Nonnegativity of a Discrete Poisson Kernel for the Hahn Polynomials

GEORGE GASPER*

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

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1. INTRODUCTION

For real $\alpha > -1$, $\beta > -1$, $N$ a nonnegative integer, and $n = 0, 1, \ldots, N$, we define the Hahn polynomials of degree $n$ by

$$Q_n(x) = Q_n(x; \alpha, \beta, N)$$

$$= \binom{-n, n + \alpha + \beta + 1, -x}{\alpha + 1, -N}$$

$$= \sum_{k=0}^{n} (-n)_k (n + \alpha + \beta + 1)_k (-x)_k$$

$$\frac{1}{k!(x + 1)_k (-N)_k}$$

where $(a)_0 = 1$, $(a)_k = a(a + 1) \cdots (a + k - 1)$ for $k \geq 1$. These polynomials satisfy the orthogonality relation

$$\sum_{x=0}^{N} \rho(x) Q_n(x) Q_m(x) = \begin{cases} 
0 & \text{if } n \neq m, \\
1/\pi_n & \text{if } n = m, 
\end{cases}$$

for $n, m = 0, 1, \ldots, N$, where

$$\rho(x) = \rho(x; \alpha, \beta, N) = \frac{(x + \alpha) (N - x + \beta)}{(N + \alpha + \beta + 1)}$$

$$\pi_n = \pi_n(\alpha, \beta, N)$$

$$= \frac{(-1)^n (-N)_n (\alpha + 1)_n (\alpha + \beta + 1)_n}{n! (N + \alpha + \beta + 2)_n (\beta + 1)_n} \cdot \frac{2n + \alpha + \beta + 1}{\alpha + \beta + 1}$$.

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and, in particular, \( r_0 = 1 \). The Hahn polynomials are a discrete analog of the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \), and, in fact, the Jacobi polynomials may be obtained as limiting forms of the Hahn polynomials by means of the relation

\[
\frac{P_n^{(\alpha, \beta)}(1 - 2x)}{P_n^{(\alpha, \beta)}(1)} = \lim_{N \to \infty} Q_n(Nx; \alpha, \beta, N). \tag{1.3}
\]

For these and other properties of the Hahn polynomials, we refer the reader to Karlin and McGregor [10], where a slightly different notation is used.

In [9] it was found necessary to have a discrete analog for Hahn polynomials of the following formula, which is Bailey's representation [1] for the Poisson kernel for Jacobi series in terms of a positive \( F_4 \) Appell function:

\[
\sum_{n=0}^{\infty} h_n x^n P_n^{(\alpha, \beta)}(\cos 2\varphi) P_n^{(\alpha, \beta)}(\cos 2\Phi)
= \frac{\Gamma(\alpha + \beta + 2)(1 - t)}{2^{\alpha + \beta + 1}\Gamma(\alpha + 1)\Gamma(\beta + 1)(1 + t)^{\alpha + \beta + 2}}
\times F_4\left(\frac{(\alpha + \beta + 2)/2, (\alpha + \beta + 3)/2; \alpha + 1, \beta + 1; a^2/k^2, b^2/k^2}\right),
\tag{1.4}
\]

where

\[
h_n = (2n + \alpha + \beta + 1) \frac{n!\Gamma(n + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)},
\]

and

\[
a = \sin \varphi \sin \Phi, \quad b = \cos \varphi \cos \Phi, \quad k = (t^{-1/2} + t^{1/2})/2.
\]

At a first glance, in obtaining a discrete analog of (1.4), we would consider the kernel

\[
\sum_{n=0}^{z} \frac{(-z)_n}{(-N)_n} \pi_n(\alpha, \beta, N) Q_n(x; \alpha, \beta, N) Q_n(y; \alpha, \beta, N).
\]

However, for the applications in [9] we needed a formula which implies the nonnegativity of the sum

\[
\sum_{n=0}^{N+y-x} \frac{(z - y - N)_n}{(\alpha, \beta, N) Q_n(x; \alpha, \beta, N) Q_n(y; \alpha, \beta, N + y - z), \tag{1.5}}
\]
when \(x, y, z; N\) are nonnegative integers and
\[
0 \leq x, y, z \leq N, \quad y \leq z, \quad \alpha > -1, \quad \beta > -1. \tag{1.6}
\]
Therefore, in this paper we shall consider the more general kernel,
\[
S_2(x, y; \alpha, \beta, N, M)
= \sum_{n=0}^{z} \frac{(-z)_n}{(-N)_n} \pi_n(\alpha, \beta, N)Q_n(x; \alpha, \beta, N)Q_n(y; \alpha, \beta, M),
\tag{1.7}
\]
and derive a double series representation (formula (3.3)) for (1.7), which has Bailey’s formula (1.4) as a limiting form and which implies the nonnegativity of (1.5) under the restrictions (1.6). In Section 3 we also derive a discrete analog for Hahn polynomials of the following formula, which is Eq. (2.1) in Bailey [1]:
\[
\sum_{n=0}^{\infty} \frac{n!(\alpha + \beta + 1)_n}{(\alpha + 1)_n (\beta + 1)_n} t^n P_n^{(\alpha, \beta)}(\cos 2\varphi) Q_n(\cos 2\varphi)
= (1 + t)^{-\alpha-\beta-1}
\times F_4[(\alpha + \beta + 1)/2, (\alpha + \beta + 2)/2; \alpha + 1; \beta + 1; \alpha^2/k^2, \beta^2/k^2],
\tag{1.8}
\]
where \(a, b, \) and \(k\) are the same as in (1.4).

Since Bailey’s main tool for deriving (1.4) was Watson’s formula [12] for the product of two terminating hypergeometric functions,
\[
\frac{\pi_n(\alpha, \beta, c; z)}{(\alpha + 1)_n (\beta + 1)_n} F_4[-n, n + a; c, a - c + 1 + 2^n; (1 - z)(1 - z)],
\tag{1.9}
\]
we also needed a generalization of this formula to the product of two terminating \(\tilde{F}_2(1)\) functions in a form general enough to include the product \(Q_n(x; \alpha, \beta, N) Q_n(y; \alpha, \beta, M).\) The required generalization of (1.9) will be presented in Section 2. Some limiting forms of our formulas for orthogonal polynomials of a discrete variable (Charlier, Krawtchouk, and Meixner polynomials) are considered in Section 4.

It is worthwhile noting that formulas of the above types have important applications to the harmonic analysis of orthogonal polynomials. For example, in [13, Section 11.6], Watson uses (1.9) to prove Bateman’s expansion for the product of two Bessel functions, which he then uses in [13, p. 413] to evaluate
an infinite sum of the product of three hypergeometric functions in terms of an integral of a triple product of Bessel functions. This latter formula is the main tool used in [8] to derive an integral representation for the product of two Jacobi polynomials. This integral representation was then used to obtain a convolution structure for Jacobi series and to extend Bochner's results in [3] to Jacobi polynomials. It will be interesting to see what applications will be given for the formulas in the following sections, as they become better known.

2. PRODUCT OF TWO \( _3F_2(1) \) FUNCTIONS

Employing the following notation of Burchnall and Chaundy [4],

\[
\begin{align*}
 F \left[ \frac{a_1, \ldots, a_l; b_1, \ldots, b_k; c_1, \ldots, c_m}{d_1, \ldots, d_n; e_1, \ldots, e_p; f_1, \ldots, f_q; x, y} \right] \\
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a_1)_{r+s} \cdots (a_l)_{r+s} (b_1)_r \cdots (b_k)_r (c_1)_s \cdots (c_m)_s}{r! (d_1)_{r+s} \cdots (d_n)_{r+s} (e_1)_r \cdots (e_p)_r (f_1)_s \cdots (f_q)_s x^r y^s}
\end{align*}
\]  

our generalization of (1.9) may be stated in the form

\[
\begin{align*}
 _3F_2 \left[ \frac{-n, n + a, b}{c, d} \right] _3F_2 \left[ \frac{-n, n + a, e}{c, f} \right] \\
= \frac{(-1)^n (a - c + 1)_n}{(c)_n} \\
\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (n + a)_{r+s} (b)_r (e)_s (d - b)_r (f - c)_s}{r! s! (d)_{r+s} (f)_{r+s} (c)_r (a - c + 1)_s}
\end{align*}
\]

where \( n = 0, 1, \ldots \), and, as elsewhere, the arguments \( x \) and \( y \) in (2.1) are not displayed when they are both equal to 1. The \( F \)-function in (2.2) must be interpreted in terms of the double sum displayed in (2.2). To obtain (1.9) from (2.2) one need but replace \( b, d, e, f \) in (2.2) by \( N \alpha, N, N \beta, N \) respectively, and then let \( N \to \infty \).
Note that, for Hahn polynomials, (2.2) implies the identity

\begin{equation}
Q_n(x; \alpha, \beta, N)Q_n(y; \alpha, \beta, M) = \frac{(-1)^n(\beta + 1)_n}{(\alpha + 1)_n} \times F\left[\begin{array}{c}
-n, n + \alpha + \beta + 1: -x, -y; x - N, y - M; \\
-N, -M: \alpha + 1; \beta + 1;
\end{array}\right].
\end{equation}

In [12] Watson states that he discovered his formula by considering various types of normal solutions of the wave-equation in four dimensions. At this time we lack such a method for discovering (2.2). It was only by doing the discrete analog of Watson's argument in reverse order that we were able to derive (2.2).

Our first step is to observe that a discrete analog of

\begin{equation}
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qandaq
and then we use Vandermonde's theorem to find that the double sum in
braces is equal to

\[
\sum_{j=0}^{n} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-n)_j (n + a)_j (c)_j (b)_q (c + q + j)_n-j (-1)^{j+q}}{r!(j-r)! (q-r)! (c+d-a)_q (a-c+1)_n-q (c)_r (f)_j} \times \frac{(c)_r (f)_j (a-c+1)_r (r+s-j)!}{(c)_r (f)_j (a-c+1)_r (r+s-j)!}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-n)_{r+s} (n + a)_{r+s} (c)_r (d)_s (f)_r (a-c+1)_s (c+d-a)_q (a-c+1)_r (r+s-q)!}{r!(j-r)! (q-r)! (c+d-a)_q (a-c+1)_r (r+s-j)! (f)_j} \times \frac{(b)_r (-1)^r}{(q-r)! (c+d-a)_q (a-c+1)_{r+s-q} (r+s-j)! (f)_j}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-n)_{r+s} (n + a)_{r+s} (c)_r (d)_s (f)_r (a-c+1)_s (c+d-a)_q (a-c+1)_{r+s-q} (r+s-j)!}{r!(j-r)! (q-r)! (c+d-a)_q (a-c+1)_{r+s-q} (r+s-j)! (f)_j} \times \frac{(b)_{r+s} (-1)^{r+s}}{p!(c+d-a)_{r+s} (a-c+1)_{s-p}}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-n)_{r+s} (n + a)_{r+s} (b)_r (c)_r (d)_s (f)_r (a-c+1)_s (c+d-a)_q (a-c+1)_{r+s-q} (r+s-j)!}{r!(j-r)! (q-r)! (c+d-a)_q (a-c+1)_{r+s-q} (r+s-j)! (f)_j}
\]

\[
\times \frac{\Gamma(d-b) \Gamma(c+d-a)}{\Gamma(d) \Gamma(c+d-a-b)}
\]

\[
= \frac{\Gamma(d-b) \Gamma(c+d-a)}{\Gamma(d) \Gamma(c+d-a-b)}
\]

\[
= \frac{\Gamma(d-b) \Gamma(c+d-a)}{\Gamma(d) \Gamma(c+d-a-b)} \times \frac{\Gamma(d-b) \Gamma(c+d-a) \Gamma(c+d-a-b)}{\Gamma(d) \Gamma(c+d-a) \Gamma(c+d-a-b)} \times \Gamma(d-b) \Gamma(c+d-a) \Gamma(c+d-a-b)
\]

which, combined with (2.4) and (2.5), gives (2.2). The argument requires
that Re(d - b) > 0 in order that the infinite series involved be absolutely
convergent and the use of Gauss's theorem be justified. This restriction can
be removed in (2.2) since the series involved terminate.

Generalizations of Watson's general formula [12, p. 194] and of Bailey's
product formula [2, p. 81, (1)] to products of \( \mathcal{F}_2 \) functions will be given
elsewhere.
3. The Function $S_z(x, y; \alpha, \beta, N, M)$

From $(1.1)$ and $(1.7)$ we have

$$S_z(x, y; \alpha, \beta, N, M)$$

(3.1)

$$= \sum_{n=0}^{z} \frac{(-z)_n (\alpha + 1)_n (\alpha + \beta + 1)_n (-1)^n}{n!(N + \alpha + \beta + 2)_n (\beta + 1)_n} \cdot \frac{2n + \alpha + \beta + 1}{\alpha + \beta + 1}
\times \, {}_3F_2 \left[ \begin{array}{c} -n, n + \alpha + \beta + 1, -x; \\ \alpha + 1, -N \end{array} \right] \, {}_3F_2 \left[ \begin{array}{c} -n, n + \alpha + \beta + 1, -y; \\ \alpha + 1, -M \end{array} \right].$$

In deriving our formula for this sum we permit $x, y, N, M, \alpha, \beta$ to be complex variables. However, it is still assumed that $z$ is a nonnegative integer. Letting $2a = \alpha + \beta + 1$, and using $(2.2)$ and $(n + a)/a = (a + 1)_n/(a)_n$, we find that $S_z(x, y; \alpha, \beta, N, M)$ is equal to

$$= \sum_{n=0}^{z} \frac{(-z)_n (2a)_n (a + 1)_n}{n!(N + 2a + 1)_n (a)_n} \times \left( \frac{-n}{r+1} \frac{(n + 2a)_r}{(\alpha + 1)_r (\beta + 1)_r} \right)$$

(3.2)

$$= \sum_{n=0}^{z} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-z)_m (2a)_m (a + 1)_m}{r! m! (N + 2a + 1)_m (\alpha + 1)_r (\beta + 1)_s} \times \left( \frac{(a + 1)_m (x - N)_r (y - M)_s (-1)^{r+s}}{(a)_m (\alpha + 1)_r (\beta + 1)_s} \right)$$

From [2, p. 30 (1.1)] it follows that the above $3F_2$ is equal to

$$\frac{(N - z)(N + 1 - r - s)}{(N + r + s + a + 1)}.$$
which can be written in the form

\[
\frac{(N - z) \Gamma(N + z) (-N)_{r+s} (N + 2a + 1)_{r+s} (-1)^{r+s}}{\Gamma(N + 1) (N + 2a + 1)_{r} (1 - N - z)_{2r+2s}}.
\]

Combining the above we obtain the identity

\[
S_{n}(x, y; \alpha, \beta, N, M)
\]

\[
= \frac{(N - z) \Gamma(N + z)}{\Gamma(N + 1)(N + \alpha + \beta + 2)_{z}}
\]

\[
\times \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(\alpha + \beta + 2)_{2r+2s} (-z)_{r+s} (-y)_{r} (x - N)_{s} (y - M)_{s}}{r! s! (1 - N - z)_{2r+2s} (-M)_{r+s} (\alpha + 1)_{s} (\beta + 1)_{s}}
\]

\[
= \frac{(N - z) \Gamma(N + z)}{\Gamma(N + 1)(N + \alpha + \beta + 2)_{z}}
\]

\[
\times F \left[ \left( \frac{\alpha + \beta + 2}{2}, \frac{\alpha + \beta + 3}{2}, \frac{-z}{2}, -x, -y; x - N, y - M; \right) \right]
\]

\[
\left( \frac{1 - N - z}{2}, \frac{2 - N - z}{2}, -M; \alpha + 1; \beta + 1; \right).
\]

for \( z = 0, 1, \ldots \). Bailey's formula (1.4) can be obtained from (3.3) by replacing \( x \) by \( N(1 - \cos \phi)/2 \), \( y \) by \( M(1 - \cos \Phi)/2 \), \( z \) by \( tN \), and then letting \( N, M \to \infty \) and using (1.3).

Since

\[
\lim_{N \to \infty} \frac{(N - z) \Gamma(N + z)}{(1 - N - z)_{2z}} = \begin{cases} 1 & \text{if } k = z, \\ 0 & \text{if } k < z, \end{cases}
\]

it follows from (3.3) that

\[
S_{N}(x, y; \alpha, \beta, N, M)
\]

\[
= \frac{(-x)_{r} (-y)_{r} (x - N)_{N-r} (y - M)_{N-r}}{\alpha + 1, (\beta + 1)_{N-r}}
\]

\[
\frac{(-1)^{N} (-M)_{N}}{(-1)^{r} (N - r)! \alpha + 1, (\beta + 1)_{N-r}}
\]

for \( N = 0, 1, \ldots \). If \( x = 0, 1, \ldots \), \( N \) and \( N = 0, 1, \ldots \), then (3.4) gives

\[
S_{N}(x, y; \alpha, \beta, N, M) = \frac{(x + \beta + 2)_{N} (-y)_{z} (y - M)_{N-z}}{(-M)_{N} (\alpha + 1)_{z} (\beta + 1)_{N-z}}
\]

which implies that

\[
S_{N}(x, y; \alpha, \beta, N, N) = \begin{cases} 0 & \text{if } x \neq y, \\ 1/\rho(x) & \text{if } x = y, \end{cases}
\]
when \( x, y = 0, 1, \ldots \), \( N, N = 0, 1, \ldots \), and \( \rho(x) = \rho(x; \alpha, \beta, N) \neq 0 \). Formula (3.5) gives the dual orthogonality relation in [10, (1.5)].

From the previous formulas it is obvious that the sum in (1.5) is non-negative if (1.6) holds and \( x, y, z, N \) are nonnegative integers. In fact, we clearly have the following theorem.

**Theorem 1.** If \( \alpha > -1, \beta > -1 \) and if \( x, y, z, N, M \) are nonnegative integers such that

\[
0 \leq x \leq N, \quad 0 \leq y \leq M, \quad 0 \leq z \leq \min(N, M),
\]

then

\[
S_z(x, y; \alpha, \beta, N, M) \geq 0.
\]

From this it easily follows that we also have another theorem.

**Theorem 2.** Let \( \alpha > -1, \beta > -1 \) and \( N, M = 0, 1, \ldots \). If \( a_0, \ldots, a_N \) are constants such that

\[
0 < x < N, \quad 0 < y < M, \quad 0 < z < \min(N, M),
\]

then

\[
\sum_{n=0}^{N} a_n Q_n(x; \alpha, \beta, N) \geq 0, \quad x = 0, 1, \ldots, N,
\]

\[
\sum_{n=0}^{N} \frac{(-x)^n}{(N)^n} a_n Q_n(y; \alpha, \beta, M) \geq 0,
\]

\[
y = 0, 1, \ldots, M, \quad z = 0, 1, \ldots, \min(N, M).
\]

If we delete the factor \((2n + \alpha + \beta + 1)/(\alpha + \beta + 1)\) from the coefficient of the sum in (3.1), we obtain the sum,

\[
T_z(x; y; \alpha, \beta, N, M)
\]

\[
= \sum_{n=0}^{N} \frac{(-x)^n (\alpha + 1)_n (\alpha + \beta + 1)_n (-1)^n}{n!(N + \alpha + \beta + 2)_n (\beta + 1)_n}
\]

\[
\times {}_3 F_2 \left[ -n, n + \alpha + \beta + 1, -x; \alpha + 1, -N \right] \cdot {}_3 F_2 \left[ -n, n + \alpha + \beta + 1, -y; \alpha + 1, -M \right],
\]

for a discrete analog of the sum on the left side of (1.8). Proceeding as in (3.2) we find that if \( z = 0, 1, \ldots \), then

\[
T_z(x; y; \alpha, \beta, N, M)
\]

\[
= \sum_{r=0}^{x} \sum_{s=0}^{z-r} \frac{(-x)^{r+s} (\alpha + \beta + 1)_{2r+2s} (-y)^r}{r!(N + \alpha + \beta + 2)_{r+s} (-N)_{r+s} (\alpha + 1)_{r+s} (\beta + 1)}
\]

\[
\times {}_2 F_1 \left[ 2r + 2s + \alpha + \beta + 1, r + s - z; \frac{N}{r}, s + \alpha, \beta + 2 \right].
\]
and so, by Vandermonde’s theorem, we have the identity

\[ T_z(x, y; \alpha, \beta, N, M) = \frac{\Gamma(N + z + 1)}{\Gamma(N + 1)(N + \alpha + \beta + 2)} \times \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(\alpha + \beta + 1)_{2r+2s} (-z)_r (-y)_r (x - N)_r (y - M)_s}{r! s! (-N - z)_{2r+2s} (-M)_{r+s} (\alpha + 1)_r (\beta + 1)_s} \]

\[ = \frac{\Gamma(N + z + 1)}{\Gamma(N + 1)(N + \alpha + \beta + 2)} \times \sum_{n=0}^{z} \frac{(-z)_n (\alpha + 1)_n (\alpha + \beta + 1)_n (-1)^n}{n! (N + \alpha + \beta + 2)_n (\beta + 1)_n} Q_n(x; \alpha, \beta, N) \]

\[ = \frac{\Gamma(N + z + 1)}{\Gamma(N + 1)(N + \alpha + \beta + 2)} \times \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(\alpha + \beta + 1)/2, (\alpha + \beta + 2)/2, -z; -x, -y; x - N, y - M; \alpha + 1; \beta + 1}{{}_4F_3} \]

\[ = \frac{\Gamma(N + z + 1)}{\Gamma(N + 1)(N + \alpha + \beta + 2)} \times \sum_{n=0}^{z} \frac{(-z)_n (\alpha + 1)_n (\alpha + \beta + 1)_n (-1)^n}{n! (N + \alpha + \beta + 2)_n (\beta + 1)_n} Q_n(x; \alpha, \beta, N) \]

for \( z = 0, 1, \ldots \). From (3.7) it is obvious that \( T_z(x, y; \alpha, \beta, N, M) \geq 0 \) if \( \alpha > -1, \beta > -1, \alpha + \beta + 1 \geq 0 \) and \( x, y, z, N, M \) are nonnegative integers satisfying the restrictions in (3.6).

If one of the variables \( x, y, N - x, M - y \) equals zero then the double sums in (3.3) and (3.7) reduce to single sums. In particular, setting \( y = 0 \) in (3.7), we get

\[ \sum_{n=0}^{z} \frac{(-z)_n (\alpha + 1)_n (\alpha + \beta + 1)_n (-1)^n}{n! (N + \alpha + \beta + 2)_n (\beta + 1)_n} Q_n(x; \alpha, \beta, N) \]

\[ = \frac{\Gamma(N + z + 1)}{\Gamma(N + 1)(N + \alpha + \beta + 2)} \times \sum_{r=0}^{z} \sum_{s=0}^{z-r} \frac{(\alpha + \beta + 1)/2, (\alpha + \beta + 2)/2, -z; -x, -y; x - N, y - M; \alpha + 1; \beta + 1}{{}_4F_3} \]

which is a discrete analog of [7, p. 264 (10)].

4. Discrete Orthogonal Polynomials

The formulas in sections 2 and 3 can be used to obtain analogous formulas for the Krawtchouk, Meixner, and the Charlier polynomials, which are
limits of the Hahn polynomials [6, 10]. If these polynomials are normalized to equal 1 at $x = 0$, then they may be defined, respectively, by:

$$K_n(x; p, N) = \lim_{t \to \infty} Q_n(x; pt, (1 - p) t, N)$$

$$= _2F_1 \left[ -n, -x; -N; 1/p \right],$$

$$M_n(x; \beta, c) = \lim_{N \to \infty} Q_n(x; \beta - 1, N(1 - c)/c, N)$$

$$= \lim_{t \to \infty} Q_n(x; ct/(c - 1), t/(1 - c), -\beta)$$

$$= _2F_1 \left[ -n, -x; \beta; 1 - c^{-1} \right],$$

$$c_n(x; a) = \lim_{N \to \infty} K_n(x; a/N, N)$$

$$= \lim_{\beta \to x} M_n(x; \beta, a/\beta)$$

$$= _2F_0 \left[ -n, -x; \beta; -1/a \right].$$

The Krawtchouk polynomials $K_n(x; p, N)$, $n = 0, 1, \ldots, N$, $0 < p < 1$, are orthogonal on $[0, 1, \ldots, N]$ with respect to the jump function $(x^p(1 - p)^N - x)$. The Meixner polynomials $M_n(x; \beta, c)$, $n = 0, 1, \ldots, \beta > 0$, $0 < c < 1$, and the Charlier polynomials $c_n(x; a)$, $n = 0, 1, \ldots, a > 0$, are orthogonal on $[0, 1, \ldots]$ with respect to the jump functions $e^{\lambda(x)}x!$ and $e^{-a^2/x!}$, respectively (see [6, pp. 224–226]).

Using (2.3) and (4.1) we obtain

$$K_n(x; p, N) K_n(y; p, M)$$

$$= (1 - p^{-1})^n F \left[ -n: -x, -y; x - N, y - M; \beta; p^{-1}, (1 - p)^{-1} \right],$$

where the $F$-function is as defined in (2.1). From (4.1) and (4.2) it is clear that

$$M_n(x; -N, p/(p - 1)) = K_n(x; p, N);$$

so, from (4.4), which is valid even if $N$ and $M$ are not integers, we have

$$M_n(x; \beta, c) M_n(y; b, c)$$

$$= c^{-n} F \left[ -n: -x, -y; x + \beta, y + b; \beta; 1 - c^{-1}, 1 - c \right].$$

This also follows by the equivalent procedure of applying the second limit operation in (4.2) to (2.3). Note, however, that if the first limit operation in (4.2) is applied to (2.3), then the coefficient of the $F$-function diverges.
Divergence also occurs if the two limit operations in (4.3) are applied directly to (4.4) and (4.6). It should be noted that (4.4) and (4.6) are equivalent to the identity
\[2F_1[-n, a; b; t] \sum_{n=0}^{\infty} \binom{N}{n} p^n(1 - p)^{N-n} K_n(x; p, N) K_n(y; p, M)\]
where \(n = 0, 1, \ldots\)

If we apply the limit operation in (4.1) to (3.3) or to (3.7), then we obtain the identity
\[\sum_{n=0}^{\infty} \binom{N}{n} \frac{(-z)_n}{(-N)_n} \left(\frac{1 - p}{p}\right)^{N-n} K_n(x; p, N) K_n(y; p, M)\]
\[= \frac{(1 - p)^{N-N}}{(1 - p)^{n-n}} \sum_{j=0}^{\infty} \frac{(-z)_j}{(-N)_j} \frac{(x - N)_j (x - M)_j}{j! p^j (p - 1)^{-j}} \times 3F_2\left[-x, -y, -z; 1 + N - x - z, 1 + M - y - z; (p - 1)/p\right], \tag{4.7}\]
for \(z = 0, 1, \ldots\), which gives a discrete Poisson kernel for the Krawtchouk polynomials. Since \(K_n(x; p, N) = K_n(n; p, N)\), renaming the parameters in (4.7) gives
\[\sum_{n=0}^{\infty} \binom{N}{n} \frac{(-z)_n}{(-N)_n} \left(\frac{1 - p}{p}\right)^{N-n} K_n(x; p, N) K_n(y; p, M)\]
\[= \frac{(n - N)_x (m - M)_x}{(1 - p)^{n-n}} \times 3F_2\left[-x, -y, -z; 1 + N - n - z, 1 + M - m - z; (p - 1)/p\right], \tag{4.8}\]
which for \(N = M\) may be thought of as a dual discrete Poisson kernel. When \(z = N = M\), (4.8) gives the orthogonality relation \([6, p. 224 (4)]\) and (4.7) gives the dual orthogonality relation.

From (4.7) it is clear that the kernel is nonnegative if \(0 < p < 1\) and and \(x, y, z, N, M\) satisfy the same conditions as in Theorem 1. This yields an analog of Theorem 2 and also yields the following dual result.
Theorem 3. Let $0 < p < 1$ and $N, M = 0, 1, \ldots$. If $a_0, \ldots, a_N$ are constants such that
\[
\sum_{x=0}^{N} a_x K_n(x; p, N) \geq 0, \quad n = 0, 1, \ldots, N,
\]
then
\[
\sum_{x=0}^{M} \frac{(-z)_x}{(-N)_x} a_x K_m(x; p, M) \geq 0,
\]
\[m = 0, 1, \ldots, M, \quad z = 0, 1, \ldots, \min(N, M).
\]

For Meixner polynomials, application of the second limit operation in (4.2) to formula (3.3) gives a result which is the same as setting $p = c/(c - 1), N = -\beta, M = -b$ in formula (4.7).

It should also be noted that (3.3) can be employed to obtain a Poisson kernel for the Charlier polynomials which has $t^n$ in the coefficient instead of $(-z)_n/(-N)_n$. Since (4.7) is a limiting case of (3.3), it will shorten the argument if we apply the first limit operation in (4.3) to (4.7) with $z = btN$ and $M = Na/b$ to get Meixner's bilinear generating function [11]:
\[
\sum_{n=0}^{\infty} \frac{a^n b^n}{n!} t^n c_n(x; a) c_n(y; b) = e^{abt(1 - at)} (1 - bt)^x c_n(y; - (1 - at) (1 - bt)/t), \tag{4.9}
\]
which is valid for $x, y = 0, 1, \ldots, |t| < \infty$. Note that (4.9) implies the dual formula
\[
\sum_{x=0}^{\infty} \frac{a^x b^x}{x!} t^x c_n(x; a) c_m(x; b) = e^{abt(1 - at)^m (1 - bt)^n} c_n(m; - (1 - at) (1 - bt)/t), \tag{4.10}
\]
since $c_n(x; a) = c_n(n; a)$. When $a = b = t^{-1}$, formulas (4.9) and (4.10) degenerate to the dual orthogonality relation and the orthogonality relation [6, p. 226 (3)], respectively. Analogos of (4.9) and (4.10) for the Krawtchouk and the Meixner polynomials follow directly from Meixner's formula in [5, p. 85 (12)].

References