# The existence of countably many positive solutions for nonlinear singular $m$-point boundary value problems on the half-line ${ }^{\star \pi}$ 

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#### Abstract

In this paper, by introducing a new operator, improving and generating a $p$-Laplace operator for some $p>1$, we study the existence of countably many positive solutions for nonlinear boundary value problems on the half-line $$
\begin{aligned} & \left(\varphi\left(u^{\prime}\right)\right)^{\prime}+a(t) f(u(t))=0, \quad 0<t<+\infty \\ & u(0)=\sum_{i=1}^{m-2} \alpha_{i} u\left(\xi_{i}\right), \quad u^{\prime}(\infty)=0 \end{aligned}
$$ where $\varphi: R \rightarrow R$ is the increasing homeomorphism and positive homomorphism and $\varphi(0)=0$. We show the sufficient conditions for the existence of countably many positive solutions by using the fixed-point index theory and a new fixed-point theorem in cones. (c) 2008 Published by Elsevier B.V.

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[^0]
## 1. Introduction

In this paper, we consider the existence of countable many positive solutions of the following boundary value problem on a half-line

$$
\begin{align*}
& \left(\varphi\left(u(t)^{\prime}\right)\right)^{\prime}+a(t) f(u(t))=0, \quad 0<t<+\infty  \tag{1.1}\\
& u(0)=\sum_{i=1}^{m-2} \alpha_{i} u\left(\xi_{i}\right), \quad u^{\prime}(\infty)=0 \tag{1.2}
\end{align*}
$$

where $\varphi: R \rightarrow R$ is the increasing homeomorphism and positive homomorphism and $\varphi(0)=0, \xi_{i} \in(0,+\infty)$ with $0<\xi_{1}<\xi_{2}<\cdots<\xi_{m-2}<+\infty$ and $\alpha_{i}$ satisfy $\alpha_{i} \in[0,+\infty), 0<\sum_{i=1}^{m-2} \alpha_{i}<1, f \in$ $C([0,+\infty),[0,+\infty)), a(t):[0,+\infty) \rightarrow[0,+\infty)$ and has countably many singularities in $[0,+\infty)$.

A projection $\varphi: R \rightarrow R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:
(1) if $x \leq y$, then $\varphi(x) \leq \varphi(y)$, for all $x, y \in R$;
(2) $\varphi$ is a continuous bijection and its inverse mapping is also continuous;
(3) $\varphi(x y)=\varphi(x) \varphi(y)$, for all $x, y \in[0,+\infty)$.

In the above definition, we can replace condition (3) by the following stronger condition:
(4) $\varphi(x y)=\varphi(x) \varphi(y)$, for all $x, y \in R$, where $R=(-\infty,+\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then it implies that $\varphi$ is homogeneous generating a $p$-Laplace operator, i.e. $\varphi(x)=|x|^{p-2} x$, for some $p>1$.

In this paper, some of the following hypotheses are satisfied:
$\left(\mathrm{C}_{1}\right) f \in C([0,+\infty),[0,+\infty)), f(u(t)) \not \equiv 0$ on any subinterval of $(0,+\infty)$ and when $u$ is bounded $f((1+t) u(t))$ is bounded on $[0,+\infty)$;
$\left(\mathrm{C}_{2}\right)$ There exists a sequence $\left\{t_{i}\right\}_{i=1}^{\infty}$ such that $1 \leq t_{i+1}<t_{i}, \lim _{i \rightarrow \infty} t_{i}=t_{0}<+\infty$, and $t_{0}>1 . \lim _{t \rightarrow t_{i}} a(t)=$ $\infty, i=1,2, \ldots$, and

$$
\begin{equation*}
0<\int_{0}^{+\infty} a(t) \mathrm{d} t<+\infty, \int_{0}^{+\infty} \varphi^{-1}\left(\int_{\tau}^{+\infty} a(s) \mathrm{d} s\right) \mathrm{d} \tau<+\infty \tag{1.3}
\end{equation*}
$$

$\left(\mathrm{C}_{3}\right)$ There exists a sequence $\left\{t_{i}\right\}_{i=1}^{\infty}$ such that $0<t_{i+1}<t_{i}<1, \lim _{i \rightarrow \infty} t_{i}=t_{0}<+\infty$, and $0<t_{0}<1$. $\lim _{t \rightarrow t_{i}} a(t)=\infty, i=1,2, \ldots$, and (1.3) holds. Moreover, $a(t)$ does not vanish identically on any subinterval of $[0,+\infty)$.
In recent years, the existence and multiplicity of positive solutions for the $p$-Laplacian operator, i.e. $\varphi(x)=$ $|x|^{p-2} x$, for some $p>1$ have received wide attention, please see $[6-9,11,13,16,18]$ and references therein. However for the increasing homeomorphism and positive homomorphism operator the research has proceeded very slowly. In [2], Liu and Zhang study the existence of positive solutions of quasilinear differential equation

$$
\begin{aligned}
& \left(\varphi\left(x^{\prime}\right)\right)^{\prime}+a(t) f(x(t))=0, \quad 0<t<1 \\
& x(0)-\beta x^{\prime}(0)=0, \quad x(1)+\delta x^{\prime}(1)=0,
\end{aligned}
$$

where $\varphi: R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0)=0$. They obtain the existence of one or two positive solutions by using a fixed-point index theorem in cones. But, whether or not we can obtain countable many positive solutions of $m$-point boundary value problem on the half-line (1.1) and (1.2) still remain unknown. So the goal of the present paper is to improve and generate $p$-Laplacian operator and establish some criteria for the existence of countable many solutions.

The motivation for the present work stems from both the practical and theoretical aspects. In fact, boundary value problems on the half-line occur naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see [4,12], and various physical phenomena [3,10], such as unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, determining the electrical potential in an isolated neutral atom. In all these applications, it is frequent that only solutions that are positive are useful. Recently, many papers have been published that investigate the positive solutions of boundary value problem on the half-line, see [1,14-17]. They discuss the
existence and multiplicity (of at least three) positive solutions to the nonlinear differential equation. However, to the best knowledge of the authors, there is no paper concerned with the existence of countable many positive solutions to the boundary value problems of differential equation on infinite intervals so far.

So in this paper, we use fixed-point index theory and a new fixed-point theorem in cones to investigate the existence of countable solutions to boundary value problems (1.1) and (1.2).

The plan of the paper is as follows. In Section 2, for the convenience of the reader we give some definitions. In Section 3, we present some lemmas in order to prove our main results. Section 4 is developed to presenting and proving our main results. In Section 5 we present the example of the increasing homeomorphism and positive homomorphism operators.

## 2. Some definitions and fixed-point theorems

In this section, we provide some background definitions cited from cone theory in Banach spaces.
Definition 2.1. Let $(E,\|\|$.$) be a real Banach space. A nonempty, closed, convex set P \subset E$ is said to be a cone provided the following are satisfied:
(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y=0$.

If $P \subset E$ is a cone, we denote the order induced by $P$ on $E$ by $\leq$, that is, $x \leq y$ if and only if $y-x \in P$.
Definition 2.2. A map $\alpha$ is said to be a nonnegative, continuous, concave functional on a cone $P$ of a real Banach space $E$, if

$$
\alpha: P \rightarrow[0, \infty)
$$

is continuous, and

$$
\alpha(t x+(1-t) y) \geq t \alpha(x)+(1-t) \alpha(y)
$$

for all $x, y \in P$ and $t \in[0,1]$.
Definition 2.3. Given a nonnegative continuous functional $\gamma$ on a cone $P$ of $E$, for each $d>0$ we define the set

$$
P(\gamma, d)=\{x \in P: \gamma(x)<d\}
$$

The following fixed-point theorems are fundamental and important to the proofs of our main results.
Theorem 2.1 ([5]). Let $E$ be a Banach space and $P \subset E$ be a cone in $E$. Let $r>0$ define $\Omega_{r}=\{x \in P:\|x\|<r\}$. Assume that $T: P \bigcap_{\Omega_{r}} \rightarrow P$ is a completely continuous operator such that $T x \neq x$ for $x \in \partial \Omega_{r}$.
(i) If $\|T x\| \leq\|x\|$ for $x \in \partial \Omega_{r}$, then

$$
i\left(A, \Omega_{r}, P\right)=1
$$

(ii) If $\|T x\| \geq\|x\|$ for $x \in \partial \Omega_{r}$, then

$$
i\left(A, \Omega_{r}, P\right)=0
$$

Theorem 2.2 ([18]). Let $P$ be a cone in a Banach space E. Let $\alpha, \beta$ and $\gamma$ be three increasing, nonnegative and continuous functionals on $P$, satisfying for some $c>0$ and $M>0$ such that

$$
\gamma(x) \leq \beta(x) \leq \alpha(x), \quad\|x\| \leq M \gamma(x)
$$

for all $x \in \overline{P(\gamma, c)}$. Suppose that there exists a completely continuous operator $T: \overline{P(\gamma, c)} \rightarrow P$ and $0<a<b<c$ such that
(i) $\gamma(T x)<c$, for all $x \in \partial P(\gamma, c)$;
(ii) $\beta(T x)>b$, for all $x \in \partial P(\beta, b)$;
(iii) $P(\alpha, a) \neq \emptyset$, and $\alpha(T x)<a$, for all $x \in \partial P(\alpha, a)$.

Then $T$ has at least three fixed points $x_{1}, x_{2}, x_{3} \in \overline{P(\gamma, c)}$ such that

$$
0 \leq \alpha\left(x_{1}\right)<a<\alpha\left(x_{2}\right), \quad \beta\left(x_{2}\right)<b<\beta\left(x_{3}\right), \quad \gamma\left(x_{3}\right)<c
$$

## 3. Preliminaries and lemmas

In this paper, we will use the following space $E$ to the study of (1.1) and (1.2), which is denoted by

$$
E=\left\{u \in C[0,+\infty): \sup _{0 \leq t<+\infty} \frac{|u(t)|}{1+t}<+\infty\right\} .
$$

Then $E$ is a Banach space, equipped with the norm $\|u\|=\sup _{0 \leq t<+\infty} \frac{|u(t)|}{1+t}<+\infty$.
Define cone $K \subset E$ by

$$
K=\{u \in E: u(t) \text { is a nondecreasing and nonnegative concave function on }[0,+\infty)\} .
$$

Lemma 3.1. Suppose that $\left(\mathrm{C}_{2}\right)$ holds. Then
(i) for any constant $\theta \in(1,+\infty)$ which satisfies

$$
0<\int_{\frac{1}{\theta}}^{\theta} a(t) \mathrm{d} t<+\infty
$$

(ii) the function

$$
H(t)=\int_{t}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{\frac{1}{t_{0}}}^{t} \varphi^{-1}\left(\int_{s}^{t} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}
$$

is continuous and positive on $\left[\frac{1}{t_{0}}, t_{0}\right]$. Furthermore,

$$
L=\min _{t \in\left[\frac{1}{t_{0}}, t_{0}\right]} H(t)>0
$$

Proof. Firstly we can easily obtain (i) from the condition $\left(\mathrm{C}_{2}\right)$.
Next we prove that conclusion (ii) is also true. It is easily seen that $H(t)$ is continuous on $\left[\frac{1}{t_{0}}, t_{0}\right]$. Let

$$
\begin{aligned}
& H_{1}(t)=\int_{t}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s \\
& H_{2}(t)=\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{\frac{1}{t_{0}}}^{t} \varphi^{-1}\left(\int_{s}^{t} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}
\end{aligned}
$$

Then from condition $\left(\mathrm{C}_{2}\right)$, we know that $H_{1}(t)$ is strictly monotone decreasing on $\left[\frac{1}{t_{0}}, t_{0}\right]$ and $H_{1}\left(t_{0}\right)=0$. Similarly function $H_{2}(t)$ is strictly monotone increasing on $\left[\frac{1}{t_{0}}, t_{0}\right]$ and $H_{2}\left(\frac{1}{t_{0}}\right)=0$. Since $H_{1}(t)$ and $H_{2}(t)$ are not equal to zero at the same time. So the function $H(t)=H_{1}(t)+H_{2}(t)$ is positive on $\left[\frac{1}{t_{0}}, t_{0}\right]$, which implies that $L=\min _{t \in\left[\frac{1}{t_{0}}, t_{0}\right]} H(t)>0$.

Lemma 3.2. Let $u \in K$ and $[a, b]$ is any finite closed interval of $(0,+\infty)$, then $u(t) \geq \lambda(t)\|u\|$, where

$$
\lambda(t)= \begin{cases}\sigma, & t \geq \sigma, \\ t, & t \leq \sigma,\end{cases}
$$

and $\sigma=\inf \left\{\xi \in[0,+\infty): \sup _{t \in[0,+\infty)} \frac{|u(t)|}{1+t}=\frac{u(\xi)}{1+\xi}\right\}$.

Proof. From the definition of $K$, we know that $u(t)$ is increasing on $[0,+\infty)$. Moreover the function $\frac{u(t)}{1+t}$ achieves its maximum at $\xi \in[0,+\infty)$. So we divide the proof into three steps:
Step (1). If $\sigma \in[0, a]$, then we have $t \geq \sigma$, for $t \in[a, b]$. Since $u(t)$ is increasing on $[0,+\infty)$. So we have

$$
u(t) \geq u(\sigma)=(1+\sigma)\|u\|>\sigma\|u\|, \quad \text { for } t \in[a, b] .
$$

Step (2). If $\sigma \in[a, b]$, then we have $t \leq \sigma$, for $t \in[a, \sigma]$. By the concavity of $u(t)$, we can obtain

$$
\frac{u(t)-u(0)}{t} \geq \frac{u(\sigma)-u(0)}{\sigma}
$$

i.e.,

$$
\frac{u(t)}{t} \geq \frac{u(\sigma)}{\sigma}-\frac{u(0)}{\sigma}+\frac{u(0)}{t} \geq \frac{u(\sigma)}{1+\sigma}=\|u\| .
$$

Therefore $u(t) \geq t\|u\|$, for $a \leq t \leq \sigma$. If $t \in[\sigma, b]$, similar to Step (1), we have

$$
u(t) \geq \sigma\|u\|, \quad \text { for } \sigma \leq t \leq b
$$

Step (3). If $\sigma \in[b,+\infty)$. Similarly by the concavity of $u(t)$, we also have

$$
\frac{u(t)-u(0)}{t} \geq \frac{u(\sigma)-u(0)}{\sigma}
$$

which yields $u(t) \geq t\|u\|$, for $a \leq t \leq b \leq \sigma$. The proof is complete.
Remark 3.1. It is easy to see that
(i) $\lambda(t)$ is nondecreasing on $[a, b]$;
(ii) $0<\lambda(t)<1$, for $t \in[a, b] \subset(0,1)$.

Now, we define an operator $T: K \rightarrow C[0,+\infty)$ by

$$
\begin{equation*}
(T u)(t)=\int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \tag{3.1}
\end{equation*}
$$

Obviously, $(T u)(t) \geq 0$, for $t \in(0,+\infty)$ and $(T u)^{\prime}(t)=\varphi^{-1}\left(\int_{t}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \geq 0$, furthermore $\left(\varphi(T u)^{\prime}(t)\right)^{\prime}=-a(t) f(u(t)) \leq 0$. This shows that $(T K) \subset K$.

To obtain the complete continuity of $T$, the following lemma is still needed.
Lemma 3.3 ([17]). Let $W$ be a bounded subset of $K$. Then $W$ is relatively compact in $E$ if $\left\{\frac{W(t)}{1+t}\right\}$ are equicontinuous on any finite subinterval of $[0,+\infty)$ and for any $\varepsilon>0$, there exists $N>0$ such that

$$
\left|\frac{x\left(t_{1}\right)}{1+t_{1}}-\frac{x\left(t_{2}\right)}{1+t_{2}}\right|<\varepsilon
$$

uniformly with respect to $x \in W$ as $t_{1}, t_{2} \geq N$, where $W(t)=\{x(t): x \in W\}, t \in[0,+\infty)$.
Lemma 3.4. Let $\left(\mathrm{C}_{1}\right)$ and $\left(\mathrm{C}_{2}\right)$ or $\left(\mathrm{C}_{3}\right)$ hold. Then $T: K \rightarrow K$ is completely continuous.
Proof. Firstly, it is easy to check that $T: K \rightarrow K$ is well defined. From the definition of $E$, we can choose $r_{0}$ such that $\sup _{n \in N \backslash\{0\}}\left\|u_{n}\right\|<r_{0}$. Let $B_{r_{0}}=\sup \left\{f((1+t) u), t \in[0,+\infty), u \in\left[0, r_{0}\right]\right\}$ and $\Omega$ be any bounded subset of $K$. Then there exist $r>0$ such that $\|u\| \leq r$, for all $u \in \Omega$. Therefore we have

$$
\|T u\|=\sup _{t \in[0,+\infty)} \frac{1}{1+t}\left|\int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right|
$$

$$
\begin{aligned}
& \leq \sup _{t \in[0,+\infty)} \frac{1}{1+t} \int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& \quad+\sup _{t \in[0,+\infty)} \frac{1}{1+t} \frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{m-2}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
& \leq \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)\left(1+\frac{\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right) \\
& \leq C \varphi^{-1}\left(B_{r}\right), \quad \forall u \in \Omega .
\end{aligned}
$$

So $T \Omega$ is bounded. Moreover for any $T \in(0,+\infty)$ and $t_{1}, t_{2} \in[0, T]$, we have

$$
\begin{aligned}
& \left|\frac{(T u)\left(t_{1}\right)}{1+t_{1}}-\frac{(T u)\left(t_{2}\right)}{1+t_{2}}\right| \leq \frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{\left(1-\sum_{i=1}^{m-2} \alpha_{i}\right)}\left|\frac{1}{1+t_{1}}-\frac{1}{1+t_{2}}\right| \\
& \quad+\left|\frac{1}{1+t_{1}} \int_{0}^{t_{1}} \varphi^{-1}\left(\int_{\tau}^{+\infty} a(s) f(u(s)) \mathrm{d} s\right) \mathrm{d} \tau-\frac{1}{1+t_{2}} \int_{0}^{t_{2}} \varphi^{-1}\left(\int_{\tau}^{+\infty} a(s) f(u(s)) \mathrm{d} s\right) \mathrm{d} \tau\right| \\
& \quad \leq C \varphi^{-1}\left(B_{r}\right)\left|\frac{1}{1+t_{1}}-\frac{1}{1+t_{2}}\right|+C \varphi^{-1}\left(B_{r}\right)\left|t_{1}-t_{2}\right| \\
& \rightarrow 0, \quad \text { uniformly as } t_{1} \rightarrow t_{2} .
\end{aligned}
$$

We can get that $T \Omega$ is equicontinuous on any finite subinterval of $[0,+\infty)$.
Next we prove for any $\varepsilon>0$, there exists sufficiently large $N>0$ such that

$$
\begin{equation*}
\left|\frac{(T u)\left(t_{1}\right)}{1+t_{1}}-\frac{(T u)\left(t_{2}\right)}{1+t_{2}}\right|<\varepsilon, \quad \text { for all } t_{1}, t_{2} \geq N, \forall u \in \Omega \tag{3.2}
\end{equation*}
$$

Since $\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau<+\infty$. Therefore we can choose $N_{1}>0$ such that

$$
\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{N_{1}\left(1-\sum_{i=1}^{m-2} \alpha_{i}\right)}<\frac{\varepsilon}{5} .
$$

We can also select $N_{2}, N_{3}>0$ are satisfied respectively

$$
N_{2}>\frac{5 \int_{0}^{+\infty} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{\varepsilon}, \varphi^{-1}\left(\int_{N_{3}}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)<\frac{\varepsilon}{5}
$$

Then let $N=\max \left\{N_{1}, N_{2}, N_{3}\right\}$. Without loss generality, we assume $t_{2}>t_{1} \geq N$. So it follow that

$$
\begin{aligned}
& \left|\frac{(T u)\left(t_{1}\right)}{1+t_{1}}-\frac{(T u)\left(t_{2}\right)}{1+t_{2}}\right| \\
& \quad \leq \int_{0}^{+\infty} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s\left|\frac{1}{1+t_{1}}-\frac{1}{1+t_{2}}\right|+\frac{\int_{t_{1}}^{t_{2}} \varphi^{-1}\left(\int_{t_{1}}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1+t_{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \quad+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{\left(1+t_{1}\right)\left(1-\sum_{i=1}^{m-2} \alpha_{i}\right)}+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{\left(1+t_{2}\right)\left(1-\sum_{i=1}^{m-2} \alpha_{i}\right)} \\
& \leq \frac{2 \varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}=\varepsilon .
\end{aligned}
$$

That is (3.2) holds. By Lemma 3.3, $T \Omega$ is relatively compact. Therefore we know that $T$ is a compact operator.
Thirdly we prove that $T$ is continuous. Let $u_{n} \rightarrow u$ as $n \rightarrow+\infty$ in $K$. Then by the Lebesgue dominated convergence theorem and continuity of $f$, we can get

$$
\left|\int_{t}^{+\infty} a(s) f\left(u_{n}(s)\right) \mathrm{d} s-\int_{t}^{+\infty} a(s) f(u(s)) \mathrm{d} s\right| \leq \int_{t}^{+\infty} a(s)\left|f\left(u_{n}(s)\right)-f(u(s))\right| \mathrm{d} s \rightarrow 0, \quad \text { as } n \rightarrow+\infty,
$$

i.e.,

$$
\int_{t}^{+\infty} a(s) f\left(u_{n}(s)\right) \mathrm{d} s \rightarrow \int_{t}^{+\infty} a(s) f(u(s)) \mathrm{d} s, \quad \text { as } n \rightarrow+\infty .
$$

Moreover

$$
\varphi^{-1}\left(\int_{t}^{+\infty} a(s) f\left(u_{n}(s)\right) \mathrm{d} s\right) \rightarrow \varphi^{-1}\left(\int_{t}^{+\infty} a(s) f(u(s)) \mathrm{d} s\right), \quad \text { as } n \rightarrow+\infty .
$$

So

$$
\begin{aligned}
& \left\|T u_{n}-T u\right\| \\
& \quad \leq \sup _{t \in[0,+\infty)} \frac{1}{1+t} \int_{0}^{t}\left|\varphi^{-1}\left(\int_{\tau}^{+\infty} a(s) f\left(u_{n}(s)\right) \mathrm{d} s\right)-\varphi^{-1}\left(\int_{\tau}^{+\infty} a(s) f(u(s)) \mathrm{d} s\right)\right| \mathrm{d} \tau \\
& \quad+\frac{\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\left|\varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f\left(u_{n}(\tau)\right) \mathrm{d} \tau\right)-\varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)\right| \\
& \rightarrow 0, \quad \operatorname{as} n \rightarrow+\infty .
\end{aligned}
$$

Therefore $T$ is continuous. In summary $T: K \rightarrow K$ is completely continuous.

## 4. Main results

For notational convenience, we denote by

$$
\lambda_{1}=\frac{1+t_{0}}{L}>0, \quad \lambda_{2}=\frac{1-\sum_{i=1}^{m-2} \alpha_{i}}{\varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(1-\sum_{i=1}^{m-2} \alpha_{i}+\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}\right)}>0
$$

The main results of this paper are the following.
Theorem 4.1. Suppose that conditions $\left(\mathrm{C}_{1}\right)$ and $\left(\mathrm{C}_{2}\right)$ hold. Let $\left\{\theta_{k}\right\}_{k=1}^{\infty}$ be such that $\theta_{k} \in\left(t_{k+1}, t_{k}\right)(k=1,2, \ldots)$. Let $\left\{r_{k}\right\}_{k=1}^{\infty}$ and $\left\{R_{k}\right\}_{k=1}^{\infty}$ be such that

$$
R_{k+1}<\frac{\lambda\left(\frac{1}{\theta_{k}}\right)}{1+\theta_{k}} r_{k}<r_{k}<m r_{k}<R_{k}, \quad k=1,2,3, \ldots
$$

Furthermore, for each natural number $k$ we assume that $f$ satisfies:
( $\left.\mathrm{C}_{4}\right) f((1+t) u) \geq \varphi\left(m r_{k}\right)$, for all $\frac{\lambda\left(\frac{1}{k_{k}}\right)}{1+\theta_{k}} r_{k} \leq u(t) \leq r_{k}$;
( $\left.\mathrm{C}_{5}\right) f((1+t) u) \leq \varphi\left(M R_{k}\right)$, for all $0 \leq u(t) \leq R_{k}$,
where $m \in\left(\lambda_{1},+\infty\right), M \in\left(0, \lambda_{2}\right)$. Then the boundary value problem (1.1) and (1.2) has infinitely many solutions $\left\{u_{k}\right\}_{k=1}^{\infty}$ such that

$$
r_{k} \leq\left\|u_{k}\right\| \leq R_{k}, \quad k=1,2, \ldots
$$

Proof. Since $1<t_{0} \leq t_{k+1}<\theta_{k}<t_{k}<+\infty, k=1,2, \ldots$, then, for any $k \in N$ and $u \in K$, by Lemma 3.2, we have

$$
\begin{equation*}
u(t) \geq \lambda(t)\|u\|, \quad t \in\left[\frac{1}{\theta_{k}}, \theta_{k}\right] . \tag{4.1}
\end{equation*}
$$

We define the sequences $\left\{\Omega_{1, k}\right\}_{k=1}^{\infty}$ and $\left\{\Omega_{2, k}\right\}_{k=1}^{\infty}$ of open subsets of $E$ as follows:

$$
\begin{aligned}
& \Omega_{1, k}=\left\{u \in K:\|u\|<r_{k}\right\}, \quad k=1,2, \ldots, \\
& \Omega_{2, k}=\left\{u \in K:\|u\|<R_{k}\right\}, \quad k=1,2, \ldots .
\end{aligned}
$$

For a fixed $k$ and $u \in \partial \Omega_{1, k}$. From (4.1) we have

$$
\begin{aligned}
r_{k} & =\|u\|=\sup _{0 \leq t<+\infty} \frac{|u(t)|}{1+t} \geq \sup _{\frac{1}{\theta_{k}} \leq t<\theta_{k}} \frac{u(t)}{1+t} \geq \frac{u(t)}{1+t} \geq \inf _{\frac{1}{\theta_{k}} \leq t<\theta_{k}} \frac{u(t)}{1+t} \\
& \geq \frac{u\left(\frac{1}{\theta_{k}}\right)}{1+\theta_{k}} \geq \frac{\lambda\left(\frac{1}{\theta_{k}}\right)}{1+\theta_{k}}\|u\|=\frac{\lambda\left(\frac{1}{\theta_{k}}\right)}{1+\theta_{k}} r_{k}, \quad \text { for all } t \in\left[\frac{1}{\theta_{k}}, \theta_{k}\right] .
\end{aligned}
$$

By condition ( $\mathrm{C}_{4}$ ), we have

$$
f(u) \geq \varphi\left(m r_{k}\right), \quad \text { for all } t \in\left[\frac{1}{\theta_{k}}, \theta_{k}\right] .
$$

Since $\left(\frac{1}{t_{0}}, t_{0}\right) \subset\left[\frac{1}{\theta_{k}}, \theta_{k}\right]$, if $\left(\mathrm{C}_{2}\right)$ holds, in the following, we consider three cases:
(i) If $\xi_{1} \in\left[\frac{1}{t_{0}}, t_{0}\right]$. In this case, from (3.1), condition ( $\mathrm{C}_{4}$ ) and Lemma 3.1, we have

$$
\begin{aligned}
\|T u\| & =\sup _{t \in[0,+\infty)} \frac{1}{1+t}\left|\int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right| \\
& \geq \frac{1}{1+t_{0}} \int_{\xi_{1}}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{1}{1+t_{0}} \frac{\sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}} \int_{\frac{1}{t_{0}}}^{\xi_{1}} \varphi^{-1}\left(\int_{s}^{\xi_{1}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& \geq \frac{1}{1+t_{0}}\left(m r_{k}\right)\left[\int_{\xi_{1}}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}} \int_{\frac{1}{t_{0}}}^{\xi_{1}} \varphi^{-1}\left(\int_{s}^{\xi_{1}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s\right] \\
& =\frac{m r_{k}}{1+t_{0}} H\left(\xi_{1}\right)>\frac{L m r_{k}}{1+t_{0}} \\
& >r_{k}=\|u\| .
\end{aligned}
$$

(ii) If $\xi_{1} \in\left(0, \frac{1}{t_{0}}\right.$ ). In this case, from (3.1), condition ( $\mathrm{C}_{4}$ ) and Lemma 3.1, we have

$$
\|T u\| \geq \sup _{t \in[0,+\infty)} \frac{1}{1+t} \int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s
$$

$$
\begin{aligned}
& \geq \frac{1}{1+t_{0}} \int_{\frac{1}{t_{0}}}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& \geq \frac{m r_{k}}{1+t_{0}} \int_{\frac{1}{t_{0}}}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s \\
& =\frac{m r_{k}}{1+t_{0}} H\left(\frac{1}{t_{0}}\right)>\frac{L m r_{k}}{1+t_{0}} \\
& >r_{k}=\|u\|
\end{aligned}
$$

(iii) If $\xi_{1} \in\left(t_{0},+\infty\right)$. In this case, from (3.1), condition $\left(\mathrm{C}_{4}\right)$ and Lemma 3.1, we have

$$
\begin{aligned}
\|T u\| & \geq \frac{1}{1+t_{0}} \frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{\frac{1}{t_{0}}}^{t_{0}} \varphi^{-1}\left(\int_{s}^{t_{0}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
& \geq \frac{m r_{k}}{1+t_{0}} H\left(t_{0}\right)>\frac{L m r_{k}}{1+t_{0}} \\
& >r_{k}=\|u\|
\end{aligned}
$$

Thus in all cases, an application of Theorem 2.1, implies that

$$
\begin{equation*}
i\left(T, \Omega_{1, k}, K\right)=0 \tag{4.2}
\end{equation*}
$$

On the another hand, let $u \in \partial \Omega_{2, k}$, we have $\frac{u(t)}{1+t} \leq \sup _{0 \leq t<+\infty} \frac{|u(t)|}{1+t}=\|u\|=R_{k}$, by $\left(\mathrm{C}_{5}\right)$ we have

$$
f(u(t)) \leq \varphi\left(M R_{k}\right), \quad \text { for all } t \in[0,+\infty)
$$

So,

$$
\begin{aligned}
\|T u\| & =\sup _{t \in[0,+\infty)} \frac{1}{1+t}\left|\int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right| \\
& \leq \sup _{t \in[0,+\infty)} \frac{1}{1+t} \int_{0}^{t} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{m-2}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
& \leq \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)+\frac{\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}}{1-\sum_{i=1}^{m-2} \alpha_{i}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \\
& \leq M R_{k}\left[\varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(1+\frac{\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right)\right] \\
& \leq R_{k}=\|u\| .
\end{aligned}
$$

Thus Theorem 2.1 implies that

$$
\begin{equation*}
i\left(T, \Omega_{2, k}, K\right)=1 \tag{4.3}
\end{equation*}
$$

Hence since $r_{k}<R_{k}$ for $k \in N$, (4.2) and (4.3), it follows from the additivity of the fixed-point index that

$$
i\left(T, \Omega_{2, k} \backslash \bar{\Omega}_{1, k}, K\right)=1, \quad \text { for } k \in N .
$$

Thus $T$ has a fixed point in $\Omega_{2, k} \backslash \bar{\Omega}_{1, k}$ such that $r_{k} \leq\left\|u_{k}\right\| \leq R_{k}$. Since $k \in N$ was arbitrary, the proof is completed.

In order to use Theorem 2.2, let $\frac{1}{\theta_{k}}<r_{k}<\theta_{k}$ and $\theta_{k}$ of Theorem 4.1, we define the nonnegative, increasing, continuous functionals $\gamma_{k}, \beta_{k}$, and $\alpha_{k}$ by

$$
\begin{aligned}
& \gamma_{k}(u)=\max _{\frac{1}{\theta_{k}} \leq t \leq r_{k}} u(t)=u\left(r_{k}\right), \\
& \beta_{k}(u)=\min _{r_{k} \leq t \leq \theta_{k}} u(t)=u\left(r_{k}\right), \\
& \alpha_{k}(u)=\max _{\frac{1}{\theta_{k}} \leq t \leq \theta_{k}} u(t)=u\left(\theta_{k}\right) .
\end{aligned}
$$

It is obvious that for each $u \in K$,

$$
\gamma_{k}(u) \leq \beta_{k}(u) \leq \alpha_{k}(u) .
$$

In addition, by Lemma 3.2, for each $u \in K$,

$$
\gamma_{k}(u)=u\left(r_{k}\right) \geq \frac{1}{\theta_{k}}\|u\| .
$$

Thus

$$
\|u\| \leq \theta_{k} \gamma_{k}(u), \quad \text { for all } u \in K .
$$

In the following, we denote by

$$
\begin{aligned}
& \rho_{k}=\varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(\theta_{k}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right), \\
& \eta_{k}=\frac{1}{\theta_{k}} \varphi^{-1}\left(\int_{r_{k}}^{\theta_{k}} a(\tau) \mathrm{d} \tau\right) .
\end{aligned}
$$

Theorem 4.2. Suppose that conditions $\left(\mathrm{C}_{1}\right)$ and $\left(\mathrm{C}_{2}\right)$ hold. Let $\left\{\theta_{k}\right\}_{k=1}^{\infty}$ be such that $\theta_{k} \in\left(t_{k+1}, t_{k}\right)(k=1,2, \ldots)$. Let $\left\{a_{k}\right\}_{k=1}^{\infty},\left\{b_{k}\right\}_{k=1}^{\infty}$ and $\left\{c_{k}\right\}_{k=1}^{\infty}$ be such that

$$
c_{k+1}<a_{k}<\frac{\lambda\left(\frac{1}{\theta_{k}}\right)}{\theta_{k}+1} b_{k}<\lambda\left(\frac{1}{\theta_{k}}\right) b_{k}<c_{k}, \quad \text { and } \quad \rho_{k} b_{k}<\eta_{k} c_{k}, \quad \text { for } k=1,2, \ldots .
$$

Furthermore for each natural number $k$ we assume that $f$ satisfies:
( $\left.\mathrm{C}_{6}\right) f((1+t) u)<\varphi\left(\frac{c_{k}}{\rho_{k}}\right)$, for all $0 \leq u(t) \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} c_{k}$;
(C $\mathrm{C}_{7} f((1+t) u)>\varphi\left(\frac{b_{k}}{\eta_{k}}\right)$, for all $\frac{b_{k}}{1+\theta_{k}} \leq u(t) \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} b_{k}$;
(C $\left.\mathrm{C}_{8}\right) f((1+t) u)<\varphi\left(\frac{a_{k}}{\rho_{k}}\right)$, for all $0 \leq u(t) \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} a_{k}$.
Then the boundary value problem (1.1) and (1.2) has three infinite families of solutions $\left\{u_{1 k}\right\}_{k=1}^{\infty},\left\{u_{2 k}\right\}_{k=1}^{\infty}$ and $\left\{u_{3 k}\right\}_{k=1}^{\infty}$ satisfying

$$
0 \leq \alpha_{k}\left(u_{1 k}\right)<a_{k}<\alpha_{k}\left(u_{2 k}\right), \quad \beta_{k}\left(u_{2 k}\right)<b_{k}<\beta_{k}\left(u_{3 k}\right), \quad \gamma\left(u_{3 k}\right)<c_{k}, \quad \text { for } k \in N .
$$

Proof. We define the completely continuous operator $T$ by (3.1). So it is easy to check that $T: \overline{K\left(\gamma_{k}, c_{k}\right)} \rightarrow K$, for $k \in N$.

We now show that all the conditions of Theorem 2.2 are satisfied. To make use of property (i) of Theorem 2.2, we choose $u \in \partial K\left(\gamma_{k}, c_{k}\right)$. Then $\gamma_{k}(u)=\max _{\frac{1}{\theta_{k}} \leq t \leq r_{k}} u(t)=u\left(r_{k}\right)=c_{k}$, this implies that $0 \leq u(t) \leq c_{k}$ for $t \in\left[0, r_{k}\right]$, i.e., $0 \leq \frac{u(t)}{1+t} \leq c_{k}$. If we recall that $\|u\| \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} \gamma_{k}(u)=\frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} c_{k}$. So we have

$$
0 \leq \frac{u(t)}{1+t} \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} c_{k}, \quad 0 \leq t<+\infty
$$

Then assumption $\left(\mathrm{C}_{6}\right)$ implies that

$$
f(u)<\varphi\left(\frac{c_{k}}{\rho_{k}}\right), \quad 0 \leq t<+\infty
$$

Therefore

$$
\begin{aligned}
\gamma_{k}(T u)= & \max _{\frac{1}{\theta_{k} \leq t \leq r_{k}}}(T u)(t)=(T u)\left(r_{k}\right)=\int_{0}^{r_{k}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& +\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
\leq & \int_{0}^{\theta_{k}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{m-2}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
= & \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)\left(\theta_{k}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right) \\
< & \frac{c_{k}}{\rho_{k}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(\theta_{k}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right) \\
= & c_{k} .
\end{aligned}
$$

Hence condition (i) is satisfied.
Secondly, we show that (ii) of Theorem 2.2 is fulled. For this we select $u \in \partial K\left(\beta_{k}, b_{k}\right)$. Then, $\beta_{k}(u)=$ $\min _{r_{k} \leq t \leq \theta_{k}} u(t)=u\left(r_{k}\right)=b_{k}$, this means $u(t) \geq b_{k}$, for $r_{k} \leq t \leq \theta_{k}$. So we have $\|u\| \geq \frac{u(t)}{1+t} \geq \frac{b_{k}}{1+t} \geq \frac{b_{k}}{1+\theta_{k}}$, for $r_{k} \leq t \leq \theta_{k}$. Noticing that $\|u\| \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} \gamma_{k}(u) \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} \beta_{k}(u)=\frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} b_{k}$, we have

$$
\frac{b_{k}}{1+\theta_{k}} \leq \frac{u(t)}{1+t} \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} b_{k}, \quad \text { for } r_{k} \leq t \leq \theta_{k}
$$

By ( $\mathrm{C}_{7}$ ), we have

$$
f(u)>\varphi\left(\frac{b_{k}}{\eta_{k}}\right), \quad \text { for } r_{k} \leq t \leq \theta_{k} .
$$

Therefore

$$
\begin{aligned}
\beta_{k}(T u)= & \min _{r_{k} \leq t \leq \theta_{k}}(T u)(t)=(T u)\left(r_{k}\right)=\int_{0}^{r_{k}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& +\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{\theta_{k}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
> & \int_{0}^{r_{k}} \varphi^{-1}\left(\int_{s}^{\theta_{k}} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
\geq & \frac{b_{k}}{\eta_{k}} \int_{0}^{r_{k}} \varphi^{-1}\left(\int_{r_{k}}^{\theta_{k}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s \\
= & \frac{b_{k}}{\eta_{k}} r_{k} \varphi^{-1}\left(\int_{r_{k}}^{\theta_{k}} a(\tau) \mathrm{d} \tau\right) \geq b_{k} .
\end{aligned}
$$

Hence condition (ii) is satisfied.
Finally, we verify that (iii) of Theorem 2.2 is also satisfied. We note that $u(t) \equiv \frac{a_{k}}{4}, 0 \leq t<+\infty$ is a member of $K\left(\alpha_{k}, a_{k}\right)$ and $\alpha_{k}(u)=\frac{a_{k}}{4}<a_{k}$. So $K\left(\alpha_{k}, a_{k}\right) \neq \emptyset$. Now let $u \in \partial K\left(\alpha_{k}, a_{k}\right)$. Then $\alpha_{k}(u)=\max _{\frac{1}{\theta_{k}} \leq t \leq \theta_{k}} u(t)=$ $u\left(\theta_{k}\right)=a_{k}$. This implies that $0 \leq u(t) \leq a_{k}$, for $0 \leq t \leq \theta_{k}$, for $t \in\left[0, \theta_{k}\right]$, we have $0 \leq \frac{u(t)}{1+t} \leq \frac{a_{k}}{1+t}<a_{k}$. Together with $\|u\| \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} \gamma_{k}(u) \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} \alpha_{k}(u)=\frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} a_{k}$. Then we get

$$
0 \leq \frac{u(t)}{1+t} \leq \frac{1}{\lambda\left(\frac{1}{\theta_{k}}\right)} a_{k}, \quad 0 \leq t<+\infty
$$

By ( $\mathrm{C}_{8}$ ) we have

$$
f(u)<\varphi\left(\frac{a_{k}}{\rho_{k}}\right) .
$$

As before, we get

$$
\begin{aligned}
\alpha_{k}(T u)= & \max _{\frac{1}{\theta_{k}} \leq t \leq \theta_{k}}(T u)(t)=(T u)\left(\theta_{k}\right)=\int_{0}^{\theta_{k}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s \\
& +\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} \varphi^{-1}\left(\int_{s}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
\leq & \int_{0}^{\theta_{k}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{m-2}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}} \\
\leq & \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) f(u(\tau)) \mathrm{d} \tau\right)\left(\theta_{k}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& <\frac{a_{k}}{\rho_{k}} \varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(\theta_{k}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2} \alpha_{i}}\right) \\
& =a_{k} .
\end{aligned}
$$

Thus (iii) of Theorem 2.2 is satisfied. Since all the hypotheses of Theorem 2.2 are satisfied, the assertion follows.
Now we deal with the case in which condition $\left(\mathrm{C}_{1}\right)$ and $\left(\mathrm{C}_{3}\right)$ holds. The method is just similar to what we have done above.

Lemma 4.1. Suppose that $\left(\mathrm{C}_{3}\right)$ holds. Then
(i) for any constant $\theta \in(0,1)$, we have

$$
0<\int_{\theta}^{\frac{1}{\theta}} a(t) \mathrm{d} t<+\infty,
$$

(ii) the function

$$
H(t)=\int_{t}^{\frac{1}{\tau_{0}}} \varphi^{-1}\left(\int_{s}^{\frac{1}{\tau_{0}}} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s+\frac{\sum_{i=1}^{m-2} \alpha_{i} \int_{t_{0}}^{t} \varphi^{-1}\left(\int_{s}^{t} a(\tau) \mathrm{d} \tau\right) \mathrm{d} s}{1-\sum_{i=1}^{m-2} \alpha_{i}}
$$

is continuous and positive on $\left[t_{0}, \frac{1}{t_{0}}\right]$. Furthermore,

$$
\bar{L}=\min _{t \in\left[t_{0}, \frac{1}{t_{0}}\right]} H(t)>0 .
$$

Let

$$
\begin{aligned}
& \overline{\lambda_{1}}=\frac{1+\frac{1}{t_{0}}}{L}>0, \quad \overline{\lambda_{2}}=\frac{1-\sum_{i=1}^{m-2} \alpha_{i}}{\varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(1-\sum_{i=1}^{m-2} \alpha_{i}+\sum_{i=1}^{m-2} \alpha_{i} \xi_{m-2}\right)}>0, \\
& \rho_{k}=\varphi^{-1}\left(\int_{0}^{+\infty} a(\tau) \mathrm{d} \tau\right)\left(\frac{1}{\theta_{k}}+\frac{\xi_{m-2} \sum_{i=1}^{m-2} \alpha_{i}}{1-\sum_{i=1}^{m-2}}\right), \quad \eta_{k}=\theta_{k} \varphi^{-1}\left(\int_{r_{k}}^{\frac{1}{\theta_{k}}} a(\tau) \mathrm{d} \tau\right) .
\end{aligned}
$$

Theorem 4.3. Suppose that conditions $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{3}\right)$ hold. Let $\left\{\theta_{k}\right\}_{k=1}^{\infty}$ be such that $\theta_{k} \in\left(t_{k+1}, t_{k}\right),(k=1,2, \ldots)$. Let $\left\{r_{k}\right\}_{k=1}^{\infty}$ and $\left\{R_{k}\right\}_{k=1}^{\infty}$ be such that

$$
R_{k+1}<\frac{\lambda\left(\theta_{k}\right)}{1+\frac{1}{\theta_{k}}} r_{k}<r_{k}<m r_{k}<R_{k}, \quad k=1,2,3, \ldots
$$

Furthermore for each natural number $k$ we assume that $f$ satisfies:
(C9) $f((1+t) u) \geq \varphi\left(m r_{k}\right)$, for all $\frac{\lambda\left(\theta_{k}\right)}{1+\frac{1}{\partial_{k}}} r_{k} \leq u(t) \leq r_{k}$;
$\left(\mathrm{C}_{10}\right) f((1+t) u) \leq \varphi\left(M R_{k}\right)$, for all $0 \leq u(t) \leq R_{k}$,
where $m \in\left(\overline{\lambda_{1}},+\infty\right), M \in\left(0, \overline{\lambda_{2}}\right)$. Then the boundary value problem (1.1) and (1.2) has infinitely many solutions $\left\{u_{k}\right\}_{k=1}^{\infty}$ such that

$$
r_{k} \leq\left\|u_{k}\right\| \leq R_{k}, \quad k=1,2, \ldots
$$

Let $\theta_{k}<r_{k}<\frac{1}{\theta_{k}}$ and $\theta_{k}$ of Theorem 4.3, we define the nonnegative, increasing, continuous functional $\gamma_{k}, \beta_{k}$, and $\alpha_{k}$ by

$$
\begin{aligned}
& \gamma_{k}(u)=\max _{\theta_{k} \leq t \leq r_{k}} u(t)=u\left(r_{k}\right), \\
& \beta_{k}(u)=\min _{r_{k} \leq t \leq \frac{1}{\theta_{k}}} u(t)=u\left(r_{k}\right), \\
& \alpha_{k}(u)=\max _{\theta_{k} \leq t \leq \frac{1}{\theta_{k}}} u(t)=u\left(\frac{1}{\theta_{k}}\right) .
\end{aligned}
$$

It is obvious that for each $u \in K$,

$$
\gamma_{k}(u) \leq \beta_{k}(u) \leq \alpha_{k}(u)
$$

and

$$
\gamma_{k}(u)=u\left(r_{k}\right) \geq \lambda\left(r_{k}\right)\|u\| \geq \lambda\left(\theta_{k}\right)\|u\| .
$$

Therefore

$$
\|u\| \leq \frac{1}{\lambda\left(\theta_{k}\right)} \gamma_{k}(u), \quad \text { for all } u \in K .
$$

Theorem 4.4. Suppose that conditions $\left(\mathrm{C}_{1}\right)$ and $\left(\mathrm{C}_{3}\right)$ hold. Let $\left\{\theta_{k}\right\}_{k=1}^{\infty}$ be such that $\theta_{k} \in\left(t_{k+1}, t_{k}\right)(k=1,2, \ldots)$. Let $\left\{a_{k}\right\}_{k=1}^{\infty},\left\{b_{k}\right\}_{k=1}^{\infty}$ and $\left\{c_{k}\right\}_{k=1}^{\infty}$ be such that

$$
c_{k+1}<a_{k}<\frac{\lambda\left(\theta_{k}\right)}{\theta_{k}+1} b_{k}<\lambda\left(\theta_{k}\right) b_{k}<c_{k}, \quad \text { and } \quad \rho_{k} b_{k}<\eta_{k} c_{k}, \quad \text { for } k=1,2, \ldots
$$

Furthermore for each natural number $k$ we assume that $f$ satisfies:
$\left(\mathrm{C}_{11}\right) f((1+t) u)<\varphi\left(\frac{c_{k}}{\rho_{k}}\right)$, for all $0 \leq u(t) \leq \frac{1}{\lambda\left(\theta_{k}\right)} c_{k}$;
$\left(\mathrm{C}_{12}\right) f((1+t) u)>\varphi\left(\frac{b_{k}}{\eta_{k}}\right)$, for all $\frac{b_{k}}{1+\theta_{k}} \leq u(t) \leq \frac{1}{\lambda\left(\theta_{k}\right)} b_{k}$;
$\left(\mathrm{C}_{13}\right) f((1+t) u)<\varphi\left(\frac{a_{k}}{\rho_{k}}\right)$, for all $0 \leq u(t) \leq \frac{1}{\lambda\left(\theta_{k}\right)} a_{k}$.
Then the boundary value problem (1.1) and (1.2) has three infinite families of solutions $\left\{u_{1 k}\right\}_{k=1}^{\infty},\left\{u_{2 k}\right\}_{k=1}^{\infty}$ and $\left\{u_{3 k}\right\}_{k=1}^{\infty}$ satisfying

$$
0 \leq \alpha_{k}\left(u_{1 k}\right)<a_{k}<\alpha_{k}\left(u_{2 k}\right), \quad \beta_{k}\left(u_{2 k}\right)<b_{k}<\beta_{k}\left(u_{3 k}\right), \quad \gamma\left(u_{3 k}\right)<c_{k}, \quad \text { for } k \in N .
$$

Remark 4.1. If we add the condition of $a(t) f(u(t)) \not \equiv 0, t \in[0,+\infty)$, to Theorems 4.2 and 4.4 , we can get three infinite families of positive solutions $\left\{u_{1 k}\right\}_{k=1}^{\infty},\left\{u_{2 k}\right\}_{k=1}^{\infty}$ and $\left\{u_{3 k}\right\}_{k=1}^{\infty}$ satisfying

$$
0<\alpha_{k}\left(u_{1 k}\right)<a_{k}<\alpha_{k}\left(u_{2 k}\right), \quad \beta_{k}\left(u_{2 k}\right)<b_{k}<\beta_{k}\left(u_{3 k}\right), \quad \gamma\left(u_{3 k}\right)<c_{k}, \quad \text { for } n \in N .
$$

Remark 4.2. The same conclusions of Theorems 4.1-4.4 hold when conditions (1), (2) and (4) are satisfied. Especially for $p$-Laplacian operator $\varphi(x)=|x|^{p-2} x$, for some $p>1$, our conclusions are also true and new.

## 5. Examples and remark

Example 5.1. As an example we mention the boundary value problem

$$
\begin{aligned}
& \left(\varphi\left(u^{\prime}\right)\right)^{\prime}+a(t) f(u(t))=0, \quad 0<t<1, \\
& u(0)=\sum_{i=1}^{m-2} \alpha_{i} u\left(\xi_{i}\right), \quad u^{\prime}(\infty)=0,
\end{aligned}
$$

where

$$
\varphi(u)= \begin{cases}\frac{u^{5}}{1+u^{2}}, & u \leq 0 \\ u^{2}, & u>0\end{cases}
$$

and $\xi_{i} \in(0,+\infty)$ with $0<\xi_{1}<\xi_{2}<\cdots<\xi_{m-2}<+\infty$ and $\alpha_{i}$ satisfy $\alpha_{i} \in[0,+\infty), 0<\sum_{i=1}^{m-2} \alpha_{i}<1$, $f \in C([0,+\infty),[0,+\infty)), a(t):[0,+\infty) \rightarrow[0,+\infty)$ and has countably many singularities in $[0,+\infty)$ and $f$ satisfy the conditions of Theorems 4.1-4.4. It is clear that $\varphi: R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0)=0$.

Remark 5.1. From the Example 5.1, we can see that $\varphi$ is not odd, then the boundary value problem with $p$-Laplacian operator do not apply to Example 5.1. So we generalize a $p$-Laplace operator for some $p>1$ and the function $\varphi$ which we defined above is more comprehensive and general than $p$-Laplace operator.

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