The Isaacs equation for differential games, totally optimal fields of trajectories and related problems

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Abstract

Some aspects of the Isaacs principle of transition and his main equation for differential games are considered to demonstrate that those equations contain some implicit assumptions, and are valid under certain contiguity condition which is defined and analyzed for differential games. The notion of total optimality is defined, and the totally optimal fields of trajectories and control curves are introduced and studied in relation to the Isaacs principle of transition, Bellman principle of optimality, maximum principle of Pontryagin, and variational principles of mechanics. It is demonstrated that the Isaacs, Bellman and Pontryagin theories are valid if and only if the optimal trajectories and optimal control curves generated by those methods are totally optimal. In this context, the Hamilton–Jacobi partial differential equation can be used for sequential solution of multi-games defined as n-person games with m controls, r cost functionals and multiple min, max, min–max, etc., operators in fixed order of application and not creating multi-objective game problems. Over totally optimal fields, the structure of controls is invariant under time uncertainty. Parallel and series games are considered, so the Isaacs procedure can be reduced to the application of the Bellman equations twice.

Control systems with incomplete information or structural limitations on controls do not, in general, satisfy the contiguity condition, thus, are not totally optimal. Game problems for such systems may have optimal solutions which, however, cannot be obtained by the Isaacs equations. This fact is shown in an example of a widely used engineering system for which an optimal trajectory has all its parts non-optimal and non-contiguous to the optimal trajectory. The paper presents theoretical justification of the Isaacs equations for contiguous systems, comparison of optimal control principles with variational principles of mechanics, the consideration of total optimality and totally optimal fields of trajectories as necessary and sufficient conditions for validity of the three major optimal control theories, and some other results important for applications.

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1. Introduction

There is vast literature on differential games, mainly concerned with the application of the Isaacs equation to various kinds of problems, the derivation of necessary or sufficient conditions of optimality and the corresponding equations, their solution by different methods under certain assumptions and conditions, investigation of convergence and the existence of solutions for differential games, stability properties, numerical studies and computational experiments, etc. Here we cite only references of immediate relation to the problems considered.

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In [1,2, Ch. 2], the formulation of a differential game problem is presented which we briefly reproduce in the author’s notation with some notational replacements (\(u\) for \(\varphi\), \(v\) for \(\psi\), \(X\) for \(\varphi \subset \mathbb{R}^n\)) to comply with notation currently used in the literature. The following equations are considered, see (2.1.1) in [1,2, Ch. 2, Sec. 2.1]:

\[
\frac{dx}{dt} = f(x, u, v), \quad t \geq 0, \ x \in X \subset \mathbb{R}^n, \ X \text{ closed, } x(0) = x_0 \text{ given, } u \in \mathbb{R}^m, \ v \in \mathbb{R}^k,
\]

\[
a_i(x) \leq u_i \leq b_i(x), \quad a_j^x(x) \leq v_j \leq b_j^x(x), \quad (1.1)
\]

which are called equations of motion. Vector-function \(f\) is assumed smooth and having all (continuous) partial derivatives that may be needed. Variables \(u, v\) are called controls that are under restrictions of (1.1) and can be changed by players at any moment. Thus, the motion is defined by the will and goals of both players, and the case of conflicting goals is of main interest. Player \(u\) is trying to minimize an objective function (the payoff, or cost) and player \(v\) is trying to maximize the same function which is chosen in the form:

\[
P = \int_0^T G(x, u, v) dt + H(s), \quad H(s) = K(x(T)) \quad (1.2)
\]

where \(T > 0\) is a fixed time of termination, and \(H(s)\) represents a terminal cost at the arrival onto some surface of interception represented by the \((n-1)\)-vector parameter \(s\), and in the case of collision or short distance capture it simply depends on some final position \(x(T)\), see [1,2, Sec. 2.4]. It is mentioned that \(T\) can be considered as a phase coordinate, thus, being a variable with the equation \(dT/dt = 1\). Then a function is introduced:

\[
Q = G(x, u, v) + p f(x, u, v), \quad p \in \mathbb{R}^n, \quad (1.3)
\]

where \(p\) is a vector of multipliers (instead of \(u\) in [1,2]) and the saddle point condition is assumed:

\[
\min_u \max_v Q = \max_v \min_u Q \quad \text{for all } p \text{ and all } x. \quad (1.4)
\]

Upon termination, the min–max of the Payoff (objective function) is called the Value which depends on a starting or intermediate point, and is denoted

\[
V(x(t)) = \min_x P(x(t)) = \min_{u(x), v(x)} \int_0^T G(x, u, v) dt + K(x(T)), \quad t \geq 0, \quad (1.5)
\]

where the min–max is taken with respect to all feasible \(u(x), v(x)\), an indication whereof we shall drop in the following. Note that if \(T\) varies, it must be included as a variable at the left in (1.5).

Possible difficulties with integration of system (1.1) are circumvented by the use of \(K\)-strategies [1–3] for integration over a sequence of segments \([t_i, t_{i+1}], i = 0, 1, 2, \ldots\), over which \(u(x), v(x)\) are approximated as piecewise constant functions called tactics. Restrictions on controls in (1.1) are considered constant and vector functions \(u(x), v(x)\) are supposed to belong to convex hulls which are closed and bounded, and thus, compact, see [1, 2, Sec. 2.6, 2.7].

2. Principle of transition and the main equation

In [1] SIAM, 1967, a re-edited version of RAND Report RM-1411 (21 December 1955), p. 64, with our notation \(u^*, v^*\) for optimal controls and \(X^*\) for “terminal surface”, \(X^* \subset X\), we read

“4.1. The nature of a solution

When a particular differential game has been solved, the results will generally embody such entities as

1. The Value: the function \(V\) defined over \(X\).

2. The optimal strategies: (vector) functions \(u^*(x)\) and \(v^*(x)\) defined over \(X\). They may or may not be unique. In the latter event we might be interested in obtaining a complete set of all optimal strategies or we might be satisfied with just one.

3. The optimal paths: \(\ldots\) These paths should fill \(X\) and each should terminate on \(X^*\).”

We denote optimal trajectories by \(x^*(x_0, t_0, t), t_0 \geq 0\) or simply \(x^*(t)\), and they are terminated on \(X^* \subset \partial X\) (terminal surface that, in general, belongs to the boundary of \(X\)). In some cases \(X^*\) may belong to the interior of \(X\), \(\text{int } X\). The surface \(X^*\) can be represented by \(n - 1\) parameters in the form \(x^* = h(s_1, \ldots, s_{n-1}) = h(s)\), giving rise to the term \(H(s)\) in (1.2), cf. [1,2, (2.3.1) in Sec. 2.3; (2.4.1) in Sec. 2.4].
The closed set $X$ in (1.1) is usually subdivided (by surfaces called *singular*) into a finite number of domains within each of which the value $V(x)$ is continuously differentiable (class $C^1$), so that for small $\Delta x$ we have

$$
\Delta V = V(x + \Delta x) - V(x) = \nabla V(x) \Delta x + o(\Delta x) = dV + o(dx),
$$

(2.1)

where gradient $\nabla V = \text{grad } V$ is continuous, see (4.1.1) in [1,2] where $u$ is used instead of $\Delta x$. Note that $x$ in (2.1) represents an initial point of an *optimal* trajectory since $V(x)$ is the value of the game, see (1.5), corresponding to optimal controls $u^*(x)$ and $v^*(x)$. To solve differential games, the following idea from [4] is used, see [1,2, Sec. 4.2], which we reproduce from [2, p. 69] in French, followed by our translation into English from Russian [1] and French [2] translations:

“Principe de transition: Si la partie progresse d’une position a une autre, $V$ etant supposee connue pour la seconde, la premiere est determinee par l’exigence des joueurs d’optimiser (c’est-a-dire rendre minimax) l’accroissement de $V$ pendant la transition.”

“Principle of transition. If the game has passed from one position to another and if in the second position the value of $V$ is known, then in the first position this value is defined by the requirement: the players must optimize the increase of $V$ (i.e., render it minimax) for the time of the passage.”

In [1, the second printing, SIAM, 1967, pp. 67–68], the Principle is described as follows:

“We utilize what might be called the tenet of transition... Let us consider an interlude of time in midplay. At its commencement the path has reached some definite point of $X$. We consider all possible $x$ which may be reached at the end of the interlude for all possible choices of the control variables by both players. We suppose that, for each endpoint, the game beginning there has already been solved; in other words, $V$ is known there. Then the payoff resulting from each choice, $u$, $v$ during the interlude will be known, and the control variables are to be so chosen as to render it minimax. When we let the duration of the interlude approach zero, the result yields a differential equation.”

The principle can be analytically described as follows:

$$
V(x_1) = \min \max \Delta V[x_1, x_2] + V(x_2),
$$

(2.2)

or, referring to (2.1), for small $\Delta x$

$$
V(x + \Delta x) = \min \max \Delta V(\Delta x) + V(x),
$$

(2.3)

where $x_1 = x + \Delta x, x_2 = x$ are arbitrary points in $X$. By definition of $V(x)$ in (1.5), semi-trajectories starting at $x_1 = x + \Delta x, x_2 = x$ in (2.2) and (2.3) are considered optimal.

Assuming that the value of the game in (1.5) exists, the main partial differential equation for $V(x)$ is derived as follows. Let $x(t) \in X$ be the state of a game at a moment $t$ with a known value $V(x)$ according to (1.5). In a small time increment $h$ (not to be confused with $h(s)$ above), the state changes to $x^0 = x(t + h) = x + \Delta x$, and the cost at $x(t)$ becomes

$$
P(x) = \int_t^{t+h} G(x, u, v)dt + V(x^0).
$$

(2.4)

Here the same letter $t$ is used as a dummy variable of integration and in the limits of the integral; for the simplicity of writing formulas, we shall use this habit throughout the paper. Also, the same letters $f, G$ will be used to denote original functions in (1.1) and (1.2) and different functions obtained from $f(\cdot), G(\cdot)$ by transformation of their arguments, see (2.9) below; this is customary, preserves the line of thought and should not cause confusion. Expanding the integral of (2.4) in the Taylor series with the remainder in the form of Lagrange implies, due to (1.1),

$$
\int_t^{t+h} G(x, u, v)dt = hG(x, u, v) + 0.5h^2 \nabla G f(x, u, v)|_{t+\theta h}, \quad 0 < \theta < 1.
$$

(2.5)

Due to (2.1), we have

$$
V(x^0) = V(x) + \Delta V = V(x) + dV + o(dx) = V(x) + \nabla V dx + o(dx)
= V(x) + \nabla V f(x, u, v)dt + o(dx) = V(x) + \nabla V f(x, u, v)h + o(h),
$$

(2.6)
and from (2.4)–(2.6) it follows that

\[ V(x) = \min \max P(x) = \min \max [\int_t^{t+h} G(x, u, v) \, dt + V(x^0)] \]

\[ = V(x) + h \min \max [\nabla V f(x, u, v) + G(x, u, v) + 0.5h \nabla G f(x, u, v) |_{t+oh} + o(h)]. \]

Now, \( V(x) \) on the left and the right cancels out, so dividing by \( h \) and letting \( h \to 0 \), the following main equation is obtained for differential games \([1, 2, \text{Sec. } 4.2, \text{Eq. (4.2.1)}] \):

\[ \min \max [\nabla V f(x, u, v) + G(x, u, v)] = 0. \]  

(2.8)

Comparing (2.8) with (1.3), we see that in (1.3) the multiplier \( p = \nabla V \), and in (1.4) we have, in fact, \( \min \max Q = 0 \). The saddle point condition (1.4) can be applied to (2.8) to facilitate the computation of optimal \( \min \max \) controls \( u, v \) making use of favorable structure of some particular games.

Optimal \( \min \max \) controls \( u^*, v^* \), in general, depend on the gradient \( \nabla V \), denoted in \([1, 2] \) as \( \nabla V = V_x = (V_j) \); thus, Eq. (2.8) can be rewritten as follows:

\[ \nabla V f(x, u^*(x, V_x), v^*(x, V_x)) + G(x, u^*(x, V_x), v^*(x, V_x)) = 0, \]

(2.9)

which is the partial differential equation for \( V \), see \([1, 2, \text{Sec. } 4.2, \text{Eq. (4.2.3)}]. \) For its solution, the path equations and the retrogression principle \([1, \text{SIAM, pp. 80–85 \text{ \{}Le Principe de Regression\}} \text{ in } [2, \text{pp. 83–87}\}] \) are generally used, though not always, as illustrated in Exercise 4.2.1, also called Example 4.4.1 (in the re-edited version \([1, \text{SIAM, 1967, p. 75} \text{ \}} \text{ and in } [2, \text{p. 77}] \) which we solve here to discuss some aspects pertaining to the method.

Exercise 4.2.1 (Example 4.4.1). Let the payoff \( P(x) \) in (1.2) and (2.4) be terminal, that is \( G(x, u, v) \equiv 0 \), and consider the equations of motion

\[ \frac{dx}{dt} = a(x, y) v + b(x, y) \sin u, \quad \frac{dy}{dt} = -1 + b(x, y) \cos u; \quad a, b > 0, \ v \in [-1, 1]. \]

(2.10)

Eq. (2.8) has the form

\[ \min \max [\partial V / \partial x (av + b \sin u) + \partial V / \partial y (-1 + b \cos u)] = 0. \]

(2.11)

Opening the parentheses in (2.11), we get

\[ \min \max [av \sin u + v \cos y - V_y] = 0. \]

(2.12)

Controls are separated, and it is clear that \( v^* = V_y \), thus, we have from (2.12):

\[ b \min (V_x \sin u + V_y \cos u) + a V_x v^* - V_y = 0. \]

(2.13)

Denoting \( \rho^2 = V_x^2 + V_y^2 \), we see that scalar product \( (V_x, V_y) (\sin u, \cos u) \) in the parentheses of (2.13) attains its min on the circumference of radius \( \rho \) (Lemma 2.8.1) and equals \( -\rho \), so that \( \min (V_x \sin u + V_y \cos u) = V_x \sin u^* + V_y \cos u^* = -\rho \) with \( \sin u^* = -V_x / \rho, \cos u^* = -V_y / \rho \), and Eq. (2.9) is of the form: \( a \vert V_x \vert = \rho - V_y = 0 \). This equation does not depend on initial conditions (not specified in (2.10)); thus, \( \min \max \) optimal trajectories in the \((x, y)\)-plane form a field over which optimal controls retain the same form and depend only on the current state (feedback control), one of them being \textit{bang–bang} control, \( v^* \). Moreover, by (2.9) we have

\[ \frac{dV}{dt} = -G(.) \equiv 0, \quad \text{thus } V(x, y) = \text{const}, \]

(2.14)

representing the surface of arrival, further called a semi-permeable surface under condition (2.8) with \( G(.) \equiv 0 \), cf. Eq. (2.11) above, (4.3.1) in \([1, 2, \text{Sec. } 4.3 \text{ and Theorem 4.4.1, with gradient } \nabla V \text{ normal to } V \equiv \text{const, see Example 4.4.1 to 4.4.5 in } [1, 2, \text{Sec. } 4.4] \text{ for the same system (2.10).} \]

The \textit{principle of transition} and the \textit{main equation} (2.8) have been used since to solve many specific practical problems. In \([1, 2]\), they are applied to discrete differential games, to games of kind and degree, to games with incomplete information; various applications are considered, mainly to warfare and some to economy (optimal program of steel production, Sec. 5.6), and extensive investigation of different surfaces (switching, singular, dispersive, universal, integral constraints) to be used as tools for solution is presented. However, the \textit{principle} and the \textit{main equation} are important statements in their own right, and it is interesting to study them as such, without an accent on their numerous applications, solution methods, and technical details.
3. Observations

The formulation of the principle of transition and of the main equation as presented in [1,2, Sec. 4.2], see Section 2 above, contains some explicit and implicit assumptions, such as the existence of optimal policies and trajectories which includes optimal first stage decisions and trajectories as its parts, the existence of the state resulting from the first stage decisions, and the existence of non-optimal policies and corresponding trajectories emanating from the same initial and intermediate states along an optimal trajectory. The value function \( V(x) \) is defined after a concrete differential game has been solved, thus, \( V(x) \) is already optimal (the value of the game). Among the most important suppositions, we can list the following properties assumed a priori.

3.1. The congruence property

Consider analytical formulation (2.2) of the principle, where terms at extreme left and right are already optimal. For simplicity, we drop the star as indication of optimality, denote times of departure from \( x_1 \) and \( x_2 \) as \( t_1 \) and \( t_2 \), and suppose first that optimal policies are unique and defined by piecewise continuous functions \( u_1(x) \), \( v_1(x) \) for a trajectory departing from \( x_1 \) \( (t_1 \leq t < T, T \leq \infty) \), then by \( u_2(x) \), \( v_2(x) \) for a path departing from \( x_2 \) \( (t_2 \leq t < T, T \leq \infty) \), and then by \( u_0(x) \), \( v_0(x) \) for a segment between \( x_1 \) and \( x_2 \) \( (t_1 \leq t < t_2) \). The congruence property is the requirement that the curves \( u_1(x) \), \( v_1(x) \) defined at \( x_1 \) \( (t_1 \leq t < T, T \leq \infty) \) for the entire period of the game coincide with the curve obtained by joining the segment of curves \( u_0(x) \), \( v_0(x) \) between \( x_1 \) and \( x_2 \) \( (t_1 \leq t < t_2) \), with the entire curves \( u_2(x) \), \( v_2(x) \) defined for a path departing from \( x_2 \) \( (t_2 \leq t < T, T \leq \infty) \), and this—for all \( t_1, t_2 \) within \([0, T)\):

\[
\{u_1(x), v_1(x)\} \equiv \{u_0(x), v_0(x)\} \cup \{u_2(x), v_2(x)\}, \quad \forall t_1, t_2, \ 0 \leq t_1 < t_2 < T. \tag{3.1}
\]

Indeed, otherwise, the principle of transition becomes self-contradictory for some \( t_1, t_2 \). In this case, the optimal \( V(x_1) = V(t_1, T) \) and \( V(x_2) = V(t_2, T) \) may well exist, but the principle of transition is invalid. Note that the principle of transition is postulated in [1] for any choice of \( x_1 \) and \( x_2 \), that is, for any fragmentation of the optimal trajectory.

If optimal policies are not unique, then the congruence property means the existence of such choices for which the identity (3.1) holds, and only those choices can be used in the main equation. For this and other reasons, see Section 6, we prefer to consider congruence as a separate assumption meaning the identity (3.1), without requirement of optimality for the entries of (3.1). There are different terms expressing this notion in the literature, e.g., “joining controls” in [5, p. 83], or “concatenated” pieces of admissible trajectory in [6, p. 87].

3.2. Additivity of costs in the optimality criterion \( P(.) \) and value function \( V(x) \)

This assumption can be seen from the functional equation (2.2) and other similar equations derived from the principle of transition. In some sources, e.g., in [5, pp. 84–85], it is postulated separately. In integral criteria, see (1.2), it follows from the additivity of the integral with respect to intervals of integration. This assumption represents a restriction on the class of problems to which the principle of transition is applied.

3.3. Feedback structure of optimal policies

This property is explicitly postulated in [1,2, Ch. 4] as reproduced above in definition of optimal strategies, and it can be seen from various equations in [1,2] that define optimal policies.

3.4. Independence on the preceding data and/or decisions (the Markov property)

The optimality of optimal min–max controls does not explicitly depend on a preceding state nor preceding controls. This requirement is manifested by the notations \( u^*(x), v^*(x) \) throughout the book [1,2]. In practice, it means that feedback controls obtained from equations implied by the principle of transition explicitly depend on the current state, and not on preceding states, nor preceding decisions.
3.5. Invariance of the value function \( V(x) \)

This hypothesis is not mentioned in the principle of transition as formulated in [1,2], and it does not follow from the principle nor from any of the preceding observations. However, it is used in derivation of all equations and PDEs in [1,2], see Section 2 above, and it is tacitly assumed in applications and research works that followed in subsequent publications. This hypothesis is supported by a number of solved examples, though without any theoretical justification. The invariance of the function \( V(x) \) may be violated with modification of parameters or arguments at subsequent parts of optimal trajectories.

3.6. The saddle point condition (1.4)

Though convenient for some problems, this condition (1.4) is actually not needed and represents a tribute to the classical game theory and to celebrated von Neumann theorem on mixed strategies for bilinear forms [7], see also [8, Ch. III, Sec. 17.6], that assure a saddle point. In problems with separated controls, see (2.12) above, it is trivial. Differential game problems without assumed saddle point in (1.4) can be solved by the application of the Bellman equations twice, Theorem 4.1 below; or by set contraction methods, see [9, Part IV Global Differential Games, pp. 203–267] where global solution by the Integral Global Optimization Method (Q. Zheng) is presented; or by the minmax and maximin Cubic Algorithms that are developed in [10] and applied to the solution of the homicidal chauffeur game, see [1, SIAM, 1967, pp. 30, 279] or [2, pp. 30, 299], for the realistic situation of a ship–torpedo collision-avoidance game under maneuverability restrictions on control systems and with some initial position of rudders at the moment the torpedo hits the water. With non-separable cost function (minimax intercept time), the global saddle point appears naturally without any preconditions [10].

4. Comparison with Bellman’s principle of optimality

In the ten line section from [11, page 83] we read

“§3. The principle of optimality

In each process, the functional equation governing the process was obtained by an application of the following intuitive:

Principle of Optimality. An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The mathematical transliteration of this simple principle will yield all the functional equations we shall encounter throughout the remainder of the book. A proof by contradiction is immediate.”

In control literature, the mathematical transliteration of the above principle for deterministic processes is related to certain types of functional equations, see, e.g., [11–16]. We reproduce some of them in the author’s notation from [11, Ch. IX, Sec. 13] where a new formalization of the calculus of variations is proposed for non-autonomous problems

\[
    f(a, c) = \max_u J(u) = \max_u \int_0^T F(x, u, t)dt, \quad dx/dt = G(x, u, t), \quad 0 \leq a \leq t \leq T, \quad x(a) = c, \quad (4.1)
\]

parameterized by \( a, c \) with variable \( a \in [0, T) \). Splitting the integral over two segments \([a, a + S]\) and \([a + S, T]\), we have, according to the principle of optimality,

\[
    f(a, c) = \max_{u[a, a+S]} \left[ \int_a^{a+S} F(x, u, t)dt + f(a + S, c(a + S)) \right], \quad (4.2)
\]

for all \( a \) and all \( S \) such that \( 0 \leq a < a + S \leq T \). For small \( S \) this yields

\[
    f(a, c) = \max_{u[a, a+S]} [F(c, u(a), a)S + f(a, c) + Sf_a + Sf_c dc(S)/dS + o(S)]. \quad (4.3)
\]
In (4.3) the terms \( f(a, c) \) cancel out and, dividing by \( S \), we get, to the first order as \( S \to 0 \)

\[
0 = \max_{u(a)}[F(c, u(a), a) + f_a + f_c G(c, u(a), a)],
\]

\[ (4.4) \]

\[
-f_a = \max_{v} [F(c, v, a) + f_c G(c, v, a)], \quad v = v(a, c) = u(a),
\]

\[ (4.5) \]

and for optimal \( v = v^0 \) in the interior of the region, we get, cf. [11, Ch. IX, Sec. 13, (8)] where \( \dim x = \dim c = 1 \):

\[
-f_a = F(c, v^0, a) + f_c G(c, v^0, a),
\]

\[ (4.6) \]

\[
0 = F_v(c, v^0, a) + f_c G_v(c, v^0, a).
\]

\[ (4.7) \]

Solving this system for \( f_a, f_c \) and equating \( f_{ac} = f_{ca} \) yields a partial differential equation to determine \( f(a, c) \) and \( v^0(a, c) \) which is the extremal control \( u^a(t, x(t)) \). Note that for \( \dim x > 1 \), the procedure of solving (4.6) and (4.7) should be modified, cf. (9.4) and (9.10) for \( \dim x = \dim c = 2, \dim u = 1 \) in Example 9.1 for an autonomous system.

**Remark 4.1.** If the extremal (supposed to be optimal) \( v^0 \) of (4.5) is not in the interior of the admissible region, then PDEs (4.6) and (4.7) must be replaced by extremality (stationarity) conditions of the Karush–Kuhn–Tucker type. Moreover, in this case the classical theorem on mixed partial derivatives to find \( f(a, c), v^0(a, c) \) may be inapplicable.

In engineering and economics, optimal policies (controls) usually belong to boundaries of admissible regions for controls and state variables, which necessitates the replacement of those PDEs (not the Bellman equation (4.5)).

**Remark 4.2.** Both the principle of transition (Isaacs) and the principle of optimality (Bellman) postulate the semi-group property for optimal policies (control curves); the first principle does it indirectly and the second in a straightforward and explicit statement. In simple terms, for problem (1.1) to (1.5) the semi-group property means the following. If we shut off the controls: \( u \equiv 0, v \equiv 0 \), then in (1.1) we have an autonomous system \( dx/dt = f(x), t \geq t_0 \geq 0, x(t_0) = x_0 \) given, which defines a field of tangent directions \( f(x) = dx/dt \) to trajectories \( x(t), t \geq t_0 \). This field is fixed within \( X \subseteq \mathbb{R}^n \) and does not depend on time which implies the congruence of all semi-trajectories of the system. Indeed, given \( x(t_0) = x_0 \), we have the corresponding trajectory \( x(t) = x(t_0, x_0, t), t \geq t_0 \). To any moment \( t^* > t_0 \), there corresponds a point \( x^* = x(t^*) = x(t_0, x_0, t^*) \) on that trajectory.

If we consider a new trajectory starting at \( (t^*, x^*) \) for the same system, i.e., the curve \( x(t^*, x^*, \tau), \tau = t - t^*, t^* = \text{const} \), then for this trajectory we would have the same tangent directions as for the semi-trajectory \( x(t) = x(t_0, x_0, t), t \geq t^* > t_0 \), because \( dx/d\tau = dx/d(t - t^*) = f(x) \) depends only on \( x \in X \). By the theorem of uniqueness of solutions for ODEs, this means that they coincide completely: \( x(t^*, x^*, \tau) = x(t_0, x_0, t) \) for \( t = \tau + t^* \), yielding the semi-group property (congruence in the sense of observation (3.1)) for trajectories of the field. However, for a non-autonomous system \( dx/dt = f(x, t) \), the tangent directions \( dx/d\tau = dx/d(t - t^*) = f(x, t - t^*) \) depend on \( t^* \) and will not, in general, be the same: for \( \tau = 0 \) we have \( t = t^* \) and \( x^* = x(t^*) \) directions are \( dx/d\tau = f(x^*, 0) \neq f(x^*, t^*) \) at \( t = t^* \). Hence, \( x(t^*, x^*, \tau) \neq x(t_0, x_0, t) \), for \( t = \tau + t^* \), semi-trajectories deviate, and the semi-group property (congruence) does not hold. The problem is that both principles postulate the semi-group property not for trajectories, but for policies (control curves) which is not always true even for autonomous closed loop systems, see Section 10. Moreover, the Bellman optimality principle postulates this property for non-autonomous control systems, as can be seen from (4.1) and (4.2) above.

There is an analogy and a difference between Isaacs’ principle of transition and Bellman’s principle of optimality. Both principles consider an entire optimal trajectory split in two parts by an arbitrary intermediate point. The Isaacs principle of transition requires optimization (minimax) of the starting part of the trajectory whereas the Bellman principle of optimality requires optimization (max or min only) of the remaining part of the trajectory. However, it is easy to see that Bellman’s equation (4.4) coincides with Isaacs’ main equation (2.8) with min-operator discarded.

For comparison, we reproduce here Eq. (2.8) from Section 2 denoting \( V \) by \( V_x \) as in [1], and Eq. (4.4) above where adjustment to the case in Section 2 implies \( a = 0, f_0 = 0, c = x \in \mathbb{R}^n, u \in \mathbb{R}^m \), as modified below:

\[
\min_{u} \max_{v} [V_x f(x, u, v) + G(x, u, v)] = 0, \quad u \in \mathbb{R}^m, \quad v \in \mathbb{R}^k,
\]

\[ (2.8^*) \]

\[
0 = \max_{u} [F(x, u) + f_x G(x, u)], \quad x \in \mathbb{R}^n, \quad f_x = \nabla f.
\]

\[ (4.4^*) \]
Comparing notations in (1.1), (1.2) and (1.5) with (4.1) and (4.2), we see that in (4.1) the letter $F$ is used as $G$ in (1.2), the letter $G$ is used as $f$ in (1.1), the letter $f$ in (4.2) is used as $V$ in (1.5), and $u$ in (4.4) plays the role of $v$ in (2.8*). So, excluding $u$ and the min operation with respect to $u$ from (2.8*) above, and considering $v$ instead of $u$ in (4.4*), we obtain identical equations.

Now, considering $u$ as a parameter in (2.8*), and noting that Bellman’s equations for min retain the same form as for max, we get the following procedure.

**Theorem 4.1.** If the Isaacs equation is valid for the case and admits sequential optimization uniform with respect to controls, then its solution can be obtained by applying the Bellman equations twice.

**Proof.** Consider the control function $u(x)$ as a parameter in (2.8*), removing the min operation for the moment. If the optimal $v = v^0$ is in the interior of the region uniformly with respect to $u(\cdot)$, then we can write similarly to (4.6) and (4.7), reversing the order of terms and using the Isaacs notations:

\[
V_x f(x, u, v^0) + G(x, u, v^0) = 0, \tag{4.8}
\]

\[
V_x f_v(x, u, v^0) + G_v(x, u, v^0) = 0. \tag{4.9}
\]

1. **Scalar case $n = m = k = 1$.** In this case $V_x = dV/dx$, $f_v = \partial f/\partial v$, $G_v = \partial G/\partial v$, so that all entries in (4.8) and (4.9) are scalars. System (4.8) and (4.9) is homogeneous and, if $V_x$ exists, the determinant in (4.8) and (4.9) is zero, yielding the equation that defines the extremal feedback control $v^0(x, u(\cdot))$, cf. [11, Ch. IX, Sec. 3, Eqs. (13)–(14); or Sec. 6, Eqs. (6)–(7) with $f_v = 0$]. If this $v^0(x, u(\cdot))$ is indeed optimal (max), then substituting it into (2.8*) removes the max operation, so regarding (2.8*) as a min-problem with respect to $u(\cdot)$ in the interior of the region uniformly with respect to $v^0(\cdot)$, we can write (4.8) again

\[
V_x f(x, u, v^0(x, u)) + G(x, u, v^0(x, u)) = 0, \tag{4.10}
\]

adding its full derivative with respect to $u$, as follows

\[
V_x f_u(x, u, v^0) + G_u(x, u, v^0) + V_x f_v(x, u, v^0)\frac{\partial v^0}{\partial u} + G_v(x, u, v^0)\frac{\partial v^0}{\partial u} = 0. \tag{4.11}
\]

These are two scalar equations with all scalar entries again. Excluding $V_x$ from (4.10) and (4.11) yields the equation that defines the extremal feedback control $u^0(x)$. If this $u^0(x)$ is indeed optimal (min), then it remains to compute the integral, using (4.10):

\[
dV f(x, u^0(x), v^0(x)) + G(x, u^0(x), v^0(x))dx = 0, \quad V(x(T)) = K(x(T)), \tag{4.12}
\]

which defines the value function $V(x)$ of (1.5), where we used the same notation $v^0(\cdot)$ for different functions in (4.10) and (4.12) produced by the procedure.

2. **Vector case.** In this case $V_x = \nabla V$ and $f(\cdot)$ in (4.8) and (4.9) are $n$-dimensional vectors, $G_v = \nabla_v G$ in (4.9) is a $k$-dimensional vector, and $f_v$ is the $n \times k$-Jacobian matrix in (4.9). Hence, we have one Eq. (4.8) and $k$ equations in (4.9) for a total of $k + 1$ equations to find $k$ components of the extremal feedback control $v^0(x, V_x, u(\cdot))$, which would normally depend on components of the gradient vector $V_x = \nabla V$. If this control $v^0(x, V_x, u(\cdot))$ is indeed optimal (max), then substituting it into (2.8*) removes the max operation, so regarding (2.8*) as a min-problem with respect to $u(\cdot)$ in the interior of the region uniformly with respect to $v^0(\cdot)$, we can write (4.8) again, in a different form:

\[
V_x f(x, u, v^0(x, V_x, u)) + G(x, u, v^0(x, V_x, u)) = 0, \tag{4.13}
\]

adding its full derivative with respect to $u$, as follows

\[
V_x f_u(x, u, v^0) + G_u(x, u, v^0) + V_x f_v(x, u, v^0)\frac{\partial v^0}{\partial u} + G_v(x, u, v^0)\frac{\partial v^0}{\partial u} = 0, \tag{4.14}
\]

where $f_u$, $f_v$, $\frac{\partial v^0}{\partial u}$ are Jacobian matrices of corresponding dimensions. Now we have one equation of (4.13) and $m$ equations in (4.14), for a total of $m + 1$ equations to find $m$ components of the extremal feedback control $u^0(x, V_x)$. If this control $u^0(x, V_x)$ is indeed optimal (min), then it remains to solve the nonlinear partial differential equation

\[
V_x f(x, u^0(\cdot), v^0(\cdot)) + G(x, u^0(\cdot), v^0(\cdot)) = 0, \tag{4.15}
\]

which defines the value function $V(x)$ of (1.5) and the resulting min–max controls $u^0(x)$, $v^0(x)$, where we used the same notations $u^0(\cdot)$, $v^0(\cdot)$ for different functions produced by the procedure. Of course, one cannot expect to
obtain formula-like solutions, although they do exist in many cases. The equations are amenable to both symbolic and numerical computations. In certain games, singularities may happen that would prevent the computations. Such cases are excluded by the general assumption of the theorem. □

Note that if \( u^o(.) \) or \( v^o(.) \) are not in the interior of admissible region, the Karush–Kuhn–Tucker type conditions should be applied instead of simple derivatives in (4.9), (4.11) and (4.14). Also, sufficient conditions for optimality should be used if necessary. If controls are independent of each other, then \( \partial v^o/\partial u = 0 \) and Eqs. (4.11) and (4.14) are simplified, see Example 4.1 below, and Example 9.1, Case 2, in Section 9.

**Example 4.1.** In Exercise 4.2.1 (Example 4.4.1) above, Eq. (2.13) has been surmised from (2.12) using its specific structure. Let us solve this problem by the general procedure given in the proof of Theorem 4.1 with Eq. (2.11) in place of (2.8*). We have, similarly to (4.8) and (4.9) with \( G(.) \equiv 0, n = 2, m = k = 1 \):

\[
\begin{align*}
\frac{\partial V}{\partial x} (a u^o + b \sin u) + \frac{\partial V}{\partial y} (-1 + b \cos u) &= 0, \\
\frac{\partial V}{\partial x} a(x, y) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\text{to maximize with respect to } v \\
\text{and that both terms in the parenthesis of } (4.18) \\
\text{be negative, yielding}
\end{align*}
\]

\[
\begin{align*}
\sin u^o(x) &= -V_x/\rho, \\
\cos u^o(x) &= -V_y/\rho,
\end{align*}
\]

\[
\begin{align*}
u^o(x) &= 2\pi - \arcsin(V_x/\rho), \\
u^o(x) &= \pi - \arccos(V_y/\rho).
\end{align*}
\]

With optimal controls defined by (4.18), we get from (4.18) the equation

\[
\frac{\partial V}{\partial x} a(x, y) - b(x, y)\rho - V_y = 0
\]

with \( \rho \) from (4.20), which presents the partial differential equation for the value function \( V(x, y) = \min \max P(x, y) \), the same as in Exercise 4.2.1 in Section 2. Comparing (4.20) and (4.22), we see that one should be careful with formal unique arc solutions for trigonometric equations but rather make a good choice from multiple arc solutions thereof.

**Remark 4.3.** Saddle point condition (1.4) for the function \( Q(.) \) of (1.3) has not been used in the proof of Theorem 4.1. In fact, for min–max problems one can use the protocol described in the proof of Theorem 4.1, and for max–min problems this protocol must be reversed: first solve for min with respect to \( u \) using \( v \) as a parameter, and then solve for max with respect to \( v \) using minimal \( u^o(x, V_x, v) \) already computed. Such protocols fully correspond to the definition of min–max and max–min, and to the intuitive common sense of the game.

### 5. Proof by contradiction

Geometrically, the principle of transition and the principle of optimality can be illustrated as follows. Consider a period of time \([t_0, T]\), \( T \leq \infty \), an admissible region \( X \) (state space), and denote by \( x(t) \in X \) a point on a trajectory \( x(t_0, T) \) which can be discrete or continuous and may depend on several policies (controls) applied to the process to assure the optimality of some criterion. For clarity, consider first the Bellman setting with one control \( u(x, t) \), see [11, 12]. Admissible controls are piecewise continuous functions, and the cost function (optimality criterion) max \( J(u(\cdot), \cdot) \) is assumed to be additive [11–16], that is

\[
J[t_1, t_3] = J[t_1, t_2] + J[t_2, t_3], \quad t_0 \leq t_1 < t_2 < t_3 \leq T.
\]
The policy (control)
\[ u[t_1, t_3] = u^o(x, t) = \arg \max J(u, \cdot), \quad t \in [t_1, t_3] \] (5.2)
and the corresponding piece of trajectory \( x[t_1, t_3] \) are called optimal. The statement of Bellman’s principle is simple: if \( u[t_o, T] \) is optimal for the whole trajectory \( x[t_o, T] \), then the remaining parts \( u[t, T], x[t, T] \) are also optimal for any \( t \in (t_o, T) \), provided that \( x(t) \in x[t_o, T] \). Shortly: remaining parts of an optimal policy and trajectory are themselves optimal with respect to the same criterion.

In some sources, see, e.g., [16, footnote to (4.2.16)], this condition is qualified as sufficient condition. Other sources present a proof of the principle as a necessary condition of optimality. In [5, pp. 86–87], or [6, p. 87], a simple proof of the principle is given which can be summarized as follows.

**Proof.** Setting \( t_1 = t_o, t_3 = T \) in (5.1) and assuming the existence of optimal policies (trajectories) starting at any point \( x(t) \in x[t_o, T] \), one can see that if \( J[t_o, T] = J^o \) is optimal, say, \( J^o = \max J[t_o, T] \) for some \( u^o(t), t \in [t_o, T] \), then it implies
\[ \max J[t_o, T] = J[t_o, t_2] + \max J[t_2, T]. \] (5.3)
Indeed, if \( J[t_2, T] \) in (5.3) is not maximal, then there is a better policy \( u[t_2, T], \) thus, \( J[t_o, T] \) at the left in (5.3) is not optimal, contradicting the assumption of its optimality. □

Using Theorem 4.1, the reader can extend this proof, adding the second stage, for the Isaacs principle of transition. We prefer to give an independent proof based on the same idea. According to the Isaacs notations, we have to use \( P(x, u, v) \), see (1.2), instead of \( J \) in (5.1). Then the Isaacs principle of transition implies, cf. (5.3):
\[ V[t_o, T] = \min \max P[t_o, T] = \min \max P[t_o, t_2] + V[t_2, T]. \] (5.4)
Indeed, if \( P[t_o, t_2] \) in (5.4) is not min–max optimal, then there are better policies \( u[t_o, t_2], v[t_o, t_2] \), thus, \( V[t_o, T] \) at the left in (5.4) is not optimal, contradicting the assumption of its min–max optimality. Thereby, the value of optimal \( V[t_2, T] \) at the right in (5.4) may be known or not. Note that in (5.4), in contrast with (5.3), the “remaining” part is the first (starting) part, not the second part. Also, relations (5.3) and (5.4) can be considered with respect to many controls corresponding to operators min and max taken in any order.

### 6. Total optimality and optimal (extremal) fields

With the same idea, a much stronger result can be proved.

**Theorem 6.1.** If for an optimal trajectory, all remaining (or starting) decisions are optimal and congruent with the original optimal policy, then every part of that trajectory is itself optimal.

**Proof.** Consider relation (5.1) again with \( t_3 = T \), not fixing \( t_1 \) nor \( t_2 \). By assumption, \( J[t_1, T], J[t_2, T] \) are optimal, say, maximal, as corresponding to optimal remaining decisions, thus, instead of (5.1), we can write
\[ \max J[t_1, T] = J[t_1, t_2] + \max J[t_2, T], \quad t_o \leq t_1 < t_2 \leq T. \] (6.1)

Now, if \( J[t_1, t_2] \) in (6.1) is not maximal, there is a better policy \( u[t_1, t_2] \), thus, \( J[t_1, T] \) at the left in (6.1) is not optimal, contradicting the assumption of its optimality. The same conclusion follows for remaining (starting) decisions in (5.4) with all starting (remaining) decisions of an optimal trajectory being optimal and congruent with the original optimal policy. □

The conclusion holds not only for max-optimality as in (6.1), but also for min–max or any other optimality in (6.1). It is worth noting that the same conclusion is true, if max-values in (6.1) are substituted with zero values. Indeed, if \( J[t_1, T] = J[t_2, T] = 0 \), then necessarily \( J[t_1, t_2] = 0 \). This is important for some problems with inflection points, see Remark 6.1 below, and also for variational problems. Indeed, if \( \delta J[t_1, T] = \delta J[t_2, T] = 0 \), then \( \delta J[t_1, t_2] = 0 \) which implies extremality of parts of trajectory with respect to controls, or to the cost functional itself, or to neighboring arcs of the trajectory, see Section 12. All these cases we shall connote in the following with the term “extremal(ity)”. This term includes extremality in the sense of (4.7), see Remark 4.1, as a particular case.
Corollary 6.1. If two parts \( x[t_1, t_3], x[t_2, t_3], t_1 < t_2 < t_3, \) of a trajectory, not necessarily optimal (extremal), are optimal (extremal), then the piece \( x[t_1, t_2] \) of that trajectory is also optimal (extremal). In general, under assumptions 3.1, 3.2, and 3.5, Section 3, if two terms in (5.1) are optimal (extremal), then the third term is also optimal (extremal).

Definition 6.1. If all parts of a trajectory including the whole trajectory itself are optimal (extremal), then such a trajectory (process) is called totally optimal (extremal).

We see that, according to the proof by contradiction, both Isaacs’ and Bellman’s principles under assumptions 3.1 to 3.5 of Section 3 imply total optimality. It is well known that certain optimal autonomous systems are totally optimal. Certain optimal non-autonomous systems may also be totally optimal.

The notion of total optimality (extremality) of a trajectory extends to the notion of totally optimal (extremal) field of trajectories. Consider again system (1.1) with the cost (1.2), (1.5). For a given set \( X_0 \subseteq X \), trajectories corresponding to different \( x_0 \in X_0 \) and optimal \( u^*(x), v^*(x) \) fill certain region in \( X \), and they do not intersect if (1.1) has a unique solution for any fixed \( x_0, u^*(x), v^*(x) \), which is assumed in [1]. Such a set of trajectories is usually called a field \( \Phi \) of trajectories. Consider any point \( x = x(t) \) on any trajectory of this field \( \Phi \).

Definition 6.2. If every trajectory in \( \Phi \) is totally optimal (extremal) with respect to the same cost functional \( V(x) \), then \( \Phi \) is called a totally optimal (extremal) field of trajectories with respect to \( V(u) \).

Totally optimal fields of trajectories, surfaces, or manifolds do exist in any dimension. Over such fields, different principles of optimality can be formulated for max, min, min–max, etc., based on fragmentation, which in a continuous case would render functional differential equations for the corresponding value function.

Example 6.1. The existence of totally min–max optimal controls and trajectories can be seen without any optimality arguments. Indeed, for the problem:

\[
V(x) = \min_u \max_v \int_t^T [(x + u)^2 - v^2]dt, \quad \frac{dx}{dt} = f(x, t, u, v), \quad t \geq 0, \ x(0) = x_0, \quad (6.2)
\]

the global min–max solution is \( V(x) = 0 \) for \( u = -x, v = 0 \), and all \( t, T, x_0 \). The set of those trajectories is obviously a totally optimal field of trajectories corresponding to a given \( f(.) \) for which solutions of \( \frac{dx}{dt} = f(x, t, -x, 0) \) exist over \([0, T] \) for any \( x_0 \).

Example 6.2. Consider the problem: find \( u(x) \) and \( F(x) \) such that

\[
W(\Omega) = \min_u \int_\Omega \sum_{i=1}^n (x_i + u_i)^2dx_i \ldots dx_n, \quad u\nabla F = 0; \quad x \in R^n, \ \Omega \subseteq R^n. \quad (6.3)
\]

The global optimal solution for (6.3) is \( W(\Omega) = 0 \) for \( u_i = -x_i \neq 0 \). \( F(x) = \varphi(x_1/x_2, \ldots, x_i/x_{i+1}, \ldots, x_{n-1}/x_n, x_n/x_1) \), with arbitrary function \( \varphi \) and any \( \Omega, n \geq 2 \).

Remark 6.1. In (6.2) and (6.3), the global optimality is assured by the choice of squares in the costs, making the functional in (6.2) globally convex–concave in \( u, v \), and the functional in (6.3) globally convex in \( u_i \). If one takes odd powers in (6.2) and (6.3), there will be inflection points at zero for both cost functionals. In this case, there will be no min–max nor min optimality, and both problems will present totally extremal fields of trajectories in (6.2) and surfaces in (6.3). Hence, there are systems which present totally extremal, not necessarily optimal fields of manifolds, see Section 11 below.

Remark 6.2. From the above considerations, it is clear that a totally optimal (extremal) field represents a structure with a sort of semi-group property. Autonomous systems of ODEs and PDEs, or other Markovian systems may have the semi-group property without being optimal. Consideration of some kind of optimality for such a system may conserve or destroy the semi-group property that it originally possessed. Moreover, in the optimized system, a semi-group property, if it exists, relates to a specific entity. The semi-group property embodied in the principles of Isaacs and Bellman was postulated for policies (controls) with congruence tacitly implied, and then checked on examples where resulting equations had a solution. As we shall see below, this semi-group property regarding optimality of policies (controls) holds over totally optimal trajectories, and only over such trajectories.
There is another noteworthy property of totally extremal (optimal) fields, concerning the reception and use of measured data. Denote by \( z(t) \) some quantity (position, velocity, mass, energy, charge, temperature, etc.) that changes with time. To avoid confusion with physical uncertainty (Heisenberg’s relation), suppose for a moment that, when measuring the value of \( z(t) \) with some supernatural device, we do not interfere with its state or magnitude by the external action of the measuring device; thus, the measure of \( z(t) \) is precise and made at the very moment \( t \). To receive and use this information about \( z(t) \), we have to transmit it to some other device(s) which we assume to be precise and free of errors in reception and action too. Upon reception, it is usually said that \( z(t) \) is observed or “known” (the measuring action is concentrated upon \( z(t) \) at a moment \( t \), but its conception, utilization, value or quality appears somewhere else, at a distance).

**Time-uncertainty statement.** The value \( z(t) \) is not known at time \( t \).

Indeed, since the speed of information transmittal is finite (by the postulate of Einstein, it is less than the speed of light), so the value \( z(t) \) is received at a moment \( t + \delta, \delta > 0 \). Hence, \( z(t) \) is not known and cannot be used at time \( t \), but only later. It implies a finite time error \( \Delta z = z(t + \delta) - z(t) \), to which other errors due to physical uncertainty and measurement imprecision add up. This delay of information can be felt in everyday life. It can cause a car accident: if a driver in front of you applies brakes, you see his red lights but can react only in a second or two, even later if you are talking on a cell phone. Let us compare the error in location of a particle due to time uncertainty with the error in location of the same particle due to physical uncertainty implied by Heisenberg’s relation. Using data from [17, p. 55] for helium, the lightest monatomic gas, under normal conditions (0 °C and 1 atm.) we have in c.g.s. °C system the following data:

- Planck’s constant \( h = 6.6242 \times 10^{-27} \)
- Boltzmann constant \( k = 1.3805 \times 10^{-16} \)
- Atomic mass of helium \( m = 1.6725 \times 10^{-24} \)
- Absolute temperature (Kelvin) \( T = 273 \).

With these data, the Heisenberg uncertainty relation (physical uncertainty) gives “a lower limit of the uncertainty \( \Delta x \) in the location of the particle” [17, p.64]:

\[
\Delta x > \frac{h}{2\pi} (3mkT)^{1/2} = 24.2345 \times 10^{-10} \text{ (cm)},
\]

(6.4)

where \((3mkT)^{1/2} = mv = p\) is the momentum of the particle and \( v = (3kT/m)^{1/2} \cong 2.6 \times 10^5 \text{ cm/s}\) is the root-mean-square velocity of the haphazard thermal motion. Now, assuming the speed of information transmittal equal to the speed of light in a vacuum \( c = 2.997925 \times 10^{10} \cong 3 \times 10^{10} \text{ (cm/s)}\), we obtain a lower limit of the error \( \Delta^x x = \Delta z \) due to time uncertainty \( \delta > 0 \) for the location of the same particle \( x = z \):

\[
\Delta^x x = \Delta z = w\delta = wl/c > 0.867 \times 10^{-3} \text{ (cm)}.
\]

(6.5)

Here \( w = \Delta z/\delta \cong dz/dt \) denotes the mean velocity of \( z(t) = x(t) \) during the time increment \( \delta = l/c \) with \( l \) being the length of information transmittal in cm. If “information transmittal” means establishing a steady current in a circuit of a measuring device, that is, an electric field to be set up along the circuit for ordered motion of the electrons to begin (propagation of electric field), then its velocity is the speed of light \( c \) in a vacuum. In this case, delay for the signal of a change in location \( x \) of a particle for \( l = 100 \text{ cm} \) is \( \delta = l/c = 0.333564 \times 10^{-8} \), so that, with \( w = v \cong 2.6 \times 10^5 \text{ cm/s} \), we have a lower bound for the uncertainty in the location of \( x \) due to time delay as given in (6.5), which is much greater than measurement uncertainty in the location of \( x \) presented in (6.4). However, if “information transmittal” meant measuring with a steady current for which, at the maximum permissible current densities, the average velocity of the ordered motion of the electrons would be \( v^* \cong 10^{-2} \text{ cm/s} \), so using this velocity instead of the speed of light \( c \), we would get \( \delta = l/v^* \cong 10^4 \text{ s} \), yielding the estimate \( \Delta^x x = \Delta z = w\delta \cong 2.6 \times 10^9 \text{ cm} \), which means that steady current cannot be used for such experiments. \( \square \)

Of course, we cannot but ignore \( \Delta z \) in physical measurements: there is no other option. However, for greater (noticeable) values of \( \Delta z \), delay equations should be used instead of ODEs or ordinary PDEs. In contrast, whatever \( \Delta z \), the conservation of feedback controls, postulated by optimality principles discussed above, actually holds over totally optimal (extremal) fields and presents an important property of such a field with respect to time-uncertainty:

**The structure (type, formula) of optimal (extremal) policy (controls, forces) is not affected by time-uncertainty (time delay) over totally optimal (extremal) fields.**
Unfortunately, the nature is not composed of optimal (extremal) fields only, and this is why the Isaacs, Bellman and similar principles of optimality (extremality) do not hold for all systems in science, or for all applications in engineering and economy. It puts a limit on their use in science and technology.

As a matter of fact, the time-uncertainty shifts our knowledge to the past. With a small shift, it makes no harm. With a greater shift, it has to be taken into account. In such cases, care should be taken when verifying abstract theories by experimental data. With large shift, we should recognize that our knowledge pertains to a distant past only. For example, certain stars are known to be many light years afar from the Earth. It means that what we know from our astronomical observations about distant parts of the Universe is nothing more than past time slices distant from our time of several thousand years by many light years to the past. Natural time delay is not just a question of history—some beautiful theories dealing with motion of small particles at high velocities may need an adjustment to take into account the time uncertainty.

7. Redundancy of the saddle point condition for differential games

The saddle point condition for differential games defined by (1.1), (1.2) and (1.5) is written as follows

\[ V(x(t)) = \min_{u(x)} \max_{v(x)} P(x(t)) = \min_{u(x)} \max_{v(x)} \int_{t}^{T} G(x, u, v) dt + K(x(T)) \]

\[ = \max_{v(x)} \min_{u(x)} P(x(t)) = \max_{v(x)} \min_{u(x)} \int_{t}^{T} G(x, u, v) dt + K(x(T)), \quad t \geq 0, \quad (7.1) \]

but it is valid for deterministic differential games only under strong convexity–concavity, separation or symmetry conditions, cf. Examples 4.1 and 6.1 above. It is comfortable to assume this condition which, if it holds, may help to solve the game or to obtain certain approximations. However, the general condition which always holds and presents the upper or lower bound for those games is

\[ V(x(t)) = \min_{u(x)} \max_{v(x)} P(x(t)) \geq \max_{v(x)} \min_{u(x)} P(x(t)). \quad (7.2) \]

Since a two player game can usually be solved by application of the Bellman equations twice, see Theorem 4.1, there is no need to impose condition (7.1). However, the application of the Bellman equations yields only local min–max or max–min solutions since those equations are gradient-based, thus, render only local optimality. Usually, it is the global min–max or max–min optimality which is required, especially in warfare, in long term planning, and in large scale projects with massive investments. In non-convex–concave situations, gradient based theories fail to deliver the global solution which obviously exists in many real life problems under realistic budget or other constraints. Randomized problems, or modified problems presenting convex–concave cases may help to some extent without a guarantee to obtain a viable solution. In non-convex–concave situations, set contractive methods can be used which guarantee the global optimal solution, exact in the limit, or approximate in a finite number of iterations depending on the required precision prescribed in advance, see [9,10,18].

8. Contiguity conditions for the Isaacs equation

The basic congruence identity (3.1) holds for totally optimal systems with respect to Isaacs’ principle of transition, to Bellman’s principle of optimality, and to any other principle for split (fragmented) optimality that can be postulated for a combination of min and max operators in any order. For not totally optimal systems, it does not hold. However, not holding for fragmented finite pieces of control curves in (3.1), it would be enough if it held in the limit, which is sufficient for validity of the Isaacs and Bellman equations irrespective of the principles from which those equations have been derived.

Consider again Eq. (2.2) for a general case where (3.1) does not hold. To clarify the situation, let us write Eq. (3.1) with non-congruence sign (instead of identity sign) under Eq. (2.2), as follows:

\[ V(x_1) = \min_{u_1(x)} \max_{v_1(x)} \Delta V[x_1, x_2] + V^*(x_2), \quad (2.2^*) \]

\[ \{u_1(x), v_1(x)\} \not\subset \{u_0(x), v_0(x)\} \quad \cup \quad \{u_2(x), v_2(x)\}, \quad (8.1) \]
with the meaning that control functions in (8.1) acting over time periods indicated in brackets below assure the optimal values for the terms in (2.2*) located right above them. Notation \( V^* \) in (2.2*) is used to allow for a possible case of non-invariance of \( V(x) \) along the fragmented optimal trajectory. In order that Isaacs’ principle of transition (2.2) have a sense, relation (2.2*) must be assured with one value function \( V(x) \) and one single pair of controls. This not being the case and keeping in mind that the goal is to optimize the entire trajectory, one has to use the optimal pair of controls on the left of (8.1) which will produce the optimal value \( V(x_1) \), see (1.5) for \( x = x_1 \), with at least one non-optimal value on the right in (2.2*). This will be true for any \( t_1 = t, x_1 = x(t), t_2 = t + h \) in (2.2) and (2.7) for which controls are non-congruent. But it means that in this case the Isaacs principle of transition (2.2) is invalid for one single piecewise continuous optimal control over the whole trajectory, so that Eq. (2.2) should be corrected as follows:

\[
V(x(t)) = \min \max \Delta V[x(t), x_2(t)] + g_0(t, t + h) + V(x_2(t)) + g_2(t, t + h), \quad h > 0,
\]

where \( x_2(t) = x(t + h), 0 < t < t + h < T, h > 0 \) arbitrary, and \( g_0(t, t + h), g_2(t, t + h) \) account for possible non-optimality of the pair on the left of (8.1), taken as the single control along the whole trajectory, with respect to the terms on the right of (2.2*). If this control happens to be optimal for all \( t \) and \( h \) for one of the terms at the right of (2.2*), say, for the second one, then \( g_2(t, t + h) \equiv 0 \). Now it is clear that the term

\[
g^*(t, t + h) = g_0(t, t + h) + g_2(t, t + h) \neq 0, \quad \forall t > 0,
\]

is missing on the right of (2.6) for not totally optimal systems. Adding this term to (2.6), we get in the bracket of (2.7) an additional discrepancy \( g^*(t, t + h)/h \neq 0 \) which has to be taken into account. Suppose that the following condition holds

\[
\lim_{h \to 0} [g^*(t, t + h)/h] = 0, \quad \text{for all } t \in (0, T),
\]

which we call the contiguity condition. If (8.4) holds, then transition from (2.7) to (2.8) is correct, hence, Eqs. (2.8) and (2.9) are true. If the limit in (8.4) is nonzero or does not exist, then transition from (2.7) to (2.8) fails. Moreover, in this case Eqs. (2.8) and (2.9) do not admit a single pair of controls as their solutions. Hence, the following statement is proved.

**Theorem 8.1.** Under the assumptions cited in observations 3.2 to 3.5, Section 3, the Isaacs equations (2.8) and (2.9) are valid if and only if the contiguity condition (8.4) is satisfied, irrespective of the principle of transition. \( \square \)

For important results, it is good to have more than one proof. Let us check again Theorem 8.1 using the Isaacs equations (2.7) to (2.9) without the principle of transition. **Second proof.** With non-congruent control curves, the principle of transition does not apply, and the transition from (2.6) to (2.7) is not valid. Moreover, an optimal cost \( V(x) \) in (1.5) for the optimal policies (controls) over fragmented trajectory need not be preserved as the same function for all \( x(t) \). Keeping this in mind, consider again Eqs. (2.7) to (2.9) without implicit assumption that one single optimal control is implicated in all terms of all equations. To obtain the main equation (2.8) from Eq. (2.2), the following notations are used in [1] for the variables in (2.2):

\[
x_1 = x(t), x_2 = x^0 = x(t + h), \text{ see (2.4) to (2.7)}.
\]

Let us write down Eq. (2.7) from Section 2 indicating under its terms the (possibly incongruent) controls that optimize those terms:

\[
V(x(t)) = \min \max P(x(t)) = \min \max \left[ \int_{t}^{t+h} G(x, u, v) \, dt + V^*(x(t+h)) \right]
\]

\[
= V^*(x(t), h) + h \min \max [\nabla V f(x, u, v) + G(x, u, v) + 0.5h \nabla^2 G f(x, u, v)|_{t+th} + o(h)].
\]

If \( h = 0 \), then (8.5) becomes a trivial identity with \( V^*(x) \equiv V(x) \) and the same controls. If \( t = 0, h > 0 \), then \( x(0) = x_0 \), so in the limit as \( h \to 0 \), and due to uniqueness of the value function, \( V^*(x_0, 0) = V(x_0) \) cancel out of (8.5) and (8.6), and we get from (8.6) the Isaacs equations (2.8) with the same optimal controls (if unique) for \( t = 0 \). However, if \( t > 0 \), then \( \{u_2(x), v_2(x)\} \) depend on \( h \), and \( V^*(x(t), h) \), may depend also on \( x_0 \), which is really the case
for systems with structural constraints, see Section 10 below. In this case, dividing by $h$ and letting $h \to 0$, we get from (8.5) and (8.6):

$$\min \max [\nabla V f(x, u, v) + G(x, u, v)] = \lim_{h \to 0} \frac{[V(x(t)) - V^*(x(t), h)]}{h} = \lim_{h \to 0} \frac{g^*(t, t + h)}{h},$$

(8.7)

which is the contiguity measure, the same as in (8.4), that must be zero for all $t \in (0, T)$ in order that Isaacs’ equations (2.8) and (2.9) be valid. □

**Remark 8.1.** It is worth noting that for totally optimal systems we have $g^*(t, t + h) \equiv 0$ for all $t, h$ which means precisely the congruence of control curves under the principle of transition. Also, for $t = 0$ condition (8.4) is satisfied for all systems by uniqueness of the value function at $x_0$, as mentioned in the second proof. □

To compute the value in (8.3), one can use Theorem 4.1 performing two Bellman optimizations successively. In this case, if we remove the min operator from (2.2), regarding $u(x)$ in (2.2*), (8.1) above as a parameter, then discrepancy in (8.3) will be parameterized by $u(x(t))$, and condition (8.4) must be stronger:

$$\lim_{h \to 0} \frac{[g^*_1(t, t + h, u, v)]}{h} = 0, \quad \text{for all } t \in (0, T), \text{ uniformly in } u, v,$$

(8.8)

assuring the validity of (4.8) and (4.9). Now, with the optimal $v^*(x, V, u(\cdot))$ found, we substitute it into (2.2) with the min operator with respect to $u$ reinstated, and consider (2.2*), (8.1) again, getting another condition:

$$\lim_{h \to 0} \frac{[g^*_2(t, t + h, u, v^*(x(t)), V, u)]}{h} = 0, \quad \text{for all } t \in (0, T), \text{ uniformly in } u,$$

(8.9)

which assures the validity of (4.13) and (4.14), yielding the PDE (4.15) for the value function $V(x)$. Two Bellman optimizations require two conditions of contiguity, (8.8) and (8.9), each corresponding to one optimization operator min or max in the formulation of a differential game. Those conditions can be computed using the procedure developed for the Bellman equations in [19, pp. 249–250, 254–255].

Contiguity condition in (8.4) and (8.7) can be simplified. Indeed, due to the first equality in its measure in (8.7), we can write (8.4) as follows:

$$\lim_{h \to 0} \frac{[V(x(t)) - V^*(x(t), h)]}{h} = 0, \quad \text{for all } t \in (0, T),$$

(8.10)

where the value in brackets is a measure of $V$-proximity of neighboring semi-trajectories produced by $\{u_1(x), v_1(x)\}$ and $\{u_2(x), v_2(x)\}$ at times $t, t + h$ which need not coincide, but must be touching at $x(t + h)$ as $h \to 0$ to the second or higher order, i.e., lie in a weak neighborhood of one another (cf. the curvature in the Weierstrass form of the Euler–Lagrange equations). Denoting the expression in the brackets of (8.10) by $C(t, t + h)$, we get an equivalent representation for (8.10):

$$\lim_{h \to 0} \frac{C(t, t + h)}{h} = 0, \quad \text{for all } t \in (0, T).$$

(8.11)

Under assumptions of continuous partial derivatives adopted in [1], the functional $C(t, t + h)$ is continuously differentiable with respect to $h$. Consider the limit

$$C(t) = \lim_{h \to 0} C(t, t + h) = \lim_{h \to 0} \frac{[V(x(t)) - V^*(x(t), h)]}{h}.$$ 

(8.12)

If $C(t) \neq 0$, then in (8.11) we have $\pm \infty$, and non-contiguity follows. If $C(t) = 0$, the l’Hôpital rule applies, and (8.11) is equivalent to the condition

$$\lim_{h \to 0} \frac{dC(t, t + h)}{dh} = 0, \quad \text{for all } t \in (0, T),$$

(8.13)

which is the contiguity condition (8.4) for the Isaacs equations (2.8) and (2.9). For $t = 0$, the conditions (8.4), (8.13) hold for all systems by construction, under the differentiability assumptions adopted in [1].

### 9. Parallel and series games

To further discuss the application of the Isaacs main equation, we consider an engineering problem of optimal stabilization of a stationary linear oscillator (engines, turbines, vibratory machines, etc.) in the presence of conflicting
controls, which we study in several distinct settings to discuss some specific problems that arise when information available to players is restricted, or asymmetric, or unavailable at all. Such cases may present differences in regard to optimality that occur with modifications in the structure of information.

**Example 9.1.** In all textbooks on undergraduate mathematics, mechanics, or control theory, the following linear oscillator equation is considered where we added stabilizing and destabilizing controls \( u, w \):

\[
d^2x/dt^2 + x = u + w, \quad t \geq 0, \quad x(0) = x_0, \quad dx(0)/dt = v_0. \tag{9.1}
\]

Denoting \( dx/dt = v \) (velocity), we convert \( (9.1) \) into the normal form:

\[
dx/dt = v, \quad dv/dt = -x + u + w, \quad t \geq 0, \quad x(0) = x_0, \quad v(0) = v_0. \tag{9.2}
\]

Vibrations in \( (9.1) \) and \( (9.2) \) should be optimally stabilized (decreased) by control \( u \) against destabilizing influence of control \( w \) regarding the functional

\[
V(x_0, v_0) = \min_u \max_w J(u, w) = \int_0^\infty (ax^2 + bv^2 + cu^2 + lw^2)dt, \quad a, b, c = \text{const} > 0. \tag{9.3}
\]

**Case 1. Classical single control solution (Bellman).** With \( w \equiv 0 \), the problem \( (9.1)–(9.3) \) is considered in the literature to illustrate the optimal regulator design by dynamic programming. The system is autonomous, its optimal trajectories are totally optimal, the principle of optimality applies; thus, denoting \( V(x, v) = \min_u J(u) \) for \( t > 0 \), the Bellman equation follows with \( V_T = 0 \) for \( T = \infty \) (fixed), cf. [10, Ch. IX, Sec. 3, (12), with \( f \) instead of \( V \) below]:

\[
V_T(x, v) = \min_u [ax^2 + bv^2 + cu^2 + v\partial V/\partial x + (-x + u)\partial V/\partial v] = 0. \tag{9.4}
\]

With the control space open, it implies for optimal \( u^o \),

\[
2cu^o + \partial V/\partial v = 0. \tag{9.5}
\]

Substituting optimal control \( u^o \) from \( (9.5) \) into \( (9.4) \) (which removes the \( \min \) operation) and solving the nonlinear PDE, we obtain the minimal cost (note that \( J(u) \) of \( (9.3) \) is strictly convex with \( w \equiv 0 \)) and the unique optimal feedback:

\[
V(x, v) = (c + a)^1/2(b - 2cp)^1/2x^2 - 2cpxv + (bc - 2c^2p)^1/2v^2, \tag{9.6}
\]

\[
u^o = px - (b/c - 2p)^1/2v, \quad p = 1 - (1 + a/c)^1/2. \tag{9.7}
\]

We see that feedback \( (9.7) \) does not depend on initial conditions in \( (9.2) \). It means that every remaining part of an optimal trajectory is itself optimal. Also, for fixed \( T \), the Eq. \( (9.4) \) does not depend on boundary conditions \( x(t_0), x(T) \), \( 0 \leq t_0 < T < \infty \), which means that every intermediate part \( x[t_0, T] \) of the optimal trajectory is itself optimal (total optimality) with the same control \( (9.7) \) (congruence), cf. Theorem 6.1.

**Case 2. The Isaacs min–max solution (a parallel game).** Consider two controls \( u(x, v), w(x, v) \) in \( (9.1)–(9.3) \), with \( l > 0 \) in \( (9.3) \) for concavity with respect to \( w \). To avoid diverging integral in \( (9.3) \) when maximizing by \( w(x, v) \), replace \( \infty \) by some large \( T = \text{const} \) in the upper limit, as suggested in [1, Sec. 2.3]. Instead of \( (9.4) \), we have, according to the Isaacs main equation \( (2.8) \):

\[
\min_u \max_w [ax^2 + bv^2 + cu^2 + lw^2 + v\partial V/\partial x + (-x + u + w)\partial V/\partial v] = 0. \tag{9.8}
\]

With the control space open, it implies for optimal \( u^o, w^o \)

\[
\partial[\ldots]/\partial u = 2cu^o + \partial V/\partial v = 0, \quad \partial[\ldots]/\partial w = 2lw^o + \partial V/\partial v = 0. \tag{9.9}
\]

Substituting optimal controls \( u^o, w^o \) from \( (9.9) \) into \( (9.8) \) (which removes the \( \min–\max \) operation), we obtain the nonlinear partial differential equation:

\[
ax^2 + bv^2 - 0.25(c^{-1} + l^{-1})(\partial V/\partial v)^2 + v\partial V/\partial x - x\partial V/\partial v = 0. \tag{9.10}
\]

Substituting into \( (9.10) \) a quadratic form with undetermined coefficients:

\[
V = Ax^2 + Bv^2 + 2Dxv, \tag{9.11}
\]
and collecting terms with $x^2$, $v^2$, $xv$, we obtain the equations for $A$, $B$, $D$:

$$rD^2 + 2D - a = 0, \quad rB^2 = 2D + b, \quad A = B + rBD, \quad \text{where } r = c^{-1} + l^{-1}. \quad \tag{9.12}$$

Solving quadratic equations (9.12), we get

$$D = r^{-1}[1 + (1 + ar)^{1/2}], \quad B = (2r^{-1}D + r^{-1}b)^{1/2}, \quad A = B + rBD, \quad \tag{9.13}$$

where unsuitable roots are discarded. Substituting (9.11) into (9.9), we obtain optimal controls with values of $B$, $D$ from (9.13):

$$u^o = -c^{-1}(Bv + Dx), \quad w^o = -l^{-1}(Bv + Dx), \quad u^o/w^o = l/c, \tag{9.14}$$

and with $c > 0$, $l < 0$, we have min–max in (9.8), which can be seen by checking second derivatives of the brackets in (9.9). Using for $u^o$ in (9.14) the values of $B$, $D$ from (9.13) with $r = c^{-1}$, the reader can verify that $u^o$ from (9.14) coincides with $u^o$ in (9.7), and the same for $w^o$ with $l$ substituted by $c$. This is natural since Isaacs’ min–max solution carries two Bellman’s optimizations (max or min makes no difference) which are identical due to separation in $u$, $w$ for the problem (9.1)–(9.3).

A distinctive feature of this game is that both players, defender $u$ who wants to decrease the vibrations in (9.1) and attacker $w$ who wants to increase them, are equal with respect to input information (9.1)–(9.3) which is known to and used by both players. Such games we call parallel games. The only parameter that provides some dependency between players is $r = c^{-1} + l^{-1}$, see (9.10) and (9.12). If $r = 0$ for $c = -l \neq 0$, then, subtracting equalities in (9.9), we obtain $u^o + w^o = 0$, so conflicting controls neutralize each other and disappear from Eqs. (9.1)–(9.3). This value $r = 0$ is the boundary point between elimination of vibrations in (9.1) and destruction of the system (engine, turbine, etc.) subject to those vibrations.

It seems reasonable to consider a realistic case when players are not assumed to have equal knowledge of all game ingredients. In fact, the model Eqs. (9.1) and (9.2) and/or preferences (9.3) of a defender may be unknown to the attacker. The same situation exists in the economy where industrial secrets (confidentiality) are commonplace.

Case 3. The series game. Setting $w = 0$, consider again Eqs. (9.1)–(9.3), which are known to player $u$ and represent his model. Player $u$ can solve the problem in Case 1 to determine his optimal control (9.7) which is linear in $x$, $v$ and then plug it into (9.1) or (9.2). The closed-loop system (9.1) with $w = 0$ and $u$ from (9.7) is unknown to the attacker $w$, so he must first identify its trajectory $x(t)$ and possibly its dynamics (see, e.g., [20]). If dynamics of the open-loop system in (9.1) is known to both players, then player $w$ can identify the control in (9.7). So, it is plausible to assume that Eq. (9.1) with $u$ from (9.7) is known to the attacker who can choose his functional in (9.3), with $w = 0$. Then he can solve Case 1 for himself (with a different functional), or simply use in (9.1) a linear control $w(x, v)$ to get in the closed-loop system (9.1) $\Re \lambda > 0$ as large as possible. In reality, resources of the attacker are bounded, so positive $\Re \lambda$ are bounded, and the linear control $w(x, v)$ can be identified by player $u$ who can solve the problem (9.1)–(9.3) again to determine an additional linear control $u(x, v)$ which would neutralize $w(x, v)$ and decrease vibrations in closed-loop system (9.1), if player $u$ has enough resources to supply in (9.1) a necessary regulator $u(.)$, more powerful than destructive control $w$. Such problems require application of successive Bellman equations, together with a proper identification protocol.

10. Non-contiguous systems

It is tempting to think that the Isaacs equation, applicable even without the principle of transition, might be universally applicable to almost all zero sum differential games which most likely satisfy the contiguity condition, with the exception of, maybe, some pathological examples. Unfortunately, this is not the case.

Non-contiguity (i.e., non-satisfaction of the contiguity condition) in optimal dynamical systems is a rather fine property that invalidates the extension of optimal controls over some part of a trajectory for use over another part of the same trajectory. Strange as it may seem, non-contiguous systems are common in practice. In such cases, the Isaacs main equation is inapplicable; some other methods can be used, see, e.g., [5,6,9,10,15,16,18].

Systems with incomplete information and/or structure constraints are generally non-contiguous, even if they are autonomous. To illustrate the existence of such systems, we consider the same engineering problem of optimal stabilization of a linear oscillator, see Example 9.1, Case 1 with $w \equiv 0$, to discuss some specific properties, and
to demonstrate the fundamental difference in regard to optimality that may occur with modifications in the structure of a controller.

**Example 10.1** ([21]). Suppose that the space coordinate \( x(t) \) cannot be measured (fluid friction which is common in engineering); thus, the control must be of the form \( u = u(v) \). If \( a = 0 \) in \((9.3)\), then \( p = 0 \) and \( u^o \) of \((9.7)\) is feasible. However, in \((9.3)\) we have \( a > 0 \); thus, in \((9.7)\) \( p < 0 \) and \( u^o \) is infeasible. In the class of controls \( u(v) \) depending on velocity, Eq. \((9.4)\) does not have a solution, though “approximate” solutions with small \( |p| \) do exist, all non-optimal since they actually correspond to small \( a > 0 \) in \((9.3)\).

In the spirit of dynamic programming and noting the result \((9.7)\), let us look for the optimal feedback in the class of controls

\[
  u = -2qv, \quad q > 0, \tag{10.1}
\]

which represents a nonsingular Lagrange problem of the calculus of variations with two constraints \((9.2)\) if we eliminate \( u \) from \((9.3)\) by the substitution \( u = x + v' \) from \((9.2)\), and then substitute \((10.1)\) into \((9.2)\).

For this class of controls, one can also write the Bellman equation by simply substituting \((10.1)\) into \((9.4)\), and then considering \( q(\cdot) \) as a control. Denoting \( u = -2q(\cdot)v = u^c(\cdot) \), one gets the same Eqs. \((9.4)\) and \((9.5)\) with the solution \((9.7)\)—nothing new.

However, the optimal regulator of the form \((10.1)\) with \( q = \text{const} \) (optimal dampening) does exist. For system \((9.2)\), \((9.3)\) and \((10.1)\), the characteristic equation is \( \lambda^2 + 2q \lambda + 1 = 0 \), so that \( \text{Re} \lambda_{1,2} < 0 \), the integral \((9.3)\) is convergent and, for \( q \neq 1 \), it has the value

\[
  J(x_o, v_o, q) = [(a + c)x_o^2 + cv_o^2]q + 0.25(a + b)(x_o^2 + v_o^2)q^{-1} + ax_o v_o. \tag{10.2}
\]

From the equation \( \partial J/\partial q = 0 \), we find the extremal value

\[
  q_o^2 = 0.25(a + b)(x_o^2 + v_o^2)/[(a + c)x_o^2 + cv_o^2], \quad q_o > 0, \tag{10.3}
\]

for which \( \partial^2 J/\partial q^2 > 0 \), yielding the minimum in \((10.2)\), see [20, formulae (5.7)–(5.12)]. Since \( J(\cdot) \) of \((10.2)\) is continuously differentiable at \( q = 1 \), for all values of parameters in \((10.2)\), formulae \((10.3)\), \((10.2)\) are valid also for \( q = 1 \), covering the whole range of possible friction fluid densities.

Comparing \((9.7)\), \((10.1)\) and \((10.3)\), one can see that

1. \( u^o \) of \((9.7)\) does not depend on initial data meaning that \( u^o \) is optimal not only for any remaining part of an optimal trajectory (Bellman’s principle of optimality) but also for all trajectories in the feasible space \( X \). It means that Eqs. \((9.6)\) and \((9.7)\) generate a totally optimal field in the sense of Definition 6.2, so that for the system \((9.1)\)–\((9.3)\) there exists one single optimal control \((9.7)\) such that every trajectory starting from any point \( x \in X = R^2 \) is optimal. Moreover, there is a unique surface \((9.6)\) which renders the optimal value \( V(x, v) \) of the functional \((9.3)\) on the trajectory starting from that point \((x, v)\).

2. \( u = -q_o v \) of \((10.1)\) and \((10.3)\) does depend on initial data \( x_o, v_o \), meaning that, being optimal for the whole trajectory corresponding to those data, it is not optimal for any part thereof, hence, Bellman’s principle of optimality is invalid. In the Isaacs principle of transition \((2.2)\), the invariance of the value function \( V(x) \) is violated since \( q(\cdot) \) for \( V(x_1), V(x_2) \) in \((2.2)\) is different due to dependence on initial data in \((10.2)\) and \((10.3)\). Thus, Isaacs’ principle of transition is also invalid in the class of controls \((10.1)\), if \( a > 0 \) in \((9.3)\).

3. If \( a = 0 \) in \((9.3)\), we have \( u = u^o \), see \((9.7)\), \((10.1)\), \((10.3)\); thus, for this class of functionals in \((9.3)\), Bellman’s and Isaacs’ principles are valid in the class of controls \((10.1)\) for the particular system \((9.2)\). Moreover, field optimality is preserved.

Nonexistence of solutions of the Bellman equation \((9.4)\) for linear system \((9.2)\), \((9.3)\) and \((10.1)\), which equation can be formally written as proposed in [11], means that this autonomous system with incomplete information is non-contiguous. Its non-contiguity invalidates the derivation of Bellman’s equations presented in [11] and, by analogy displayed in \((2.8*)\), \((4.4*)\). Section 4, it invalidates the Isaacs main equation for differential games due to non-congruent controls. Strategies in such games depend on preceding data (cf. observation 3.4 in Section 3) and do not admit fragmentation. Approximate suboptimal solutions may exist, but a global optimal solution requires other methods.
11. The maximum principle and total optimality

Consider an autonomous control system:

\[ \frac{dx}{dt} = f(x, u), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad x(0) = x_0 \text{ given, } u(t) \in \Omega \subseteq \mathbb{R}^m. \]  

\[ J(u) = \int_0^T f_0(x, u)dt, \]  

where \( f_0(x, u) \) and \( f(x, u) = [f_1, \ldots, f_n] \) are continuously differentiable, and \( u(t) \) is piecewise continuous. A final point \( x(T) \) is given so that system (11.1) is supposed to be controllable from \( x_0 \) to \( x(T) \) by a multitude of feasible controls from which such control(s) \( u(t) \) should be chosen that would optimize (max or min) the cost functional (11.2). Together with (11.1) and (11.2) consider the adjoint system:

\[ \frac{d\varphi_k}{dt} = -\sum_{i=0}^n \varphi_i \frac{\partial f_i}{\partial x_k}, \quad k = 0, 1, \ldots, n. \]  

This is a system of homogeneous ODEs defined for any choice of \( u(t) \) in (11.1) and (11.2). The system has a unique solution \( \varphi(t) = [\varphi_0, \varphi_1, \ldots, \varphi_n], t \in [0, T] \), for any given initial data \( \varphi(0) \), and every such solution corresponds to a chosen control function \( u(t) \) and to the trajectory \( x(t) \) generated by this \( u(t) \) in the closed-loop system (11.1). If we define a function (called a Hamiltonian)

\[ H(\varphi, x, u) = \varphi_0 f_0(x, u) + \varphi f(x, u) = \sum_{i=0}^n \varphi_i f_i(x, u), \]  

then Eqs. (11.1) and (11.3) can be written in the symmetric form:

\[ \frac{dx_j}{dt} = \partial H/\partial \varphi_j, \quad j = 0, 1, \ldots, n; \quad \frac{dx_0}{dt} = \frac{dJ}{dt} = f_0(x, u), \]  

\[ \frac{d\varphi_j}{dt} = -\partial H/\partial x_j = -\sum_{i=0}^n \varphi_i \frac{\partial f_i}{\partial x_j}. \]  

If we take arbitrary feasible control \( u(t) \), that is, such control that steers \( x(0) \) to \( x(T) \) and satisfies the condition \( u \in \Omega \) in (11.1), we can integrate (11.1) and find its trajectory \( x(t) \), and then, given \( u(t), x(t) \), integrate (11.3), yielding \( \varphi(t) \) corresponding to \( u(t), x(t) \), for any initial condition \( \varphi(0) \). Note that \( a\varphi(t) \) with any \( a = \text{const} \neq 0 \) also fits (11.3). Now we translate the citation from [22, Part II, pp. 188–190], in our notation.

“For fixed (constant) values of \( \varphi \) and \( x \), the function \( H \) of variables \( \varphi_0, \ldots, \varphi_n, x_1, \ldots, x_n, u_1, \ldots, u_m \) becomes a function of parameter \( u \in \Omega \); denote the exact upper bound of values of this function by \( M(\varphi, x) \):

\[ M(\varphi, x) = \sup_{u \in \Omega} H(\varphi, x, u). \]  

If the exact upper bound of values of a continuous function \( H \) is attained at some point of \( \Omega \), then \( M \) is the maximum of \( H \) under fixed \( \varphi \) and \( x \). Therefore, the theorem cited below (a necessary condition of optimality) of which the main statement is the equality (11.8) is called by its author the maximum principle.

**Theorem I (The Maximum Principle of Pontryagin [23])**. Let \( u(t), 0 \leq t \leq T, \) be a feasible control that steers \( x(0) = x_0 \) to \( x(T) \). For optimality of control \( u(t) \) and of trajectory \( x(t) \), it is necessary that there exist such nonzero continuous vector-function \( \varphi(t) \) corresponding to the functions \( u(t) \) and \( x(t) \) that:

1° For any \( t, 0 \leq t \leq T, \) the function \( H(\varphi(t), x(t), u) \) of the variable \( u \in \Omega \) attain its maximum at the point \( u = u(t) \):

\[ H(\varphi(t), x(t), u(t)) = M(\varphi(t), x(t)). \]  

2° At the final moment \( T \) the relations hold:

\[ \varphi_0(T) \leq 0, \quad M(\varphi(T), x(T)) = 0. \]
Furthermore, it appears that, if the values ϕ(t), x(t), u(t) satisfy the system (11.5) and (11.6) and the condition 1°, the functions q0(t) and M(ϕ(t), x(t)) of the variable t are constant, so that verification of relations (11.9) can be made at any moment t, 0 ≤ t ≤ T, not necessarily at the moment T.”

For non-autonomous systems, time t explicitly enters the right-hand sides of (11.1) and (11.2), and also H(\(t\)) and M(\(t\)) in (11.8) and (11.9), but the statements 1° and 2° of Theorem I remain the same with M defined by the integral:

\[
M(\varphi(t), x(t), t) = \int_0^t \sum_{i=0}^n \varphi_i(t)(\partial f_i/\partial t)dt,
\]

whereby \(\varphi_0(t) = \text{const}\) and \(M(\varphi(t), x(t), t)\) can only differ by a constant from the value in (11.9), so that it is sufficient to check conditions (11.9) at any moment \(t, 0 \leq t \leq T\), for example at \(t = T\), see [23–25,6,15].

Sufficient conditions of optimality in the form of the maximum principle were obtained in [24,25], establishing the equivalence between general optimality of (11.1) with respect to the functional in (11.2) and a sort of special point-wise 1°-optimality in the sense of conditions 1°–2° valid for each and every piece of an optimal trajectory. In this sense, any optimal trajectory is totally 1°-optimal, by construction.

### 11.1. Initial conditions for ϕ(t)

Many researchers noted “finding the initial value \(\varphi(0)\) of the vector-function \(\varphi(t)\) is the main difficulty in the general problem of synthesis of the optimal controls by the maximum principle” [22, Part II, p. 209]. Sometimes, this difficulty can be bypassed, as is the case for the standard example of the vertical soft landing (moon landing) problem [22, Part II, pp. 207–212], see also [15], or [6, pp. 50–55]. In the general case (11.1) and (11.2), determining \(\varphi(0)\) can be reduced to a global maximization problem as follows.

For \(\tau > 0\) small, \(\varphi(0) = c\) (unknown), we have by the Euler scheme:

From (11.1): \(x(t) = x_0 + tf(x_0, u), x_0 = x(0)\) given in (11.1), \(u = u(0)\).

From (11.3): \(q_k(t) = c_k + tdq_k/\partial x_k = c_k - t \sum_{i=0}^n c_i\partial f_i(x_0, u)/\partial x_k, k = 0, 1, \ldots, n.\)

From (11.4): \(H(\varphi(t), x(t), u) = \sum_{k=0}^n c_k - t \sum_{i=0}^n c_i\partial f_i(x_0, u)/\partial x_k|f_k(x(t), u)\).

Since (11.7) to (11.9) must be satisfied for any \(t\), we have as \(t \to 0\) the global maximization problem:

\[
\sup_{c,u} \sum_{k=0}^n c_k f_k(x_0, u) = 0, \quad c_0 \leq 0, \quad u \in \Omega, \quad x_0 = x(0)\text{ given.} \tag{11.11}
\]

If \(c\) and \(u\) are subject to box constraints (\(\Omega\) is a box), then the problem (11.11) can be solved by the method presented in [26], otherwise, by the Beta algorithm [27]. Both methods yield the set of all global optimizers, as required in (11.7). Since (11.3), (11.11) are homogeneous, so if \(c\) renders sup in (11.11), so does ac for any \(a \neq 0\). Thus, \(c_0 \leq 0\) is automatic, and normalization can be done to obtain a unique solution in (11.11).

### 11.2. Problems with scarce resources

When solving optimal control problems with scarce resources, the multi-objective problems are to be solved. Consider the standard example illustrating the use of the maximum principle in the soft landing (moon landing [6, pp. 4-5]) min-time problem:

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = u, \quad t \geq 0, \quad |u| \leq M, \quad x(0) = x_0 > 0, \quad v(0) = 0,
\]

\[x(T) = v(T) = 0, \quad T = \text{min}\]

Its solution by the maximum principle for \(M = 1\) yields \(T = 2x_0^{1/2}\) with bang–bang control, switching at \(t = 0, 0.5T, T\) and spending the maximum fuel (energy) \(E = \int_0^T u^*(t)^2 dt = T = 2x_0^{1/2} = \max E(u)\). However, with less energy the same soft landing can be made in a slightly longer time \(T\) (which is safer for a moon landing), and an approximate balance equation (for the first two Fourier axes) is \(ET^3 = 2\pi^2 x_0^2\), see [28, pp. 144–147].
11.3. Total optimality and the maximum principle

The fixing of time \( t \) in \( \Omega^0 \) makes a \( t \)-cut in the fields of trajectories \( x(t), \varphi(t), t \in [0, T] \), producing a collection of geometric curves \( x_i(u), \varphi_j(u) \), which are parameterized by \( u \) for each fixed \( t \). This fact is recognized in [6, p. 35] by replacing \( u(t) \) with \( v \in \Omega \), i.e., by taking max \( H(x, v, \varphi) \), not with respect to \( u(t) \in \Omega \) with some frozen (fixed) \( t \), but with respect to an external unrelated parameter \( v \in \Omega \). This mode operandi of freezing \( t \) in order to fix \( \varphi \) and \( x \) while taking max \( H(u) \) in (11.8), then unfreezing \( t \) to obtain smooth curves \( \varphi(t), x(t) \) optimized by the choice of \( u(t) \) which appears piecewise continuous albeit generated by a point-wise procedure (fixed \( t \)), actually implies the congruence of controls in the sense of observation 3.1, Section 3, leading to the important result: the policy \( u(t) \) and trajectory \( x(t) \) optimal by virtue of the maximum principle are totally optimal in the sense of Definition 6.1.

**Theorem 11.1.** Suppose that the maximum principle is valid for an optimal trajectory \( x(t) \) of \( (11.1) \), that is, conditions \( 1^0 - 2^0 \) are satisfied on it. Then this optimal trajectory is totally optimal with respect to the functional (11.2).

**Proof.** Consider pieces of trajectory within time intervals \( (t^*, t^* + \Delta t) \subset [0, T] \) over which \( u(t) \) is continuous, which intervals are considered closed at left and/or at right at points of continuity of \( u(t) \). Since conditions \( 1^0 - 2^0 \) are sufficient [24,25] and point-wise (\( t \) fixed), so the pieces of curves \( u(t^*, t^* + \Delta t), x(t^*, t^* + \Delta t) \) over those time intervals are optimal. By additivity of the integral in (11.2) with respect to intervals of integration and by continuity of trajectory in (11.1), all those pieces can be joined together forming the whole control curve \( u(t) \) and trajectory \( x(t), t \in [0, T] \), which are optimal too (note that condition \( 2^0 \) holds at any moment \( t \) as being constant within \( [0, T] \)). Since the choice of intervals \( (t^*, t^* + \Delta t) \) is arbitrary, total optimality follows. \( \Box \)

**Corollary 11.1.** Trajectories optimized by the maximum principle are totally optimal.

**Corollary 11.2.** If an optimal trajectory \( x(t) \) and corresponding optimal control \( u(t) \) are not totally optimal, the maximum principle is not applicable to the case. \( \Box \)

This means that the maximum principle is valid only for totally optimal fields of trajectories. It is not universal. If its application fails for a concrete system, it does not necessarily mean a mistake in writing corresponding equations, nor an error in software, nor even the nonexistence of an optimal control under incomplete information and/or structural restrictions on controls. It may simply mean that the optimal trajectory of the system is not totally optimal, see \# 11.5 below.

11.4. Connection of the maximum principle with Isaacs’ and Bellman’s theories

If the reader compares the expression for the Hamiltonian \( H(\cdot) \) in (11.4) with the brackets in (2.8) and (4.5), he will notice striking similarities: with the change of notations \( \varphi = \nabla V, \dot{\varphi}_0 f_0 = G, f \) the same, \( H(\cdot) \) coincides with the bracket in (2.8) if \( v \equiv 0 \); with the change of notations \( \varphi = f, \dot{\varphi}_0 f_0 = F, f = G \) in (11.4), one gets the bracket in (4.5). Indeed, it is well known that the Principle of transition (Isaacs), Principle of optimality (Bellman), and the Maximum principle (Pontryagin et al. [23]) are intimately related, see, e.g., [5,6,29–32]. A difference is that the first two principles are formulated as general self-evident axioms (which is not true, see, e.g., [33,21,19]), and the third one is derived from a multitude of conditions and lemmas [23–25]. To illustrate their interrelation, a simple derivation of the adjoint equations and the necessary conditions of the maximum principle from the Isaacs main equation can be made as follows.

Consider again system (1.1), the value function (1.5), and the main equation (2.8), which we rewrite for reference, with the natural boundary condition from (1.2) and (1.5):

\[
\min \max [\nabla f(x, u, v) + G(x, u, v)] = 0, \quad V(x(T)) = K(x), \quad x \in H(s). \tag{2.8*}
\]

To include also non-autonomous systems when (1.1)–(1.5) contain \( t \) explicitly, thus, \( V = V(t, x(t)) \) in (1.5), a new state variable can be defined:

\[
dx_{n+1}/dt = 1, \quad x_{n+1}(0) = 0 \quad (\text{thus}, \ x_{n+1} \equiv t), \quad H = \{x = (x_1, \ldots, x_n, T), x_i \in R\}. \tag{11.12}
\]
The new variable in (11.12) extends the gradient $\nabla V$ of (2.8*) to $\nabla V^*$ below, and implies, according to (1.1) and (1.5) for $V = V(t, x)$, $t = x_{n+1}$:

$$\nabla V^* = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n, \partial V/\partial x_{n+1}), \quad \partial V/\partial x_{n+1} = V_t, \quad f_{n+1}(.) \equiv 1. \quad (11.13)$$

Substituting (11.13) into (2.8*) and denoting for symmetry of notation $G = f_0$, the following relations are obtained instead of (2.8*):

$$-V_t = \min \max [\nabla V f(x, u, v) + f_0(x, u, v)], \quad V(T, x) = K(x), \ x \in H(x) \subset R^n, \quad (11.14)$$

where $K(x)$ from (1.5) presents the boundary condition in (11.14). Note that if $u$ and min operator are removed, relation (11.14) coincides with the Bellman functional equation (4.5) for non-autonomous systems, if in (4.1) we denote $f(a, c) = V(t, x)$, $F = f_0$, $G = f$, to align the notations in (4.1) and (11.14). For autonomous problems in (1.1)–(1.5), we have $V_t = 0$, returning to the main equation (2.8*).

Denoting in (11.13) and (11.14) $\partial V/\partial x_j = -\varphi_j$ ($j = 1, \ldots, n$), the bracket in (11.14), (2.8*) is represented as the Hamiltonian:

$$H = -[\nabla V f(x, u, v) + f_0(x, u, v)] = \sum_{j=0}^{n} \varphi_j f_j, \quad \varphi_0 = -1, \varphi = -\nabla V, \quad (11.15)$$

so that we obtain from (11.15) the adjoint equations, cf. (11.3)–(11.6):

$$dx_i/dt = \partial H/\partial \varphi_i = f_1(x, u, v), \quad (11.16)$$

$$d\varphi_i/dt = -\partial H/\partial x_i = -\sum_{j=0}^{n} \varphi_j \partial f_j/\partial x_i, \quad i = 0, \ldots, n. \quad (11.17)$$

Moreover, since $\min z = -\max(-z)$, so with $v \equiv 0$, $V_t = 0$, the relation $\min z = \min[\ldots] = 0$ with respect to $u$ in (2.8*) corresponds to $\max H = M = 0$ in (11.15), (11.8) and (11.9), and $\varphi_0 = -1$ in (11.15) agrees with condition 2° of Theorem I above.

11.5. Validity of the maximum principle, and software test

The maximum principle is stated for piecewise continuous bounded controls in (11.1). Since no other restrictions on controls are mentioned, the principle is generally perceived to be valid for all such controls. Unfortunately, this is not true. In systems with incomplete information, controls usually depend or may be required to depend on some part of the coordinates only. In the economy, urban development, long term planning and management, financial regulation, environmental policies, it is always the case. In engineering, it is usually the case too. Depending on a part of the coordinates, controls can still be piecewise continuous and bounded, but the maximum principle for such a system may be invalid. For example, problem (9.2) and (9.3) considered as control problem, $w \equiv 0$, has the solution (9.6) and (9.7); the same problem considered as game problem, has the solution (9.14). Both problems can be solved by the maximum principle. For restricted control (10.1): $u = -2qv$, $q = \text{const} > 0$, continuous and bounded, both problems also have the solution, cf. (10.2) and (10.3), which depends on initial conditions. However, this solution cannot be obtained by the maximum principle.

Indeed, if the equations of the adjoint system (11.3) are actually written for (9.2) and (9.3) with $w \equiv 0$, the control case, one obtains

$$\varphi_0' = 0, \quad \text{thus } \varphi_0(t) = \text{const}, \quad \varphi'_1 = -2ax\varphi_0 + \varphi_2, \quad \varphi'_2 = -2bv\varphi_0 - \varphi_1,$$

$$M(\varphi, x) = \sup_u H = \sup_u [\varphi_0(ax^2 + bv^2 + cu^2) + \varphi_1 v + \varphi_2(-x + u)],$$

and with control space open, we have, cf. (9.5): $2cu\varphi_0 + \varphi_2 = 0$, $\varphi_2 = -2uc\varphi_0 = 4qvc\varphi_0$, whence $\varphi'_2 = 4q'vc\varphi_0 = 4qcv\varphi_0x''$, which is non-causal, see Remark 12.2 below, and contradicting definition of $\varphi'_2 = -2bx'\varphi_0 - \varphi_1$ above. If we consider $q = q(t, x) \neq \text{const}$, the same contradiction remains, so it is not due to the case of optimization with respect to a parameter. It means that the maximum principle may be inapplicable for systems with incomplete information, and it really is, if the optimal trajectory is not totally optimal, cf. Corollary 11.2 above.
Another point of interest is worth noting. The problem (9.2) and (9.3) has the explicit solution (9.6) and (9.7) produced by Bellman’s equations (9.4) and (9.5). This solution defines the totally optimal field of trajectories for the problem; thus, the maximum principle is also applicable to the case. If adjoint variables $\varphi_i(t)$ of (11.3) and (11.4) are chosen as feedback $\varphi(t) = \nabla V(x(t), v(t))$, then the same solution (9.6) and (9.7) is obtained. However, if an open-loop representation $\varphi(t)$ is chosen, as specified by the maximum principle in (11.3)–(11.9), it will be difficult to retrieve the formula-like solution (9.6) and (9.7) which is unique for the problem due to convexity of the functional (9.3) with respect to $x, v, u$. The solution will be in the form of computed curves $x(t), v(t), u(t)$ which must satisfy Eq. (9.7). If it is really observed computationally, it means that the software is correct. This is a test for software of the maximum principle.

11.6. Total optimality in the classical optimal control theories

If we consider (2.8*) as it is, retaining max with respect to $v$ and with the game ending at a terminal set at time $T$, then the Hamiltonian (11.15) and adjoint Eqs. (11.16) and (11.17) are the same, and the game formulation in the form of the maximum principle is obtained, with two conflicting feedback controls and with the same necessary condition for min–max optimality with two players, as presented in Theorem 1 (condition $\textbf{1}^\circ$) for simple optimality with one player, see [31]. Certain sufficient conditions are developed in [29,30], and also in [34–35] in terms of a piecewise-smooth field (regular synthesis). Different methods are considered in [36–38], and there is vast literature on game theory of which citations can be found elsewhere. Here we are concerned with just one point: the connection of the principal game theoretic methods with the totally optimal fields of trajectories and controls.

**Theorem 11.2.** For an optimal trajectory, if the Isaacs principle of transition, or the Bellman principle of optimality, or the maximum principle of Pontryagin, together with corresponding equations, is valid, then two other principles are also valid, and the trajectory is totally optimal.

**Proof.** If the maximum principle of Pontryagin is valid, then total optimality of an optimal trajectory is proved in Theorem 11.1. Since the Isaacs and Bellman equations can be transformed into the Hamiltonian formulation of the maximum principle for differential games or optimal control, so those equations are valid too, the optimal trajectory is totally optimal, and the corresponding principle is valid on it. Vice versa, if Isaacs’ or Bellman’s principle is valid, then its trajectory is totally optimal by Theorem 6.1; thus, the corresponding maximum principle holds, whereby the total optimality of the trajectory is confirmed again by Theorem 11.1. □

**Theorem 11.3.** If for an optimal trajectory the contiguity condition is satisfied, this trajectory is totally optimal.

**Proof.** If the contiguity condition is satisfied, then the Isaacs or Bellman equation is valid, and the optimal trajectory is totally optimal by Theorem 6.1 or 11.2. □

We see that all three principles of optimal control and game theory define totally optimal trajectories, if they exist for the problem under consideration. There are fields of totally optimal trajectories, and over such fields all three theories are valid. However, there exist fields of optimal trajectories such that any part of any trajectory is not optimal, see Example 10.1. Problems generating such fields cannot be solved by those theories of optimality. In some cases, the space can be subdivided into regions within each (or some) of which there exists a totally optimal field of trajectories. Finding such a subdivision is the problem of so called regular synthesis [24,25,34,35], which is beyond the scope of this paper. There are fields (and field theorems) serving other purposes [5,34–36]. It is worth noting that totally optimal field structures may appear and disappear due to some changes of parameters of the system and/or the cost functional at some moments of ongoing process. For example, the system (9.2) and (9.3), Case 1 ($w = 0$), generates a fixed totally optimal field of trajectories in the whole time–space. If at some moment $t^* > 0$ a sensor in the control $u$ fails, so $u$ is not affected by $x(t)$, i.e. $p = 0$ in (9.7), then from that moment the trajectory will have non-optimal parts, see Example 10.1 with control (10.1) acting from the moment $t^*$. However, if at that same moment $t^*$, the quality manager has changed the functional (9.3), putting $a = 0$ in it, the field of trajectories remains totally optimal, but different from the initial field (9.6) and (9.7). In economics, finance, ecology and medicine, such situations are commonplace.
12. Variational principles of mechanics and composite optimality

Let us consider some properties of differential systems in Newtonian mechanics in their relation to totally optimal (extremal) fields of trajectories, to study interesting analogies with the results in previous sections. For clarity, we reproduce, in a simple and concise manner, some well known concepts of analytical mechanics [39,40].

Newtonian equations of a constrained mechanical system of \( N \) point-wise masses can be written in the form:

\[
m_i x_i'' = F_i(t, x, v) + R_i(t, x, v), \quad i = 1, \ldots, N; \quad x'' = v' = d^2x/dt^2,
\]

where \( m_i \) are masses, \( F_i \) are active forces and \( R_i \) reactions of constraints acting on the masses \( m_i \); \( x_i, v_i, x_i'' \) are state, velocity and acceleration vectors in a Cartesian coordinate system (phase space). Since the mass \( m_i \) can be subject to forces acting from other masses, the \( 3N \)-vector \( x \) composed of \( N \) 3D-vectors \( x_i \) denoting coordinates of masses \( m_i \) is included in the forces \( F_i \) and \( R_i \) together with velocity \( v \). This is a short form to avoid a double index writing \( m_i x_i'' = F_i(t, x_k, v_k) + R_i(t, x_k, v_k) \), cf. (12.5) below, where \( x_k \) means \( [x_1, \ldots, x_N] \) with index \( k \) not included in subsequent summations. If Eqs. (12.1) are divided by masses and reduced to the normal form by writing \( d^2v_i/dt^2 \) instead of \( x_i'' \) with vector equations \( dx_i/dt = v_i \) added, we obtain \( 6N \) dimensional vector equation (1.1) of the first order whereby \( F_i \) can be regarded as controls (or containing controls). Constraints are assumed ideal which means that the total work of constraint reactions is zero, \( \sum R_i \delta x_i = 0 \), where \( \delta x_i \) are any possible, i.e., allowed by the constraints (virtual, \( t \) fixed) displacements. Using this equation to exclude the unknown reactions of constraints yields the general equation of motion (principle of D’Alembert):

\[
\sum_{i=1}^{N} [m_i x_i'' - F_i(t, x, v)] \delta x_i = 0,
\]

for the \( N \)-mass system (12.1). At rest \( x_i'' \equiv 0 \), and in this case Eq. (12.2) renders the criterion (necessary and sufficient condition) for the equilibrium of active forces \( F_i \) (principle of virtual displacements, J. Bernoulli, 1717).

12.1. The principle of minimum forcing and totally optimal fields in dynamics

From (12.2), it follows the least curvature principle of Gauss and Hertz [39, #105, pp. 254–256], also known as the minimum forcing principle [40, Ch. IV, # 8]. To present it in the form of [39], drop “\( \delta x_i = 0 \)” in (12.2), then take \( m_i \) out of the bracket, and square the new bracket yielding

\[
\sum_{i=1}^{N} m_i [x_i'' - F_i(t, x, v)/m_i]^2 = \min.
\]

Indeed, taking \( t \) as initial moment, we have \( x_i(t + s) = x_i(t) + x_i'(t)s + 1/2x_i''(t)s^2 + \ldots \) so for small \( s \) we have \( \delta x_i(t + s) = 1/2\delta x_i''(t)s^2 \) since initial \( x_i(t), v_i(t) \) are not varied [40]. Putting this into (12.2) and noting that masses and active forces do not vary too, we get

\[
\sum_{i=1}^{N} [m_i x_i'' - F_i(t, x, v)] \delta [m_i x_i'' - F_i(t, x, v)] / m_i = \delta \sum_{i=1}^{N} m_i [x_i'' - F_i(t, x, v)/m_i]^2 = 0,
\]

which is equivalent to (12.3) due to convexity of the square bracket with respect to accelerations, and attains its global minimum equal to zero for a free system when the bracket is zero (the second law of Newton). For another derivation see [39, p. 256].

The principle of minimum forcing is valid for any kind of constraints. It relates to any moment \( t \) (compare with condition 1° of the maximum principle), thus, establishing a totally optimal (with respect to neighboring arcs) field of trajectories for any constraints and forces which are considered given and not depending on accelerations and higher order derivatives. With consideration of reactive forces, it is valid also for variable masses depending on time, position and velocity, thus applicable to reactive motion of a spacecraft, presenting a really universal principle in this sense.
Remark 12.1. The cost functional (12.3) relates to accelerations, since masses and active forces in (12.1) are supposed to be given. It means that other objectives regarding forces and/or masses viewed as controls can be imposed on the motion, which is indeed the case for soft landing (see Section 11) and other problems considered in the literature. Introducing objectives with respect to masses and forces preserves total optimality with respect to neighboring arcs. However, optimized trajectories may not be totally optimal with respect to added cost functional, especially in systems with non-holonomic constraints which is the case in Example 10.1.

Remark 12.2. Active forces often contain controls, $F_i(t, x, v, u(\cdot))$, which may depend on left higher order derivatives, $F_i(t, x, v, u(t, x, v, x^{''}_i, \ldots, x^{(k)}))$. Such forces are not considered in the second law of Newton, $m_i x^{''}_i = F_i(t, x, v)$, though they have been noticed by Kirchhoff and Sir W. Thomson (1871), as forces of fluid pressure linearly dependent on the acceleration of a solid in an ideal liquid, see [41, pp. 168,169,190] (the author is grateful to V.V. Rumyantsev for this reference). Forces depending on higher order derivatives are being used in acceleration assisted control. Clearly, the min-forcing principle (12.3) and other principles of mechanics in their standard formulation are not applicable to such cases. However, for “effective” forces, see [42, pp. 34–35], the second law of Newton and the parallelogram law still hold [42, pp. 36–39]. This suggests that variational principles of mechanics could possibly be transformed for use with effective forces of the general case. In this paper, only standard formulations are considered.

12.2. Holonomic systems and Hamilton’s principle

Analytically, constraints are expressed by several independent equations:

$$l_k(t, x_j, v_j) = 0, \quad k = 1, \ldots, s. \quad (12.4)$$

If Eqs. (12.4) do not contain velocities $v_j$ or can be integrated not to contain them, constraints are called geometric and system (12.1)–(12.4) is called holonomic. In this case, from equations $l_k(t, x_j) = 0$ one can express certain $s$ coordinates as functions of $3N - s$ other coordinates and time $t$ and consider those $3N - s$ coordinates as independent variables that define the state of the system at time $t$. However, it is not binding to take Cartesian coordinates as independent variables. It may be convenient to express all $3N$ Cartesian coordinates as functions of $n = 3N - s$ independent parameters $q_1, \ldots, q_n$ and time $t$ which define the so called configuration space. Substituting thus obtained $x_j = \beta_i(t, q_1, \ldots, q_n)$ into (12.1), a new system of second order equations with respect to independent parameters $q_1, \ldots, q_n$ and time $t$ is derived. Those parameters (called generalized coordinates) do not have a transparent meaning of Cartesian coordinates, but they present a differential system without constraints, of minimal order with respect to independent variables $q_i(t)$ which define all Cartesian coordinates and velocities, thus, the state of original system (12.1) subject to constraints (12.4). Using (12.2) and noting that elementary work of active forces $\delta A = \sum_{j=1}^{N} F_j \delta x_j = \sum_{i=1}^{n} Q_i \delta q_i$, the minimal order system of Lagrange equations of the second kind is obtained in the form:

$$\frac{d}{dt} \frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i, \quad Q_i = Q_i(t, q_j, q_j'), \quad T = 0.5 \sum a_{ik} q'_i q'_k + \sum a_{ij} q'_i + a_0, \quad i = 1, \ldots, n, \quad (12.5)$$

where $Q_i$ are generalized forces, $T$ is kinetic energy and sums are taken from 1 to $n$. If constraints are stationary, i.e., (12.4) does not depend on $t$, then $a_0, a_i$ are zero in (12.5). Substituting the expression of $T$ in (12.5) into the Lagrange equations on the left yields

$$\sum_{k=1}^{n} a_{ik} q''_k + (\cdot) = Q_i(t, q_j, q'_j), \quad Q_i'' = Q_i^*(t, q_j, q'_j), \quad i = 1, \ldots, n, \quad (12.6)$$

where $(\cdot)$ stands for terms not containing second derivatives, and the second equation is the unique solution of the first one for $q''_i$, since determinant $\det(a_{ik})_{k=1}^{n} \neq 0$. Thus, the motion of the system is determined by initial values $q_i(0), q'_i(0)$. If generalized forces $Q_i$ in (12.5) do not depend on generalized velocities, $Q_i = Q_i(t, q_1, \ldots, q_n)$, then there exists a potential function $P(t, q_1, \ldots, q_n)$ such that $Q_i = -\partial P / \partial q_i$. Introducing kinetic potential (the
Lagrange function) \( L = T - P \), the system (12.5) at left can be written in the form:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots, n.
\]  

(12.7)

Using the \( L \)-function, \textit{Hamilton’s principle} for conservative holonomic systems in consideration of two adjacent arcs \( AB \) and \( CD \) over the same time period \([t, t^*] \subset [0, T] \), \( t, t^* \) arbitrary, with \( \delta \) denoting the variation by which we pass from a position on \( AB \) to the contemporaneous position on \( CD \) [39, # 99, pp. 245–246], can be stated as follows:

\[
\delta \int L dt = \int_C D t - \int_A B L dt = 0; \quad \text{or} \quad \int_{t}^{t^*} (\delta T + \sum Q_i \delta q_i) dt = 0,
\]

which is true also for non-potential non-conservative forces in (12.5), see its derivation from (12.5) in [39, # 101, p. 249].

We cite from [39, pp. 246, 250]: “the integral \( \int L dt \) has a stationary value for any part of an actual trajectory \( AB \), as compared with neighboring paths \( CD \) which have the same terminal points as the actual trajectory and for which the time has the same terminal values” and “... it is only for holonomic systems that the varied motion is a possible motion; so that if we compare the actual motion with adjacent motions which obey the kinematical equations of constraints, Hamilton’s principle is true only for holonomic systems.”

The principle defines not just extremal (stationary) but actually minimal (with respect to the corresponding stationary integral) trajectories, if there are no kinetic foci on the way, see [39, pp. 248–253]. Moreover, from the above citations, it is clear that Hamilton’s principle defines \textit{totally extremal}, and in the absence of kinetic foci, \textit{totally minimal} trajectories. The same is true also for the principle of \textit{Least Action} (Euler, Maupertuis, Lagrange), see [39, # 100, pp. 247–248].

Eqs. (12.7) suggest new coordinates proposed by Hamilton. Denote \( \partial L/\partial q_i' = p_i \) (generalized impulses), so that by (12.7) \( dp_i/\partial t = p_i' = \partial L/\partial q_i \), and consider new variables \( p_1, \ldots, p_n \) which together with old variables \( q_1, \ldots, q_n \) constitute the set of 2\( n \) variables of Hamilton. Since \( \partial^2 L/\partial q_i' \partial q_k' = \det(a_{ik})_{k=1}^n \neq 0 \), see expression of \( T \) in (12.5), so the Jacobian of \( \partial L/\partial q_i' \) is nonzero, and equations \( \partial L/\partial q_i' = p_i \) can be resolved for \( q_i' \) yielding \( q_i' = \varphi_i(t, q_k, p_k) \), which together with \( p_i' = \partial L/\partial q_i = \theta_i(t, q_k, p_k) \) present a Hamiltonian system of 2\( n \) equations of the first order equivalent to a Lagrangian system of \( n \) equations (12.7) of the second order, cf. (12.1) and (12.6). If the quantity \( \sum p_i q_i' - L \) is expressed as a function of \( [t, q_i, p_i] \) and denoted by \( H \), then equations of motion (12.7) in the Lagrangian form can be represented also in \textit{Hamiltonian} or \textit{canonical} form as follows [39, pp. 263–264]:

\[
\delta H = \delta \left[ \sum p_i q_i' - L \right] = \sum (q_i' \delta p_i - p_i' \delta q_i), \quad \text{thus} \; q_i' = \partial H/\partial p_i, \; p_i' = -\partial H/\partial q_i.
\]

(12.8)

The theory of contact-transformations (Jacobi) leads to the Hamilton–Jacobi partial differential equation which has a \textit{specific} integral that is intimately related to the Isacacs and Bellman equations. In fact, for the integral \( \int L dt \) mentioned in Hamilton’s principle above (for details see [39, pp. 314–318; and 40, Ch. VIII, Sec. 2]), we have due to (12.8):

\[
W = \int_0^t L dt, \quad L = dW/\partial t = \partial W/\partial t + \sum (\partial W/\partial q_i)(dq_i/\partial t)
\]

(12.9)

\[
= \partial W/\partial t + \sum \left( \int_0^t \partial L/\partial q_i \right) q_i'
\]

\[
= \partial W/\partial t + \sum \left( \int_0^t p_i' dt \right) q_i' = \partial W/\partial t + \sum p_i q_i'
\]

\[
= \partial W/\partial t + H + L, \quad \text{thus} \; \partial W/\partial t + H = 0.
\]

(12.10)

Since \( H = H(t, q_1, \ldots, q_n, p_1, \ldots, p_n) = H(t, q_1, \ldots, q_n, \partial W/\partial q_1, \ldots, \partial W/\partial q_n) \), so (12.10) can be written in the form:

\[
\partial W/\partial t + H(t, q_1, \ldots, q_n, \partial W/\partial q_1, \ldots, \partial W/\partial q_n) = 0,
\]

(12.11)
The operators min and/or max in any specified sequence can be applied to the appropriately
if we set
and
added as an additional coordinate in
of Section
corresponding to optimal controls
we have
– (11.1)–(11.3)–(11.4) can be used
of Section
corresponding to the optimal control
is identical to
is its solution provided that
in Section
V
and/or constraints in
and the Bellman
with optimal controls in place are, in fact, special cases of the Hamilton–Jacobi equation
for the case of a holonomic system.

12.3. Composite optimality over totally optimal fields of trajectories

The forces in (12.1) and/or constraints in (12.4) may depend on some piecewise continuous functions called
controls. In practice, it is often required to optimize some objective(s) with respect to those controls. From (12.9) and
(12.10), we see that integral of kinetic potential, \( W \), which is stationary (and minimal in the absence of kinetic foci) is
itself a solution of (12.11). This leads to important conclusions, since the Isaacs main equation (2.9) and the Bellman
equation (4.6) with optimal controls in place are, in fact, special cases of the Hamilton–Jacobi equation (12.11).
Indeed, “in the second position the value of \( V \) is known” [1, Sec. 4.2], the optimized over \([t, T]\) payoff function
(1.5) \( V(x(t)) = \int_0^T G(x, u^*, v^*) dt + K(x(t)) \), so we have \( dV/dt = -G(x, u^*, v^*) \). Taking \( W = V \) and noting
that (1.1), (1.2), (1.5) and (2.9) do not depend on \( t \) explicitly, we have \( \partial W/\partial t = \partial V/\partial t = 0 \), \( dW/dt = dV/dt = -G(., \partial W/\partial q_1) = \nabla V \), and \( dq/dt = f(q, u^*, v^*) \) from (1.1) with \( x = q \), which terms fit identically into the third equality of (12.9). This implies that Eq. (2.9) of Section 2 is identical to (12.11) corresponding to optimal controls \( u^*, v^* \), with the substitution \( H := \sum (\partial W/\partial x_j)(dx_j/dt) - dW/dt = \sum (\partial V/\partial x_j) f_j + G(.) = \nabla V f + G(.), \) cf. (2.9), and the value of the game \( V(x) = W(q, c) \), cf. (12.12), is its solution provided that \( c = x_0 \) is determined from initial
data of (1.1). In the same way, Eq. (4.6) of Section 4 is identical to (12.11) if we set \( a = t, c = x = q \in R^n, v^o = u^*, W = f(t, x) \) of (4.1), so \( \partial W/\partial t = f_t, dW/dt = -F(x, u^*, t), \partial W/\partial q_1 = \nabla f = f_c, dq/dt = G(q, u^*, t) \) from (4.1) with \( x = q \), which terms fit identically into the third equality of (12.9). This implies that Eq. (4.6) is identical to (12.11) corresponding to the optimal control \( u^* \) with the substitution \( H := f_t G + F, \) cf. (4.6), and the value \( f(t, x) = W(t, q, c) \) of (12.12) is its solution provided that \( c = x(a) \in R^n \) is determined from initial data of (4.1). Since Eqs. (12.5)–(12.8) are of order \( 2n \), so to complete the analogy, one can add to differential equations (1.1) and (4.1) another \( n \) trivial equations: \( x'' = dx'/dt \in R^n \), with initial conditions \( x'(0) = f(x_0, u^*(x_0), v^*(x_0)) \) for (1.1), and \( x'(0) = G(x_0, u^*(x_0), t_0) \) with \( t_0 = 0 \) for the non-autonomous case in (4.1). These extra equations added
to fit the order \( 2n \) are trivial and of no use in the Isaacs and Bellman problems.

As to the maximum principle, it is already in the form of Hamilton. Indeed, for the Hamiltonian of (11.4), the
systems (11.1)–(11.3), (11.5) and (11.6) of order \( 2n + 2 \) are already in canonical coordinates with \( x = q, \varphi = p \)
(generalized impulses) and the functional (11.2) added as an additional coordinate in (11.5). Since the problem is
autonomous, so in (12.10) and (12.11) we have \( \partial W/\partial t = 0 \), and Eq. (12.11) takes the form \( H(.) = 0 \). The maximum
principle (Theorem I in Section 11) states that max-operator before \( H \) in (12.11) interpreted in the sense of (11.8)
solves the optimal control problem.

We argue that the same practice of putting optimization operators min and/or max before a
multi-game function \( H \) in (12.11), with respect to controls contained in differential equations and cost
functionals, to solve a multi-game problem if and only if the resulting trajectories are totally optimal at every stage of
the specified sequence.

**Theorem 12.1.** The operators min and/or max in any specified sequence can be applied to the appropriately
formulated (if possible) function \( H \) in (12.11), with respect to controls contained in differential equations and cost
functionals, to solve a multi-game problem if and only if the resulting trajectories are totally optimal at every stage of
the specified sequence.
Proof. Hamilton’s principle for $W$ in (12.9) admits for comparison neighboring arcs over any time interval of an actual trajectory, given any fixed forces and controls, and under any holonomic constraints, thus involving totally extremal (minimal with respect to $W$ in the absence of kinetic foci) trajectories. If those trajectories can be further optimized by controls that may be contained in the forces and constraints while maintaining total extremality (with respect to the same arcs) of optimized trajectories and producing totally optimal trajectories at every stage with respect to added cost functionals contained in the appropriately formulated function $H$, then optimal controls are congruent, so that the resulting HJ-equations generate totally optimal trajectories at every stage.

Vice versa, if the maximum principle (or Bellman’s, or Isaacs’ formalisms) can be applied to a multi-game problem in the specified sequence, it yields the appropriate formulations of the $H$-function in (12.11) at every stage, hence the application of min and/or max operators respecting the controls and cost functionals is valid, and the collection of corresponding solutions of (12.12), $W$-functions, presents the successive value functions at every stage. 

This theorem does not specify how to solve the sequential HJ-equations, for which a formula-like solution may not exist. However, approximate solutions can be obtained. Theorem 4.1 in Section 4 is a special case of Theorem 12.1. If controls are separable, the parallel solution can be attempted, as illustrated in Example 9.1, Case 2. Also, resonances between different controls must be avoided since they may create singularities, bifurcations, and other undesirable effects (cf. kinetic foci mentioned above).

12.4. Non-holonomic systems: consequences for the classical control theories

There is not much current literature on non-holonomic systems, see [43,44] and references therein. Hamilton’s principle can be extended to non-conservative and non-holonomic systems with significant additions to the integrands to account for the work of forces on possible variations of generalized coordinates $dq_i$ and generalized velocities $dq_i^t$, see [39, pp. 248–250]. This means that Eqs. (12.8) to (12.11) do not hold in the same form for such systems. Since the functional equations of Isaacs and Bellman (with aligned notations and optimal controls in place), and the maximum principle represent the Hamilton–Jacobi equation (12.11), which equation may require substantial modifications for non-conservative or non-holonomic systems, the same may be required with respect to the Isaacs, Bellman, and Pontryagin et al. theories for such systems. Hence, these theories, in their original form, are not universally applicable, contrary to popular belief. Also, totally extremal (optimal) fields of trajectories defined (in regard to certain integrals) by the three methods may be incompatible with some realistic constraints encountered in engineering and the economy. All these questions need thorough consideration which is beyond the scope of this paper.

13. Conclusions

In this paper, some aspects of the Isaacs principle of transition and his main equation for differential games [1,2, Sec. 4.2] are considered to demonstrate that those equations contain some implicit assumptions (such as congruence of min–max policies and trajectories, the Markov property, invariance of the value function over fragmented trajectory), and are valid under certain contiguity condition which is defined and analyzed for differential games. The notion of total optimality is defined, and totally optimal fields are introduced and studied in relation to the Isaacs principle of transition, the Bellman principle of optimality, the maximum principle of Pontryagin et al., and to variational principles of mechanics for optimally controlled motions. It is demonstrated that the Isaacs, Bellman and Pontryagin theories are valid if and only if the optimal trajectories and optimal control curves generated by those methods are totally optimal. In this context, the Hamilton–Jacobi partial differential equation can be used for sequential solution of multi-game problems (composite optimality). Over totally optimal fields of trajectories, the structure of controls is invariant under time uncertainty which always exists due to a finite speed of information transmittal. Parallel and series games are considered, and a theorem is proved that the Isaacs procedure can be reduced to the application of the Bellman equations twice.

Control systems with incomplete information or structural limitations on controls do not, in general, satisfy the contiguity condition, thus, are not totally optimal. Game problems for such systems may have optimal solutions which, however, cannot be obtained by the Isaacs equation. This fact is shown in an example of a widely used engineering system for which an optimal trajectory has all its parts non-optimal and non-contiguous to the optimal trajectory. The paper presents theoretical justification of the Isaacs equations for contiguous systems, comparison of optimal control principles with variational principles of mechanics, the consideration of total optimality and totally optimal fields of
References


