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Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function

Amjad Alipanah [a,*], Shahrokh Esmaeili [b]

[a] Department of Applied Mathematics, University of Kurdistan, 66159-87875, Sanandaj, Iran

[b] Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No 424, Hafez Ave., 15914 Tehran, Iran

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ABSTRACT

In this paper, we introduce a numerical method for the solution of two-dimensional Fredholm integral equations. The method is based on interpolation by Gaussian radial basis function based on Legendre–Gauss–Lobatto nodes and weights. Numerical examples are presented and results are compared with the analytical solution to demonstrate the validity and applicability of the method.

1. Introduction

In this paper, we are concerned with Gaussian radial basis function (RBF) method for the two-dimensional Fredholm integral equations of the second kind,

$$u(\tau,\mu)=f(\tau,\mu)+\int_a^b\int_c^d K(\tau,\mu,\lambda,\eta,u(\lambda,\eta))\mathrm{d}\lambda\mathrm{d}\eta,\quad(\tau,\mu)\in D \tag{1.1}$$

where $f(\tau,\mu)$ and $K(\tau,\mu,\lambda,\eta,u)$ are given continuous functions defined, respectively on $D=[a,b]\times[c,d]$, $E=D\times D\times(-\infty,\infty)$ and u is unknown on D.

A few number of methods for the solution of the Fredholm integral equations have been given in the literature [1]. The Galerkin and collocation methods are the two commonly used methods for the numerical solutions of the two-dimensional integral equations. The analysis for convergence of these methods is well documented in the literature [2–7]. In [8], Han and Wang approximated the two-dimensional Fredholm integral equations by the Galerkin iterative method. In [9], Hadizadeh and Asgary by using the bivariate Chebyshev collocation method solved the linear Volterra–Fredholm integral equations of the second kind.

In this paper, we approximate the solution of the two-dimensional Fredholm integral equation using Gaussian radial basis function. Also we approximate its corresponding integral by the Legendre–Gauss–Lobatto (LGL) points and weights.

2. Introduction to RBFs

RBFs were introduced in [10] and they form a primary tool for multivariate interpolation. They are also receiving increased attention for solving PDE in irregular domains. Hardy [11] showed that multiquadrics RBF is related to a consistent

* Corresponding author. Tel.: +98 871 6624965.
E-mail addresses: A.Alipanah@uok.ac.ir (A. Alipanah), esmaeili@aut.ac.ir (Sh. Esmaeili).

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solution of the biharmonic potential problem and thus has a physical foundation. Buhmann and Micchelli [12] and Chiu et al. [13] have shown that RBF are related to prewavelets (wavelets that do not have orthogonality properties). Also Alipanah and Dehghan [14], used RBF for the solution of a nonlinear integral equation in one dimensional case. A radial basis function (RBF) interpolant of multivariate data $((x_k, y_k), z_k),\ k = 1, 2, \dots, N$ is as follows

$$F(x, y) = \sum_{k=1}^{N} c_k \phi(\|(x, y) - (x_k, y_k)\|), \tag{2.1}$$

where $\|(x, y)-(x_k, y_k)\| = ((x-x_k)^2+(y-y_k)^2)^{\frac{1}{2}}$. Here $\phi(r)$ is some radial basis function, and coefficients c_k are determined in such a way that

$$F(x_k, y_k) = z_k, \quad k = 1, 2, \dots, N,$$

i.e. as the solution to the linear system

$$[A]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

where the entries of the matrix A are $A_{i,j} = \phi(\|(x_i, y_i) - (x_j, y_j)\|),\ i, j = 1, 2, \dots, N$.

Here $\phi(r)$ is some positive definite functions [11]. Numerous choices for $\phi(r)$ have been used in the past. Table 1 shows a few cases for which existence and uniqueness of the interpolants $f(x)$ have been discussed in the literature; see for e.g. [15,16,11,17,18]. For many of the radial functions in Table 1, existence and uniqueness are ensured for arbitrary point distributions. However, there are some that require the form of (1.1) to be augmented by some low-order polynomial terms.

3. Strictly positive definite functions

Definition. A function ϕ on X is said to be positive definite on X, if for any set of points $x_1, x_2, \dots, x_N$ in X the $N \times N$ matrix $A_{ij} = \phi(x_i - x_j)$ is nonnegative definite, i.e.

$$V^T AV = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i v_j A_{ij} \geq 0,$$

for all nonzero $V \in R^N$. If $V^T AV > 0$ whenever the points x_i are distinct and $V \neq 0$, then we say that $\phi(r)$ is a strictly positive definite function [19,16].

If $\phi(r)$ be strictly positive definite function on a linear space, then the eigenvalues and determinant of A are positive. Therefore we can use a linear combination translation of $\phi(r)$ to interpolate [16].

Definition. A function $f(r)$ is said to be completely monotone on $[0, \infty)$, if for any $t > 0$ we have that

$$1.\quad f \in C^\infty[0, \infty), \qquad 2.\quad (-1)^k f^{(k)}(t) \geq 0.$$

A real-valued function F on an inner-product space is said [16] to be radial if $F(x) = F(y)$ whenever $\|x\| = \|y\|$. Now we present a theorem that introduce a large number of strictly positive definite or radial basis functions.

Theorem (*Bochner's Theorem [16]*)**.** *Let f be a nonnegative Borel function on R, if $0 < \int_R f < \infty$, then $\hat{f}$ is strictly positive definite, where $\hat{f}$ is the Fourier transform of function f, which*

$$\hat{f}(x) = \int_{-\infty}^{+\infty} f(y) e^{ixy} dy.$$

We can find many strictly positive definite functions by using this theorem. In Table 1 we give some positive definite functions (RBFs) by using Bochner's Theorem.

Thus for any set of distinct points $x_0, x_1, \dots, x_N$ on $[a, b]$, the matrix $A_{ij} = \hat{f}(\|x_i - x_j\|^2)$ is strictly positive definite.

4. Legendre–Gauss–Lobatto nodes and weights

Let $L_N(x)$ be the shifted Legendre polynomial of order N on [0, 1]. Then the Legendre–Gauss–Lobatto nodes are

$$x_0 = 0 < x_1 < \cdots < x_{N-1} < x_N = 1, \tag{4.1}$$

and x_m, $1 \leq m \leq N - 1$ are the zeros of $\dot{L}(x)$, where $\dot{L}(x)$ is the derivative of $L_N(x)$ with respect to $x \in [0, 1]$. No explicit formulas are known for the points x_m, and so they are computed numerically using subroutines [20].

Table 1
Strictly positive definite functions that satisfy Bochner's Theorem.

$f_1(x) = \frac{1}{\pi(1+x^2)}$	$\hat{f}_1(x) = e^{-\lvert x\rvert}$
$f_2(x) = \frac{e^{-\lvert x\rvert}}{2}$	$\hat{f}_2(x) = \frac{1}{1+x^2}$
$f_3(x) = \pi^{-\frac{1}{2}} e^{-x^2}$	$\hat{f}_3(x) = e^{-\frac{x^2}{4}}$
$f_4(x) = \frac{1+x^{-2}}{2\pi}$	$\hat{f}_4(x) = \lvert x\rvert^{-1}(1 - e^{-\lvert x\rvert})$
$f_5(x) = \text{sech}(\pi x)$	$\hat{f}_5(x) = \text{sech}(\frac{x}{2})$
$f_6(x) = \frac{1-x\,\text{csch}(x)}{2x^2}$	$\hat{f}_6(x) = \log(1 + e^{-\frac{\pi}{\lvert x\rvert}})$

Also we approximate the integral of f on [0, 1] as

$$\int_0^1 f(x)\mathrm{d}x = \sum_{i=0}^{N} w_i f(x_i), \tag{4.2}$$

where x_i are the Legendre–Gauss–Lobatto nodes in Eq. (4.1) and the weights w_i given in [20, p. 76]

$$w_i = \frac{2}{N(N+1)} \cdot \frac{1}{[L_N(x_i)]^2}, \quad i = 0, 1, \ldots, N. \tag{4.3}$$

It is well known [20] that the integration in Eq. (4.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N + 1$.

5. Discretizing the two-dimensional Fredholm integral equations

Changing the variables $\tau = (b-a)x + a,\ \mu = (d-c)y + c,\ \lambda = (b-a)t + a$ and $\eta = (d-c)s + c$, Eq. (1.1) can be written as

$$u(x, y) = f_1(x, y) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x, y, t, s, u(t, s))\mathrm{d}t\mathrm{d}s, \quad (x, y) \in D, \tag{5.1}$$

where $f_1(x, y) = f((b-a)x+a, (d-c)y+c),\ K_1(x, y, t, s, u(t, s)) = K((b-a)x+a, (d-c)y+c, (b-a)t+a, (d-c)s+c, u((b-a)t+a, (d-c)s+c))$ and $D = [0, 1] \times [0, 1]$.

Let $\phi(r)$ be a strictly positive definite function or RBF and we approximate $u(x, y)$ with interpolation by function $\phi(r)$ i.e.,

$$u(x, y) \simeq \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x, y) = C^T\Psi(x, y), \tag{5.2}$$

where

$$\phi_{ij} = \phi_{ij}(x, y) = \phi(\|(x, y) - (x_i, y_j)\|),$$
$$\Psi(x, y) = [\phi_{00}, \phi_{10}, \ldots, \phi_{N0}; \phi_{01}, \phi_{11}, \ldots, \phi_{N1}; \cdots; \phi_{0M}, \phi_{1M}, \ldots, \phi_{NM}]^T,$$

and

$$C^T = [c_{00}, c_{10}, \ldots, c_{N0}; c_{01}, c_{11}, \ldots, c_{N1}; \ldots; c_{0M}, c_{1M}, \ldots, c_{NM}].$$

Also (x_i, y_j) are the Legendre–Gauss–Lobatto nodes. Now by substituting Eq. (5.1) in Eq. (1.1) we have that

$$C^T\Psi(x, y) = f_1(x, y) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x, y, t, s, C^T\Psi(t, s))\mathrm{d}t\mathrm{d}s.$$

For obtaining $c_{ij}, i = 0, 1, \ldots, N,\ j = 0, 1, \ldots, M$ in the above equation, by collocating at the points $(x, y) = (x_i, y_j)$ for $i = 0, 1, \ldots, N,\ j = 0, 1, \ldots, M$ we have that

$$C^T\Psi(x_i, y_j) = f_1(x_i, y_j) + (b-c)(d-c)\int_0^1\int_0^1 K_1(x_i, y_j, t, s, C^T\Psi(t, s))\mathrm{d}t\mathrm{d}s. \tag{5.3}$$

By applying numerical integration method given in Eq. (4.2), we can approximate the integral in Eq. (5.1) and hence the above equation can be written as follow

$$C^T\Psi(x_i, y_j) = f_1(x_i, y_j) + (b-a)(d-c)\sum_{r_1=0}^{N}\sum_{r_2=0}^{M} w_{r_1}w_{r_2}K_1\left(x_i, y_j, t_{r_1}, s_{r_2}, C^T\Psi(t_{r_1}, s_{r_2})\right), \tag{5.4}$$

for $i = 0, 1, \ldots, N,\ j = 0, 1, \ldots, M$ and w_r are given in Eq. (4.3).

This is a nonlinear system of equations that can be solved via Newton's iteration method to obtain unknown vector C^T.

Table 2
Numerical results for different RBFs of Example 1.

N	$\lambda = \frac{1}{2}$	$\lambda = 1$	$\lambda = 2$
2	4.1×10^{-2}	8.0×10^{-2}	1.2×10^{-1}
3	1.2×10^{-2}	3.0×10^{-2}	1.0×10^{-1}
4	2.1×10^{-3}	6.0×10^{-3}	3.0×10^{-2}
5	5.1×10^{-4}	1.4×10^{-3}	1.0×10^{-2}
6	1.1×10^{-5}	2.5×10^{-4}	3.0×10^{-3}
7	1.6×10^{-6}	7.2×10^{-5}	8.0×10^{-4}
8	3.1×10^{-7}	3.2×10^{-6}	1.0×10^{-5}

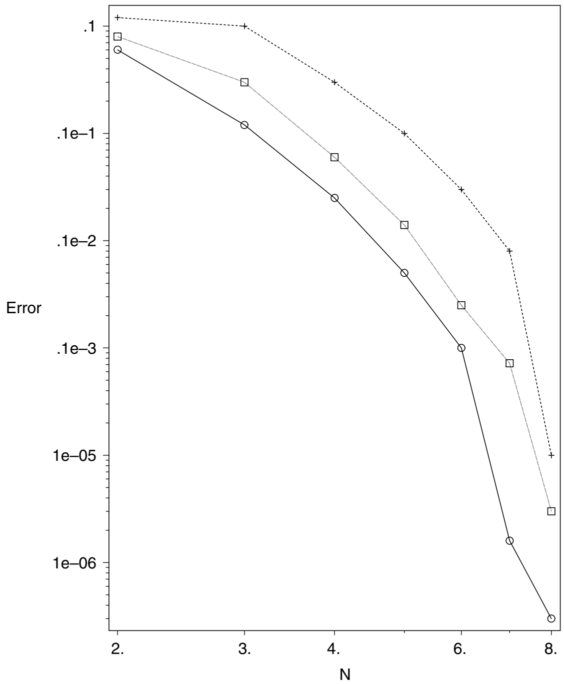

Fig. 1. Circle, box and cross are respectively for $\lambda = \frac{1}{2}$, 1 and 2 using Gaussian RBFs of Example 1.

6. Numerical examples

We used the method presented in this paper using Gaussian radial basis function $\phi(r) = e^{-\lambda^2 r^2}$ for solving the two-dimensional Fredholm integral equation given in [4,5]. Our method differs from the methods given in [4,5], since this method is simple and involve less computation. In all examples, we use the Gaussian RBFs for $\lambda = 1,\ \lambda = 2$ and $\lambda = \frac{1}{2}$, also we use the maximum errors for different N which is given as

$$E_\infty = \max\left\{\left|u(x, y) - \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x, y)\right| : (x, y) \in [0, 1] \times [0, 1]\right\}.$$

6.1. Example 1

Firstly, consider the following Fredholm nonlinear integral equation [7,8],

$$u(x, y) = \frac{1}{(1 + x + y)^2} - \frac{x}{6(1 + y)} + \int_0^1\int_0^1 \frac{x}{1 + y}(1 + t + s)u^2(t, s)\mathrm{d}t\mathrm{d}s,$$
$$(x, y) \in [0, 1] \times [0, 1],$$

whose exact solution is $u(x, y) = \frac{1}{(1+x+y)^2}$.

Errors for the numerical solution by RBFs for different values of $N = M$ are given in Table 2 and Fig. 1.

Table 3
Numerical results of different RBFs for Example 2.

N	$\lambda = \frac{1}{2}$	$\lambda = 1$	$\lambda = 2$
2	6.0×10^{-2}	1.1×10^{-2}	8.0×10^{-2}
3	1.2×10^{-2}	5.3×10^{-2}	9.0×10^{-2}
4	5.0×10^{-3}	5.0×10^{-3}	8.0×10^{-2}
5	9.0×10^{-5}	1.1×10^{-3}	8.0×10^{-2}
6	1.2×10^{-6}	2.5×10^{-4}	2.0×10^{-2}
7	1.2×10^{-6}	3.0×10^{-5}	9.0×10^{-3}
8	2.3×10^{-10}	9.8×10^{-6}	8.0×10^{-3}

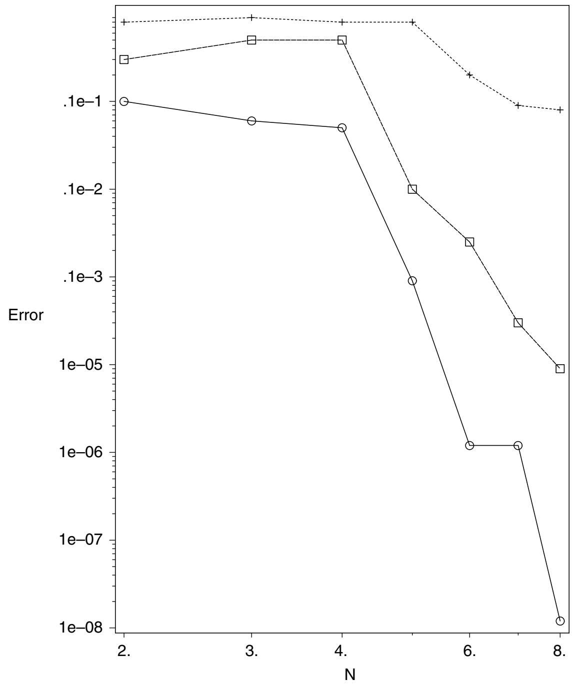

Fig. 2. Circle, box and cross are respectively for $\lambda = \frac{1}{2}$, 1 and 2 using Gaussian RBF of Example 2.

6.2. Example 2

Consider linear Fredholm integral equation given in,

$$5u(x, y) = \int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} u(t, s) \cos(xt) \cos(sy) \mathrm{d}s\mathrm{d}t + 5 - \frac{\sin\left(\sqrt{\pi} y\right) \sin\left(\sqrt{\pi} x\right)}{xy} \tag{6.1}$$

which the exact solution is $u(x, y) = 1$.

Errors of the numerical results are given in Table 3 and Fig. 2.

7. Conclusion

In this paper, we have investigated the application of interpolation by radial basis function for solving the nonlinear Fredholm integral equations. This technique is very simple and involves less computation. Also we can expand this method to higher dimensional problems and other classes of integral equations such as integro-differential and nonlinear equations. Note that the final system extracted from the nonlinear equations will be nonlinear and a proper technique such Newton method could be applied.

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