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Isolation number versus Boolean rank

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ABSTRACT

Let $\mathbb{B}$ be the binary Boolean algebra. The Boolean rank, or factorization rank, of a matrix A in $\mathcal{M}_{m,n}(\mathbb{B})$ is the smallest k such that A can be factored as an $m \times k$ times a $k \times n$ matrix. The isolation number of a matrix, A, is the largest number of entries equal to 1 in the matrix such that no two ones are in the same row, no two ones are in the same column, and no two ones are in a submatrix of A of the form $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It is known that the isolation number of A is always at most the Boolean rank. This paper investigates for each k, if the isolation number of A is k what are some of the possible values of the Boolean rank of A.

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1. Introduction

Let $\mathbb{B}$ be the binary Boolean algebra, that is, $\mathbb{B} = \{0, 1\}$ with addition and multiplication defined as in the real numbers except that $1 + 1 = 1$. Let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ $(0, 1)$-matrices with matrix addition and multiplication following the usual rules and using Boolean arithmetic on the entries. Let $\mathcal{M}_n(\mathbb{B}) = \mathcal{M}_{m,n}(\mathbb{B})$ if $m = n$, let I_m denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{B})$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{B})$. The subscripts are usually omitted if the order is obvious, and we write I, O, J. Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$, $A \neq O$. The *Boolean rank* of A, denoted $\beta(A)$, is the smallest k such that for some $B \in \mathcal{M}_{m,k}(\mathbb{B})$ and $C \in \mathcal{M}_{k,n}(\mathbb{B})$, $A = BC$. Also, $\beta(O) = 0$, and is the only matrix of rank 0. The Boolean rank of a matrix has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is the adjacency matrix of the bipartite graph G with bipartition (X, Y) (so that $|X| = m$ and $|Y| = n$), the Boolean rank of A is the minimum number of bicliques that cover the edges of G, called the biclique covering number.

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We say that a matrix A *dominates* matrix B if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix X, we let $\mathbf{x}^{(j)}$ denote the jth column of X and $\mathbf{x}_{(i)}$ denote the ith row. Now if $\beta(A) = k$ and $A = BC$ is a factorization of A, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(k)}\mathbf{c}_{(k)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a rank one matrix, the Boolean rank of A is also the minimum number of rank one matrices whose sum is A. This is sometimes called the *one-rank* of A. The study of the Boolean rank and its application and interface with bipartite graphs has been the object of some research in the past 30 years. Among the earliest work are articles by DeCaen, Gregory, Pullman, Jones, Lundgren and others. See [1,2] and other articles by these authors.

Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$, a set of *isolated ones* is a set of locations, usually written as $S = \{a_{i,j}\}$ such that $a_{i,j} = 1$, no two are in the same row, no two ones in the same column, and, if $a_{i,j}, a_{k,l} \in S$ then, $a_{i,l}a_{k,j} = 0$, that is, isolated ones are independent ones and no two isolated ones lie in a submatrix of A of the form $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The *isolation number of* A, $\iota(A)$, is the maximum size of a set of isolated ones. Note that $\iota(A) = 0$ if and only if $A = O$.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{B})$ and $\beta(A) = k$. Then there are k rank one matrices A_i such that $A = A_1 + A_2 + \cdots + A_k$. Because each rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} J & O \\ O & O \end{bmatrix}$, it is easily seen that the matrix consisting of two isolated ones of A cannot be dominated by any one of the rank one matrices. Thus $\iota(A) \leqslant \beta(A)$, see [2, Lemma 2.4]. For $A = A_1 + A_2 + \cdots + A_k$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of A_i and $\mathcal{C}_j$ denote the indices of the nonzero columns of A_j, $i, j = 1, \ldots, k$. Let $r_i = |\mathcal{R}_i|$ and $c_j = |\mathcal{C}_j|$. We shall use the symbol $\subset$ to denote proper inclusion, and $\subseteq$ when we wish to allow equality.

So, if $\beta(A) = 1$ then $\iota(A) = 1$. Is the converse true?

2. Main results

Theorem 2.1. *If $A \in \mathcal{M}_{m,n}(\mathbb{B})$, then $i(A) = 1$ if and only if $\beta(A) = 1$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$. If $\beta(A) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) \leqslant \beta(A)$, $\iota(A) = 1$.

Now, suppose that $\iota(A) = 1$ and that $\beta(A) \geqslant 2$. Then, for some P and Q, permutation matrices of the appropriate orders, $PAQ = \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix} + A_2$ for some r, s with either $r < m$ or $s < n$. Partition A_2 as $A_2 = \begin{bmatrix} A_{2,1} & A_{2,2} \\ A_{2,3} & A_{2,4} \end{bmatrix}$ where $A_{2,1}$ is $r \times s$. Since $\beta(PAQ) = \beta(A) \geqslant 2$, $A \neq J$, and hence, one of $A_{2,2}, A_{2,3}, A_{2,4}$ has a zero entry. Further, one of $A_{2,2}, A_{2,3}, A_{2,4}$ has an entry of 1 since $PAQ \neq \begin{bmatrix} J_{r,s} & O \\ O & O \end{bmatrix}$. Thus, in PAQ there is some location that is zero and lies in a nonzero column and a nonzero row. Then, any of the ones in that column together with a one in the nonzero row form a set of two isolated ones, a contradiction, thus $\beta(A) = 1$. $\square$

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{B})$ of matrices with isolation number one is the same as the set of matrices of Boolean rank one.

Lemma 2.2. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$. Then: if $\beta(A) = 2$ then $\iota(A) = 2$; if $\iota(A) = 2$ then $\beta(A) \neq 3$.*

Proof. If $\beta(A) = 2$ then $\iota(A) > 1$ by Theorem 2.1. Since $\iota(A) \leqslant \beta(A)$, we have that $\iota(A) = 2$.

Now, suppose that $\iota(A) = 2$ and that $\beta(A) = 3$. Let $A = A_1 + A_2 + A_3$ where $\beta(A_i) = 1$.

Permute the rows so that $\mathcal{R}_1 = \{1, 2, \ldots, r_1\}$ and permute the columns of A so that $\mathcal{C}_2 = \{1, 2, \ldots, c_2\}$ and $\mathcal{C}_3 = \{k + 1, k + 2, \ldots, k + c_3\}$ where $k \leqslant c_2$.

Note that $\mathcal{R}_i \neq \mathcal{R}_j$ and $\mathcal{C}_i \neq \mathcal{C}_j$ unless $i = j$ otherwise $A_i + A_j$ would be rank 1.

Suppose that $\mathcal{R}_1 \subset \mathcal{R}_2$. Permute the remaining rows so that $\mathcal{R}_2 = \{1, 2, \ldots, r_2\}$, and $\mathcal{R}_3 = \{a + 1, a + 2, \ldots, a + b + c, r_2 + 1, r_2 + 2, \ldots, w\}$ where $a + b \leqslant r_1$, $a + b + c \geqslant r_1$ and $w \geqslant r_2$.

Thus, we have that

$$A = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & J_{b,v} & O_{b,w} \\ J_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,g} & O_{d,h} & O_{d,u} & O_{d,v} & O_{d,w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v} & O_{e,w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v} & O_{f,w} \end{bmatrix},$$

for some $a, b, c, d, e, f, g, h, k, u, v$ and w. Thus, with this notation,

$$A_1 = \begin{bmatrix} J_{a,k} & J_{a,g} & J_{a,h} & O_{a,u} & J_{a,v} & O \\ J_{b,k} & J_{b,g} & J_{b,h} & O_{b,u} & J_{b,v} & O \\ O & O & O & O & O & O \end{bmatrix},$$

$$A_2 = \begin{bmatrix} J_{a,k} & J_{a,g} & O \\ J_{b,k} & J_{b,g} & O \\ J_{c,k} & J_{c,g} & O \\ J_{d,k} & J_{d,g} & O \\ O & O & O \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} O_{a,k} & O_{a,g} & O_{a,h} & O_{a,u} & O_{a,v+w} \\ O_{b,k} & J_{b,g} & J_{b,h} & J_{b,u} & O_{b,v+w} \\ O_{c,k} & J_{c,g} & J_{c,h} & J_{c,u} & O_{c,v+w} \\ O_{d,k} & O_{d,g} & O_{d,h} & O_{d,u} & O_{d,v+w} \\ O_{e,k} & J_{e,g} & J_{e,h} & J_{e,u} & O_{e,v+w} \\ O_{f,k} & O_{f,g} & O_{f,h} & O_{f,u} & O_{f,v+w} \end{bmatrix}.$$

Now, if $A[r_1 + 1, \ldots, m|1, \ldots, n] = A_2[r_1 + 1, \ldots, m|1, \ldots, n] + A_3[r_1 + 1, \ldots, m|1, \ldots, n]$ has rank 1 then $d = e = 0$ and hence $\beta(A) = 2$, a contradiction. Thus, $A[r_1 + 1, \ldots, m|1, \ldots, n]$ must have rank 2, and hence has two isolated ones, say i_2 and i_3. If $\mathcal{C}_1 \nsubseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then without loss of generality we have that $a_{1,x} \neq 0$ for $x = k + g + h + u + 1$, but then, $\{a_{1,x}, i_2, i_3\}$ is a set of three isolated ones, a contradiction. Thus, $v = 0$ and hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, A would have rank 2.

Note that $a, u, d \neq 0$, for otherwise $\beta(A) = 2$. If $e = 0$ then $b+c \neq 0$ so that $\{a_{1,c_1}, a_{a+1,k+c_3}, a_{r_2,1}\}$ is a set of three isolated ones. If $e \neq 0$, then $\{a_{1,c_1}, a_{r_2,1}, a_{r_2+e,k+c_3}\}$ is a set of three isolated ones, contradicting that $\iota(A) = 2$. Thus, $\mathcal{R}_1 \not\subset \mathcal{R}_2$.

By renumbering and/or transposing we have proven that $\mathcal{R}_i \not\subset \mathcal{R}_j$ and $\mathcal{C}_i \not\subset \mathcal{C}_j$ for any pair i and j.

Now, permute the rows and columns of A so that $\mathcal{R}_1 = \{1, 2, \ldots, r_1\}$, $\mathcal{R}_2 = \{a+1, a+2, \ldots, a+b, a+b+c+1, a+b+c+2, \ldots, a+b+c+d+e+f\}$, and $\mathcal{R}_3 = \{a+b+1, a+b+2, \ldots, a+b+c+d+e, a+b+c+e+f+1, a+b+c+e+f+2, \ldots, a+b+c+e+f+g\}$ for some a, b, c, d, e, f, g where $a + b + c + d = r_1$, so that A has the form:

$$A = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & J_{d,v} & O_{d,w} \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O_{e,v} & O_{e,w} \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O_{f,r} & O_{f,s} & O_{f,v} & O_{f,w} \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O_{g,v} & O_{g,w} \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O_{h,v} & O_{h,w} \end{bmatrix}, \tag{1}$$

for some $a, b, c, d, e, f, g, h, k, l, p, q, r, s, v$, and w, so that

$$A_1 = \begin{bmatrix} J_{a,k} & O_{a,l} & J_{a,p} & O_{a,q} & J_{a,r} & O_{a,s} & J_{a,v} & O_{a,w} \\ J_{b,k} & O_{b,l} & J_{b,p} & O_{b,q} & J_{b,r} & O_{b,s} & J_{b,v} & O_{b,w} \\ J_{c,k} & O_{c,l} & J_{c,p} & O_{c,q} & J_{c,r} & O_{c,s} & J_{c,v} & O_{c,w} \\ J_{d,k} & O_{d,l} & J_{d,p} & O_{d,q} & J_{d,r} & O_{d,s} & J_{d,v} & O_{d,w} \\ O & O & O & O & O & O & O & O \end{bmatrix},$$

$$A_2 = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O \\ J_{b,k} & J_{b,l} & J_{b,p} & J_{b,q} & O \\ O_{c,k} & O_{c,l} & O_{c,p} & O_{c,q} & O \\ J_{d,k} & J_{d,l} & J_{d,p} & J_{d,q} & O \\ J_{e,k} & J_{e,l} & J_{e,p} & J_{e,q} & O \\ J_{f,k} & J_{f,l} & J_{f,p} & J_{f,q} & O \\ O & O & O & O & O \end{bmatrix}, \quad \text{and}$$

$$A_3 = \begin{bmatrix} O_{a,k} & O_{a,l} & O_{a,p} & O_{a,q} & O_{a,r} & O_{a,s} & O \\ O_{b,k} & O_{b,l} & O_{b,p} & O_{b,q} & O_{b,r} & O_{b,s} & O \\ O_{c,k} & O_{c,l} & J_{c,p} & J_{c,q} & J_{c,r} & J_{c,s} & O \\ O_{d,k} & O_{d,l} & J_{d,p} & J_{d,q} & J_{d,r} & J_{d,s} & O \\ O_{e,k} & O_{e,l} & J_{e,p} & J_{e,q} & J_{e,r} & J_{e,s} & O \\ O_{f,k} & O_{f,l} & O_{f,p} & O_{f,q} & O_{f,r} & O_{f,s} & O \\ O_{g,k} & O_{g,l} & J_{g,p} & J_{g,q} & J_{g,r} & J_{g,s} & O \\ O_{h,k} & O_{h,l} & O_{h,p} & O_{h,q} & O_{h,r} & O_{h,s} & O \end{bmatrix}.$$

Suppose that $v \neq 0$ and $A[r_1+1, \ldots, m|1, \ldots, n] = A_2[r_1+1, \ldots, m|1, \ldots, n] + A_3[r_1+1, \ldots, m|1, \ldots, n]$ has rank 1. Then, $f = g = 0$ and we must have $l, s \neq 0$, for otherwise $\beta(A) = 2$, a contradiction. Further, if $b = c = 0$ the rank of A is two, again a contradiction. Thus, using a 1 from each of the blocks subscripted (a, v), (b, l) and (e, s) or a 1 from each of the blocks subscripted (a, v), (e, l) and (c, s) we have three isolated ones, a contradiction. Thus the rank of $A[r_1+1, \ldots, m|1, \ldots, n]$ must have rank 2, and hence has two isolated ones, say i_2 and i_3. If $\mathcal{C}_1 \not\subseteq \mathcal{C}_2 \cup \mathcal{C}_3$ then $a_{1,x} \neq 0$ for

$x = k + l + p + q + r + s + 1$ then, $\{a_{1,x}, i_2, i_3\}$ is a set of three isolated ones, a contradiction. Thus, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \cup \mathcal{C}_3$. Further, $\mathcal{C}_1 \neq \mathcal{C}_2 \cup \mathcal{C}_3$, otherwise, A would have rank 2. Thus, $v = 0$, and hence, $\mathcal{C}_1 \subset \mathcal{C}_2 \cup \mathcal{C}_3$.

Since the choice of rows versus columns can be changed by transposition and the index of $\mathcal{R}_i$ and $\mathcal{C}_j$ by renumbering, we have shown that if $\{i, j, k\} = \{1, 2, 3\}$ then $\mathcal{R}_i \subset \mathcal{R}_j \cup \mathcal{R}_k$ and $\mathcal{C}_i \subset \mathcal{C}_j \cup \mathcal{C}_k$.

Consider the matrix (1). Since $\mathcal{R}_1 \subset \mathcal{R}_2 \cup \mathcal{R}_3$ we have that $a = 0$; since $\mathcal{R}_2 \subset \mathcal{R}_1 \cup \mathcal{R}_3$ we have that $f = 0$; since $\mathcal{C}_2 \subset \mathcal{C}_1 \cup \mathcal{C}_3$ we have that $l = 0$; and since $\mathcal{C}_3 \subset \mathcal{C}_1 \cup \mathcal{C}_2$ we have that $s = 0$. That is, since $a = f = l = s = v = 0$, A has the form

$$A = \begin{bmatrix} J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ J & J & J & J & O \\ O & J & J & J & O \\ O & O & O & O & O \end{bmatrix}.$$

where the indices have been omitted. Thus $\beta(A) = 2$ a contradiction. Thus, if $\iota(A) = 2$ then $\beta(A) \neq 3$. $\square$

Theorem 2.3. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$. Then, $\iota(A) = 2$ if and only if $\beta(A) = 2$.*

Proof. By virtue of Lemma 2.2, we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{B})$ with $\iota(A) = 2$ and $\beta(A) > 2$. By Lemma 2.2, $\beta(A) \neq 3$ so choose A such that if $\beta(A) > \beta(B) > 2$ then $i(B) > 2$. Suppose that $A = A_1 + A_2 + \cdots + A_k$ for $k = \beta(A)$ where each A_i is rank one, i.e., k is the minimum k such that $\beta(A) = k$ and $\iota(A) = 2$.. Suppose that A_1 has the fewest number of nonzero rows of the A_i's. As in the above lemma, permute the rows of A so that A_1 has nonzero rows $1, 2, \ldots, r_1$. For $j = 1, \ldots, r_1$, let B_j be the matrix whose first j rows are the first j rows of A and whose last $m - j$ rows are all zero. Let C_j be the matrix whose first j rows are all zero and whose last $m-j$ rows are the last $m-j$ rows of A. Then $A = B_j + C_j$. Further any set of isolated ones of C_j is a set of isolated ones for A. Now, since $\beta(A) \leqslant \beta(B_j) + \beta(C_j)$, and the fact that $\beta(C_j) = \beta(C_{j-1})$ or $\beta(C_j) = \beta(C_{j-1}) - 1$, there is some j such that $\beta(C_j) = \beta(A) - 1$. Since $\beta(C_j) < k$, by the choice of A, for this j, we have that $\iota(C_j) > 2$, since $\beta(C_j) \geqslant 3$. That is $\iota(A) > 2$, a contradiction. $\square$

Note, as can be seen in the following example, there is a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $\iota(A) = 3$ and $\beta(A)$ is relative large, depending on m and n.

Example 2.4. For $n \geqslant 3$, let $D_n \in \mathcal{M}_n(\mathbb{B})$ denote the matrix $J \setminus I$. Then, it is easily shown that $\iota(D_n) = 3$ while $\beta(D_n) = k$ where $k = min\left\{k : n \leqslant \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix}\right\}$, see [1, Corollary 2]. So, $\iota(D_{20}) = 3$ while $\beta(D_{20}) = 6$.

A tournament matrix is the adjacency matrix of a directed graph called a tournament, T. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$ where $[T]$ is the adjacency matrix of T.

Another problem we wish to address is: For each $k = 1, 2, \ldots, \min\{m, n\}$ characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ for which $\iota(A) = \beta(A)$. Of course this is done if $k = 1$ or $k = 2$, but only in those cases. For $k = m$ we can also find a characterization:

Theorem 2.5. *Let $1 \leqslant m \leqslant n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B})$. Then, $\iota(A) = \beta(A) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $PAQ = [B|C]$ where $B = I_m + T$ where T is a matrix that is dominated by a tournament matrix. (There are no restrictions on C.)*

Proof. Suppose that $\iota(A) = m$, then by permuting by P and Q, so that the set of isolated ones are in the (i, i) positions, i.e., if $X = PAQ$ then $S = \{a_{1,1}.a_{2,2}, \ldots, a_{m,m}\}$, then $X = [B|C]$, with $b_{i,i} = 1$ and $b_{i,j}b_{j,i} = 0$ for every i and $j \neq i$. Thus, $B = I_m + T$ where T is a submatrix of a tournament matrix. Thus, $PAQ = [B|C]$ where $B = I_m + T$ and clearly there are no conditions on C.

If $PAQ = [B|C]$ and $B = I_m + T$ where T is a submatrix of a tournament matrix, then the diagonal entries of B form a set of isolated vertices for PAQ and hence , A has a set of m isolated vertices. $\square$

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