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Oscillatory Behavior of Delay Partial Difference Equations with Positive and Negative Coefficients

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Abstract—We construct an important transform to obtain sufficient conditions for the oscillation of all solutions of delay partial difference equations with positive and negative coefficients of the form

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + p_{mn}A_{m-k,n-l} - q_{mn}A_{m-k',n-l'} = 0,$$

where $m, n = 0, 1, \dots$, and k, k', l', l are nonnegative integers, $p, q \in (0, \infty)$, the coefficients $\{q_{mn}\}$ and $\{p_{mn}\}$ are sequences of nonnegative real numbers.

Keywords—Delay partial difference equation, Oscillation, Positive and negative coefficients.

1. INTRODUCTION

Partial difference equations are difference equations that involve functions with two or more independent integer variables. Such equations arise from considerations of random walk problems, molecular structure problems [1], and numerical difference approximation problems [2]. Recently, the problem of oscillation and nonoscillation of solutions of delay partial difference equations is receiving much attention. See papers [3–10] by Zhang and Liu.

In this paper, we consider the delay partial difference equation with positive and negative coefficients of the form

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + p_{mn}A_{m-k,n-l} - q_{mn}A_{m-k',n-l'} = 0, \tag{1.1}$$

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and let $N_i = \{i, i+1, i+2, \dots \mid i = 0, 1, 2, \dots\}$, where $m, n \in N_0$ and

$$k, k', l', l \in N_0, \qquad p_{mn}, q_{mn} \in \left[N_0^2, (0, \infty)\right], \qquad k \ge k' + 1, \qquad l \ge l' + 1. \tag{1.2}$$

We note that $q_{mn} = 0$, the results for the oscillation of (1.1) have been obtained in [3,4]. To the best of our knowledge, there are not any results for the oscillation of (1.1) in the literature. In this paper, an important transform is being used to obtain sufficient conditions for the oscillation of all solutions of equation (1.1).

A solution $\{A_{mn}\}$ of (1.1) is said to be eventually positive if $A_{mn} > 0$ for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Regarding definition of the initial value problem of (1.1), see [3] or [4].

In fact, we remark further that equation (1.1) may also be regarded as a discrete analog of partial differential equations of the form

$$\frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + A(x, y) + p(x, y)A(x - \sigma, y - \tau) - q(x, y)A(x - \xi, y - \eta) = 0.$$

Therefore, qualitative properties of (1.1) may yield useful information for this delay partial differential equation.

2. LEMMAS

Consider the delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + p_{mn}A_{m-k,n-l} = 0. \tag{2.1}$$

From [3, pp. 217–223] and [4, pp. 482–486], we have the following lemmas.

LEMMA 1. *Assume that one of the following two conditions is satisfied.*

(i) *(See [3].)*

$$\liminf_{m,n \to \infty} \left(\frac{1}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p_{ij} \right) > \frac{\omega^\omega}{(\omega + 1)^{\omega+1}}. \tag{2.2}$$

(ii) *(See [4].) For all large m and n, there exists a positive number ξ such that*

$$p_{mn} \ge \xi > \frac{(k + l)^{(k+l)}}{(k + l + 1)^{(k+l+1)}}, \tag{2.3}$$

where $k, l \in N_1$, $\omega = 2kl/(k + l)$.

Then every solution of equation (2.1) oscillates.

LEMMA 2. *(See [5, pp. 65–71].) Assume that either (2.2) or (2.3) is satisfied. Then partial difference inequalities*

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + p_{mn}A_{m-k,n-l} \le 0$$

cannot have eventually positive solutions and

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + p_{mn}A_{m-k,n-l} \ge 0$$

cannot have eventually negative solutions.

LEMMA 3. *(See [3].)*

$$\sum_{i=m-k}^{m} \sum_{j=n-l}^{n} (A_{i+1,j} + A_{i,j+1} - A_{ij}) = \sum_{i=m+1-k}^{m+1} \sum_{j=n+1-l}^{n} A_{ij} + \sum_{i=m-k}^{m} A_{i,n+1} - A_{m-k,n-l} + A_{m+1,n-l}.$$

Assume that there exist positive integers s, t such that

$$s \geq m, \qquad t \geq n \tag{2.4}$$

and

$$\begin{aligned} C_{st} = A_{st} &- (3)^{s+t-m-n} \left(\sum_{i=s}^{m+k'} q_{in} A_{i-k',n-l'} + \sum_{j=t}^{n+l'} q_{mj} A_{m-k',j-l'} \right) \\ &- \frac{1}{2} \left(\sum_{i=s}^{m+k'} q_{i+k'-k,n+l'-l} A_{i-k,n-l} + \sum_{j=t}^{n+l'} q_{m+k'-k,j+l'-l} A_{m-k,j-l} \right). \end{aligned} \tag{2.5}$$

Let

$$\alpha_{mn} = p_{mn} - q_{m+k'-k,n+l'-l} > 0, \qquad \text{for } m \geq k - k', \quad n \geq l - l'. \tag{2.6}$$

From (2.5), we obtain the following results.

LEMMA 4. *Assume that (1.2) holds and $\{A_{mn}\}$ is an eventually positive solution of (1.1), then there exist positive integers M, N such that $A_{mn} > 0$ as $m \geq M$, $n \geq N$. Then*

(i) *C_{mn} is monotone decreasing in m, n, that is,*

$$C_{m+1,n} \leq C_{mn}, \qquad C_{m,n+1} \leq C_{mn}, \tag{2.7}$$

(ii) $C_{mn} \leq A_{mn}$,

(iii) $C_{m+1,n} + C_{m,n+1} - C_{mn} = -\alpha_{mn} A_{m-k,n-l} - \beta_{mn}(A)$,

where

$$\begin{aligned} \beta_{mn}(A) &= 3q_{mn} A_{m-k',n-l'} + 5\Delta_1 + \frac{1}{2}\Delta_2, \\ \Delta_1 &= \sum_{i=m+1}^{m+k'} q_{in} A_{i-k',n-l'} + \sum_{j=n+1}^{n+l'} q_{mj} A_{m-k',j-l'}, \\ \Delta_2 &= \sum_{i=m}^{m+k'} q_{i+k'-k,n+l'-l} A_{i-k,n-l} + \sum_{j=n}^{n+l'} q_{m+k'-k,j+l'-l} A_{m-k,j-l}. \end{aligned} \tag{2.8}$$

PROOF.

(i) From (2.5), we obtain

$$\begin{aligned} C_{m+1,n} &= A_{m+1,n} - 3\Delta_1 - \frac{1}{2}\Delta_2 - 3q_{mn} A_{m-k',n-l'} + \frac{1}{2} q_{m+k'-k,n+l'-l} A_{m-k,n-l}, \\ C_{mn} &= A_{mn} - \Delta_1 - \frac{1}{2}\Delta_2 - 2q_{mn} A_{m-k',n-l'}. \end{aligned} \tag{2.9}$$

We note that $A_{mn} > 0$, thus we have

$$\begin{aligned} C_{m+1,n} - C_{mn} &\leq A_{m+1,n} + A_{m,n+1} - A_{mn} - 2\Delta_1 - q_{mn} A_{m-k',n-l'} \\ &\quad + \frac{1}{2} q_{m+k'-k,n+l'-l} A_{m-k,n-l} \\ &< -p_{mn} A_{m-k,n-l} + q_{mn} A_{m-k',n-l'} - 2\Delta_1 \\ &\quad - q_{mn} A_{m-k',n-l'} + q_{m+k'-k,n+l'-l} A_{m-k,n-l} \\ &= -\alpha_{mn} A_{m-k,n-l} - 2\Delta_1 \leq -\alpha_{mn} A_{m-k,n-l} \leq 0, \end{aligned} \tag{2.10}$$

that is, $C_{m+1,n} - C_{mn} < 0$. Similarly, we have also $C_{m,n+1} - C_{mn} < 0$.

(ii) From (2.9), we immediately obtain (ii).

(iii) From (2.5), we have also

$$C_{m,n+1} = A_{m,n+1} - 3\Delta_1 - \frac{1}{2}\Delta_2 - 3q_{mn}A_{m-k',n-l'} + \frac{1}{2}q_{m+k'-k,n+l'-l}A_{m-k,n-l},$$

and using above $C_{m+1,n}$ and C_{mn}, we obtain that

$$\begin{aligned} C_{m+1,n} + C_{m,n+1} - C_{mn} &= -\alpha_{mn}A_{m-k,n-l} - 3q_{mn}A_{m-k',n-l'} - 5\Delta_1 - \frac{1}{2}\Delta_2 \\ &= -\alpha_{mn}A_{m-k,n-l} - \beta_{mn}(A). \end{aligned}$$

Hence, $C_{m+1,n} + C_{m,n+1} - C_{mn} = -\alpha_{mn}A_{m-k,n-l} - \beta_{mn}(A)$. Note that $\beta_{mn}(A) > 0$, thus we have also

$$C_{m+1,n} + C_{m,n+1} - C_{mn} < -\alpha_{mn}A_{m-k,n-l} < 0. \tag{2.11}$$

LEMMA 5. *Assume that (1.2) and (2.6) hold, and for $m \geq k - k'$, $n \geq l - l'$, we have*

$$\left(\sum_{i=m}^{m+k'} q_{in} + \sum_{j=n}^{n+l'} q_{mj}\right) + \frac{1}{2}\left(\sum_{i=m}^{m+k'} q_{i+k'-k,n+l'-l} + \sum_{j=n}^{n+l'} q_{m+k'-k,j+l'-l}\right) < 1. \tag{2.12}$$

Let $\{A_{mn}\}$ be an eventually positive solution of equation (1.1). Then $\{C_{mn}\}$ by the definition of (2.5) is decreasing and eventually positive in m, n.

PROOF. By Lemma 4, $\{C_{mn}\}$ is decreasing in m, n. Next, we shall show that the $\{C_{mn}\}$ is eventually positive in m, n. Because $\{A_{mn}\}$ is an eventually positive solution of equation (1.1) and the $\{C_{mn}\}$ is monotone decreasing in m, n, thus $\{C_{mn}\}$ exists limit as $m, n \to \infty$. If $\lim_{m,n\to\infty} C_{mn} = -\infty$, as $m, n \to \infty$, then $\{A_{mn}\}$ must be unbounded. There exists $\{(m_k, n_k)\}$ such that $\lim_{k\to\infty} m_k = \infty$, $\lim_{k\to\infty} n_k = \infty$, and $A_{m_k,n_k} = \max_{\substack{M\leq m\leq m_k+k, \\ N\leq n\leq n_k+l}} \{A_{m-k,n-l}\} \to \infty$ as $k \to \infty$. On the other hand,

$$\begin{aligned} C_{m_k n_k} &= \Bigg[A_{m_k n_k} - \left(\sum_{i=m_k}^{m_k+k'} q_{i n_k} A_{i-k',n_k-l'} + \sum_{j=n_k}^{n_k+l'} q_{m_k j} A_{m_k-k',j-l'}\right) \\ &\quad -\frac{1}{2}\left(\sum_{i=m_k}^{m_k+k'} q_{i+k'-k,n_k+l'-l} A_{i-k,n_k-l} + \sum_{j=n_k}^{n_k+l'} q_{m_k+k'-k,j+l'-l} A_{m_k-k,j-l}\right)\Bigg] \\ &\geq A_{m_k n_k}\Bigg[1 - \left(\sum_{i=m_k}^{m_k+k'} q_{i n_k} + \sum_{j=n_k}^{n_k+l'} q_{m_k j}\right) - \frac{1}{2}\Bigg(\sum_{i=m_k}^{m_k+k'} q_{i+k'-k,n_k+l'-l} \\ &\quad + \sum_{j=n_k}^{n_k+l'} q_{m_k+k'-k,j+l'-l}\Bigg)\Bigg] \\ &\geq 0, \end{aligned}$$

a contradiction. Hence, $\lim_{m,n\to\infty} C_{mn} = \beta$ exists. As before, if $\{A_{mn}\}$ is unbounded, then $\beta \geq 0$. Now we consider the case that $\{A_{mn}\}$ is bounded. Let $\bar{\beta} = \limsup_{m,n\to\infty} A_{mn} =$

$\lim_{m',n'\to\infty} A_{m',n'}$. Then

$$\begin{aligned} A_{m',n'} - C_{m',n'} &= \left(\sum_{i=m'}^{m'+k'} q_{in'} A_{i-k',n'-l'} + \sum_{j=n'}^{n'+l'} q_{m'j} A_{m'-k',j-l'}\right) \\ &\quad + \frac{1}{2}\left(\sum_{i=m'}^{m'+k'} q_{i+k'-k,n'+l'-l} A_{i-k,n'-l}\right. \\ &\quad \left. + \sum_{j=n'}^{n_\prime+l'} q_{m'+k'-k,j+l'-l} A_{m'-k,j-l}\right) \\ &\le A(\xi_m,\eta_n)\left[\left(\sum_{i=m'}^{m'+k'} q_{in'} + \sum_{j=n'}^{n'+l'} q_{m'j}\right)\right. \\ &\quad \left. +\frac{1}{2}\left(\sum_{i=m'}^{m'+k'} q_{i+k'-k,n'+l'-l} + \sum_{j=n'}^{n_\prime+l'} q_{m'+k'-k,j+l'-l}\right)\right] \\ &\le A(\xi_m,\eta_n), \end{aligned} \tag{2.13}$$

where $A(\xi_m,\eta_n) = \max\{A_{i-k,j-l}, \mid i = m', m'+1, \dots, m'+k',\ j = n', n'+1, \dots, n'+l'\}$. Taking superior limit on both sides of the above inequality, we have $\bar{\beta} - \beta \le \bar{\beta}$, therefore, $\beta \ge 0$. Hence, $C_{mn} > 0$ for $m \ge M$, $n \ge N$.

3. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF (1.1)

The next result provides sufficient conditions so that every nonoscillatory solution of equation (1.1) tends to zero as $m,n \to \infty$.

THEOREM 3.1. *Assume that either*

(i) *(2.12) holds and there exists a positive integer α_0 such that*

$$p_{mn} - q_{m-k+k',n-l+l'} \ge \alpha_0, \qquad \text{for } m \ge k-k', \quad n \ge l-l', \tag{3.1}$$

or

(ii) *there exists a positive constant $\beta_0 \in (0,1)$ such that*

$$\begin{gathered} \left(\sum_{i=m}^{m+k} q_{in} + \sum_{i=n}^{n+l} q_{mj}\right) + \frac{1}{2}\left(\sum_{i=m}^{m+k} q_{i+k'-k,n+l'-l} + \sum_{j=n}^{n+l} q_{m+k'-k,j+l'-l}\right) \\ \le 1-\beta_0, \qquad \text{for } m \ge k-k', \quad n \ge l-l', \end{gathered} \tag{3.2}$$

and

$$\sum_{i=k+k'}^{\infty} \sum_{j=l+l'}^{\infty} (p_{ij} - q_{i-k+k',j-l+l'}) = \infty. \tag{3.3}$$

Then every nonoscillatory solution of equation (1.1) tends to zero as $m,n \to \infty$.

PROOF. By Lemma 5, the sequence $\{C_{mn}\}$ is eventually decreasing and positive. Hence,

$$\lim_{m,n\to\infty} C_{mn} = \xi \in R^+, \tag{3.4}$$

where $R^+ = [0,\infty)$. By Lemma 4, it is easy to see

$$C_{m+1,n} + C_{m,n+1} - C_{mn} \le -(p_{mn} - q_{m-k+k',n-l+l'}) A_{m-k,n-l}. \tag{3.5}$$

Taking m_1, n_1 sufficiently large, and summing both sides of (3.5) from m_1, n_1 to infinity, we find

$$\lim_{m,n\to\infty} \sum_{i=m_1}^{m} \sum_{j=n_1}^{n} (C_{i+1,j} + C_{i,j+1} - C_{ij}) \le - \sum_{i=m_1}^{\infty} \sum_{j=n_1}^{\infty} (p_{ij} - q_{i-k+k',j-l+l'}) A_{i-k,j-l}. \tag{3.6}$$

By Lemma 3, we have

$$\begin{aligned} \lim_{m,n\to\infty} \left(\sum_{i=m_1+1}^{m+1} \sum_{j=n_1+1}^{n} C_{ij} + \sum_{i=m_1}^{m} C_{i,n+1} - C_{m_1,n_1} + C_{m+1,n_1} \right) \\ \le - \sum_{i=m_1}^{\infty} \sum_{j=n_1}^{\infty} (p_{ij} - q_{i-k+k',j-l+l'}) A_{i-k,j-l}. \end{aligned} \tag{3.7}$$

In view of (3.4), we obtain

$$\lim_{m,n\to\infty} \left(\sum_{i=m_1+1}^{m+1} \sum_{j=n_1+1}^{n} C_{ij} + \sum_{i=m_1}^{m} C_{i,n+1} + C_{m+1,n_1} \right) = L,$$

L is finite. Therefore, from (3.7), we have

$$L - C_{m_1,n_1} \le - \sum_{i=m_1}^{\infty} \sum_{j=n_1}^{\infty} (p_{ij} - q_{i-k+k',j-l+l'}) A_{i-k,j-l}. \tag{3.8}$$

First assume that (3.1) holds. Then (3.8) implies that

$$\sum_{i=m_1}^{\infty} \sum_{j=n_1}^{\infty} (p_{ij} - q_{i-k+k',j-l+l'}) A_{i-k,j-l} < \infty.$$

Hence,

$$\lim_{m,n\to\infty} A_{m,n} = 0.$$

Next assume that (3.2) and (3.3) hold. From (3.8), it follows that

$$\liminf_{m,n\to\infty} A_{m,n} = 0.$$

Also (2.9) implies that $C_{mn} \le A_{mn}$, and in view of (3.4), $\xi = 0$. Now we claim that $\{A_{mn}\}$ is bounded. Otherwise, there exists a subsequence $\{A_{m_r,n_r}\}$ of $\{A_{mn}\}$ such that

$$\begin{gathered} A_{m_r,n_r} = \max \{A_{m-k,n-l} \mid m \le m_r + k,\ n \le n_r + l,\ \text{for } r = 1, 2, \ldots\} \\ \text{and} \qquad \lim_{r\to\infty} A_{m_r,n_r} = \infty. \end{gathered}$$

Then by (2.9) and (3.2), we have

$$\begin{aligned} C_{m_r n_r} &= \Bigg[A_{m_r n_r} - \left(\sum_{i=m_r}^{m_r+k} q_{in_r} A_{i-k',n_r-l'} + \sum_{i=n_r}^{n_r+l} q_{m_r j} A_{m_r-k',j-l'} \right) \\ &\quad - \frac{1}{2} \left(\sum_{i=m_r}^{m_r+k} q_{i+k'-k,n_r+l'-l} A_{i-k,n_r-l} + \sum_{j=n_r}^{n_r+l} q_{m_r+k'-k,j+l'-l} A_{m_r-k,j-l} \right) \Bigg] \\ &\ge A_{m_r n_r} \Bigg[1 - \left(\sum_{i=m_r}^{m_r+k} q_{in_r} + \sum_{i=n_r}^{n_r+l} q_{m_r j} \right) \\ &\quad - \frac{1}{2} \left(\sum_{i=m_r}^{m_r+k} q_{i+k'-k,n_r+l'-l} + \sum_{j=n_r}^{n_r+l} q_{m_r+k'-k,j+l'-l} \right) \Bigg] \\ &\ge \beta_0 A_{m_r,n_r} \to \infty, \qquad \text{as } r \to \infty, \end{aligned}$$

which contradicts the fact that $\xi = 0$, and hence, $\{A_{mn}\}$ is bounded. Set

$$\lambda = \limsup_{m,n\to\infty} A_{m,n}$$

and let $\{A_{m_s,n_s}\}$ be a subsequence of $\{A_{mn}\}$ such that

$$\lim_{s\to\infty} A_{m_s,n_s} = \lambda.$$

Then for any $\varepsilon > 0$, there exists sufficiently large s, it follows from (2.9) and (3.2) that

$$\begin{aligned} C_{m_s n_s} &= \left[A_{m_s n_s} - \left(\sum_{i=m_s}^{m_s+k} q_{in_s} A_{i-k',n_s-l'} + \sum_{i=n_s}^{n_s+l} q_{m_s j} A_{m_s-k',j-l'} \right) \right. \\ &\quad \left. -\frac{1}{2} \left(\sum_{i=m_s}^{m_s+k} q_{i+k'-k,n_s+l'-l} A_{i-k,n_s-l} + \sum_{j=n_s}^{n_s+l} q_{m_s+k'-k,j+l'-l} A_{m_s-k,j-l} \right) \right] \\ &\geq A_{m_s,n_s} - (\lambda+\varepsilon)(1-\beta_0). \end{aligned}$$

By taking limits as $s \to \infty$ and by using the fact that $\xi = 0$, we obtain

$$0 \geq \lambda - (\lambda+\varepsilon)(1-\beta_0).$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\lambda = 0$, and the proof is complete.

4. OSCILLATION OF EQUATION (1.1)

In this section, we will establish sufficient conditions for the oscillation of all solutions of equation (1.1).

THEOREM 4.1. *Assume that (1.2), (2.5), and (2.6) hold, and assume that either*

(i)

$$\liminf_{m,n\to\infty} \left(\frac{1}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} \left(p_{ij} - q_{i-k+k',j-l+l'}\right) \right) > \frac{\omega^\omega}{(\omega+1)^{\omega+1}}, \tag{4.1}$$

or

(ii)

$$\sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} \left(p_{ij} - q_{i-k+k',j-l+l'}\right) > 0, \qquad \text{for all large } m, n$$

$$\text{and} \tag{4.2}$$

$$\limsup_{m,n\to\infty} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} \left(p_{ij} - q_{i-k+k',j-l+l'}\right) > 1.$$

Then every solution of equation (1.1) oscillates.

PROOF. Assume, for the sake of contradiction, that equation (1.1) has an eventually positive solution $\{A_{mn}\}$. By Lemmas 4 and 5, it follows that the sequence $\{C_{mn}\}$ is eventually decreasing and positive and

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \left(p_{mn} - q_{m-k+k',n-l+l'}\right) A_{m-k,n-l} \leq 0. \tag{4.3}$$

Also,

$$0 < C_{mn} \leq A_{mn}, \tag{4.4}$$

thus, $C_{m-k,n-l} < A_{m-k,n-l}$, using (4.3), we obtain

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + (p_{mn} - q_{m-k+k',n-l+l'})C_{m-k,n-l} \le 0. \tag{4.5}$$

However, by Lemmas 1 and 2, inequality (4.5) cannot have an eventually positive solution. This contradicts (4.4), and the proof is complete.

EXAMPLE 1. Consider the partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{mn} + \left(\frac{3}{4} - \frac{1}{2n}\right) A_{m-2,n-1} - \frac{1}{n} A_{m-1,n} = 0. \tag{4.6}$$

In this example, $m \ge 2$, $n \ge 4$, $p_{mn} = 3/4 - 1/2n$, $q_{mn} = 1/n$, $k = 2$, $k' = l = 1$, $l' = 0$. Since $k = 2 > 1 = k'$, $l > l'$ and for $m \ge 2$, $n \ge 4$, we have

1^0.
$$p_{mn} - q_{m-k+k',n-l+l'} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{n-1} > 0.$$

2^0.
$$\begin{aligned}
&\liminf_{m,n\to\infty} \left[\frac{1}{kl} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} (p_{ij} - q_{i-k+k',j-l+l'})\right] \\
&= \liminf_{m,n\to\infty} \left[\frac{1}{2} \sum_{i=m-2}^{m-1} \sum_{j=n-1}^{n-1} \left(\frac{3}{4} - \frac{1}{2j} - \frac{1}{j-1}\right)\right] \\
&= \liminf_{m,n\to\infty} \left(\frac{3}{4} - \frac{1}{2(n-1)} - \frac{1}{n-2}\right) = \frac{3}{4} > \frac{12}{49}\left(\frac{4}{7}\right)^{1/3} = \frac{\omega^\omega}{(\omega+1)^{\omega+1}}.
\end{aligned}$$

Hence, all the hypotheses of Theorem 4.1 are satisfied. Therefore, all solutions of equation (4.6) are oscillatory. In fact, (4.6) has an oscillatory solution $\{A_{mn}\} = \{(-1)^m(1/2^n)\}$ for $m \ge 2$, $n \ge 4$.

Before we establish the next oscillation theorem, we need the following result about partial difference inequalities which is interesting in its own right.

LEMMA 6. *Assume that for $s = 1, 2, \dots, p$, $k_s, l_s \in N_0$ and $\{r_{mn}^{(s)}\}$ are sequences of nonnegative real numbers such that for every $m_0, n_0 \in N_0$, there exists an $s_0 \in \{0, 1, 2, \dots, p\}$ with the property that*

$$\sum_{i=m_0}^{m_0+k_{s_0}} \sum_{j=n_0}^{n_0+l_{s_0}} r_{i+1,j}^{(s_0)} > 0. \tag{4.7}$$

Let $\bar{k} = \max\{k_0, k_1, \dots, k_p\}$, $\bar{l} = \{l_0, l_1, l_2, \dots, l_p\}$ and assume that the inequality

$$\sum_{s=0}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} r_{i+1,j}^{(s)} b_{i-k,j-l_s} \le b_{mn}, \qquad \text{for } \begin{cases} m \ge m_1, \\ n \ge n_1, \end{cases} \tag{4.8}$$

has a positive solution $b = \{b_{mn}\}_{m_1-\bar{k},n_1-\bar{l}}^{\infty,\infty}$ such that

$$b_{m_1,n_1} < b_{mn}, \qquad \text{for } m_1 - \bar{k} \le m \le m_1, \quad n_1 - \bar{l} \le n \le n_1. \tag{4.9}$$

Then there exists a positive solution $c = \{c_{mn}\}_{m_1-\bar{k},n_1-\bar{l}}^{\infty,\infty}$ of the equation

$$\sum_{s=0}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} r_{i+1,j}^{(s)}\, c_{i-k_s,j-l_s} = c_{mn}, \qquad \text{for } \begin{cases} m \ge m_1, \\ n \ge n_1. \end{cases} \tag{4.10}$$

PROOF. Define the set of nonnegative sequences

$$\Lambda = \left\{ \bar{c} = \{\bar{c}_{mn}\}_{m=m_1,n=n_1}^{\infty,\infty} \mid 0 \le \bar{c}_{mn} \le b_{mn} \text{ for } m \ge m_1,\ n \ge n_1 \right\}.$$

For every $\bar{c} \in \Lambda$, define the sequence $c = \{c_{mn}\}_{m_1-\bar{k},n_1-\bar{l}}^{\infty,\infty}$ by

$$c_{mn} = \begin{cases} \bar{c} & : m,n \in [m_1,\infty) \times [n_1,\infty), \\ \bar{c}_{m_1n_1} + b_{mn} - b_{m_1n_1} & : m,n \in \left[m_1 - \bar{k}, m_1\right) \times \left[n_1 - \bar{l}, n_1\right). \end{cases}$$

Clearly,

$$0 \le c_{mn} \le b_{mn}, \qquad \text{for } m \ge m_1 - \bar{k}, \quad n \ge n_1 - \bar{l},$$

and in view of (4.9),

$$c_{mn} > 0, \qquad \text{for } m,n \in \left[m_1 - \bar{k}, m_1\right) \times \left[n_1 - \bar{l}, n_1\right). \tag{4.11}$$

Now define the mapping T on Λ as follows: for every $\bar{c} = \{\bar{c}_{mn}\} \in \Lambda$, let the term of the sequence $T\bar{c}$ be

$$\sum_{s=0}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} r_{i+1,j}^{(s)} c_{i-k_s,j-l_s}.$$

Then one can see that T is monotone in the sense that if $\bar{c}^{(1)}, \bar{c}^{(2)} \in \Lambda$ and $\bar{c}^{(1)} \le \bar{c}^{(2)}$ (that is, $\bar{c}_{mn}^{(1)} \le \bar{c}_{mn}^{(2)}$ for $m \ge m_1$, $n \ge n_1$), then $T\bar{c}^{(1)} \le T\bar{c}^{(2)}$. From (4.8), $Tb \le b$, from which it follows that

$$T : \Lambda \to \Lambda.$$

Define

$$\bar{c}^{(0)} = \{b_{mn}\}_{m_1,n_1}^{\infty,\infty} \quad \text{and} \quad \bar{c}^{(r)} = T\bar{c}^{(r-1)}, \qquad \text{for } m,n = 1,2,\dots.$$

Then one can see by induction that the sequence $\{\bar{c}^{(r)}\} \subset \Lambda$ satisfies

$$0 \le \bar{c}^{(r+1)} \le \bar{c}^{(r)} \le b_{mn}, \qquad \text{for } m \ge m_1, \quad n \ge n_1.$$

Thus,

$$\bar{c}_{mn} = \lim_{r\to\infty} \bar{c}_{mn}^{(r)}, \qquad m \ge m_1, \quad n \ge n_1$$

exists and $\bar{c} = \{\bar{c}_{mn}\}_{m=m_1,n=n_1}^{\infty,\infty}$ belongs Λ. Also $T\bar{c} = \bar{c}$ and so c is a solution of equation (4.10). It remains to show that

$$c_{mn} > 0, \qquad \text{for } m \ge m_1 - \bar{k}, \quad n \ge n_1 - \bar{l}. \tag{4.12}$$

If (4.12) is false, then there exist some $m_2 \ge m_1$, $n_2 \ge n_1$ such that

$$c_{m_2,n_2} = 0 \quad \text{and} \quad c_{mn} > 0, \qquad \text{for } m,n \in \left[m_1 - \bar{k}, m_2\right) \times \left[n_1 - \bar{l}, n_2\right).$$

Then from (4.10),

$$\sum_{s=0}^{p} \sum_{i=m_2}^{\infty} \sum_{j=n_2}^{\infty} r_{i+1,j}^{(s)} c_{i-k_s,j-l_s} = 0. \tag{4.13}$$

But by (4.7), there exists an $s_2 \in \{0,1,2,\dots,p\}$ such that

$$\sum_{i=m_2}^{m_2+k_{s_2}} \sum_{j=n_2}^{n_2+l_{s_2}} r_{i+1,j}^{(s_2)} > 0.$$

Hence,

$$\sum_{s=0}^{p}\sum_{i=m_2}^{\infty}\sum_{j=n_2}^{\infty} r_{i+1,j}^{(s_2)} c_{i-k_{s_2},j-l_{s_2}} \geq \sum_{i=m_2}^{m_2+k_{s_2}}\sum_{j=n_2}^{n_2+l_{s_2}} r_{i+1,j}^{(s_2)} c_{i-k_{s_2},j-l_{s_2}} > 0,$$

which contradicts (4.13) and completes the proof.

LEMMA 7. *Using equation (1.1) and the transform (2.5), we have*

$$\begin{aligned} &C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} C_{m-k,n-l} \\ &+ \frac{1}{2}\alpha_{mn} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l} \leq 0. \end{aligned} \tag{4.14}$$

PROOF. From (iii) of (2.7), we have

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} A_{m-k,n-l} + \beta_{mn}(A) = 0$$

or

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} A_{m-k,n-l} \leq 0. \tag{4.15}$$

From (2.5), we have

$$\begin{aligned} A_{mn} = C_{mn} &+ \left(\sum_{i=m}^{m+k'} q_{in} A_{i-k',n-l'} + \sum_{i=n}^{n+l'} q_{mj} A_{m-k',j-l'}\right) \\ &+ \frac{1}{2}\left(\sum_{i=m}^{m+k'} q_{i+k'-k,n+l'-l} A_{i-k,n-l} + \sum_{j=n}^{n+l'} q_{m+k'-k,j+l'-l} A_{m-k,j-l}\right). \end{aligned}$$

We can improve it for the following case:

$$\begin{aligned} A_{mn} = C_{mn} &+ \left(\sum_{i=0}^{k'} q_{m+i,n} A_{m+i-k',n-l'} + \sum_{i=0}^{l'} q_{m,n+j} A_{m-k',n+j-l'}\right) \\ &+ \frac{1}{2}\left(\sum_{i=0}^{k'} q_{m+i+k'-k,n+l'-l} A_{m+i-k,n-l} + \sum_{j=0}^{l'} q_{m+k'-k,n+j+l'-l} A_{m-k,n+j-l}\right) \end{aligned}$$

and

$$\begin{aligned} A_{m-k,n-l} &= C_{m-k,n-l} \\ &+ \left(\sum_{i=0}^{k'} q_{m-k+i,n-l} A_{m-k+i-k',n-l-l'} + \sum_{i=0}^{l'} q_{m-k,n-l+j} A_{m-k-k',n-l+j-l'}\right) \\ &+ \frac{1}{2}\left(\sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l} + \sum_{j=0}^{l'} q_{m-2k+k',n-2l+j+l'} A_{m-2k,n-2l+j}\right), \end{aligned} \tag{4.16}$$

therefore, we have

$$\begin{aligned} &\text{(a)} \quad A_{m-k,n-l} \geq C_{m-k,n-l} + \frac{1}{2}\sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l}, \\ &\text{(b)} \quad A_{m-k,n-l} \geq C_{m-k,n-l} + \frac{1}{2}\sum_{j=0}^{l'} q_{m-2k+k',n-2l+j+l'} A_{m-2k,n-2l+j}, \\ &\text{(c)} \quad A_{m-k,n-l} \geq C_{m-k,n-l} + \sum_{i=0}^{k'} q_{m-k+i,n-l} A_{m-k+i-k',n-l-l'}, \\ &\text{(d)} \quad A_{m-k,n-l} \geq C_{m-k,n-l} + \sum_{j=0}^{l'} q_{m-k,n-l+j} A_{m-k-k',n-l+j-l'}. \end{aligned} \tag{4.17}$$

Substituting (a) into (4.15), we obtain

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} C_{m-k,n-l} + \frac{1}{2}\alpha_{mn} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l} \le 0.$$

Similarly, using (b)–(d) in (4.17), we will also obtain similar results as (4.14). The proof is complete.

Let

$$\begin{aligned} H(m,n) = {} & \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} C_{m-2k+i,n-2l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \\ & \times \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} C_{m-3k+2i,n-3l} + \left(\frac{1}{2}\right)^2 \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \\ & \times \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} \sum_{i=0}^{k'} q_{m-4k+3i+k',n-4l+l'} C_{m-4k+3i,n-4l} + \cdots \\ & + \left(\frac{1}{2}\right)^{(p-1)} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} \times \cdots \\ & \times \sum_{i=0}^{k'} q_{m-(p+1)k+pi+k',n-(p+1)l+l'} A_{m-(p+1)k+pi,n-(p+1)l}. \end{aligned}$$

Then, we obtain that the following results.

LEMMA 8.

$$\sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l} \ge H(m,n). \tag{4.18}$$

PROOF. From (a) of (4.17), we obtain

$$A_{m-2k+i,n-2l} \ge C_{m-2k+i,n-2l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} A_{m-3k+2i,n-3l}, \tag{4.19}$$

hence,

$$\begin{aligned} & \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} A_{m-2k+i,n-2l} \qquad \text{(substituting (4.19) into } A_{m-2k+i,n-2l}) \\ & \quad \ge \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \left(C_{m-2k+i,n-2l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} A_{m-3k+2i,n-3l} \right) \\ & \quad = \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} C_{m-2k+i,n-2l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \\ & \qquad \times \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} A_{m-3k+2i,n-3l} \qquad \text{(substituting (4.19) into } A_{m-3k+2i,n-3l}) \\ & \quad \ge \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} C_{m-2k+i,n-2l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \\ & \qquad \times \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} \left(C_{m-3k+2i,n-3l} + \frac{1}{2} \sum_{i=0}^{k'} q_{m-4k+3i+k',n-4l+l'} A_{m-4k+3i,n-4l} \right) \end{aligned}$$

$$\begin{aligned}&\geq \\ &\vdots \quad \begin{pmatrix}\text{Repeating the above arguments with } (p-1)^{\text{th}} \text{ step} \\ \text{and by induction we find that.}\end{pmatrix} \\ &\geq H(m,n).\end{aligned}$$

Therefore, (4.18) holds and the proof is complete.

LEMMA 9.

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} C_{m-k,n-l} + \frac{1}{2}\alpha_{mn} H(m,n) \leq 0. \tag{4.20}$$

PROOF. Substituting (4.18) into (4.14), then we obtain (4.20).

THEOREM 4.2. *Assume that (1.2) holds and there exist positive numbers α_0, p and the nonnegative number Q such that*

(i)

$$\alpha_{mn} = p_{mn} - q_{m-k+k',n-l+l'} \geq \alpha_0, \qquad \text{for } m \geq k-k', \quad n \geq l-l', \tag{4.21}$$

(ii)

$$\sum_{i=0}^{k'} q_{m-(r+1)k+ri+k',n-(r+1)l+l'} \geq Q, \qquad \text{for } m,n \text{ sufficiently large}, \quad r = 1,2,\dots,p, \tag{4.22}$$

(iii) *every solution of the delay difference equation*

$$B_{m+1,n} + B_{m,n+1} - B_{mn} + \sum_{r=1}^{p} \alpha_{mn} \left(\frac{Q}{2}\right)^r B_{m-k,n-rl} = 0$$

oscillates.

Then every solution of equation (1.1) also oscillates.

PROOF. Assume, for the sake of contradiction, that equation (1.1) has an eventually positive solution $\{A_{mn}\}$. By Lemmas 4 and 5, for m,n sufficiently large, the sequence $\{C_{st}\}$ satisfies

$$0 < C_{mn}, \qquad C_{m+1,n} \leq C_{mn}, \qquad C_{m,n+1} \leq C_{mn}, \tag{4.23}$$

and

$$0 < C_{mn} \leq A_{mn}. \tag{4.24}$$

From (4.24), using Theorem 3.1, we get

$$\lim_{m,n\to\infty} C_{mn} = \lim_{m,n\to\infty} A_{m,n} = 0. \tag{4.25}$$

Using (4.24), we get

$$A_{m-(p+1)k+pi,n-(p+1)l} \geq C_{m-(p+1)k+pi,n-(p+1)l},$$

and substituting $C_{m-(p+1)k+pi,n-(p+1)l}$ into the last item of $H(m,n)$, we obtain

$$\begin{aligned}H(m,n) = {} & \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} C_{m-2k+i,n-2l} + \frac{1}{2}\sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \\ & \times \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} C_{m-3k+2i,n-3l} + \cdots \\ & + \left(\frac{1}{2}\right)^{(p-1)} \sum_{i=0}^{k'} q_{m-2k+i+k',n-2l+l'} \sum_{i=0}^{k'} q_{m-3k+2i+k',n-3l+l'} \times \cdots \\ & \times \sum_{i=0}^{k'} q_{m-(p+1)k+pi+k',n-(p+1)l+l'} C_{m-(p+1)k+pi,n-(p+1)l}.\end{aligned} \tag{4.26}$$

From (4.26), using condition (4.22) and the decreasing property of $\{C_{mn}\}$, replacing $C_{m-(p+1)k+pi,n-(p+1)l}$ by $C_{m-(r+1)k+rk,n-(r+1)l} = C_{m-k,n-(r+1)l}$, $r = 1, 2, \ldots, p$, we get

$$H(m,n) \geq \sum_{r=1}^{p} \left(\frac{1}{2}\right)^{r-1} Q^r C_{m-k,n-(r+1)l} \geq \sum_{r=1}^{p} \left(\frac{1}{2}\right)^{r-1} Q^r C_{m-k,n-rl}. \tag{4.27}$$

Substituting (4.27) into (4.20), we have

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \alpha_{mn} C_{m-k,n-l} + \sum_{r=1}^{p} \alpha_{mn} \left(\frac{Q}{2}\right)^r C_{m-k,n-rl} \leq 0$$

or

$$C_{m+1,n} + C_{m,n+1} - C_{mn} + \sum_{r=1}^{p} \alpha_{mn} \left(\frac{Q}{2}\right)^r C_{m-k,n-rl} \leq 0. \tag{4.28}$$

By summing up both sides of (4.28) from m, n to infinity, we obtain

$$\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} (C_{i+1,j} + C_{i,j+1} - C_{ij}) + \sum_{r=1}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \alpha_{mn} \left(\frac{Q}{2}\right)^r C_{m-k,n-rl} \leq 0.$$

In view of Lemma 3, we have

$$\sum_{i=m}^{\infty} \sum_{j=n+1}^{\infty} C_{i+1,j} - C_{mn} + \sum_{r=1}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \alpha_{mn} \left(\frac{Q}{2}\right)^r C_{m-k,n-rl} \leq 0.$$

Hence,

$$\sum_{r=1}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \alpha_{mn} \left(\frac{Q}{2}\right)^r C_{m-k,n-rl} \leq C_{mn}. \tag{4.29}$$

In view of (1.2), (4.23), (4.24), it is easy to see that the hypotheses of Lemma 6 are satisfied. Then the equation

$$\sum_{r=1}^{p} \sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \alpha_{mn} \left(\frac{Q}{2}\right)^r B_{m-k,n-rl} = B_{mn} \tag{4.30}$$

has a positive solution $\{B_{mn}\}$. Clearly, $\{B_{mn}\}$ is also a positive solution of the equation

$$B_{m+1,n} + B_{m,n+1} - B_{mn} + \sum_{r=1}^{p} \alpha_{mn} \left(\frac{Q}{2}\right)^r B_{m-k,n-rl} = 0, \tag{4.31}$$

which contradicts the hypothesis and completes the proof.

REMARK 2. We can also replace $C_{m-(p+1)k+pi,n-(p+1)l}$ by

$$C_{m-(r+1)k'+rk',n-(r+1)l'} = C_{m-k',n-(r+1)l'}, \qquad r = 1, 2, \ldots, p,$$

to obtain similar results.

EXAMPLE 2. Consider the partial difference equation

$$\begin{aligned} A_{m+1,n} + A_{m,n+1} - A_{mn} + \left(\frac{1+2^{10}}{3^2} + \frac{1}{n+1}\right) A_{m-2,n-2} \\ - \left(\frac{1}{3^2} + \frac{1}{3(n+2)}\right) A_{m-1,n-1} = 0, \qquad m \geq 2, \quad n \geq 3. \end{aligned} \tag{4.32}$$

In this example, take $p = 1$, $\alpha_0 = 2^{10}/3^2$, and $k > k'$, $l > l'$. Note the following.

(i) $p_{mn} - q_{m-k+k',n-l+l'} = p_{mn} - q_{m-1,n-1} = 2^{10}/3^2 + 2/3(n+1) > \alpha_0$.

(ii) $\sum_{i=0}^{k'} q_{m+k'-(r-1)i,n+l'-rl} = \sum_{i=0}^{1} q_{m+1,n-1} = 2/3^2 + 2/3(n+1) < 1$.

(iii) Taking $Q = 2/3^2$, then, $\sum_{i=0}^{k'} q_{m+k'-(r-1)i,n+l'-rl} = 2/3^2 + 2/3(n+1) > 2/3^2 = Q$.

(iv) For equation (4.32), due to $p = 1$, then (4.31) becomes

$$B_{m+1,n} + B_{m,n+1} - B_{mn} + \frac{1}{3^2}\left(\frac{2^{10}}{3^2} + \frac{2}{3(n+1)}\right) B_{m-2,n-2} = 0. \tag{4.33}$$

From (4.33), since

$$\begin{aligned}\liminf_{m,n\to\infty}\left(\frac{1}{kl}\sum_{i=m-k}^{m-1}\sum_{j=n-l}^{n-1}\alpha_{ij}\left(\frac{Q}{2}\right)^r\right) &= \liminf_{m,n\to\infty}\left(\frac{1}{4}\sum_{i=m-1}^{m-1}\sum_{j=n-1}^{n-1}\frac{1}{3^2}\left(\frac{2^{10}}{3^2}+\frac{2}{3(j+1)}\right)\right)\\ &= \frac{2^8}{3^4} > \frac{4}{27} = \frac{\omega^\omega}{(\omega+1)^{\omega+1}},\end{aligned}$$

using Lemma 1, we obtain that every solution of (4.33) is oscillatory. Therefore, from Theorem 4.2, every solution of (4.32) is also oscillatory.

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