and $\boldsymbol{W}$ are independent,*
(2) *$\boldsymbol{e}$ is distributed as $N_q(\boldsymbol{0}, \boldsymbol{G}^{-1})$,*
(3) *$\mathrm{vec}^*(\boldsymbol{U})$ and $\mathrm{vec}^*(\boldsymbol{W})$ are independently distributed as $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{P_M})$ and $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})$, respectively.*

Proof. The results are easily obtained by using Lemma 3.1 and the fact that $\mathrm{vec}^*(\boldsymbol{V})$ is distributed as $N_{p(p+1)/2}(\mathbf{0}, \boldsymbol{I}_{p(p+1)/2})$. □

Using Lemmas 3.2 and 3.3, we can write the characteristic function (3.5) as

$$\phi(t) = (1-2\theta)^{-r/2}\exp\left[\theta(1-2\theta)^{-1}\left\{\frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2)\right\}\right] \times \mathrm{E}\left[Q(\dot{\boldsymbol{V}})T(\dot{\boldsymbol{V}})\right] + O(n^{-1}), \tag{3.10}$$

where $\dot{\boldsymbol{V}}$ is given by Lemma 3.2.

Here the expectation in (3.10) is taken with respect to the distribution of $\boldsymbol{U}$ (or $\boldsymbol{e}$) and $\boldsymbol{W}$ given in Lemma 3.3. After calculation of these expectations, we obtain

$$\phi(t) = (1-2\theta)^{-r/2}\exp\left[\theta(1-2\theta)^{-1}\left\{\frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2)\right\}\right] \times \left\{1 + \frac{1}{\sqrt{n}}\sum_{j=0}^{3} c_j(1-2\theta)^{-j}\right\} + O(n^{-1}), \tag{3.11}$$

where the coefficients c_j's are given by

$$\begin{aligned}
c_0 &= \frac{1}{2}(g^{ab}+\delta^a\delta^b)\boldsymbol{K}_{ab\delta} - \frac{1}{4}(g^{ab}+\delta^a\delta^b)\boldsymbol{K}_{(ab)\delta} - \frac{1}{2}\boldsymbol{K}_{a\delta^2}\delta^a + \frac{1}{3}\boldsymbol{K}_{\delta^3},\\
c_1 &= \frac{1}{4}(g^{ab}+\delta^a\delta^b)\boldsymbol{K}_{(ab)\delta} + \left(\frac{1}{4}h_3 g^{ab} - \frac{1}{2}\delta^a\delta^b\right)\boldsymbol{K}_{ab\delta} - \frac{1}{2}\widetilde{h}_3(p+1)\boldsymbol{K}_\delta + \frac{1}{2}\boldsymbol{K}_{a\delta^2}\delta^a - \frac{1}{2}\boldsymbol{K}_{\delta^3},\\
c_2 &= \frac{1}{2}\widetilde{h}_3\left\{(p+1)\boldsymbol{K}_\delta - g^{ab}\boldsymbol{K}_{ab\delta}\right\} - \frac{1}{12}h_3\boldsymbol{K}_{\delta^3}, \qquad c_3 = \frac{1}{6}\widetilde{h}_3\boldsymbol{K}_{\delta^3}.
\end{aligned} \tag{3.12}$$

Here we use the following notations:

$$\boldsymbol{K}_{ab\delta} = \mathrm{tr}(\boldsymbol{J}_a\boldsymbol{J}_b\boldsymbol{W}_\delta), \qquad \boldsymbol{K}_{(ab)\delta} = \mathrm{tr}(\boldsymbol{J}_{(ab)}\boldsymbol{W}_\delta), \qquad \boldsymbol{K}_{\delta^k} = \mathrm{tr}(\boldsymbol{W}_\delta^k),$$

and so on. The formulas needed for calculating expectations are given in Appendix A. By inverting the characteristic function term by term, we obtain an expansion of the non-null distribution of T_h under the local alternative as in the following theorem.

Theorem 3.1. *Let T_h be the test statistic given by (1.1) with a function h satisfying* A2 *and* A3. *Suppose that a given covariance structure $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi})$ satisfies* A1. *Then under the local alternative hypothesis H_n, the distribution of T_h can be expanded for large n as*

$$\mathrm{P}(T_h \le x) = G_r(x;\tau) + \frac{1}{\sqrt{n}}\sum_{j=0}^{3} c_j G_{r+2j}(x;\tau) + O(n^{-1}), \tag{3.13}$$

where $r = p(p+1)/2 - q$, $\tau = \mathrm{tr}(\boldsymbol{W}_\delta^2)/2$, $G_k(\cdot;\tau)$ is the noncentral χ^2 distribution function with k degrees of freedom and the noncentrality parameter τ, and the coefficients c_j's are given by (3.12).

4. Applications

4.1. Power comparisons

Wakaki et al. [17] gave an asymptotic expansion of the null distribution of T_h in a general case as

$$\mathrm{P}(T_h \le x|H_0) = G_r(x) + \frac{1}{n}\sum_{j=0}^{3} a_j G_{r+2j}(x) + O(n^{-3/2}), \tag{4.1}$$

where $G_k(\cdot)$ is the χ^2 distribution function with k degrees of freedom, the coefficients a_j's are given by

$$\begin{aligned}
a_0 &= \frac{1}{72}\left\{-3p(p^2+3p-1) - 9g^{abcd}\boldsymbol{K}_{abcd} + g^{abcdef}\boldsymbol{K}_{abc,def}\right\} + \frac{1}{16}g^{ab}g^{cd}\left\{4\boldsymbol{K}_{[ab]cd} - \boldsymbol{K}_{[ab][cd]} + 2\boldsymbol{K}_{[ab][cd]}\right\},\\
a_1 &= -a_0 + \widetilde{h}_3^{\,2}C - (h_4-6)B + \widetilde{h}_3 D,\\
a_2 &= -\widetilde{h}_3^{\,2}(A+C) + (h_4-6)B - \widetilde{h}_3 D,\\
a_3 &= \widetilde{h}_3^{\,2}A,
\end{aligned} \tag{4.2}$$

and the coefficients $A, \ldots, D$ are given by

$$\begin{aligned} A &= \frac{1}{72}\Big\{6p(4p^2+9p+7) - 36q(3p+4) - 9(p^2+2p+3)g^{ab}\boldsymbol{K}_{a,b} \\ &\quad + 6(p+1)g^{abcd}\boldsymbol{K}_{abc,d} + 18g^{abcd}\boldsymbol{K}_{abcd} - g^{abcdef}\boldsymbol{K}_{abc,def}\Big\}, \\ B &= \frac{1}{48}\left\{p(p^2+5p+5) - 4q(2p+3) - 2g^{ab}\boldsymbol{K}_{a,b} + g^{abcd}\boldsymbol{K}_{abcd}\right\}, \\ C &= \frac{1}{12}\left\{p(4p^2+9p+7) - 12q(p+1) - 3g^{ab}g^{cd}\boldsymbol{K}_{abcd} - 2g^{ab}g^{cd}g^{ef}\boldsymbol{K}_{ace,bdf}\right\}, \\ D &= -\frac{1}{6}p(p^2+3p+4) + q(2p+3) + \frac{1}{2}g^{ab}\boldsymbol{K}_{a,b} - \frac{1}{4}(p+1)g^{ab}g^{cd}\boldsymbol{K}_{abc,d} \\ &\quad - \frac{1}{2}g^{abcd}\boldsymbol{K}_{abcd} + \frac{1}{36}g^{abcdef}\boldsymbol{K}_{abc,def} - \frac{1}{4}(p+1)g^{ab}\boldsymbol{K}_{[ab]} + \frac{1}{4}g^{ab}g^{cd}\boldsymbol{K}_{[ab]cd}. \end{aligned} \tag{4.3}$$

Here we use the following notations:

$$g^{abcd} = \sum_{[3]} g^{ab}g^{cd}, \qquad g^{abcdef} = \sum_{[5]} g^{ab}g^{cdef}, \qquad \boldsymbol{K}_{abc\cdots} = \mathrm{tr}(\boldsymbol{J}_a\boldsymbol{J}_b\boldsymbol{J}_c\cdots),$$
$$\boldsymbol{K}_{[ab]cd} = \mathrm{tr}(\boldsymbol{J}_{[ab]}\boldsymbol{J}_c\boldsymbol{J}_d), \qquad \boldsymbol{K}_{abc,def} = \boldsymbol{K}_{abc}\boldsymbol{K}_{def},$$

and so on.

Let t_α be the upper 100α percent point of the null distribution of T_h and χ^2_α be the upper 100α percent point of the χ^2 distribution with r degrees of freedom. By the Cornish–Fisher expansion, we obtain

$$t_\alpha = \chi^2_\alpha - \frac{1}{n}\left\{\frac{1}{g_r(\chi^2_\alpha)}\sum_{j=0}^{3} a_j G_{r+2j}(\chi^2_\alpha)\right\} + O(n^{-3/2}) = \chi^2_\alpha + O(n^{-1}). \tag{4.4}$$

Using (3.13), (4.1) and (4.4), we can calculate the power β_h,

$$\beta_h = \mathrm{P}(T_h > t_\alpha | H_1) = 1 - G_r(\chi^2_\alpha;\tau) - \frac{1}{\sqrt{n}}\sum_{j=0}^{3} c_j G_{r+2j}(\chi^2_\alpha;\tau) + O(n^{-1}). \tag{4.5}$$

We use useful formulas for reductions of (4.5). Noncentral χ^2 distribution function and χ^2 distribution can be expanded as (see e.g., [7])

$$G_r(x;\tau) = \sum_{k=1}^{\infty} P_k G_{r+2k}(x), \quad \text{where } P_k = \frac{e^{-\tau/2}\left(\frac{1}{2}\tau\right)^k}{k!}, \tag{4.6}$$
$$G_{r+2}(x) = -2g_{r+2}(x) + G_r(x), \tag{4.7}$$

respectively, where $g_k(\cdot)$ is the pdf of the χ^2 distribution with k degrees of freedom. Using (4.6) and (4.7), we can obtain

$$\begin{aligned} \sum_{j=0}^{3} c_j G_{r+2j}(\chi^2_\alpha;\tau) &= (c_1+c_2+c_3+c_4)\sum_{k=0}^{\infty} P_k G_{r+2k}(\chi^2_\alpha) - 2(c_1+c_2+c_3) \\ &\quad \times \sum_{k=0}^{\infty} P_k g_{r+2k+2}(\chi^2_\alpha) - 2(c_2+c_3)\sum_{k=0}^{\infty} P_k g_{r+2k+4}(\chi^2_\alpha) - 2c_3\sum_{k=0}^{\infty} P_k g_{r+2k+6}(\chi^2_\alpha), \end{aligned} \tag{4.8}$$

where coefficients c_j's are given by (3.12). After calculating (4.8), we can rewrite (4.5) as

$$\beta_h = \frac{1}{\sqrt{n}}\tilde{h}_3\left[\left\{(p+1)\boldsymbol{K}_\delta - g^{ab}\boldsymbol{K}_{ab\delta}\right\}g_{r+4}(\chi^2_\alpha;\tau) + \frac{1}{3}\boldsymbol{K}_{\delta^3}g_{r+6}(\chi^2_\alpha;\tau)\right] + \beta_{\mathrm{LR}} + O(n^{-1}), \tag{4.9}$$

where $g_k(\cdot;\tau)$ is the pdf of the noncentral χ^2 distribution with k degrees of freedom and the noncentrality parameter τ and β_{LR} is the power of the likelihood ratio statistic. The formula (4.9) gives us some information about the superiority of tests in our class of tests. If we have some prior information about the covariance matrix, we may be able to choose a test by calculating (4.9).

4.2. Linear structures

We consider the structure: $\boldsymbol{\Sigma}$ is a linear combination of matrices,

$$\boldsymbol{\Sigma}(\boldsymbol{\xi}) = \xi^1\boldsymbol{G}_1 + \xi^2\boldsymbol{G}_2 + \cdots + \xi^q\boldsymbol{G}_q,$$

where $\boldsymbol{G}_a$'s are given $p \times p$ symmetric matrices which are linearly independent, satisfying that

$$\boldsymbol{G}_j^2 = \boldsymbol{G}_j, \qquad \boldsymbol{G}_i\boldsymbol{G}_j = O \quad (i \neq j),$$

and ξ^a's are unknown such that $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ is positive definite. We note that this structure includes sphericity structure, intraclass correlation structure, and so on.

We can easily calculate $\boldsymbol{J}_a$ and $\boldsymbol{K}_\delta$, $\boldsymbol{K}_{ab\delta}$ in this case as

$$\boldsymbol{J}_a = -\boldsymbol{G}_a, \qquad \boldsymbol{K}_\delta = 0, \qquad \boldsymbol{K}_{ab\delta} = 0. \tag{4.10}$$

Hence we can write the power (4.9) as

$$\beta_h = \frac{1}{3\sqrt{n}}\widetilde{h_3}\boldsymbol{K}_{\delta^3}g_{r+6}(\chi^2_\alpha;\tau) + \beta_{\mathrm{LR}} + O(n^{-1}).$$

It is equivalent to

$$\sqrt{n}(\beta_h - \beta_{\mathrm{LR}}) \to \frac{1}{3}\widetilde{h_3}\boldsymbol{K}_{\delta^3}g_{r+6}(\chi^2_\alpha;\tau) \quad (n \to \infty). \tag{4.11}$$

This shows that LR statistic has greater power than statistics with negative values of $\widetilde{h_3}$ if $\boldsymbol{K}_{\delta^3} > 0$.

In a special case that $q = 1$, this structure is the sphericity structure. Since we can choose an arbitrary parameterization, we use $\boldsymbol{\Sigma}(\boldsymbol{\xi}) = \left\{\exp(\xi^1)\right\}\boldsymbol{I}_p$, then,

$$\boldsymbol{J}_1 = -\boldsymbol{I}_p, \qquad \boldsymbol{K}_{\delta^3} = \sum_{i=1}^{p}(\nu_i - \overline{\nu})^3, \tag{4.12}$$

where ν_i's are the eigenvalues of $\boldsymbol{\Delta}$ and $\overline{\nu} = \frac{1}{p}\sum_{i=1}^{p}\nu_i$. From (4.11) and (4.12), when $\boldsymbol{K}_{\delta^3} \neq 0$, power comparisons of sphericity test in the class depend on a kind of skewness of $\boldsymbol{\Delta}$'s eigenvalues. When $\boldsymbol{K}_{\delta^3} = 0$, we cannot compare the power asymptotically in the class on order $n^{-1/2}$. So we consider an asymptotic expansion of the non-null distribution of T_h under the local alternatives up to the order n^{-1} taking focus on h_3 and h_4, we have

$$\begin{aligned}\mathrm{P}(T_h \le x) &= G_r(x;\tau) + \frac{1}{\sqrt{n}}\sum_{j=0}^{1}\tilde{c}_jG_{r+2j}(x;\tau) + \frac{1}{n}\left\{\sum_{j=1}^{5}d_jG_{r+2j}(x;\tau)\right.\\ &\qquad \left. + \sum_{j=1}^{4}f_jG_{r+2j}(x;\tau) + bG_r(x;\tau)\right\} + O(n^{-3/2}),\end{aligned} \tag{4.13}$$

where b does not depend on h_3 and h_4, the coefficients $\tilde{c}_j$'s, d_j's, f_j's and the formulas needed for calculating expectations are given in Appendix A.

Using (3.13), (4.13), (4.6) and (4.7) with noting (4.4), we can also calculate the power β_h as

$$\begin{aligned}\beta_h &= \frac{1}{n}\left[\sum_{j=2}^{5}e_jg_{r+2j}(\chi^2_\alpha;\tau) + \left\{(h_3^2 + 4h_3)E + h_3D\right\}\frac{(\chi^2_\alpha)^2}{r(r+2)}g_r(\chi^2_\alpha;\tau)\right]\\ &\quad + \frac{1}{n}\left[\sum_{j=2}^{4}g_jg_{r+2j}(\chi^2_\alpha;\tau) - 2h_4B\frac{(\chi^2_\alpha)^2}{r(r+2)}g_r(\chi^2_\alpha;\tau)\right] + c + O(n^{-3/2}),\end{aligned} \tag{4.14}$$

where c which does not depend on h_3 and h_4, and E are given by

$$c = 1 - G_r(\chi^2_\alpha;\tau) - \frac{1}{n}\left(bG_r(\chi^2_\alpha;\tau)\right), \qquad E = \frac{1}{2}\left(C - \frac{\chi^2_\alpha}{r+4}A\right),$$

and the coefficients e_j's, g_j's are given in Appendix B, the coefficients $A, \ldots, D$ are given by (4.3). The difference of local powers among the class is complex. We can examine the difference numerically for specified values of p, α and $\boldsymbol{\Delta}$. Hayakawa [4], Pillai and Jayachandran [10] also gave the numerical examples about the power of T_h in some special cases.

5. Concluding remarks

We have shown that the difference of the asymptotic power in our class of the test for linear structures which depend on only $\boldsymbol{K}_{\delta^3}$. Other important covariance structures arise when we treat covariance structure analysis (system of equation model) (see e.g., [18]). In this case, we have to consider nonlinear covariance structures. Sometimes the domain Ξ of $\boldsymbol{\Sigma}(\Xi)$ is not an open set. If the minimizing point lies on the boundary, the asymptotic expansion formulas derived in this paper are not applied. The problems of deriving asymptotic expansion formulas in such cases are left for the future.

Yuan and Bentler [18] introduced several models considered in structural equation modeling. One of general models is $N_p(\boldsymbol{\mu}(\boldsymbol{\xi}), \boldsymbol{\Sigma}(\boldsymbol{\xi}))$. In this model both the mean vector and the covariance matrix have a structure with the coordinate parameter $\boldsymbol{\xi}$. The asymptotic distribution of the maximum likelihood estimator of $\boldsymbol{\xi}$ can be found in [6, Sec. 10]. Recently Ogasawara [9] derived an asymptotic expansion of the distribution of the parameter estimator which is given by

$$\hat{\boldsymbol{\xi}} = \operatorname*{argmin}_{\xi} F^*_{\mathrm{NT}},$$

$$F^*_{\mathrm{NT}} = \{\bar{\boldsymbol{x}} - \boldsymbol{\mu}(\boldsymbol{\xi})\}'\boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}\{\bar{\boldsymbol{x}} - \boldsymbol{\mu}(\boldsymbol{\xi})\} - \log|\boldsymbol{S}\boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}| + \mathrm{tr}\{\boldsymbol{S}\boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1} - I_p\}.$$

Our class of test statistics can be generalized as follows.

$$T_{k,h} = n \inf_{\xi\in\Xi} \rho_{k,h}[(\bar{\boldsymbol{x}}, \boldsymbol{S}), (\boldsymbol{\mu}(\boldsymbol{\xi}), \boldsymbol{\Sigma}(\boldsymbol{\xi}))],$$

$$\rho_{k,h}[(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1), (\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)] = \sum_{i=1}^{p}\{k(d_i) + h(\lambda_i)\},$$

where $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_2^{-1/2} = (d_1, \ldots, d_p)$ and λ_i's are the eigenvalues of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$ with smooth functions k and h which satisfy some appropriate conditions.

Other related problems are deriving asymptotic expansions of the distributions of test statistics under non-normality, and under high-dimensional setup.

By using an asymptotic expansion of the joint distribution of sample mean vector and sample covariance matrix from an elliptical population given by Wakaki [15] instead of (3.3), we can derive asymptotic expansions for our test statistics.

Recently asymptotic expansions of distributions of several statistics were derived under a high-dimensional setup: $p/n \to c \in (0, 1)$. However results on asymptotic expansions under alternative hypotheses are very few (e.g. [16]) as far as the authors know. It will be very difficult to derive asymptotic expansion formulas for our test statistics under high-dimensional setup. Recent developments on high-dimensional approximations can be found in [2].

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Appendix A. Formulas of expectations

Let $\boldsymbol{V}$ be a $p \times p$ symmetric random matrix normal with pdf $f_0(\boldsymbol{V})$ in (3.3). Let $\boldsymbol{e} = (e^1, \ldots, e^q)'$ and $\boldsymbol{W}$ be the random vector and matrix defined by (3.6). Then, it holds that any $p \times p$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\mathrm{E}[e^a e^b] = g^{ab}, \qquad \mathrm{E}[e^a e^b e^c e^d] = g^{abcd}, \qquad \mathrm{E}[e^a e^b e^c e^d e^e e^f] = g^{abcdef},$$

$$\mathrm{E}[\mathrm{tr}(\boldsymbol{AW})\mathrm{tr}(\boldsymbol{BW})] = 2\mathrm{tr}(\boldsymbol{AB}) - g^{ab}\mathrm{tr}(\boldsymbol{AJ}_a)\mathrm{tr}(\boldsymbol{BJ}_b),$$

$$\mathrm{E}[\mathrm{tr}(\boldsymbol{AWBW})] = \mathrm{tr}\boldsymbol{A}\mathrm{tr}\boldsymbol{B}' + \mathrm{tr}(\boldsymbol{AB}') - g^{ab}\mathrm{tr}(\boldsymbol{AJ}_a\boldsymbol{BJ}_b),$$

$$\begin{aligned}\mathrm{E}[\mathrm{tr}(\boldsymbol{AW}^2)\mathrm{tr}(\boldsymbol{BW}^2)] &= 4\mathrm{tr}(\boldsymbol{A}\bar{\boldsymbol{B}}) + (p^2 + 2p + 1)\mathrm{tr}\boldsymbol{A}\mathrm{tr}\boldsymbol{B} - (p + 1)g^{ab}\{\mathrm{tr}\boldsymbol{A}\mathrm{tr}(\boldsymbol{BJ}_a\boldsymbol{J}_b)\\ &\quad + \mathrm{tr}\boldsymbol{B}\mathrm{tr}(\boldsymbol{AJ}_a\boldsymbol{J}_b)\} - 8g^{ab}\mathrm{tr}(\boldsymbol{AJ}_a\bar{\boldsymbol{B}}\boldsymbol{J}_b) + g^{abcd}\mathrm{tr}(\boldsymbol{AJ}_a\boldsymbol{J}_b)\mathrm{tr}(\boldsymbol{BJ}_c\boldsymbol{J}_d),\end{aligned}$$

$$\mathrm{E}[\mathrm{tr}(\boldsymbol{AW})\mathrm{tr}\boldsymbol{W}^3] = 6(p + 1)\mathrm{tr}\boldsymbol{A} - 6g^{ab}\mathrm{tr}(\bar{\boldsymbol{A}}\boldsymbol{J}_a\boldsymbol{J}_b) - 3(p + 1)g^{ab}\mathrm{tr}(\boldsymbol{AJ}_a)\boldsymbol{K}_b + g^{abcd}\mathrm{tr}(\boldsymbol{AJ}_a\boldsymbol{K}_{bcd}),$$

$$\mathrm{E}[\boldsymbol{W}^4] = p(2p^2 + 5p + 5) - 4q(2p + 3) - 2g^{ab}\boldsymbol{K}_{a,b} + g^{abcd}\boldsymbol{K}_{abcd},$$

$$\begin{aligned}\mathrm{E}[(\boldsymbol{W}^3)^2] &= 6p(4p^2 + 9p + 7) - 24q(2p + 3) - 12g^{ab}(2p + 3)\boldsymbol{K}_{ab} - 3g^{ab}(3p^2 + 6p + 7)\boldsymbol{K}_{a,b}\\ &\quad + 6(p + 1)g^{abcd}\boldsymbol{K}_{a,bcd} + 18g^{abcd}\boldsymbol{K}_{abcd} - g^{abcdef}\boldsymbol{K}_{abc,def},\end{aligned}$$

where $\bar{\boldsymbol{A}} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}')$. The expectations are obtained by using Lemma 3.3 and the fact that $\mathrm{vec}^*(\boldsymbol{V})$ is distributed as $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{I}_{p(p+1)/2})$. The calculations can be simplified by using the properties such as

$$\mathrm{E}[\mathrm{tr}\boldsymbol{W}^2\mathrm{tr}\boldsymbol{W}^2] = \mathrm{E}[\mathrm{tr}(\boldsymbol{W}^2\ddot{\boldsymbol{W}}^2) + 2\mathrm{tr}(\boldsymbol{W}\ddot{\boldsymbol{W}})\mathrm{tr}(\boldsymbol{W}\ddot{\boldsymbol{W}})],$$

where $\ddot{\boldsymbol{W}}$ is a symmetric random matrix having the same distribution $\boldsymbol{W}$ and being independent of $\boldsymbol{W}$.

Appendix B. Coefficients

The coefficients b and $\tilde{c}_j$'s, d_j's, f_j's are given by

$$\tilde{c}_0 = \frac{1}{2}\overline{\nu}\boldsymbol{K}_{\delta^2}, \qquad \tilde{c}_1 = -\tilde{c}_0,$$

$$f_1 = -\frac{1}{2}g_2, \qquad f_2 = \frac{1}{2}(g_2 - g_3), \qquad f_3 = \frac{1}{2}(g_3 - g_4), \qquad f_4 = \frac{1}{2}g_4,$$

$$d_1 = -\frac{1}{2}e_2, \qquad d_2 = \frac{1}{2}(e_2 - e_3), \qquad d_3 = \frac{1}{2}(e_3 - e_4), \qquad d_4 = \frac{1}{2}(e_4 - e_5), \qquad d_5 = \frac{1}{2}e_5,$$

$$b = \mathrm{E}\left[Q_2(\boldsymbol{V}_1) + \theta Q_1(\boldsymbol{V}_1)\mathrm{tr}(\boldsymbol{W}_1\boldsymbol{K}) + \frac{1}{2}\theta^2 \{\mathrm{tr}(\boldsymbol{W}_1\boldsymbol{M}_1)\}^2 + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_1^2)\right],$$

where $Q_1(\cdot)$ and $Q_2(\cdot)$ are given by (3.4), and the coefficients e_j's, g_j's, $\boldsymbol{V}_1$, $\boldsymbol{W}_1$, $\boldsymbol{M}_1$ are given by

$$e_2 = -\frac{1}{24}h_3^2(4p^3 + 9p^2 - 13p - 12 + 4p^{-1}) + h_3 p\boldsymbol{K}_{\delta^2},$$

$$e_3 = -\frac{1}{4}h_3^2(p + 2 - 2p^{-1})\boldsymbol{K}_{\delta^2} + \frac{1}{2}h_3 p^{-1}\boldsymbol{K}_{\delta^2}\boldsymbol{K}_{\delta^2} - \frac{1}{2}h_3\boldsymbol{K}_{\delta^4} + \frac{1}{144}(h_3^2 + 4h_3)(6p^3 + 18p^2 - 24p - 72 + 96p^{-1}),$$

$$e_4 = \frac{1}{4}(h_3^2 + 4h_3)\boldsymbol{K}_{\delta^2} - \frac{1}{8}h_3^2\boldsymbol{K}_{\delta^4},$$

$$e_5 = \frac{1}{8}(h_3^2 + 4h_3)(\boldsymbol{K}_{\delta^4} - p^{-1}\boldsymbol{K}_{\delta^2}\boldsymbol{K}_{\delta^2}),$$

$$g_2 = \frac{1}{24}h_4(2p^3 + 5p^2 - 7p - 12 + 12p^{-1}),$$

$$g_3 = \frac{1}{12}h_4(2p + 3 - 6p^{-1})\boldsymbol{K}_{\delta^2}, \qquad g_4 = \frac{1}{24}h_4\boldsymbol{K}_{\delta^4}$$

$$\boldsymbol{V}_1 = \boldsymbol{U} + (1 - 2\theta)^{-\frac{1}{2}}\boldsymbol{W} + 2\theta(1 - 2\theta)^{-1}\boldsymbol{W}_\delta,$$

$$\boldsymbol{W}_1 = (1 - 2\theta)^{-\frac{1}{2}}\boldsymbol{W} + (1 - 2\theta)^{-1}\boldsymbol{W}_\delta,$$

$$\begin{aligned}\boldsymbol{M}_1 \; = \; & -p^{-1}\left\{\mathrm{tr}(\boldsymbol{W}_1^2)\right\}\boldsymbol{I}_p - p^{-1}\{\mathrm{tr}(\boldsymbol{V}_1\boldsymbol{\Delta})\}(e^1 + \delta^1)\boldsymbol{I}_p \\ & - p^{-1}\{\mathrm{tr}(\boldsymbol{W}_1)\}(e^1 + \delta^1)\boldsymbol{I}_p + (e^1 + \delta^1)^2\boldsymbol{I}_p - (e^1 + \delta^1)(\boldsymbol{V}_1 + \boldsymbol{\Delta}) + \frac{1}{2}(\boldsymbol{V}_1\boldsymbol{\Delta} + \boldsymbol{\Delta}\boldsymbol{V}_1).\end{aligned}$$

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