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Generalizing a definition of Lusternik and Schnirelmann to model categories

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For Stephen Halperin on his 50th birthday

Abstract

Hess, K.P. and J.-M. Lemaire, Generalizing a definition of Lusternik and Schnirelmann to model categories, Journal of Pure and Applied Algebra 91 (1994) 165–182.

Lusternik–Schnirelmann category is an important numerical homotopy invariant, defined originally for topological spaces but employed since in other categories in which there is a good notion of homotopy. In the interest of unification, Doeraene defined a *J-category* to be a category satisfying a certain set of axioms which ensure that an LS-type invariant can be reasonably defined. He provided two equivalent definitions of LS-category for J-categories and showed that these generalized definitions agree with previous definitions in specific categories. Extending his results, we provide a third equivalent definition of the LS-category of an object in a J-category, analogous to the original topological definition of Lusternik and Schnirelmann.

Introduction

In 1934 Lusternik and Schnirelmann defined a numerical invariant for topological spaces, which they called the *category* of a space [10]. Their main result—and motivation for defining this invariant—was that the category of a manifold is a lower bound for the number of critical points of a real function defined on the manifold. Later Fox introduced Lusternik–Schnirelmann category in the framework of homotopy theory and this viewpoint was extensively studied by Whitehead, Ganea,

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Hilton and others during the sixties. For an excellent and thorough introduction to LS-category, the reader is referred to James' survey article [8]. More recently, LS-catetory has turned out to be a most useful notion in rational homotopy and more generally in the study of algebraic models of homotopy types; see [9] for an introduction.

There are three standard, equivalent definitions of LS-category: the original definition of Lusternik and Schnirelmann, as well as two more recent definitions of Whitehead and Ganea. The original, and perhaps most intuitive, definition is the following:

Definition [10]. The *LS-category* of a topological space S, denoted $\operatorname{cat} S$, is the least m for which there exists a covering of S by $m + 1$ open sets, each contractible within S.

Thus, for example, the LS-category of a contractible space is zero, and the LS-category of the n-sphere is one, for all $n > 0$.

The second definition of LS-category uses the notion of *fat wedge*.

Definition. Let S be a pointed topological space with basepoint $*$. The mth *fat wedge* of S is

$$T^m S = \{(s_0, \ldots, s_m) \in S^{m+1} \mid s_i = *, \text{ for some } i\}.$$

Hence, $T^1 S = *$ and $T^2 S = S \vee S$.

Definition [13]. $\operatorname{cat} S \leq m$ if and only if there exists a map $S \to T^m S$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} & & S \\ & \swarrow & \downarrow{\scriptstyle \Delta} \\ T^m S & \hookrightarrow & S^{m+1} \end{array}$$

The third equivalent definition of LS-category is due to Ganea, and relies on the following "fibre-cofibre" construction:

Definition. Let S be a pointed space. The mth *Ganea fibration* $p_m : E^m S \to S$ of S is defined recursively as follows.

(0) $p_0 : E^0 S = P_* S \to S$ is the path fibration.

(1) Given $p_k : E^k S$, take the mapping cone $G^{k+1} S$ of the inclusion of the fibre F^k into E^k; take the canonical extension $r_{k+1} : G^{k+1} S \to S$:

$$\begin{array}{ccccc} F^k & \hookrightarrow & E^k S & \longrightarrow & G^{k+1} S \\ & & \downarrow{\scriptstyle p_k} & \swarrow{\scriptstyle r_{k+1}} & \\ & & S & & \end{array}$$

Convert r_{k+1} into a fibration to obtain $p_{k+1} : E^{k+1} S \to S$.

Remark. (1) One easily sees that $E^1S \simeq \Sigma\Omega S$.

(2) [5] $F^m \simeq \Omega S * \cdots * \Omega S$ — the $(m+1)$-fold join of ΩS with itself.

Definition [6]. $\operatorname{cat} S \leq m$ if and only if there exists a homotopy sections $s: S \to E^m S$ of p_m, i.e. $p_m \circ s \simeq 1s$.

One unfortunate fact about LS-category is that it is quite difficult to compute, no matter which of the three definitions is used, except in very special cases. For this reason, topologists have attempted since the late 1970s to approximate LS-category algebraically, in the hopes that making computations for algebras would be easier than for topological spaces. In this context, "algebraic approximation" means modelling topological spaces in some algebraic category, in which an LS-category-type homotopy invariant is defined.

In order to model topological spaces algebraically, we need an algebraic category $\mathscr{C}$ in which there is a well-defined notion of homotopy (e.g., a Quillen model category such as the category of connected chain algebras over a field) and a model $\mathscr{M}: \mathscr{TOP} \to \mathscr{C}$. A *model* is an assignment of topological spaces to objects in $\mathscr{C}$ and of continuous maps to morphisms in $\mathscr{C}$, which induces a functor from the homotopy category of $\mathscr{TOP}$ to the homotopy category of $\mathscr{C}$ (the model $\mathscr{M}$ may depend on non-canonical choices which prevent it from being a true functor $\mathscr{TOP} \to \mathscr{C}$). Examples of models include the Sullivan minimal model, which takes values in the category of commutative cochain algebras over $\mathbb{Q}$ [12], and the Adams–Hilton model, with values in the category of connected chain algebras over a ring [1].

In 1982 Félix and Halperin defined a homotopy invariant, denoted cat_0, in the category of commutative cochain algebras over $\mathbb{Q}$ [4]; they proved that cat_0 of the Sullivan minimal model of space is equal to the LS-category of the rationalization (localization at 0) of the space. Thus, cat_0 of the Sullivan minimal model of a space is a lower bound for the LS-category of the space. Later, inspired by the definition of cat_0, Halperin and Lemaire defined similar homotopy invariants in the category of 1-connected cochain algebras over a field and in the category of connected chain algebras over a field [7], which also provide lower bounds for topological LS-category.

In view of these results, one may ask whether there exists a certain structure common to all of these categories, which allows a category-theoretic definition of LS-category which would agree with the homotopy invariants mentioned above. This would provide a unified proof that the LS-category of the model bounds the LS-category of the space below; this would also allow to extend the notion of LS-category to any other categories with the appropriate structure.

In his thesis [3], Doeraene made considerable progress in answering this question. He established the set of axioms which must hold in a category in which there is a LS-category-type homotopy invariant. Categories satisfying these axioms are called *J-categories.* Doeraene also provided two equivalent definitions of LS-catetory, analogous to the definitions of Whitehead and Ganea, in an arbitrary J-category. He showed that several familiar categories are J-categories and proved that many results known to be true for the original topological LS-category hold in the general context.

The goal of this article is to present a third possible definition of LS-category in an arbitrary J-category, analogous to the original Lusternik–Schnirelmann definition. We prove that this third definition is equivalent to those of Doeraene. Beforehand, we review the necessary algebraic background material, including the axioms of a J-category, in Section 1.

1. Algebraic preliminaries

In order to generalize the notion of LS-category to categories other than $\mathcal{TOP}$, it is necessary to distinguish the relevant structure of $\mathcal{TOP}$ which enables us to define LS-category, so that it is a homotopy invariant. Doeraene investigated this question in his thesis and formulated axioms to be satisfied by a category in which LS-category can be defined in a manner analogous to the constructions of Ganea and Whitehead [3]. Such a category is called a *category with joins*. In a category with joins, $\mathscr{C}$, both constructions lead to invariants of weak homotopy type, but these invariants are not necessarily equal. If $\mathscr{C}$ satisfies one additional axiom formulated by Doeraene (the "cube" axiom), then the definitions are equivalent. We say then that $\mathscr{C}$ is a *J-category*. In this section, we give a self-contained account of these notions for the convenience of the reader.

Let $\mathscr{C}$ be any category. If

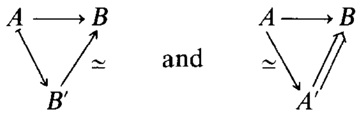
$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a fibered product (pull-back), then write $A \times_B C$ for the object E. Dually, if

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ C & \longrightarrow & S \end{array}$$

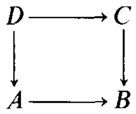
is an amalgamated sum (push-out), denote the object S by $A \vee_B C$.

A category with joins, just like a Quillen model category, has three distinguished classes of morphisms: fibrations, cofibrations, and weak equivalences. We will denote fibrations, cofibrations, and weak equivalences by arrows of the types $\twoheadrightarrow$, $\rightarrowtail$, and $\xrightarrow{\simeq}$, respectively. Moreover, as in a model category, any morphism $A \to B$ can be factored in two ways:

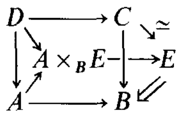


In such a category, commutative homotopy pull-backs and push-outs can be defined as follows:

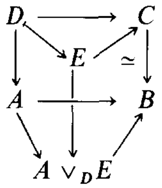
Definition. The commutative diagram



is a *commutative homotopy pull-back* when the induced map $D \to A \times_B E$ in the following diagram is a weak equivalence:



The rectangle *D-C-B-A* is a *commutative homotopy push-out* when the induced map $A \vee_D E \to B$ in the following diagram is a weak equivalence.



Let 0 denote the initial/final object of the category. An object B is called *cofibrant* if every fibration which is also a weak equivalence (also known as *trivial fibration*)

$$p: A \overset{\simeq}{\twoheadrightarrow} B$$

admits a section $s: B \to A$, i.e. $p \circ s = 1_B$. If every cofibration which is also a weak equivalence (also known as *trivial cofibration*)

$$i: B \overset{\simeq}{\rightarrowtail} C$$

admits a retract $r: C \to B$ so that $r \circ i = 1_B$, then B is *fibrant*. If the map $B \to 0$ is a fibration, then B is 0-*fibrant*. The object B is 0-*cofibrant* if $0 \to B$ is a cofibration. If fibrations have the right lifting property with respect to trivial cofibrations, then all 0-fibrant objects are fibrant. Dually, all 0-cofibrant objects are cofibrant whenever cofibrations have the left lifting property with respect to trivial fibrations.

We now present Doeraene's axioms for a category with joins. The reader familiar with the notion of a model category [11] or of a cofibration category (e.g. [2]) will recognize that a category with joins fulfills most of the requirements to be a model category or a cofibration category:

Definition. A pointed category $\mathscr{C}$ endowed with three distinguished classes of morphisms, called fibrations, cofibrations, and weak equivalences, is a *category with joins* if the following axioms are satisfied:

(J1) Any morphism $A \to B$ can be factored two ways:

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow & \nearrow_{\simeq} \\ & B' & \end{array} \qquad \text{(c-factorization)}$$

and

$$\begin{array}{ccc} A & \longrightarrow & B \\ & {}_{\simeq}\searrow & \nearrow\!\!\!\nearrow \\ & A' & \end{array} \qquad \text{(f-factorization).}$$

(J2) Given morphisms

$$\begin{array}{ccc} B & \rightarrowtail & A \\ \downarrow{\scriptstyle f} & & \\ C & & \end{array}$$

there exists a push-out

$$\begin{array}{ccc} B & \rightarrowtail & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ C & \longrightarrow & A \vee_B C \end{array}$$

Given morphisms

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle g} \\ C & \Longrightarrow & B \end{array}$$

there exists a pull-back

$$\begin{array}{ccc} A \times_B C & \Longrightarrow & A \\ \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ C & \Longrightarrow & B \end{array}$$

Furthermore, if f (respectively, g) is a weak equivalence, then so is $\bar{f}$ (respectively, $\bar{g}$).

(J3) Isomorphisms belong to all three distinguished classes of maps. The composition of cofibrations is a cofibration. The composition of fibrations is a fibration. If any two of $f: A \to B$, $g: B \to C$, and $gf: A \to C$ are weak equivalences, then the third is as well.

(J4) For every A there exists a trivial fibration $B \overset{\simeq}{\Rightarrow} A$ with B cofibrant.

Note that axioms (J1)–(J3) are self-dual; if one replaces (J4) by its dual, one obtains a *category with cojoins.*

In a category with joins, $\mathscr{C}$, one can perform the Ganea construction and define the fat wedge. To achieve this we first define the notions of join and cojoin of two objects over a third one in a category with joins.

Definition. Given two morphisms $f: A \to B$ and $g: C \to B$ in $\mathscr{C}$, their *join*, $A *_B C$, can be defined in two steps as follows.

(i) Choose an f-factorization

$$\begin{array}{ccc} A & \longrightarrow & B \\ & {\scriptstyle\simeq}\searrow & \Uparrow \\ & & D \end{array}$$

and take the pull-back

$$\begin{array}{ccc} D \times_B C & \Longrightarrow & C \\ \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ D & \Longrightarrow & B \end{array}$$

(ii) Choose a c-factorization

$$\begin{array}{ccc} D \times_B C & \longrightarrow & D \\ & \searrow & \nearrow{\scriptstyle\simeq} \\ & W & \end{array}$$

and take the push-out

$$\begin{array}{ccc} Z = D\times_B C & \rightarrowtail & W \\ \downarrow & & \downarrow \\ C & \rightarrowtail & C\vee_Z W \end{array}$$

Then $A*_B C = C \vee_Z W$.

One checks that if $\mathscr{C}$ is the category of topological spaces, and B is a point, this definition yields the standard join $A*C$. One can also check fairly easily that $A*_B C$ is well-defined up to weak equivalence.

We can now present Doeraene's fat wedge and Ganea constructions for a category with joins.

Definition. Let $\mathscr{C}$ denote a category with joins. Let B be an object of $\mathscr{C}$. Then nth *fat wedge* on B, T^nB, is defined recursively, together with a morphism $t^n: T^nB \to B^{n+1}$, as follows.

(0) $t^0: T^0B = 0 \to B$.

(1) T^nB is the join of $(1_{B^n}, 0): B^n \to B^n \times B$ and $t^{n-1}\times 1_B: T^{n-1}B\times B \to B^n \times B$; $t^n: T^nB \to B^{n+1}$ is the morphism induced from the join.

One can easily show that in $\mathscr{TOP}$ this definition of fat wedge agrees up to homotopy equivalence with the usual one.

We can now define the Ganea construction in an arbitrary category with joins:

Definition. Let $\mathscr{C}$ denote a category with joins. Let B be an object of $\mathscr{C}$. The nth *Ganea construction* on B, G^nB, is defined recursively, together with a morphism $g^n: G^nB \to B$, as follows.

(0) $g^0: G^0B = 0 \to B$.

(1) $G^nB = 0*_B G^{n-1}B$; $g^n: G^nB \to B$ is the morphism induced from the join.

Note that the definition of G^nB is equivalent to saying that G^nB is the homotopy cofiber of the homotopy fiber of g^{n-1}. Thus it is clear that the abstract definition of G^nB agrees with the original definition in $\mathscr{TOP}$.

We are now prepared to state Doeraene's two definitions of LS-category in any category with joins, $\mathscr{C}$. Doeraene proved that the definitions are equivalent in a category with joins which also satifies the following "cube" axiom:

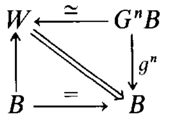
> (*Cube axiom*) Consider a commuting cubic diagram in which the vertical arrows are directed downwards. If the bottom of the cube is a commutative homotopy push-out and its sides are cummutative homotopy pull-backs, then the top is a commutative homotopy push-out.

A category with joins in which the above axiom holds is called a *J-category*.

Remark. If C is a cofibrant object of $\mathscr{C}$, then $A *_B C$ is cofibrant for all $A, B \in \mathrm{Ob}\,\mathscr{C}$ [3]. Thus $T^n A$ and $G^n A$ are cofibrant whenever A is cofibrant.

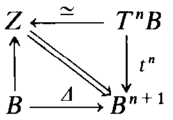
Definition. Let $B \in \mathrm{Ob}\,\mathscr{C}$ be a cofibrant and 0-fibrant. The *LS-category* of B, $\mathrm{cat}_{\mathscr{C}} B$, is the least n such that the following equivalent conditions are satisfied.

(i) There exists $W \in \mathrm{Ob}\,\mathscr{C}$ together with a diagram



in which the lower triangle commutes up to homotopy and the diagonal arrow is a fibration.

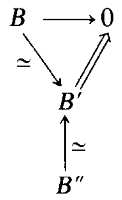
(ii) There exists $Z \in \mathrm{Ob}\,\mathscr{C}$ together with a diagram



in which the lower triangle commutes up to homotopy and the diagonal arrow is a fibration.

It is clear that in $\mathscr{TOP}$ these definitions are equivalent to the Ganea and Whitehead definitions of LS-category, respectively.

If it is true that $\mathrm{cat}_{\mathscr{C}}$ is a weak homotopy invariant in some category $\mathscr{C}$, then $\mathrm{cat}_{\mathscr{C}} B$ can be defined for arbitrary B by letting $\mathrm{cat}_{\mathscr{C}} B = \mathrm{cat}_{\mathscr{C}} B''$ where



i.e., B' is obtained via an f-factorization of $B \to 0$ and B'' is a cofibrant model of B'. Although Doeraene stated that $\mathrm{cat}_{\mathscr{C}}$ is a weak homotopy invariant in any category with joins which also satisfies the "cube" axiom, a careful examination of his proof reveals that it does not use this last axiom. The proof is thus valid in any category with joins.

2. A third abstract definition of LS-category

In this section a third possible definition of LS-category for J-categories is presented. This third definition is the analog of the original topological definition of Lusternik and Schnirelmann. We show that the new definition is equivalent to the two definitions of Doeraene. We then prove a result concerning the LS-category of a cofiber, using the new definition. The result is much harder to prove with the other two definitions.

Let $\mathscr{C}$ be a J-category. The new invariant in $\mathscr{C}$, to be called cat′ until proved equal to LS-category as defined by Doeraene, will be defined recursively.

Definition. Given a morphism $f: A \to B$ in $\mathscr{C}$, we say that

(i) $\mathrm{cat}'_{\mathscr{C}}(A; f) = 0$ if and only if $f \simeq 0$,

(ii) $\mathrm{cat}'_{\mathscr{C}}(A; f) \le m$ if and only if there exists a homotopy push-out

$$\begin{array}{ccccc} E & \longrightarrow & C & & \\ \downarrow & & \downarrow{\scriptstyle g} & & \\ D & \xrightarrow{h} & A & \xrightarrow{f} & B \end{array}$$

such that $\mathrm{cat}'_{\mathscr{C}}(C; fg) = m - 1$ and $\mathrm{cat}'_{\mathscr{C}}(D; fh) = 0$.

Let $\mathrm{cat}'_{\mathscr{C}}(A) = \mathrm{cat}'_{\mathscr{C}}(A; 1_A)$.

It is clear that the motivation for the definition of cat′ comes from the Lusternik–Schnirelmann definition of LS-category for topological spaces. Essentially, the above definition says that $\mathrm{cat}'_{\mathscr{C}} A \le m$ if A can be made up of $m + 1$ objects which are contractible in A. In the category of topological spaces, it is easy to see that cat′ is exactly LS-category as defined by Lusternik and Schnirelmann.

Remark. (1) If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of morphisms in the category $\mathscr{C}$, then

$$\mathrm{cat}'_{\mathscr{C}}(A; gf) \le \mathrm{cat}'_{\mathscr{C}}(F; f).$$

(2) Suppose there exists a weak equivalence $f: A \xrightarrow{\simeq} B$ in $\mathscr{C}$. Then, if

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ W & \xrightarrow{\delta} & A \end{array}$$

is a commutative homotopy push-out, then so is

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle f\gamma} \\ W & \xrightarrow{f\delta} & B \end{array}$$

Thus, $\mathrm{cat}'_{\mathscr{C}}(B) \leq \mathrm{cat}'_{\mathscr{C}}(A)$.

Before demonstrating that cat′ agrees with Doeraene's definition of LS-category in an arbitrary J-category, we need the following proposition describing the behavior of cat′ under retraction. An immediate consequence of this proposition is that cat′ is a homotopy invariant.

Proposition 2.1. *Let $\mathscr{C}$ be a J-category. Let $A \xrightarrow{i} B \xrightarrow{r} A$ be morphisms in $\mathscr{C}$ satisfying $r \circ i \simeq 1_A$. Then $\mathrm{cat}'_{\mathscr{C}} A \leq \mathrm{cat}'_{\mathscr{C}} B$.*

Proof. Suppose that $\mathrm{cat}'_{\mathscr{C}} B = m$. Then there is a commutative homotopy push-out

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & W \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ V & \xrightarrow{\gamma} & B \end{array}$$

where $\mathrm{cat}'_{\mathscr{C}}(W; \delta) = m - 1$ and $\mathrm{cat}'_{\mathscr{C}}(V; \gamma) = 0$, i.e., $\gamma \simeq 0$.

Choose an f-factorization of i

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ & {\scriptstyle \simeq\tau}\searrow \quad \nearrow\!\!\!\nearrow{\scriptstyle \pi} & \\ & C & \end{array}$$

and consider the diagram

$$\begin{array}{ccc} U\times_B C & \xrightarrow{\bar{\alpha}} & W\times_B C \\ \downarrow{\scriptstyle \bar{\beta}} & & \downarrow{\scriptstyle \bar{\delta}} \\ V\times_B C & \xrightarrow{\bar{\gamma}} & C \end{array} \tag{1}$$

We would like to show that this diagram is a commutative homotopy push-out and that $\pi\bar{\gamma} \simeq 0$. Note that $r\pi$ is a weak equivalence, since $r\pi\tau \simeq 1_A$. Here, we are using the fact that a morphism homotopic to a weak equivalence is a weak equivalence [2].

Since $\gamma \simeq 0$, there is a cylinder I on V

$$\begin{array}{ccc} V \vee V & \xrightarrow{\nabla} & V \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle p\simeq} & \\ & I & \end{array}$$

for which there exists a morphism $H: I \to B$ such that

$$\begin{array}{ccc} V \vee V & \xrightarrow{(\gamma, 0)} & B \\ & {\scriptstyle j}\searrow & \nearrow{\scriptstyle H} \\ & I & \end{array}$$

commutes. Take the pull-back

$$\begin{array}{ccc} I \times_B C & \xrightarrow{\bar{p}} & V \times_B C \\ \Downarrow \pi & & \Downarrow \pi \\ I & \xrightarrow{p} & V \end{array}$$

Since p is a weak equivalence, axiom (J2) implies that $\bar{p}$ is also a weak equivalence. Furthermore, $I \times_B C$ fits into the pull-back diagram below, in which both squares and the entire rectangle are pull-backs.

$$\begin{array}{ccccc} I \times_B C & \xrightarrow{\bar{p}} & V \times_B C & \xrightarrow{\bar{\gamma}} & C \\ \Downarrow \pi & & \Downarrow \pi & & \Downarrow \pi \\ I & \xrightarrow{p} & V & \xrightarrow{\gamma} & B \end{array}$$

Because

$$pj \circ (\pi \vee \pi) = \pi \nabla : (V \times_B C) \vee (V \times_B C) \to V,$$

there is an induced map

$$\bar{j} : (V \times_B C) \vee (V \times_B) \to I \times_B C$$

such that

$$\bar{p}\bar{j} = \nabla : (V \times_B C) \vee (V \times_B C) \to V \times_B C$$

and

$$\pi \bar{j} = j(\pi \vee \pi) : (V \times_B C) \vee (V \times_B C) \to I.$$

Factor $\bar{j}$ as

$$\begin{array}{ccc} (V \times_B C) \vee (V \times_B C) & \xrightarrow{\bar{j}} & V \times_B C \\ & {\scriptstyle j'}\searrow & \nearrow{\scriptstyle \sigma \simeq} \\ & J & \end{array}$$

where j' is a cofibration and σ is a weak equivalence. Then

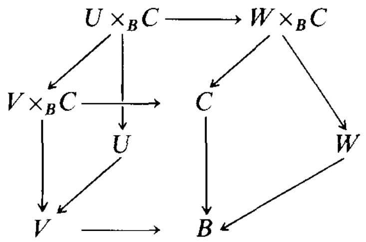
$$\begin{array}{ccc} (V\times_B C)\vee(V\times_B C) & \xrightarrow{\nabla} & V\times_B C \\ & {}_{j'}\searrow \quad \nearrow_{\bar{p}\sigma\,\simeq} & \\ & J & \end{array}$$

is a cylinder on $V\times_B C$ and

$$\begin{array}{ccc} (V\times_B C)\vee(V\times_B C) & \xrightarrow{(\pi\bar{\gamma},0)} & B \\ & {}_{j'}\searrow \quad \nearrow_{H\pi\sigma} & \\ & J & \end{array}$$

commutes. Thus $\pi\bar{\gamma}\simeq 0$.

To show that diagram (1) is a homotopy push-out, consider the cubic diagram



in which, by hypothesis, the bottom face is a homotopy push-out. Each of the sides is the pull-back of a fibration and hence a homotopy pull-back. Thus, by the cube axiom, the top face must be a homotopy push-out. Therefore, since $r\pi$ is a weak equivalence,

$$\begin{array}{ccc} U\times_B C & \xrightarrow{\bar{\alpha}} & W\times_B C \\ \downarrow{\scriptstyle \bar{\beta}} & & \downarrow{\scriptstyle r\pi\bar{\delta}} \\ V\times_B C & \xrightarrow{r\pi\bar{\gamma}} & A \end{array}$$

is a homotopy push-out in which $r\pi\bar{\gamma}\simeq 0$.

To complete the proof, note that we can demonstrate inductively that if $\mathrm{cat}'_{\mathscr{C}}(W;\bar{\delta})=m-1$, then $\mathrm{cat}'_{\mathscr{C}}(W\times_B C;\pi\bar{\delta})\leq m-1$ and hence $\mathrm{cat}'_{\mathscr{C}}(W\times_B C;\, r\pi\bar{\delta})\leq m-1$. The proof that $\pi\bar{\gamma}\simeq 0$ and that diagram (1) is a homotopy push-out is the base step in the inductive process. We also make use of the fact that if

$$\begin{array}{ccc} Z & \xrightarrow{\zeta} & W \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \theta} \\ X & \xrightarrow{\eta} & W \end{array}$$

is a commutative homotopy push-out, then

$$\begin{array}{ccc} Z\times_B C & \xrightarrow{\bar{\zeta}} & Y\times_B C \\ \downarrow{\scriptstyle \bar{\varepsilon}} & & \downarrow{\scriptstyle \bar{\theta}} \\ X\times_B C & \xrightarrow{\bar{\eta}} & W\times_B C \end{array}$$

is also a commutative homotopy push-out, according to the cube axiom.

Thus, $\mathrm{cat}'_{\mathscr{C}}A \le m = \mathrm{cat}'_{\mathscr{C}}B$. □

We wish now to compute an upper bound for cat′ of Ganea objects, as a first step in demonstrating that cat′ agrees with Doeraene's LS category.

Lemma 2.2. *Let $\mathscr{C}$ be a J-category. Suppose that $B\in \mathrm{Ob}\,\mathscr{C}$ is 0-fibrant. Then $\mathrm{cat}'_{\mathscr{C}}G^nB \le n$ for all n.*

Proof. We will prove this lemma inductively. It is obvious that $\mathrm{cat}'_{\mathscr{C}}G^0B = 0$. Suppose that $\mathrm{cat}'_{\mathscr{C}}G^{n-1}B \le n-1$.

Recall the procedure for constructing G^nB. Begin by factoring g^{n-1}, as weak equivalence followed by a fibration,

$$\begin{array}{ccc} G^{n-1}B & \xrightarrow{g^{n-1}} & B \\ & {\scriptstyle \simeq}\searrow \quad \nearrow\!\!\!\nearrow{\scriptstyle \pi} & \\ & E^{n-1} & \end{array}$$

Take the pull-back

$$\begin{array}{ccc} F^{n-1} & \xrightarrow{i} & E^{n-1} \\ \Downarrow{\scriptstyle \bar{\pi}} & & \Downarrow{\scriptstyle \pi} \\ 0 & \longrightarrow & B \end{array}$$

and then choose a c-factorization of $\bar{\pi}$,

$$\begin{array}{ccc} F^{n-1} & \xRightarrow{\bar{\pi}} & 0 \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \simeq} & \\ & W & \end{array}$$

Finally, take the push-out of j and i,

$$\begin{array}{ccc} F^{n-1} & \xrightarrow{i} & E^{n-1} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}} \\ W & \xrightarrow{\bar{i}} & G^nB \end{array}$$

It is clear, according to remarks (1) and (2) above, that

$$\mathrm{cat}'_{\mathscr{C}}(E^{n-1};\bar{j}) \leq \mathrm{cat}'_{\mathscr{C}}(E^{n-1}) \leq \mathrm{cat}'_{\mathscr{C}}(G^{n-1}B) \leq n-1$$

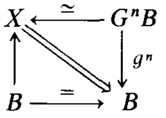
and that $\mathrm{cat}'_{\mathscr{C}}(W;\bar{\imath}) \leq \mathrm{cat}'_{\mathscr{C}}(W)$. Since $W \xrightarrow{\simeq} 0$ is a weak equivalence and $0 \to W \to 0$ is an isomorphism, $0 \to W$ is also a weak equivalence. Thus $\mathrm{cat}'_{\mathscr{C}} W \leq \mathrm{cat}'_{\mathscr{C}} 0 = 0$. Therefore, $\mathrm{cat}'_{\mathscr{C}}(G^nB) \leq n$. □

For the remainder of this section, $\mathscr{C}$ will denote a J-category satisfying in addition:
(M1) Any fibration can be factored as a cofibration followed by a trivial fibration.
(M2) Fibrations have the right lifting property with respect to trivial cofibrations.
Axioms (M1) and (M2) hold in any closed model category.

We are now ready to prove the following theorem:

Theorem 2.3. *Let $B \in \mathrm{Ob}\,\mathscr{C}$ be cofibrant and 0-fibrant. Then* $\mathrm{cat}'_{\mathscr{C}}B = \mathrm{cat}_{\mathscr{C}}B$.

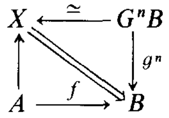
Proof. Suppose that $\mathrm{cat}'_{\mathscr{C}}B = n$. Then there is a commuting diagram



in which the lower triangle commutes up to homotopy and the diagonal arrow is a fibration. Then, by Proposition 2.1 and remark (2) above,

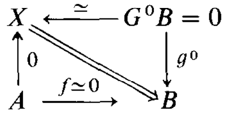
$$\mathrm{cat}'_{\mathscr{C}}B \leq \mathrm{cat}'_{\mathscr{C}}X \leq \mathrm{cat}'_{\mathscr{C}}G^nB \leq n = \mathrm{cat}'_{\mathscr{C}}B.$$

We next claim that if A and B are cofibrant and 0-fibrant, $f: A \to B$, and $\mathrm{cat}'_{\mathscr{C}}(A;f) = n$, then there is a diagram



in which the upper triangle is an f-factorization of g^n and the lower triangle commutes up to homotopy. It then follows immediately that $\mathrm{cat}'_{\mathscr{C}}A = n$ implies that $\mathrm{cat}_{\mathscr{C}}A \leq n$.

It is certainly true that if $\mathrm{cat}'_{\mathscr{C}}(A;f) = 0$, then



is the required diagram. Assume that the claim holds for $n-1$.

Suppose that $\mathrm{cat}'_{\mathscr{C}}(A; f) = n$, i.e. that there is a homotopy push-out

$$\begin{array}{ccc} U & \xrightarrow{\;i\;} & W \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\;h\;} & A \end{array} \tag{2}$$

where $\mathrm{cat}'_{\mathscr{C}}(W; fg) = n - 1$ and $\mathrm{cat}'_{\mathscr{C}}(V; fh) = 0$. Thus by induction, there are diagrams

$$\begin{array}{ccc} X & \xleftarrow{\;\simeq\;} & 0 \\ \uparrow & \searrow\!\!\searrow & \downarrow{\scriptstyle g^0} \\ W & \xrightarrow{\;fg\;} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xleftarrow{\;\simeq\;} & G^{n-1}B \\ \uparrow & \searrow\!\!\searrow & \downarrow{\scriptstyle g^{n-1}} \\ W & \xrightarrow{\;fh\;} & B \end{array}$$

in which the upper triangles are f-factorizations and the lower triangles commute up to homotopy. By Theorem 9.1 in [3], we can assume that X and Y are cofibrant and that the lower triangles actually commute exactly.

Notice that if $\alpha : A' \xrightarrow{\simeq} A$ is a weak equivalence, where A and A' are cofibrant and 0-fibrant, then there exists a homotopy inverse $\beta : A \to A'$ to α. Thus, if there is a diagram

$$\begin{array}{ccc} Z & \xleftarrow{\;\simeq\;} & G^nB \\ \uparrow{\scriptstyle \phi} & \searrow\!\!\searrow & \downarrow{\scriptstyle g^n} \\ A' & \xrightarrow{\;f\alpha\;} & B \end{array}$$

in which the upper triangle is an f-factorization and the lower triangle commutes up to homotopy, then in the diagram

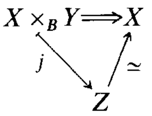
$$\begin{array}{ccc} Z & \xleftarrow{\;\simeq\;} & G^nB \\ \uparrow{\scriptstyle \phi\beta} & \searrow\!\!\searrow & \downarrow{\scriptstyle g^n} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

the lower triangle commutes up to homotopy. Thus, we can assume without loss of generality that diagram (2) is a push-out.

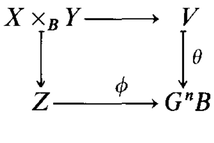
From the pull-back

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \Downarrow & & \Downarrow \\ X & \longrightarrow & B \end{array}$$

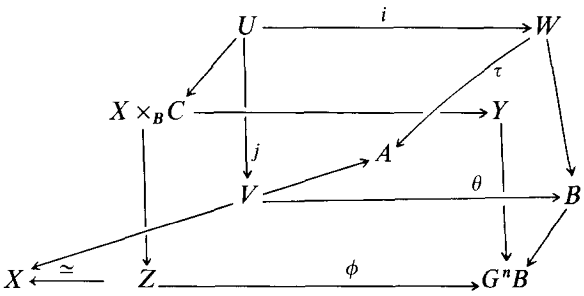
then factor



as a cofibration followed by a trivial fibration. Take the push-out

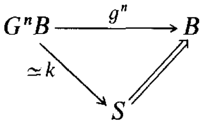


and consider the following commuting diagram:

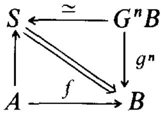


Since X is cofibrant, there is a section $X \to Z$ and hence a morphism $\sigma : V \to Z$ such that the two ways of getitng from U to G^nB, $\phi\sigma j$ and $\theta\tau i$, are homotopic.

Note that all 0-fibrant objects are fibrant, since $\mathscr{C}$ satisfies (M2) and is pointed. Thus for any choice of f-factorization



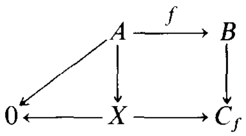
The object S is fibrant, since B, and therefore S, are 0-fibrant. Now, because S is fibrant and j is a cofibration, we can apply the homotopy extension property [2, II.2.17] to obtain a map $F : V \to S$ such that $Fj = k\theta\tau i : U \to S$. Thus, since A is the push-out of i and j, there is an induced map $A \to S$ such that the diagram



commutes. □

The following proposition is now an immediate consequence of the new LS-category definition. Doeraene proved this proposition in [3], based on the earlier two definitions, but the proof there is long and complex.

Proposition 2.4. *Given* $f: A \to B$, *a morphism in* $\mathscr{C}$, *let* $\mathscr{C}_f$ *denote the following push-out*:



Then $\text{cat}_{\mathscr{C}} C_f \leq \text{cat}_{\mathscr{C}} B + 1$. □

References

[1] J.F. Adams and P.J. Hilton, On the chain algebra of a loop space, Comment. Math. Helv. 30 (1955) 305–330.
[2] H.J. Baues, Algebraic homotopy (Cambridge University Press, Cambridge 1989).
[3] J.-P. Doeraene, LS-catégorie dans une catégorie à modèles, Doctoral Thesis, Université Catholique de Louvain, 1990.
[4] Y. Félix and S. Halperin, Rational L.-S. category and its applications, Trans. Amer. Math. Soc. 273 (1982) 1–37.
[5] T. Ganea, A generalisation of the homology and homotopy suspension, Comment. Math. Helv. 39 (1965) 295–322.
[6] T. Ganea, Lusternik–Schnirelmann category and strong category, Illinois J. Math. 11 (1967) 417–427.
[7] S. Halperin and J.-M. Lemaire, Notions of category in differential algebra, Lecture Notes in Mathematics, Vol. 1318 (1986) 138–154.
[8] I. James, On category, in the sense of Lusternik–Schnirelmann, Topology 17 (1978) 331–348
[9] J.-M. Lemaire, Lusternik–Schnirelmann category: an introduction, Lecture Notes in Mathematics, Vol. 1183 (Springer, Berlin, 1986) 259–276.
[10] L. Lusternik and L. Schnirelmann, Méthodes Topologiques dans les Problèmes Variationnels (Hermann, Paris, 1934).
[11] D. Quillen, Rational homotopy theory, Ann. Math. (2) 90 (1969) 205–295.
[12] D. Sullivan, Infinitesimal computation in topology, Publ. I.H.E.S. 47 (1977) 269–331.
[13] G. Whitehead, The homology suspension, Colloque de Topologie Algébrique de Louvain (1956) 89–95.