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Arithmêtikê stoicheiôsis: On Diophantus and Hero of Alexandria

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Two ancient mathematical works, cited in ancient sources as the "Preliminaries to the *Arithmetic Elements*" and the "Preliminaries to the *Geometric Elements*"—of which the former is no longer extant, while the latter is an alternative designation of the *Definitions*, now commonly attributed to Hero of Alexandria—are here argued to be companion works by the same author, namely Diophantus of Alexandria. This attribution has implications for the dating of Diophantus. © 1993 Academic Press, Inc.

Deux oeuvres mathématiques de l'antiquité, citées l'une comme les "Préliminaires aux *Éléments arithmétiques*," l'autre comme les "Préliminaires aux *Éléments géométriques*"—desquelles la première ne survit plus, mais la seconde désigne par ailleurs les *Définitions*, attribuées habituellement à Héron d'Alexandrie—sont assignées ici à Diophante d'Alexandrie. Cette attribution a des implications pour la question de la date de Diophante. © 1993 Academic Press, Inc.

Zwei mathematische Schriften, die im Altertum als die "Präliminarien zu den *Arithmetischen Elementen*" und die "Präliminarien zu den *Geometrischen Elementen*" zitiert werden—von denen die erste nicht mehr existiert, aber die zweite identisch mit den normalerweise dem Heron von Alexandria zugeschriebenen *Definitionen* ist—werden hier dem Diophant von Alexandria zugeschrieben. Diese Zuschreibung hat Folgerungen für die Frage der Datierung von Diophant. © 1993 Academic Press, Inc.

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A work called the *Definitions of the Terms of Geometry* (*Horoi tôn geômetrias onomatôn*), a form of introductory commentary on Euclid's *Elements*, is included in a few of the extant collections of works by Hero of Alexandria[1]. Although the attribution to Hero was questioned by Hultsch [1864] [2], it was accepted without misgiving by Heiberg [1912, iv: "Hoc opusculum, quod Heroni tribuere non dubito ..."], and that appears to have become the prevailing view. For instance, Heath [1921, II, 316] observes that the work "seems, at least in substance, to go back to Heron or earlier still," and Drachmann [1972, 311] remarks that of Hero's mathematical books "only the *Definitiones* and the *Metrica* are direct from his hand."

As I intend to argue now, however, the earlier reservations over the attribution to Hero were well founded, for the author of the *Definitions* can be identified as Diophantus.

Consider first the following remark on Diophantus that appears among the scholia to Iamblichus' commentary on Nicomachus' *Introduction to Arithmetic*:

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> The properties of the harmonic mean we shall learn more completely in the last theorem of the first book of Diophantus' *Arithmetic Elements* (*Arithmêtikê stoicheiôsis*), and the diligent ought to read these things there. [Pistelli 1894, 132; Tannery 1893–1895 II, 72]

This passage has come up for recent comment. J. Christianidis [1991, 242–244] has rightly noted that the reference cannot be to Diophantus' *Arithmetica* as such, since at the cited place (namely, Book I, Proposition 39) nothing is explicitly said about the harmonic mean. But it is observed by W. C. Waterhouse [1993] that the harmonic mean is indeed *implicitly* related to this proposition, in view of the fact that the numbers solving Diophantus' problem form a harmonic triple [3]. This ensures the connection with Diophantus' *Arithmetica,* at precisely the place indicated by the scholiast. It seems, then, that the earlier position of Tannery [1893–1895 II, 72n.] is sustained: that the scholiast encountered a discussion of the means among scholia to the *Arithmetica*. Christianidis' objection to this position [1991, 242–243] depends on the assumption that in a citation like this the scholiast would necessarily distinguish between the text and a marginal commentary of the sort proposed by Tannery. In view of the fluid relation between text and marginalia in ancient manuscripts, however, the scholiast's failure to make such a distinction is readily understandable [4].

The prior accounts of this scholium have omitted considering two related passages in the Heronian *Definitions*. The first (Sect. 122) initiates the discussion of the basic notions in Euclid's theory of proportion (*Elements*, Book V):

> What then 'part' is and 'ratio,' and what 'homogeneous' and what 'proportion,' these have been said more precisely in the Preliminaries of the *Arithmêtikê stoicheiôsis*. But now we say that as proportion obtains for the other homogeneous ⟨kinds⟩, so also does it obtain for the homogeneous in magnitudes. [Heiberg 1912, 76.21–78.2]

We encounter here a reference to an *Arithmêtikê stoicheiôsis*, or rather to the "Preliminaries" (*ta pro* . . .) to a work of that title, which we can infer, by analogy with the Heronian *Definitions,* surveyed the basic terms and conceptions relative to a treatise on arithmetic theory [5]. In a second passage from the *Definitions* (Chap. 128), relating to Euclid's theory of irrational lines (*Elements,* Book X), the same "Preliminaries" are cited in identical terms:

> What ⟨numbers⟩ are irrational and incommensurable, and what ones are rational and commensurable [6]—this has been said in the Preliminaries of the *Arithmêtikê stoicheiôsis*. But now, following Euclid, the "Elementator" (*Stoicheiôtês*, sc. author of the *Elements*), in the case of magnitudes, we say that commensurable magnitudes are defined as those measured by the same measures, while incommensurable ones are those for which there can be no common measure. [Heiberg 1912, 84.17–22]

What are we to identify as the *Arithmêtikê stoicheiôsis* cited by the Heronian writer? In an earlier examination of this passage [Knorr 1975, 235], I noted that it could not be the arithmetic books of Euclid's *Elements,* since the writer here explicitly distinguishes the cited work from Euclid, while, further, it could not be a "work like Diophantus's *Arithmetica*" either, since the latter provides no account of irrationals of the sort that the writer supposes. I thus suggested that this

Arithmêtikê stoicheiôsis was in the tradition of ancient arithmetic studies parallel and even prior to Euclid's geometric theory of irrationals.

But it is the "Preliminaries," not the *Arithmêtikê stoicheiôsis* itself, that the Heronian writer cites for the alternative account of irrationals. Further, although Diophantus does not employ the terms for "commensurables" and "incommensurables," nor even for "irrationals" (*alogoi*), he does often consider whether the solving *arithmos* is "rational" (*rhêtos*) or "not rational" (*ou rhêtos*) [7]. In all of these places, moreover, the "rational" is conceived as synonymous with the "commensurable," whereas in Euclid these classes are different [8]. Thus, a preliminary survey of Diophantus' *Arithmetica* might well include an account of irrationals like that cited by the Heronian writer.

Similarly, in connection with the earlier passage of the *Definitions,* although Diophantus' *Arithmetica* employs "proportion" (*analogia*) only in the restricted case of "geometric proportion" (V 1–2), and no instances of "homogeneous" (*homogenê*) are cited in Tannery's index, the terms "part" (*meros*) and "ratio" (*logos*) are frequent (cf. Tannery [1893–1895 II, 274–275]).

The Heronian *Definitions* has much the same relation to its parent treatise, the Euclidean *Elements.* For just as some terminology is assigned to the arithmetic "Preliminaries" that does not actually appear in Diophantus' *Arithmetica,* so also in the case of its geometric terminology, the *Definitions* goes beyond what is found in the *Elements,* even though it is intended to be an introduction to that treatise. Indeed, this is a deliberate strategy on the part of the author of the *Definitions,* as he explains in his preface (cited below).

These passages thus seem to fall into the same situation as the Iamblichus scholium: a work called the *Arithmêtikê stoicheiôsis* (or, in the Heronian case, its "Preliminaries," *ta pro tês arithmêtikês stoicheiôseôs*) can be related to materials that are pertinent to Diophantus' *Arithmetica,* but not there encountered explicitly. That is, the Iamblichus scholiast appears to have conflated the *Arithmêtikê stoicheiôsis* with the "Preliminaries," the latter being a commentary (presumably in the margins of Diophantus' *Arithmetica*), as supposed by Tannery.

But the connection between the *Definitions* and the *Arithmetica* is even closer, as we can infer from the brief passage that prefaces the *Definitions:*

> Also (*kai*) the systematized Preliminaries of the *Geomêtrikê stoicheiôsis* (*ta men pro tês geômetrikês stoicheiôseôs technologoumena*), by writing them below (*hypographôn*) for you and sketching them out (*hypotypoumenos*), in the most succinct manner, most illustrious (*lamprôtatos*) Dionysius, I shall make both the foundation and the whole arrangement in accordance with the teaching in the theory of geometry of Euclid, the "Elementator." For I think that in this wise (*houtôs*) not only will the subjects (*pragmateiai*) of that man be well surveyable (*eusynoptoi*) [9] by you, but also most of the others relating to geometry. [Heiberg 1912, 14.1–8]

By implication, the writer adopts for his own present work the description, *ta pro tês geômetrikês stoicheiôseôs,* precisely parallel to the title by which he later refers to the arithmetic compendium, *ta pro tês arithmêtikês stoicheiôseôs,* in the two passages just cited. It would appear, then, that the arithmetic "Preliminaries" were also his work [10]. This inference is supported by the remarkable opening of the preface: "Also" (*kai*); for it indicates that the present writing does not stand alone, but is the sequel to some comparable introductory work [11]. In context, we would take this to be the arithmetic compendium.

No extant work corresponds to the description of the arithmetic "Preliminaries." But comparing the Iamblichus scholium and the two passages in the *Definitions*, we have found that the *Arithmêtikê stoicheiôsis,* whose introduction it was intended to provide, is the *Arithmetica* of Diophantus. Consider, then, the opening of the preface to the *Arithmetica*:

> The solution of the problems in numbers, my most highly esteemed (*timiôtatos*) Dionysius, by recognizing that you are zealous to learn this, I have tried ⟨to organize the method⟩ [12], by starting from the fundamentals of which the subject (*ta pragmata*) consists, to lay down (*hypostênai*) the nature and power in numbers. But perhaps the matter seems quite difficult, since it is not yet familiar, for the minds of beginners despair of getting it straight, yet it will become well comprehensible (*eukatalêpton*) to you, through your eagerness and my exposition (*apodeixis*). For desire when it adds on teaching ⟨advances⟩ swiftly to learning. [Tannery, 1893–1895 I, 2.3–13] [13]

Diophantus proceeds next to the definitions of the numerical powers and their reciprocals, then describes the basic technique for problem solving. He closes with a general didactic remark:

> ⟨The propositions⟩ being very many in number and very great in bulk, and for this reason being secured only slowly by those taking them up and being hard to commit to memory, I have found it best to divide the things admitted in them, and most of all the things in the beginning that are elementary, to proceed [14], as is fit, from the simpler to the more complicated. For in this wise (*houtôs*) they will become well traversable (*euodeuta*) by beginners, and their teaching will be easily remembered. [Tannery 1893–1895 I, 14.27–16.6] [15]

Most strikingly, the prefaces to the *Arithmetica* and the *Definitions* are addressed to individuals with the same name, Dionysius. From context, one infers in both cases that the addressee, being of high social standing (indeed, *lamprôtatos*, corresponding to Latin *clarissimus,* regularly connotes nobility or high office), is a patron of mathematical studies, in which, however, he seems to be a relative beginner. Thus, even if the name is quite common [16], the coincidences of name and character would recommend viewing the two dedicatees to be the same person, if that were found to be consistent with the chronological circumstances of the associated writings.

Tannery [1896; 1912, 535–538] made a proposal to this effect, suggesting the dedicatee to be the eminent Dionysius (later saint), bishop of Alexandria (A.D. 247–265), formerly director of the Alexandrian Christian school (A.D. 231–247), to whom Diophantus can be linked via the person of Anatolius, bishop of Laodicea (ca. mid 3rd century A.D.), a writer and teacher of mathematics, and disciple of the same Dionysius (see also below). Tannery's conflation of the two Dionysii was viewed skeptically by Hultsch [1905, 993]. But Heath [1921 II, 306n.] resurrected the idea, although with reference to another Dionysius (a Roman prefect from the late third century A.D.), in accordance with a suggestion by I. Hammer-

Jensen on the identity of the dedicatee of the Heronian *Definitions*. To similar effect, Klein [1968, 247] reports a suggestion by A. Stein identifying this Dionysius as another prefect in Egypt late in the second century.

If one assumes Hero's authorship of the *Definitions*, as is generally the case, conflating the dedicatees would place Hero and Diophantus at around the same time. When Heath and the other earlier critics wrote, that seemed possible: while Diophantus was generally supposed to date from the third century (on the basis of an argument by Tannery), Hero's date was restricted only within a very wide span, sometime between Ctesibius (ca. 150 B.C.) and Pappus (ca. A.D. 320) [17]. Subsequently, however, Neugebauer argued that Hero's date must be set near the middle of the first century A.D., since the lunar eclipse that Hero employs in a computation in the *Dioptra* can be dated to A.D. 62 [1938, 21–24; 1975, 846; cf. Drachmann 1972, 310]. Neugebauer's dating for Hero, which is now the standard view, thus becomes incompatible with the conventional assignment of Diophantus to the mid third century.

The date of Diophantus, however, is not as secure as is usually supposed. It derives from another argument by Tannery, based on a remark in a letter on arithmetic by the Byzantine scholar Michael Psellus (11th century). Heath [1910, 2] renders the passage thus:

> Diophantus dealt with it [sc. the so-called Egyptian mode of arithmetic] [18] more accurately, but the very learned Anatolius collected the most essential parts of the doctrine as stated by Diophantus in a different way (*heterôs*) and in the most succinct form, dedicating his work to Diophantus. [19]

Assuming Anatolius to be the Laodicean bishop, ca. 280, as above, we must place Diophantus only slightly earlier, say ca. 250. But one immediately suspects something amiss: it seems peculiar that someone would compile an abridgment of another man's work and then dedicate it to him, while the qualification "in a different way," in itself vacuous, ought to be redundant, in view of the terms "most essential" (*synektikôtata*) and "most succinct" (*synoptikôtata*). Tannery's whole interpretation, it turns out, hinges on the word *heterôs*, which is his alteration of the manuscript's reading *heterô*. In context, the term should be read as the dative *heterôi* [20], whence the sense is this:

> . . . but the very learned Anatolius, having collected the most essential parts of that man's (*kat' ekeinon*) doctrine, to a different Diophantus (*heterôi Diophantôi*) most succinctly addressed it.

My rendering, if awkward, intends to capture a nuance of the Greek word order: that by juxtaposing the personal references, Psellus is emphasizing the distinction between Diophantus (*ekeinon*) and the dedicatee, his namesake, "a different Diophantus" (*heterôi Diophantôi*). The situation is one that Psellus could hardly have avoided commenting on, had he known of it, since the coincidence of names would be not only inherently confusing, but also tantalizingly apt [21].

Thus, the third century date derived for Diophantus from Psellus' testimony becomes only a *terminus ante quem*. Accordingly, it might be possible for Diophantus to be a contemporary of Hero, as early as the first century A.D. [22]. This notion has certain attractive features [23], and I do not know of explicit evidence that would absolutely rule it out. But the connections between the *Arithmetica* and the *Definitions* are not conclusive in its support. Even if the name "Dionysius" denotes the same individual in the two prefaces, it need not follow that he received these dedications from two different writers; for, alternatively, one and the same author might have written both: that is, the *Definitions*, whose attribution to Hero has long been questioned, might be due to the author of the *Arithmetica*.

The latter position, which is the one I advocate, receives support from agreements in the content and style of the prefaces. The principal object in both prefaces is framed in terms of pedagogy, "teaching" (*didaskalia* in the *Definitions*, *didachê* and *agôgê* in the *Arithmetica*). To this effect, both prefaces outline their strategies: in *Def.*, to follow the order of Euclid's "subjects" (*pragmateiai*) both in foundation (*archê*) and arrangement (*syntaxis*); in *Arith.*, to consider the subject (*ta pragmata*) in terms of its fundamentals (*themelia*) or elements (*ta en archêi echonta stoicheiôdôs*), progressing from simpler to more complex. Both see their approach as most convenient for the learner, and both employ verbal adjectives prefixed by *eu-* to express this notion (*eusynoptoi* in *Def.*, *eukatalêpton* and *euodeuta* in *Arith.*). Indeed, the closing thoughts in both are exactly parallel:

> *Def.*—in this wise (*houtôs*) the subject matter will be well surveyable (*eusynoptoi*) for you,
>
> *Arith.*—in this wise (*houtôs*) the elements will become well traversable (*euodeuta*) for beginners.

Further, the opening sentences have the same syntax: direct object + participial construction with the second person (*soi* in *Def.*, *se* in *Arith.*) + verb in the first person. Moreover, in both cases the style manifests a predilection for parallel expressions, sometimes euphonious, at other times redundant:

> *Def.* ... writing down for you (*hypographôn soi*) and sketching out (*hypotypoumenos*); ... the foundation and the entire arrangement; ... not only the subjects ... but also most others ...
>
> *Arith.*: ... both the nature and power in numbers; ... your eagerness (*tên sên prothymian*) and my proof (*tên emên apodeixin*); ... propositions many in number and great in bulk, ... secured slowly ... and hard to remember; ... they will become well traversable ... and their teaching will be committed to memory.

While the *Definitions* is a derivative work, in the manner of a commentary, it is hard to understand why its author (if assumed to be Hero) would imitate the writing style of Diophantus, particularly since the *Definitions* is tributary to the *Elements*, not to the *Arithmetica*.

One might think, however, that since these resemblances seem relatively pedestrian, they might be attributable to the genre of technical prefaces. But this is not the case. Many prefaces are extant from mathematical writings, and they display a diverse range of content and styles. Those from Archimedes (e.g., *Quadrature of the Parabola*, *Sphere and Cylinder*, etc.), Diocles (*Burning Mirrors*), Apollonius (*Conics*), Hypsicles (the so-called Book XIV of the *Elements*), and Pappus (*Collection*, Books III, V, VII and VIII), which all are addressed to named individuals, attempt to motivate the study of advanced materials through accounts of the history of the problems they deal with. Alternatively, Ptolemy speaks of philosophical matters to his addressee, Syrus, in the *Syntaxis* preface, while to the same addressee he remarks on the interconnections among the several treatises in his other prefaces (e.g., the *Apotelesmatica, Hypotheses planetarum,* and minor writings). Even works specifically oriented toward a teaching environment vary in their range: Theon of Alexandria describes to his "son" Epiphanius how his commentary on Ptolemy's Book I answered a request from his students; Eutocius of Ascalon presents his commentary on Archimedes for approval to his teacher, the philosopher Ammonius. Perhaps closer in manner to the prefaces of the *Definitions* and *Arithmetica,* though rather more intimate in tone, are some of Eutocius' remarks to his friend Anthemius, dedicatee of the commentaries on Apollonius (cf., in particular, the preface to *Conics,* Book IV).

But the most appropriate comparison ought to be with the works of Hero. Several of the extant Heronian writings have prefaces (e.g., *Pneumatica, Automata, Metrica, Dioptra*), and these follow a basically consistent format. Indeed, W. Schmidt [1900, 304–306] was able to argue through such agreements that a certain catoptrical tract, surviving only in its medieval Latin translation as *Ptolomei de speculis,* should actually be assigned to Hero. In general, Hero's prefaces describe the nature of the subject *(pragmateia)*, in terms of its history, its proper subdivisions, its characteristic theories, or its implementation for practical benefit. None deals with the order of exposition or the pedagogical strategy of the writing at hand, and none is directed to a specific individual, but rather to a collective readership. In all of these respects, the preface to the *Definitions* differs from the pattern set in the rest of the Heronian corpus.

To summarize: on grounds of style and genre, as well as the shared dedication to "Dionysius," the *Definitions* is nicely linked to Diophantus' *Arithmetica,* while on the very same grounds it is isolated among the body of Hero's writing. This would argue that they have the same author, Diophantus. But just as the *Definitions* is described by its author as the "Preliminaries to the *Geômetrikê stoicheiôsis*" (that is, to the *Elements* of Euclid), so it is the sequel to a work (now lost) by the same author, his "Preliminaries to the *Arithmêtikê stoicheiôsis*" (the *Arithmetica* of Diophantus). It follows that Diophantus wrote the arithmetic "Preliminaries" too, which is likely also to be the work cited by the scholiast to Iamblichus.

This view, if accepted, entails an odd scenario. By analogy with the *Definitions,* we infer that the lost "Preliminaries of the *Arithmêtikê stoicheiôsis*" provided a learner's outline for the already published *Arithmêtikê stoicheiôsis*, that is, the *Arithmetica* of Diophantus, yet both works have been dedicated to the same man, Dionysius. But surely anyone who had grappled with the systematic and demanding *Arithmetica* would find little need for the "Preliminaries." Conceivably, Dionysius found the study of the *Arithmetica* so overwhelming that he requested the introduction for assistance. But one may observe a nuance in the prefaces: in the *Arithmetica* Diophantus does indeed presuppose that the effort will tax Dionysius' powers to the limit (e.g., "but your eagerness and my exposition will make it well comprehensible to you''). On the other hand, the preface to the geometric ''Preliminaries,'' that is, the *Definitions* (which, presumably, reflects the situation of the earlier arithmetic ''Preliminaries''), says only that the Euclidean subject matter will hereby become ''well surveyable for you,'' without explicitly implying that Dionysius requires this aid for himself. It may be, then, that Dionysius has requested the companion tracts not for his own understanding, but for beginners in his charge.

This view of the relation between Diophantus and Dionysius, as master and junior colleague, fits neatly into the scheme envisaged by Tannery, who takes Dionysius to be the leader of the Alexandrian Christian school during the second quarter of the third century. At the same time, it renders all the more appropriate the link with Anatolius, whose expertise on arithmetical subjects and commentatorial efforts on Diophantus' arithmetic theory would follow from his being Dionysius' disciple. But unlike Tannery, who makes Diophantus an older contemporary of Anatolius, the present account places him a generation earlier, as an older contemporary of Dionysius. This would set Diophantus' maturity to the period around Dionysius' leadership of the school, ca. 240, and thus make Diophantus' life approximately coterminous with the first half of the third century. His involvement with Dionysius' school would also enhance the plausibility of the suggestion by Tannery [1896; 1912, 536], that Diophantus was a Christian.

Unfortunately, the identification of Dionysius must remain conjectural. Tannery's argument, based on his alteration of Psellus' passage, is undermined when its ''Diophantus'' becomes ''a different Diophantus.'' There remains only the slenderer thread, inferring from Anatolius' contributions to the study of Diophantus' work the identification of Anatolius' master Dionysius with the dedicatee of the *Arithmetica*. The more tenuous we hold that connection to be, the more seriously we must entertain the possibility that Diophantus lived earlier than the third century, possibly even earlier than Hero in the first century. Thus, it will require the identification of other chronologically significant connections between the Diophantine corpus and related works to make a firmer determination of his date.

Since Hero was known for his commentary on Euclid's *Elements*, as we learn from remarks in Pappus and Proclus and extensive selections transmitted in the Euclid commentary of al-Nairîzî, it is understandable that a survey of the *Elements* like the *Definitions* would come to be included in some Heronian collections. Further, it seems that the name of Diophantus was sometimes associated with the Heronian metrical literature, as the materials Tannery orders under the heading ''Diophantus pseudepigraphus'' reveal [1893–1895 II, 15–31]. This would not justify surmising a real contribution by Diophantus to the metrical literature, since his name was well known among the Byzantine scholars, as Tannery observes [1893–1895 II, v–vi]. But it signifies a condition of fluidity that affects the transmission of the Heronian and Diophantine writings in the later period.

As for Diophantus' two commentaries called the ''Preliminaries . . .,'' although they are ostensibly directed to a distinguished colleague, Dionysius, their real target is the audience of mathematical students. In compiling the "Geometric Preliminaries" (sc. the *Definitions*), Diophantus reveals his involvement in the teaching of Euclidean geometry. Remarkably, the conspicuous parallelism of the titles *Arithmêtikê stoicheiôsis* and *Geômetrikê stoicheiôsis*, by which he refers, respectively, to his own *Arithmetica* and to the *Elements* of Euclid, indicates his arrangement of the two great treatises, as counterparts within the mathematical curriculum [24]. This reinforces another observation by Klein [1968, 128], that by the title *Arithmetica* Diophantus establishes the context of his discipline, not as "logistic" or practical arithmetic, in the sense then usual, but as a true science. Situating the *Arithmetica* in tandem with the *Elements*, Diophantus thus conceives them together as an expanded core of basic mathematical doctrine.

NOTES

1. The manuscript evidence for the *Definitions* is late. Heiberg's edition [1912] is based on one manuscript from the 14th century (Paris ms. suppl. gr. 387) and three others, based on that same manuscript, from the 15th or 16th; another 14th century manuscript (Vatican ms. gr. 215) holds selections, amounting to about a sixth of the work. According to the summaries by Heiberg [1912, iv–x] these manuscripts typically include other portions of the Heronian corpus, such as parts of the *Geometrica*, the *Stereometrica*, the *De mensuris*, or the *Belopoeica*. However, the oldest and most important compilation of Heronian mathematical writings, the Constantinople ms. pal. vet. 1 (11th century), which includes the unique surviving copy of the *Metrica* as well as a version of the *Geometrica* and the *Stereometrica*, does not contain the *Definitions* [ibid., xii].

2. Hultsch [1905, 993] retains his skepticism, describing the *Definitions* as an anonymous compilation from Euclid, Archimedes, Geminus, and "zu einem kleinen Teile" also from Hero. But later Hultsch moderates his view somewhat, admitting that, even if the extant form of the *Definitions* derives from a later time, "in ihrem Kerne gehen sie doch wohl auf H. zurück" [1912, 1059]. Similarly, Tannery [1896; 1912, 537–538], once having rejected the *Definitions* as incompatible with the early dating of Hero then generally accepted, retracted his objection subsequently, in light of a revised dating.

3. The problem posed by Diophantus corresponds to a general proposition: three numbers a, b, c form a harmonic progression, if and only if the three numbers $a(b + c)$, $b(a + c)$, $c(a + b)$ form an arithmetic progression. This can be seen as follows: by a familiar property of the harmonic mean, $b(a + c) = 2ac$, that is, ac is arithmetic mean of ab and bc. Since, then, ab, ac, bc are an A.P., by the addition of ac to each term, $a(b + c)$, $2ac$ [$= b(a + c)$], $c(a + b)$ are an A.P., as stipulated by Diophantus. The fact that ab, ac, bc are an A.P. entails another corollary: by dividing the terms by abc, one finds that the reciprocals $1/c$, $1/b$, $1/a$ are also an A.P. This is the usual definition of the harmonic mean in modern accounts. According to an unpublished communication by Waterhouse, however, this reciprocal form does not appear in extant ancient arithmetic sources.

4. For instance, Pappus justifies a certain geometric assumption as "a lemma of the Sphaerica" in *Collection*, Book V, Prop. 12 [Hultsch 1876–1878 I, 338], but Eutocius refers to the same result simply as "among the things said in the third book of Theodosius' Sphaerica" in his commentary on Archimedes' *Plane Equilibria* I, 7 [Heiberg 1915, 270]. The lemma in fact appears as a scholium in the manuscript tradition of the *Sphaerica* [Heiberg 1927, 193–194]; for details, see Knorr [1978, 187–188, 189, 228–229]. The process whereby scholia could become incorporated into the body of their treatises can be traced out in the example of a certain text on a hyperbola construction: known to Pappus and Eutocius in the form of a scholium to Apollonius' *Conics*, the text is subsequently transmitted as a proposition in Eutocius' revised edition of the *Conics*; cf. Knorr [1982; 1989, 230–234].

5. The phrase *ta pro* ... is unusual for a title. It means, literally, "the things before ...," where the priority could be either in place or in time. Accordingly, it would designate here a commentary

that is either situated ahead of the main treatise, or intended to be studied before it. In context, the latter sense seems to apply here.

6. One should note that the subjects of these opening clauses are not nouns, but the interrogative pronoun "what" (*tines*). Following Martin, Heiberg [1912, 85n] supplies the term *arithmoi* ("numbers"). In a Euclidean geometric context, to be sure, this would be a bold proposal, indeed an intolerable one, since Euclid carefully distinguishes numbers from magnitudes. But in the present passage, the context of the opening lines is arithmetic, not geometric. Further, the adjectives are all masculine (note especially "rational," *rhêtoi*) and so would agree with *arithmoi*, but not with "magnitudes" (the neuter *megethê*) or "lines" (the feminine *eutheiai*), the only plausible geometric alternatives.

7. Tannery [1893–1895, II, 282] cites 11 such passages in his index to the *Arithmetica*.

8. In this, the Diophantine usage is consistent with pre-Euclidean passages, such as in Plato's *Republic* 546c, where the "rational" and "irrational" sides and diameters are mentioned. It appears that Euclid coopted and modified the term *rhêtos* to include not only lines commensurable with the unit line, but also lines whose squares are commensurable with the unit square (*Elements* X, def. 3).

9. This is Heiberg's emendation, certainly correct, for the manuscript reading *asynaptoi* ("disjoined"), which is altered by the second hand of one manuscript to *eusynaptoi* ("well joined together").

10. This has already been noticed by Klein [1968, 246], who, however, assigns both the "Preliminaries" to Hero.

11. Heath [1921 II, 314] omits *kai* in his translation, renders *ta pro* ... incorrectly as "the things premised ...", and is otherwise misleadingly free.

12. The phrase *organôsai tên methodon* is bracketed by Tannery for its absence from the primary manuscript (Madrid ms. 48, 13th century); presumably, he takes the phrase to be an innovation in the Diophantus edition of Maximus Planudes (late 13th century).

13. My translation is literal, to display similarities with the preface of the *Definitions*, but agrees in sense with the freer translation by Heath [1910, 128].

14. The text reads *dielein* ("to divide"), but in context *dielthein* ("to proceed") would seem more appropriate.

15. Compare the paraphrase by Heath [1910, 130].

16. In Pauly–Wissowa (Vol. V), there are 166 entries under this name; the dedicatee of the *Definitions* is 146th, that of the *Arithmetica* 147th, both entries written by Hultsch [1905].

17. Heath [1921, II, 298–307]; Heath favors Heiberg's 3rd century dating for Hero [Heath 1921 II, 306].

18. By the "Egyptian method" Psellus means simply the arithmetic techniques expounded by Diophantus, based on the special nomenclature of the *arithmos* and its powers (sc. *dynamis, kybos, dynamodynamis*, etc.); cf. also the opening of Psellus' letter [Tannery 1893–1895 II, 37–38] and his description of one of Diophantus' theorems as being "of the Egyptian analysis" [Tannery 1893–1895 II, 39, line 5]. I would suppose that it is called "Egyptian" by virtue of Diophantus' situation at Alexandria and is being contrasted with the Euclidean form of arithmetic theory. On the other hand, if Diophantus was not actually the originator of this higher arithmetic, the "Egyptian" designation might refer to its tradition among earlier Alexandrians (see note [23]).

19. The Greek text is also quoted by Heath [1910, 2n], from Tannery [1893–1895 II, 38–39]. An alternative translation, with extensive discussion, is given by Klein [1968, 244–248]; see also notes [20] and [22].

20. The omission of iota-adscript or iota-subscript is commonplace in manuscripts. Two alternative emendations of the passage may be mentioned: (1) Tannery elsewhere proposed the change of *heterô* to the dative ⟨*tôi*⟩ *hetairôi* ("friend"), which, in fact, he preferred over his other proposal [1893–1895 II, xlvii; 1896/1912, 536]. But this has not earned a favorable reception; cf. Heath [1910, 2n]. (2) Klein

[1968, 245] advocates a double change, from *heterô Diophantô* to *heterôs Diophantou,* "differently from Diophantus"; moreover, he rejects translating the verb *prosephônêse* as "he dedicated" (even though he admits this would be the natural sense in the present context), but instead construes it to mean "he named," on the view that Psellus is referring here to how Anatolius adopted a nomenclature of the arithmetic powers different from Diophantus's. But without an explicit direct object, such as "he named *the forms of the numbers* (*ta tôn arithmôn eidê*)," this verb alone would not carry the meaning supposed by Klein, for which a more explicit verb of naming (e.g., *prosonomazein* or *prosagoreuein*) would be required. Beyond this, Klein's interpretation of Psellus' intent in the passage is conjectural, since even if Anatolius had employed a variant nomenclature (see the following note), Psellus gives no indication of knowing he did so, and the passage speaks not of that, but only of Anatolius' different style of exposition (namely, that Anatolius selected the "most essential" parts from Diophantus and presented them "synoptically"). A further consideration against Klein (as, indeed, against Tannery and Heath also) is that the repetition of Diophantus' name would be superfluous, since he has already been cited twice in the two preceding lines ("*Diophantus* treated of this Egyptian method . . ., but Anatolius selected from *that man's* doctrine . . ."). On general philological grounds, moreover, one must reject Klein's ploy of drastically revising a passage to suit his preconceived sense of it, when that passage is grammatically sound without any revision, as is the case here.

21. The situation need not be considered improbable. The name Diophantus is not unusual, while up to a point such coincidences can be contrived. Since authors *choose* to whom they dedicate their books and patrons *choose* the works that interest them, one would naturally try to exploit such a namesake relationship, when opportunity arose. (If Tannery's view of the close intellectual ties among these figures is correct, this "different Diophantus" might, conceivably, have been a younger relation of the mathematician.) Presumably, Psellus knew of this situation through an explanatory preface in Anatolius' edition. His awareness of variant terminology may also come through Anatolius. For instance, the *dynamokybos* (denoting the fifth power of the unknown as the product of the second power, *dynamis*, and the third, *kybos*), Psellus says, is also called *alogos prôtos* (or "first irrational") and *arithmos pemptos* ("fifth number") [Tannery 1893–1895 II, 37–38]. It is plausible that he encountered such variants in a secondary source, such as Anatolius' abridged edition of Diophantus, as Heath [1910, 111] suggests. But Psellus says nothing at all to distinguish Diophantus' own terminology from these variants. To the contrary, our passage (beginning: "About *this* Egyptian method Diophantus treated most accurately . . .") comes immediately after Psellus' naming of the powers, and so leaves the impression that, as far as he knew, the entire system, variants and all, was due to Diophantus. It seems, then, that Psellus' source, whether Anatolius or someone else, was imprecise on this point.

22. Klein [1968, 245–248], arguing from his emendation of the Psellus passage (see note [20]), dismisses Tannery's dating of Diophantus to the time of Anatolius, but proposes instead, on the basis of the shared dedications to Dionysius, that Hero and Diophantus could be contemporaries. As of Klein's writing (1936), however, Hero's date was still debated. Accordingly, Klein offers several possibilities in the second and first centuries, although he seems most favorable to a view long ago proposed by Bachet (1621), dating Hero and Diophantus to around the time of the Emperor Nero. In the English edition of his book (1968), Klein adds a citation of the concurring position of Neugebauer. Klein and the authorities he cites, however, seem unaware of how the questions surrounding Hero's authorship of the *Definitions* affect this dating argument.

23. Specifically, it would permit us to assign to Diophantus himself what otherwise are taken to be anticipations of the special nomenclature of the *Arithmetica*. For instance, in his computation of the area of the equilateral triangle, Hero [Schöne 1903, 48] three times employs the term *dynamodynamis* as the fourth power of a line, this being Diophantus' term for the fourth power of the unknown. Moreover, the sequence: *monas, arithmos, dynamis, kybos, dynamodynamis, dynamokybos, kybokybos,* which agrees with Diophantus' terms up to the sixth power, is twice cited by Hippolytus (bishop of Rome, early 3rd century) in his *Refutation of all heresies* I.2.10 and IV.51.8 [Marcovich 1986, 59, 138]. Although Hippolytus sets this scheme in the context of traditional Pythagorean notions, like the "first monad," the gender associations of odd and even numbers, and the *tetraktys* (sc. the first tetrad of numbers—1, 2, 3, 4—which sum to the "perfect" number 10), Hippolytus might reason-

ably have been referring to contemporary sources in his account of the power sequence. Other features of it suggest a possible specific connection to Diophantus' preface. Both Hippolytus and Diophantus consider first the sequence of number *per se* that continues to infinity, Diophantus observing how "all the numbers consist of a certain multitude of units ... having existence to infinity," much as Hippolytus speaks of "all the numbers, capable of going to infinity in multitude." Both writers generate the higher powers from multiplications of the second and third powers, and both terminate their accounts with the sixth power. Thus, rather than hypothesize as their common source a centuries-old tradition of higher arithmetic (a tradition otherwise unattested in surviving documents), we could propose an arithmetic system original with Diophantus as source for Hero and Hippolytus. Admittedly, the originality of Diophantus is a hypothesis that, on present evidence, is no more or less plausible than the alternative, that Diophantus consolidated an arithmetic system initiated by others before him.

24. Diophantus' example seems to have been perpetuated by the later Byzantine teachers. For instance, scholia to the arithmetic epigrams in the *Greek Anthology* cite the *Arithmetica* as the "Elements" (*Stoicheia*), just as they do for Euclid; cf. epigrams nos. 16 and 27 in the edition of Tannery [1893–1895 II, 62, 69].

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