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Continuous and discrete least-squares approximation by radial basis functions on spheres

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Abstract

In this paper we discuss Sobolev bounds on functions that vanish at scattered points on the n-sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The Sobolev spaces involved may have fractional as well as integer order. We then apply these results to obtain estimates for continuous and discrete least-squares surface fits via radial basis functions (RBFs). We also address a stabilization or regularization technique known as *spline smoothing*.

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1. Introduction

Scattered data surface fitting on the sphere has become increasingly important by virtue of its many applications in the geosciences. A very popular method of surface fitting is to use interpolation and approximation by (*conditionally*) *positive definite* and *radial* or *zonal* functions, see for example [3–5,17,22]. It is this method that we discuss here.

Several authors have provided error estimates for such reconstruction processes (see for example [7,9–11]); these error estimates were based upon using either spherical harmonics or charts. When

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the corresponding results in Euclidean space $\mathbb{R}^n$ are known, the chart approach is the easier of the two. However, until now, deriving the correct orders for the errors involved a very technical argument (see [8,9]), especially when L_p-norms other than L_∞ were considered. Moreover, only limited information on *simultaneous approximation*, i.e. error estimates that involve also derivatives has been known [12–14].

In this paper, we will derive general *simultaneous* error estimates for interpolation by RBFs. These results are based on recent results in $\mathbb{R}^n$ for Sobolev bounds on functions having many scattered zeros [15] or with many points where the function is sufficiently small [21].

We will begin by establishing Sobolev bounds for functions that are defined on $\mathbb{S}^n$ and that are small at sufficiently many points. Using the results we get, we will derive error estimates for radial basis function interpolation and both continuous and discrete least-squares approximation.

Let us now describe the interpolation and approximation problems we want to discuss. We will restrict ourselves exclusively to the sphere, although it is clear that much of our analysis carries over to more general compact Riemannian manifolds. In particular, if the data sites are situated inside a chart then the analysis applies immediately and the results hold true.

Assume that we are given a set $X = \{x_1, \ldots, x_N\}$ of *data sites* located on the n-sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ and *data values* $f_1, \ldots, f_N \in \mathbb{R}$ which stem from a continuous function $f \in \mathcal{H} = \mathcal{H}(\mathbb{S}^n)$, where $\mathcal{H}(\mathbb{S}^n) \subset C(\mathbb{S}^n)$ is a certain function space consisting of continuous functions on the sphere. This space will later be the Sobolev space $W_2^\tau(\mathbb{S}^n)$ with $\tau > n/2$. We are interested in finding the solution of

$$\min\{\|s\|_{\mathcal{H}} : s \in \mathcal{H} \text{ with } s|X = f|X\}, \tag{1}$$

which we will denote by s_0, or, if necessary, by $s_{0,X}$.

This is just the usual *minimal norm interpolant* for the problem. However, if the data values are noisy, then it is advisable to look at the *smoothing spline* solution s_λ of

$$\min\left\{\sum_{j=1}^{N}[s(x_j) - f(x_j)]^2 + \lambda\|s\|_{\mathcal{H}}^2 : s \in \mathcal{H}\right\}, \tag{2}$$

where $\lambda > 0$ is a certain *smoothing parameter*, which has to be chosen carefully, to balance between interpolation and approximation. The determination of λ has intensively been studied in the literature, see for example [19].

To discuss the solutions to both problems we have to make two more assumptions on the function space. The first assumption is a natural one. Since we want to work with point evaluation functionals, it is reasonable to assume that point evaluation functionals are continuous on $\mathcal{H}$, i.e. that for every $x \in \mathbb{S}^n$ there exists a constant $C_x > 0$ with

$$|f(x)| \leqslant C_x\|f\|_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}.$$

Our second assumption is not that natural, but it will greatly simplify the theory and it will provide no severe restrictions in applications. We will assume that our function space $\mathcal{H}$ is a Hilbert space.

A Hilbert space $\mathcal{H}$ of functions $f : \mathbb{S}^n \to \mathbb{R}$ with continuous point evaluation functionals is known to be a *reproducing kernel Hilbert space* (RKHS) (see e.g. [1]), i.e. it possesses a unique kernel $\Phi : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ such that

1. $\Phi(\cdot, x) \in \mathcal{H}$ for all $x \in \mathbb{S}^n$,
2. $f(x) = (f, \Phi(\cdot, x))_{\mathcal{H}}$ for all $x \in \mathbb{S}^n$ and all $f \in \mathcal{H}$.

In a RKHS the reproducing kernel Φ is always symmetric and positive semi-definite; it is even positive definite if the point evaluation functionals are linearly independent, which is what we actually assume. This means that for arbitrary distinct point sets $X = \{x_1, \ldots, x_N\} \subseteq \mathbb{S}^n$, the matrices

$$A = A_{\Phi,X} = (\Phi(x_i, x_j))_{i,j}$$

are positive definite. It is well known, that in this situation the solutions of (1) and (2) have a representation of the form

$$s_\lambda(x) = \sum_{j=1}^{N} \alpha_j \Phi(\cdot, x_j),$$

where the coefficient vector $\alpha \in \mathbb{R}^N$ is uniquely determined by the linear system

$$(A + \lambda I)\alpha = f|X.$$

Besides these two reconstruction methods we will also address error estimates for least-squares fitting in both the continuous and discrete sense. To this end we introduce the space

$$V_X := \operatorname{span}\{\Phi(\cdot, x_j) : x_j \in X\}$$

and another discrete data set $Y = \{y_1, \ldots, y_M\}$, which is supposed to be “finer” than X.

Then, we are interested in the behavior of the solution of the continuous least-squares problem

$$\min \left\{ \|f - s\|_{L_2(\mathbb{S}^n)} : s \in V_X \right\} \tag{3}$$

as well as in the solution of the discrete least-squares problem

$$\min \left\{ \sum_{j=1}^{M} [f(y_j) - s(y_j)]^2 : s \in V_X \right\}. \tag{4}$$

It is our goal to state error estimates for all these approximation methods in the case of $\mathcal{H} = W_2^\tau(\mathbb{S}^n)$. This includes results on how to choose the smoothing parameter in (2) a priori (see the remarks after Corollary 3.4).

The reproducing kernel of a Hilbert space $\mathcal{H}$ of continuous functions is uniquely determined by the inner product. On the other hand, every kernel Φ defines a Hilbert space of continuous functions for which it is the reproducing kernel (see for example [20]). Hence, from now on we will use the following relaxed definition.

Definition 1.1. Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a RKHS of functions defined on $\mathbb{S}^n$ with reproducing kernel Φ. We will say that $\widetilde{\Phi} : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ is also a reproducing kernel of $\mathcal{H}$, if it generates the same space $\mathcal{H}$ and the induced norm is equivalent to the original one.

In the case of Sobolev spaces this definition means that the Fourier coefficients of the kernel $\widetilde{\Phi}$ have to satisfy a certain decay condition, which is determined by the smoothness index of the Sobolev space. Since this is rather standard we omit the details here.

We will derive our results by means of charts. Hence, in the next section we will state relevant results on subsets of $\mathbb{R}^n$. In the final section we will deal with the results derived for the sphere. We will start that section with a short review of Sobolev spaces on the sphere by means of charts.

2. Results for the Euclidean case

In this section, we will let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying an interior cone condition and having a Lipschitz boundary. We will need various Sobolev spaces; details may be found in [2]. The Sobolev space $W_p^k(\Omega)$, $k \in \mathbb{N}_0$, consists of those distributions u with distributional derivatives $D^\alpha u \in L_p(\Omega)$, $|\alpha| \leqslant k$. Associated with these spaces are the (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} \quad \text{and} \quad \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}.$$

The case $p = \infty$ is defined in the obvious way

$$|u|_{W_\infty^k(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L_\infty(\Omega)} \quad \text{and} \quad \|u\|_{W_\infty^k(\Omega)} = \sup_{|\alpha|\leqslant k} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

For fractional order Sobolev spaces, we use the norms below. Let $1 \leqslant p < \infty$, $k \geqslant 0$, $k \in \mathbb{Z}$, and let $0 < s < 1$, then

$$|u|_{W_p^{k+s}(\Omega)} := \left(\sum\nolimits_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|_2^{n+ps}} \, dx\, dy \right)^{1/p},$$

$$\|u\|_{W_p^{k+s}(\Omega)} := \left(\|u\|_{W_p^k(\Omega)}^p + |u|_{W_p^{k+s}(\Omega)}^p \right)^{1/p}.$$

We define the fractional order Sobolev spaces $W_p^{k+s}(\Omega)$ to be all u for which the last norm is finite.

Error estimates for scattered data approximation problems are usually given in terms of the mesh norm or fill distance. For a finite set $X \subset \Omega$, we define the *mesh norm* (or *fill distance*) of X in Ω to be

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2.$$

We will also need two additional geometric quantities, the *separation radius* q_X and the *mesh ratio* $\rho_X = \rho_{X,\Omega}$. They are defined by

$$q_X = \frac{1}{2} \min_{i \neq j} \|x_i - x_j\|_2, \quad \rho_X = \rho_{X,\Omega} = h_{X,\Omega}/q_X.$$

To shorten our presentation we collect several global assumptions on the indices that we will employ throughout the rest of the paper.

Assumption 2.1. Let $\tau = k + s$ with $k \in \mathbb{N}$, $0 \leqslant s < 1$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $m \in \mathbb{N}_0$ with $k > m + n/p$ if $p > 1$ or $k \geqslant m + n/p$ if $p = 1$.

The following results were established in [15, Theorems 2.12 and 2.13] and [21, Theorem 2.6]. We will need them in the sequel. However, the assumption in those papers differs from our Assumption 2.1 in the following way. In those papers the number s was supposed to satisfy $0 < s \leqslant 1$. Here, we use the improved form $0 \leqslant s < 1$. While this seems to be minor at first sight, it gives the "correct" condition $\tau > m + n/p$ for integer τ. This change in assumption has been justified in [16].

Theorem 2.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying an interior cone condition and having a Lipschitz boundary. Let $X \subset \Omega$ be a discrete set with sufficiently small mesh norm $h = h_{X,\Omega}$. Under the Assumption* 2.1, *for each $u \in W_p^\tau(\Omega)$ we have that*

$$|u|_{W_q^m(\Omega)} \leqslant C \left(h^{\tau - m - n(1/p - 1/q)_+} |u|_{W_p^\tau(\Omega)} + h^{-m} \|u|X\|_\infty \right),$$

where $C > 0$ is a constant independent of u and h, and $(x)_+ = \max\{x, 0\}$.

The next result is for the discrete least-squares problem (4). To measure the error in this case, we will employ a discrete norm, which is defined as

$$\|u\|_{\ell_q(Y)} = \begin{cases} \left(\frac{1}{M} \sum_{j=1}^M |u(y_j)|^q \right)^{1/q} & \text{for } 1 \leqslant q < \infty, \\ \max_{1 \leqslant j \leqslant M} |u(y_j)| & \text{for } q = \infty, \end{cases}$$

for a discrete set of points $Y = \{y_1, \ldots, y_M\} \subset \Omega$. Derivatives can also be included, for example, if $u \in C^k(\Omega)$ is given, we define

$$|u|_{w_q^k(Y)} = \left(\sum_{|\alpha| = k} \|D^\alpha u\|_{\ell_q(Y)}^q \right)^{1/q} \quad \text{and} \quad \|u\|_{w_q^k(Y)} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{\ell_q(Y)}^q \right)^{1/q}. \tag{5}$$

With this notation in hand, the required result on $\mathbb{R}^n$ is the following one:

Theorem 2.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying an interior cone condition and having a Lipschitz boundary. Let $X \subset \Omega$ be a discrete set with sufficiently small mesh norm $h = h_X$. Let $Y = \{y_1, \ldots, y_M\}$ be a second discrete set, with $h_Y \leqslant h$. Under the general Assumption* 2.1, *if $u \in W_p^\tau(\Omega)$ satisfies $u|X = 0$, then*

$$|u|_{w_q^m(Y)} \leqslant C \rho_Y^{n/q} h^{\tau - m - n(1/p - 1/q)_+} |u|_{W_p^\tau(\Omega)},$$

where $C > 0$ is a constant independent of X, Y and u. In particular, if $m = 0$ and $p = q = 2$, then

$$\|u\|_{\ell_2(Y)} \leqslant C \rho_Y^{n/2} h^\tau |u|_{W_2^\tau(\Omega)}.$$

3. Application to the sphere

The unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ will serve as an example of how to treat a compact manifold.

To introduce Sobolev spaces on the sphere, one can either express functions in spherical harmonics or use charts. Here, we will follow the latter approach [6,18].

Let $\mathcal{A} = \{U_j, \psi_j\}_{j=1}^m$ be an atlas of n-dimensional charts for $\mathbb{S}^n$, i.e. the open sets $U_j \subset \mathbb{S}^n$ cover the sphere $\mathbb{S}^n$ and the mappings ψ_j are homeomorphic mappings from U_j to the open unit ball $B(0, 1) \subset \mathbb{R}^n$, such that for two charts ψ_i and ψ_j having $U_i \cap U_j \neq \emptyset$ the composition

$$\psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$$

is C^∞. With such an atlas, we always have an associated family $\{\chi_j : \mathbb{S}^n \to \mathbb{R}\}_{j=1}^m$ of C^∞ functions forming a partition of unity with respect to the open covering $\{U_j\}_{j=1}^m$, i.e. they satisfy

$\chi_j \geqslant 0$, $\mathrm{supp}(\chi_j) \subseteq U_j$ and $\sum_{j=1}^m \chi_j = 1$ on $\mathbb{S}^n$. Next, for a function $f : \mathbb{S}^n \to \mathbb{R}$ we introduce the projections $\pi_j(f) : \mathbb{R}^n \to \mathbb{R}$ by

$$\pi_j(f)(x) = \begin{cases} f \circ \psi_j^{-1}(x), & x \in B(0,1), \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

We then use both the projections and the partition of unity to define the Sobolev space $W_p^\tau(\mathbb{S}^n)$ via

$$W_p^\tau(\mathbb{S}^n) := \left\{ f \in L_p(\mathbb{S}^n) : \pi_j(\chi_j f) \in W_p^\tau(\mathbb{R}^n) \text{ for } j = 1, \ldots, m \right\}.$$

This space can be equipped with the norm

$$\|f\|_{W_p^\tau(\mathbb{S}^n)} = \left(\sum_{j=1}^m \|\pi_j(\chi_j f)\|_{W_p^\tau(\mathbb{R}^n)}^p \right)^{1/p}$$

for $1 \leqslant p < \infty$. The case $p = \infty$ is defined in a similar manner.

It is important to know that even if the norm depends on the chosen atlas, the space does not. Moreover, all norms provided by different choices of atlas are equivalent.

Hence, in the rest of the paper, we can and will restrict ourselves to a specific atlas, one consisting only of the following two charts. Let $\hat{n} = (0, \ldots, 0, 1)^{\mathrm{T}}$, $\hat{s} = (0, 0, \ldots, -1)^{\mathrm{T}}$ be the north and south pole of $\mathbb{S}^n$, respectively. We denote the spherical cap with radius $\theta \in (0, \pi)$ and center z by

$$G(z, \theta) := \{\xi \in \mathbb{S}^n : d(z, \xi) < \theta\},$$

where $d(z, \xi) = \arccos(z \cdot \xi)$ denotes the usual geodesic distance. Next, we fix an angle $\theta_0 \in (\pi/2, 2\pi/3)$ and consider the following two specific spherical caps:

$$U_1 = G(\hat{n}, \theta_0), \quad U_2 = G(\hat{s}, \theta_0).$$

The homeomorphic mappings $\psi_1 : U_1 \to B(0,1)$ and $\psi_2 : U_2 \to B(0,1)$ associated with these caps are defined by

$$\psi_1(\xi) = \frac{1}{\tan(\theta_0/2)(1 + \xi_{n+1})} (\xi_1, \ldots, \xi_n)^{\mathrm{T}},$$

and

$$\psi_2(\xi) = \frac{1}{\tan(\theta_0/2)(1 - \xi_{n+1})} (\xi_1, \ldots, \xi_n)^{\mathrm{T}}.$$

Except for the scaling factor, these are simply stereographic projections (cf. [8]).

The following result is easily established. It relates the Euclidean distance between two points to the Euclidean distance between their images under the charts.

Lemma 3.1. *For $u, v \in U_j$, $j = 1, 2$, we have*

$$\sin(\theta_0)\|\psi_j(u) - \psi_j(v)\|_2 \leqslant \|u - v\|_2 \leqslant 2\tan(\theta_0/2)\|\psi_j(u) - \psi_j(v)\|_2.$$

Proof. The relation

$$\|u - v\|_2 = \frac{2\tan(\theta_0/2)\|\psi_j(u) - \psi_j(v)\|_2}{(1 + \tan^2(\theta_0/2)\|\psi_j(u)\|_2^2)^{1/2}(1 + \tan^2(\theta_0/2)\|\psi_j(u)\|_2^2)^{1/2}}$$

follows from a similar one for the stereographic mapping (see [8]). Since $\psi_j(u) \in B(0,1)$ for all $u \in U_j$, this yields the upper bound and the lower bound, after a few more steps. □

From now on, let $C_{\theta_0} = \sin(\theta_0)^{-1}$. Since the shortest path between two points in $\mathbb{R}^{n+1}$ is the line, we can conclude that

$$\|\psi_j(u) - \psi_j(v)\|_2 \leqslant C_{\theta_0}\|u - v\|_2 \leqslant C_{\theta_0} d(u, v)$$

for $j = 1, 2$ and $u, v \in U_j$. This allows us to relate the mesh norm on $\psi_j(U_j) = B(0,1)$ to the mesh norm on the sphere. The latter, of course, is now defined using the geodesic distance:

$$h_{X,\mathbb{S}^n} := \sup_{x\in\mathbb{S}^n} \min_{x_j\in X} d(x, x_j).$$

Proposition 3.2. *With the previous notations we have for* $j = 1, 2$

$$h_{\psi_j(X\cap U_j),\psi_j(U_j)} \leqslant 3C_{\theta_0} h_{X,\mathbb{S}^n}.$$

Proof. We can conclude from the definitions that

$$\begin{aligned} h_{\psi_j(X\cap U_j),\psi_j(U_j)} &= \sup_{x\in\psi_j(U_j)} \min_{x_l\in\psi_j(X\cap U_j)} \|x - x_l\|_2 \\ &= \sup_{x\in U_j} \min_{x_l\in X\cap U_j} \|\psi_j(x) - \psi_j(x_l)\|_2 \\ &\leqslant \sup_{x\in U_j} \min_{x_l\in X\cap U_j} C_{\theta_0} d(x, x_l) \\ &\leqslant C_{\theta_0} h_{X\cap U_j, U_j}. \end{aligned}$$

Finally, suppose $x \in U_j$ is given. Then, we can connect this x with the corresponding pole of U_j by a great circle and choose a y on this great circle with $d(x, y) \leqslant 2h_{X,\mathbb{S}^n}$. To this y there exists an $x_i \in X$ with $d(y, x_i) \leqslant h_{X,\mathbb{S}^n}$, and so $x_i \in U_j$. Since the triangle inequality yields that

$$d(x, x_i) \leqslant d(x, y) + d(y, x_i) \leqslant 3h_{X,\mathbb{S}^n}.$$

we can finally conclude

$$h_{X\cap U_j,U_j} \leqslant 3C_{\theta_0} h_{X,\mathbb{S}^n},$$

which settles our statement. □

With this relation at hand, the results corresponding to Theorem 2.2 can easily be established for the sphere. However, if we are considering functions u which do not vanish at X and derivative estimates, we need actually also an estimate of the form $h_{\psi_j(X\cap U_j),\psi_j(U_j)} \geqslant c h_{X,\mathbb{S}^n}$ which is equivalent to an estimate of the form $h_{X\cap U_j,U_j} \geqslant c h_{X,\mathbb{S}^n}$. Unfortunately, such an estimate can be wrong if the mesh norm on one of the U_j is much smaller than on the other one. On the other hand, it is quite natural to assume additionally that the points are similarly distributed on both spherical caps.

Theorem 3.3. *Suppose that Assumption* 2.1 *on* $\tau = k + s$, p, q, *and* m *holds, and also that* $X \subset \mathbb{S}^{n-1}$ *is finite and has a sufficiently small mesh norm* $h = h_{X,\mathbb{S}^n}$. *If* $u \in W_p^\tau(\mathbb{S}^n)$ *satisfies* $u|X = 0$ *then the following estimate holds*

$$\|u\|_{W_q^m(\mathbb{S}^n)} \leqslant C h^{\tau - m - n(1/p-1/q)_+} \|u\|_{W_p^\tau(\mathbb{S}^n)}.$$

Finally, if we assume that in case of $m > 0$ *the mesh norms on both caps* U_j, $j = 1, 2$ *are comparable to the mesh norm on* $\mathbb{S}^n$, *then for any* $u \in W_p^\tau(\mathbb{S}^n)$, *we have*

$$\|u\|_{W_q^m(\mathbb{S}^n)} \leqslant C \left(h^{\tau-m-n(1/p-1/q)_+} \|u\|_{W_p^\tau(\mathbb{S}^n)} + h^{-m} \|u|X\|_\infty \right).$$

Proof. With the abbreviation $u_j = \pi_j(\chi_j u)$, $j = 1, 2$, and Minkowski's inequality we get

$$\begin{aligned} \|u\|_{W_q^m(\mathbb{S}^n)} &= (\|u_1\|^q_{W_q^m(B(0,1))} + \|u_2\|^q_{W_q^m(B(0,1))})^{1/q} \\ &\leqslant \|u_1\|_{W_q^m(B(0,1))} + \|u_2\|_{W_q^m(B(0,1))}. \end{aligned}$$

Applying Theorem 2.2 to both summands on the right-hand side, setting $X_j = \psi_j(X \cap U_j)$ and $h_j = h_{X_j, B(0,1)}$ and using $|u_j(\psi_j(x))| \leqslant |u(x)|$ yields

$$\begin{aligned} \|u_j\|_{W_q^m(B(0,1))} &\leqslant C \left(h_j^{\tau-m-n(1/p-1/q)_+} |u_j|_{W_p^\tau(B(0,1))} + h_j^{-m} \|u_j|X_j\|_\infty \right) \\ &\leqslant C \left(h_{X,\mathbb{S}^n}^{\tau-m-n(1/p-1/q)_+} \|u_j\|_{W_p^\tau(B(0,1))} + h_{X,\mathbb{S}^n}^{-m} \|u|X\|_\infty \right) \end{aligned}$$

by Proposition 3.2. Finally, by definition we have $\|u_j\|_{W_p^\tau(B(0,1))} \leqslant \|u\|_{W_p^\tau(\mathbb{S}^n)}$. This leads to the desired estimates. □

Hence, for the solution s_λ, $\lambda \geqslant 0$, of (1) and (2) with $\mathcal{H} = W_2^\tau(\mathbb{S}^n)$ we have the following corollary:

Corollary 3.4. *Under the assumptions of Theorem* 3.3 *the following error estimate holds for all* $f \in W_2^\tau(\mathbb{S}^n)$:

$$\|f - s_\lambda\|_{W_q^m(\mathbb{S}^n)} \leqslant C \left(h^{\tau-m-n(1/2-1/q)_+} + h^{-m}\sqrt{\lambda} \right) \|f\|_{W_2^\tau(\mathbb{S}^n)}.$$

Proof. For $\lambda = 0$ this follows immediately from the norm-minimal interpolation property of s_0. For $\lambda > 0$ simply note that

$$\begin{aligned} \max\{[(f - s_\lambda)|X]^2, \lambda\|s_\lambda\|^2_{W_2^\tau(\mathbb{S}^n)}\} &\leqslant \sum_{j=1}^N [f(x_j) - s_\lambda(x_j)]^2 + \lambda\|s_\lambda\|^2_{W_2^\tau(\mathbb{S}^n)} \\ &\leqslant \lambda\|f\|^2_{W_2^\tau(\mathbb{S}^n)}, \end{aligned}$$

by using $s = f$ as an upper bound to the quadratic form. The rest follows from Theorem 3.3. □

This again gives a priori information on a good choice of $\lambda > 0$. For example, setting $q = \infty$, $p = 2$, and $m = 0$ leads to the error estimate

$$\|f - s_\lambda\|_{L_\infty(\mathbb{S}^n)} \leqslant C \left(h^{\tau-n/2} + \sqrt{\lambda} \right) \|f\|_{W_2^\tau(\mathbb{S}^n)}.$$

Hence, in this situation, a choice of the form

$$\lambda \leqslant C h^{2\tau-n}$$

is necessary to keep the optimal approximation order.

Another consequence of this result is that we now also have an error bound on the L_2-best approximation error.

Corollary 3.5. *Let s^* be the solution of the continuous $L_2(\mathbb{S}^n)$ least-squares problem* (3), *where $X \subseteq \mathbb{S}^n$, Φ is the reproducing kernel of $W_2^\tau(\mathbb{S}^n)$, and $f \in W_2^\tau(\mathbb{S}^n)$. Then, the error can be bounded by*

$$\|f - s^*\|_{L_2(\mathbb{S}^n)} \leqslant C h^\tau \|f\|_{W_2^\tau(\mathbb{S}^n)}.$$

Proof. Simply use $\|f - s^*\|_{L_2(\mathbb{S}^n)} \leqslant \|f - s_0\|_{L_2(\mathbb{S}^n)}$ and then Corollary 3.4. □

This settles the case of continuous least-squares approximation. However, we are also in the situation to bound the error for the *discrete* least-squares problem (4). To this end we remind the reader of the separation distance q_X and the mesh ratio ρ_X, which are now accordingly defined to be

$$q_X = \frac{1}{2} \min_{i \neq j} d(x_i, x_j), \quad \rho_X = \rho_{X,\mathbb{S}^n} = h_{X,\mathbb{S}^n}/q_X,$$

respectively. Theorem 2.3 yields the following result, which is proven like Theorem 3.3.

Theorem 3.6. *Under the general Assumption* 2.1 *let $X \subset \mathbb{S}^n$ be a discrete set with mesh norm $h = h_{X,\mathbb{S}^n}$. Let $Y = \{y_1, \ldots, y_M\}$ be another discrete set on the unit sphere with $h_{Y,\mathbb{S}^n} \leqslant h$.* If *$u \in W_p^\tau(\mathbb{S}^n)$ satisfies $u|X = 0$ then*

$$\|u\|_{w_q^m(Y)} \leqslant C \rho_Y^{n/q} h^{\tau - m - n(1/p - 1/q)_+} \|u\|_{W_p^\tau(\mathbb{S}^n)}$$

with a constant $C > 0$ independent of u and X.

As a consequence, we have error estimates for discrete least-squares approximation.

Corollary 3.7. *Under the assumptions of Theorem* 3.6 *let s^* be the discrete least-squares solution of* (4) *to $f \in W_2^\tau(\mathbb{S}^n)$, where Φ is the reproducing kernel of $W_2^\tau(\mathbb{S}^n)$, then there is a constant C independent of s^* and X, Y such that*

$$\|f - s^*\|_{\ell_2(Y)} \leqslant C \rho_Y^{n/2} h_{X,S^n}^\tau \|f\|_{W_2^\tau(\mathbb{S}^n)}.$$

Proof. Simply use the previous result for $u = f - s_{0,X}$. □

It remains to remark that, when working with *conditional* positive definite functions of finite smoothness (i.e. mainly functions of thin-plate spline kind) all results remain valid.

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