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Non-projectability of polytope skeleta ☆

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Abstract

We investigate necessary conditions for the existence of projections of polytopes that preserve full k-skeleta. More precisely, given the combinatorics of a polytope and the dimension e of the target space, what are obstructions to the existence of a geometric realization of a polytope with the given combinatorial type such that a linear projection to e-space strictly preserves the k-skeleton. Building on the work of Sanyal (2009), we develop a general framework to calculate obstructions to the existence of such realizations using topological combinatorics. Our obstructions take the form of graph colorings and linear integer programs. We focus on polytopes of product type and calculate the obstructions for products of polygons, products of simplices, and wedge products of polytopes. Our results show the limitations of constructions for the deformed products of polygons of Sanyal and Ziegler (2010) and the wedge product surfaces of Rörig and Ziegler (2011) and complement their results.

Keywords: Dimensional ambiguity; Projection of polytopes; Topological obstructions; Products of polygons; Products of simplices; Wedge products; Polyhedral surfaces

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1. Introduction

According to Grünbaum [4, Ch. 12], a polytope P is *dimensionally k-ambiguous* if the k-skeleton of P is isomorphic to that of a polytope Q and $\dim Q \neq \dim P$. So, not only is the k-skeleton of such a polytope *not* characteristic but, even worse, it does not even give away the dimension in which to look for it. Unfortunately, there is no effective way to decide when a polytope is dimensionally ambiguous and even the list of known instances of such polytopes is rather short. The prime example of a dimensionally $\lfloor \frac{d-3}{2} \rfloor$-ambiguous polytope is the d-simplex as is certified by the existence of *neighborly simplicial polytopes* such as the cyclic polytopes (cf. [19]). However, in recent years two more families of polytopes joined the list: the family of cubes via the existence of *neighborly cubical polytopes* [6] and the family of products of even polygons in guise of *projected deformed products of polygons* [20,17]. In both cases, the construction principle (unified in [17]) is to give a special realization of the combinatorial type and to verify that a projection to lower dimensions strictly preserves the skeleton in question.

The main motivation for this paper was to investigate the limitations of this approach. To be more precise: What are necessary conditions for the existence of a polytope $P \subset \mathbb{R}^d$ of a fixed combinatorial type and a projection $\pi : \mathbb{R}^d \to \mathbb{R}^e$ such that P and $\pi(P)$ have isomorphic k-skeleta.

Building on technology developed in [16], we devise tools that give fairly good necessary conditions for the existence of such pairs (P, π) in terms of topological combinatorics. The main observation is that if $\pi : P \to \pi(P)$ strictly preserves the k-skeleton for $k \geqslant 0$ then there is an associated pair of spaces $(\partial\mathcal{A}, \|\Sigma_k\|)$ with $\|\Sigma_k\| \hookrightarrow \partial\mathcal{A}$ where Σ_k is a simplicial complex and $\partial\mathcal{A}$ is a (polyhedral) sphere. The simplicial complex Σ_k is defined in terms of the combinatorics of P whereas the dimension of the sphere $\partial\mathcal{A}$ depends on e. Thus, the existence of (P, π) implies that Σ_k is embeddable into a sphere of a specific dimension. Obstructing the embeddability of Σ_k into a sphere of this dimension then impedes the existence of (P, π). Drawing on methods of topological combinatorics [8], our obstructions take the form of graph coloring problems and integer linear programs.

We focus on polytopes of *product type* for which the factorization of the skeleta allows us to replace the simplicial complex Σ_k by somewhat simpler subcomplexes. We apply the tools to the following three classes of polytopes:

Products of polygons. One curiosity left in connection with the deformed products of polygons of [17] is that the general construction scheme fails for *odd* polygons, i.e. polygons with an odd number of vertices. With respect to the number of *even and odd* polygons we prove necessary conditions on products of polygons to be dimensionally ambiguous via projection. Along the way, we obtain interesting byproducts. For example, it is known, though apparently nowhere written up properly, that there is no realization of a product of two odd polygons such that a projection to the plane retains all vertices. As a teaser, we generalize this result to

There is no realization of a product of r odd polygons such that a projection to r-space strictly preserves all vertices.

Products of simplices. Products of simplices are ubiquitous in geometric and topological combinatorics. Most notable are their appearances in work on tropical geometry and subdivisions [14], game theory and polynomial equations [18], and as building blocks for prodsimplicial complexes such as Hom-complexes [11]. It is known to both discrete geometers and topologists that no d-polytope is dimensionally k-ambiguous for $k \geqslant \lfloor \frac{d}{2} \rfloor$ (cf. Theorem 2.14). Essentially, the reason is that the statement is already false for the d-simplex. In Section 4, we investigate obstructions to the projectability of products of simplices – calculating these obstructions is intricately related to the coloring of *Kneser graphs*. We generalize a result in [16] that products of $r \geqslant d$ simplices of dimension d cannot strictly preserve all vertices under projection to lower dimensions.

Wedge products. The properties of (combinatorial) products that we exploit for the calculation of the obstructions hold for more general polytope constructions, most notably the wedge product. The wedge product, introduced in [12, Ch. 4] (see also [13]), is a degeneration of the product that may be described purely combinatorially. The interest for this class stems from the original context in which wedge products were introduced: The (straight) realization of (equivelar) polyhedral surfaces. The equivelar surfaces of type $\{r, 2n\}$ are topological surfaces *glued* exclusively from r-gons, $2n$ of which meet at every vertex. The discrete-geometric realization question now is to find a geometric embedding in which all the polygons are convex and flat. In [12] it is shown that certain equivelar families of type $\{r, 2n\}$ are naturally embedded into the wedge products $\mathcal{W}_{r,n-1}$ of r-gons and $(n-1)$-simplices. Techniques similar to the deformed products (cf. [17]) allow for the realization of the subfamily $\{r, 4\}$ in Euclidean 3-space. In Section 5 we show (cf. Theorem 5.8) that this is probably the only family that embeds into 3-space via projection:

For $r \geqslant 4$ and $n \geqslant 3$ there is no realization of the wedge product $\mathcal{W}_{r,n-1}$ such that a projection to 4-space retains the surface $\mathcal{S}_{r,2n}$.

Our methods do not yield an obstruction for $r = 3$ in which case the surface is triangulated and the wedge product of triangle and $(n-1)$-simplex is a simplex.

2. Combinatorial types, projections, and obstructions

In this section we develop a general framework for investigating the projectability of skeleta or more general subcomplexes of the boundary of a polytope. We briefly recap the necessary polytopal background and then proceed to reduce polytopes to their combinatorial structure – their combinatorial types. The benefit will be apparent in our results which state conditions under which there is no polytope with specific combinatorial and geometric qualities.

Throughout a (convex) *polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points $P = \mathsf{conv}\{v_1, \ldots, v_n\}$ and, equivalently, the bounded intersection of finitely many halfspaces $P = \{x \in \mathbb{R}^d\colon\ a_i \cdot x \leqslant b_i \text{ for all } i = 1, \ldots, m\}$. In both representations, we assume that the collection of *vertices* $v_1, \ldots, v_n$ and of *facet-defining inequalities* $a_i \cdot x \leqslant b_i$ is irredundant, that is, no vertex or linear inequality can be omitted. It is customary to write the system of linear inequalities succinctly as $Ax \leqslant b$. A hyperplane $H = \{x \in \mathbb{R}^d\colon\ c \cdot x = \delta\}$ is *supporting* P if $P \subseteq H^- = \{x \in \mathbb{R}^d\colon\ c \cdot x \leqslant \delta\}$ and $F = P \cap H$ is called a *face* of P – the empty set and P are also faces of P. In particular, it is clear that every $a_i \cdot x \leqslant b_i$ is a supporting hyperplane and the corresponding faces are called *facets*. The dimension $\mathsf{dim} F$ of a face $F \subseteq P$ is the dimension of its affine span. Vertices are faces of dimension 0 and facets are faces of dimension $\mathsf{dim} P - 1$. We abbreviate the notions of k-dimensional face and d-dimensional polytope with k-face and d-polytope, respectively.

Proposition 2.1. *(See [19, Prop. 2.3].) Let $P \subset \mathbb{R}^d$ be a polytope with vertex set $V \subset \mathbb{R}^d$ and facets F_i defined by $a_i \cdot x \leqslant b_i$ for $i = 1, \dots, m$. If $F \subseteq P$ is a face, then*

(1) $F = \mathsf{conv}(F \cap V)$ *and*
(2) $F = \{x \in P\colon\ a_i \cdot x = b_i \text{ for all } i \in I_P(F)\}$ *with* $I_P(F) := \{i \in [m]\colon\ F \subseteq F_i\}$.

The collection of faces $\mathcal{L}(P)$ of a polytope P ordered by inclusion is called the *face lattice* of P. The face lattice is a graded lattice of rank $\dim P + 1$ and it can be thought of as the combinatorial structure of P. We call two polytopes *combinatorially isomorphic* if $\mathcal{L}(P) \cong \mathcal{L}(Q)$ as graded lattices. It follows from Proposition 2.1 that the face lattice has two canonical representations.

Corollary 2.2. *Let P be a polytope with vertex set V and facets indexed by $[m] = \{1, 2, \dots, m\}$. Then $\mathcal{L}(P)$ is isomorphic to*

(1) $\{F \cap V\colon\ F \subseteq P \text{ a face}\} \subseteq 2^V$ *ordered by inclusion and* (vertex description)
(2) $\{I_P(F)\colon\ F \subseteq P \text{ a face }\} \subseteq 2^{[m]}$ *ordered by reverse inclusion.* (facet description)

Our main results will be concerned with the non-existence of geometric realizations of polytopes with given combinatorial features under projection. In order to avoid cumbersome formulations, we wish to abstract from the geometry of a polytope P.

Definition 2.3. A graded lattice $\mathcal{P}$ is called a *combinatorial type* of dimension d, or *d-type* for short, if $\mathcal{P} \cong \mathcal{L}(P)$ for some d-polytope P.

We want to think about combinatorial types as polytopes stripped from their geometric realization but we will nevertheless stick to our geometric terminology and, for example, call $F \in \mathcal{P}$ a face of $\mathcal{P}$. Moreover, when no confusion arises we use P and $\mathcal{P}$ interchangeably. Identifying the collection of facets of $\mathcal{P}$ with $F_1, \dots, F_m$, we write

$$I_{\mathcal{P}}(F) = \{i\colon\ F \subseteq F_i\} \subseteq [m]$$

for the *facet-incidences* of $\mathcal{P}$. The collection of all faces of $\mathcal{P}$ of dimension at most k is the *k-skeleton* of $\mathcal{P}$ and we call a d-type $\mathcal{P}$ *simple* if every k-face F is contained in exactly $d - k$ facets.

2.1. Geometry and topology of projections

Let P be a d-polytope and let $\pi : P \to \pi(P) \subseteq \mathbb{R}^e$ be an affine projection. Throughout it is understood that $d \geqslant e$ and that $\pi(P)$ is full-dimensional. We want to find conditions under which P and $\pi(P)$ have isomorphic k-skeleta. The key concept for establishing such conditions is that of faces strictly preserved under π.

Definition 2.4 *(Preserved and strictly preserved faces).* (See [16,20].) Let P be a polytope, $F \subset P$ a proper face and $\pi : P \to \pi(P)$ a projection of polytopes. The face F is *preserved* under π if

(i) $G = \pi(F)$ is a proper face of $\pi(P)$ and
(ii) F and G are combinatorially isomorphic.

If, in addition,

(iii) $\pi^{-1}(G)$ is equal to F

then F is *strictly preserved*.

With the notion of strictly preserved faces at our disposal, the task of deciding isomorphic k-skeleta of P and $\pi(P)$ can be checked one face at a time.

Lemma 2.5. *Let P be a polytope and let $\pi : P \to \pi(P)$ be a projection of polytopes. For $0 \leqslant k < \operatorname{dim} P$ the polytopes P and $\pi(P)$ have isomorphic k-skeleta if and only if every k-face of P is strictly preserved under π.*

We will therefore say that the k-skeleton is *retained* under projection whenever all k-faces are strictly preserved.

Proof. Assume that P and $\pi(P)$ have isomorphic k-skeleta. We show by induction on k that all preserved k-faces are then strictly preserved.

Since $f_l(P) = f_l(\pi(P))$ for $0 \leqslant l \leqslant k$, the 0-skeleton is strictly preserved. If for $l \geqslant 1$ the $(l-1)$-skeleton is strictly preserved under projection, then the preimage of every l-face of $\pi(P)$ is an l-face. Indeed, let $\bar{F}$ be an l-face of $\pi(P)$ and let $F = \pi^{-1}(\bar{F})$. Then the map $\pi|_F : F \to \pi(F) = \bar{F}$ is a projection of polytopes that strictly preserves the $(l-1)$-skeleton of F. Thus the $(l-1)$-skeleton of F is a subcomplex of an $(l-1)$-sphere. Hence F is an l-face of P and $\bar{F}$ is strictly preserved.

Therefore all k-faces are strictly preserved since all k-faces are preserved and P and $\pi(P)$ have isomorphic $(k-1)$-skeleta.

Conversely, since every i-face for $i \leqslant k$ is strictly preserved we have that the k-skeleton of P is isomorphic to a subposet of the k-skeleton of $\pi(P)$. Assume that the inclusion is strict and let $H \subset P$ be a proper face of dimension greater than k and $\pi(H)$ a k-face of $\pi(P)$. As a polytope, H has a proper face F of dimension k. But F is a k-face of P with $\pi(F) = \pi(H)$, since $\pi(F) \subseteq \pi(H)$ and $\operatorname{dim} \pi(F) = \operatorname{dim} \pi(H) = k$. Thus F is not strictly preserved. □

In [16], for every simple polytope P a simplicial complex $\Sigma_0 = \Sigma_0(P)$ is defined in terms of the combinatorics of the vertices of P. Furthermore, it is shown that if $\pi : P \to \pi(P)$ is a projection strictly preserving the vertices, then Σ_0 is realized as a subcomplex of a (simplicial) sphere whose dimension depends on $\operatorname{dim} \pi(P)$. Theorem 2.8 below is a generalization of this result for which we separate the technical part in the following proposition.

Proposition 2.6. *Let P be a d-polytope on m facets and let $\pi : P \to \pi(P)$ be a projection retaining all vertices of P. Then there is a polytope $\mathcal{A} = \mathcal{A}(P, \pi)$ of dimension $m - d - 1 + \operatorname{dim} \pi(P)$ with vertices $a_1, a_2, \ldots, a_m$ such that the following holds: For every strictly preserved face $G \subset P$ the set*

$$\mathcal{A}_G := \operatorname{conv}\{a_i \colon i \in [m] \setminus I_P(G)\}$$

is a simplex face of $\mathcal{A}$.

Proof. Let $e = \dim \pi(P)$. Fix a strictly preserved face G and let $I = I_P(G)$. Proposition 3.8 and Lemma 3.2 in [16] assert that there exists a polytopal Gale transform $\mathcal{G} = \{g_1, g_2, \dots, g_m\}$ in $\mathbb{R}^{d-e}$ with the property that $\mathcal{G}_I := \{g_i\colon i \in I\}$ positively spans $\mathbb{R}^{d-e}$. Let $\mathcal{A} = \operatorname{conv}\{a_1, \dots, a_m\}$ be the $(m - d - 1 + e)$-dimensional polytope Gale-dual to $\mathcal{G}$. Gale duality implies that $\mathcal{A}_G = \operatorname{conv}\{a_i\colon i \notin I\}$ is a face of $\mathcal{A}$. Clearly, every set $\mathcal{G}_J$ with $I \subseteq J \subseteq [m]$ is positively spanning as well. So Gale duality implies that every subset of the vertices of $\mathcal{A}_G$ is also a face. Hence $\mathcal{A}_G$ is a simplex face of $\mathcal{A}$. $\square$

We call the polytope $\mathcal{A}(P, \pi)$ the *projection polytope*. The collection of strictly preserved faces induces the following simplicial complex in the boundary of $\mathcal{A}$ that certifies the strict preservation.

Definition 2.7. Let P be a polytope on m facets and let $\pi : P \to \pi(P)$ be a projection of polytopes retaining all vertices. We define the *strict projection complex* $\mathsf{K}(P, \pi) \subseteq 2^{[m]}$ as the simplicial complex generated by the sets $\{[m] \setminus I_P(G)\colon G \text{ strictly preserved under } \pi\}$.

We may now rephrase Proposition 2.6 as follows.

Theorem 2.8. *Let P be a d-polytope on m facets and let $\pi : P \to \pi(P)$ be a projection strictly preserving all vertices. Then $\mathsf{K}(P, \pi)$ is embedded in a (polytopal) sphere of dimension $m - d - 2 + \dim \pi(P)$.* $\square$

Remark 2.9. The conditions of Theorem 2.8 can be weakened to the requirement that for each facet F there is a strictly preserved vertex v with $v \notin F$. The proof relies on a slight variation of [16, Proposition 3.8] which verifies that the set $\mathcal{G}$ is indeed a polytopal Gale transform.

As we are primarily interested in the preservation of full skeleta of a given dimension we introduce the following complex of a combinatorial type.

Definition 2.10. Let $\mathcal{P}$ be a combinatorial type of dimension d on m facets. For $-1 \leqslant k \leqslant d$, the *$k$-th coskeleton complex* is the simplicial complex

$$\Sigma_k(\mathcal{P}) = \bigl\{\tau \subseteq [m]\colon \tau \cap I_{\mathcal{P}}(G) = \emptyset \text{ for some } k\text{-face } G \in \mathcal{P}\bigr\} \subseteq 2^{[m]}.$$

The maximal faces of $\Sigma_k(\mathcal{P})$ are in bijection with the k-faces of $\mathcal{P}$ under the correspondence $G \mapsto [m] \setminus I_{\mathcal{P}}(G)$. The connection to $\mathsf{K}(P, \pi)$ is the following.

Observation. If $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton then

$$\{\emptyset\} = \Sigma_{-1}(P) \subset \Sigma_0(P) \subset \Sigma_1(P) \subset \cdots \subset \Sigma_k(P) \subset \mathsf{K}(P, \pi)$$

is an increasing sequence of subcomplexes.

As every k-face is contained in at least $d - k$ facets, the dimension of $\Sigma_k(\mathcal{P})$ is at most $m + k - d - 1$. If $\mathcal{P}$ is a simple d-type, then $\Sigma_k(\mathcal{P})$ is pure of this dimension. In [16], $\Sigma_0(\mathcal{P})$ was defined for simple d-types in terms of the *complement complex* of the boundary complex of the dual of $\mathcal{P}$. Here, we abandon the restriction to simple polytopes.

Every simplicial complex can be embedded in a sphere of some dimension. We will be interested in the smallest dimension of such a sphere.

Definition 2.11 *(Embeddability dimension).* Let $\mathsf{K} \subseteq 2^{[m]}$ be a simplicial complex on m vertices. The *embeddability dimension* $\mathsf{e\text{-}dim}(\mathsf{K})$ is the smallest integer d such that $\|\mathsf{K}\|$ may be embedded into the d-sphere, i.e. $\|\mathsf{K}\|$ is homeomorphic to a closed subset of the d-sphere.

Theorem 2.8 can be read as an upper bound on the embeddability dimension of the strict projection complex $\mathsf{K}(P,\pi)$. However, $\mathsf{K}(P,\pi)$ heavily depends on the geometry of the projection and hence a priori our knowledge about $\mathsf{K}(P,\pi)$ is rather limited. The virtue of the coskeleton complex is that it is a subcomplex of $\mathsf{K}(P,\pi)$ defined entirely in terms of the combinatorics of P.

Corollary 2.12. *Let $\mathcal{P}$ be a d-type on m facets and, for $0 \leqslant k < d$, let $\Sigma_k = \Sigma_k(\mathcal{P})$ be the k-th coskeleton complex of $\mathcal{P}$. If*

$$e < \mathsf{e\text{-}dim}(\Sigma_k) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton.

Proof. By contradiction, assume that P is a realization of $\mathcal{P}$ and $\pi : P \to \pi(P)$ is a projection retaining the k-skeleton with $\dim \pi(P) = e < \mathsf{e\text{-}dim}(\Sigma_k) + d - m + 2$. By Theorem 2.8, the above observation, and the fact that the embeddability dimension is monotone along subcomplexes, the complex Σ_k is realized in a sphere of dimension

$$\mathsf{e\text{-}dim}(\Sigma_k) \leqslant m - d - 2 + e < \mathsf{e\text{-}dim}(\Sigma_k). \qquad \square$$

The following well-known fact bounds the embeddability dimension of a simplicial complex in terms of its dimension.

Proposition 2.13. *(See [4, Thm. 11.1.8, Ex. 4.8.25].) Let K be a simplicial complex of dimension $\dim \mathsf{K} = \ell$. Then*

$$\ell \leqslant \mathsf{e\text{-}dim}(\mathsf{K}) \leqslant 2\ell + 1.$$

It is instructive to consider the statement of Corollary 2.12 in the extreme cases of Proposition 2.13. If the $(m + k - d - 1)$-dimensional complex Σ_k attains the lower bound $\mathsf{e\text{-}dim}(\Sigma_k) = m - d - 2 + e$, then Corollary 2.12 implies that the dimension of the target space has to be at least $e \geqslant k + 1$. This is reassuring as the projection embeds $\Sigma_k(\mathcal{P})$ into a sphere of dimension $e - 1$. Now, suppose that $\mathsf{e\text{-}dim}(\Sigma_k)$ attains the upper bound and that $\mathcal{P}$ is a simple type. Then $\dim \Sigma_k(\mathcal{P}) = m - (d - k) - 1$ and the k-skeleton is not projectable to e-space if $e < m - d + 2k + 1$. This is the linear Van Kampen–Flores result:

Theorem 2.14. *(See [3, Thm. 2].) Let $\mathcal{P}$ be a d-type and let $0 \leqslant k \leqslant \lfloor \frac{d-2}{2} \rfloor$. If*

$$e \leqslant 2k + 1$$

then there is no realization of $\mathcal{P}$ such that a projection to e-space retains the k-skeleton.

2.2. Cotype complexes of products

For our purposes we need better bounds than provided by Proposition 2.13 and so we need more sophisticated techniques to determine or at least bound the embeddability dimension $\mathsf{e\text{-}dim}(\Sigma_k)$. In this and the next section we introduce two notions that approximate the coskeleton complex as well as the embeddability dimensions and allow us to calculate bounds.

For the cases in which we want to apply Corollary 2.12, the combinatorial types under consideration are products or, at least, closely related (cf. Section 5). Let $P \subset \mathbb{R}^d$ and $P' \subset \mathbb{R}^{d'}$ be two polytopes of combinatorial types $\mathcal{P}$ and $\mathcal{P}'$, respectively. The product of P and P' is the polytope $P \times P' = \mathsf{conv}\{(p, p')\colon\ p \in P,\ p' \in P'\}$. Combinatorially we define

$$\mathcal{P} \times \mathcal{P}' := \mathcal{L}\big(P \times P'\big) = \big\{\big(F, F'\big)\colon\ F \in \mathcal{L}(P), F' \in \mathcal{L}\big(P'\big) \text{ such that } F = \emptyset \text{ iff } F' = \emptyset\big\}.$$

Note that this product of combinatorial types differs from the usual direct product of lattices inasmuch as every non-empty face of the product $\mathcal{P} \times \mathcal{P}'$ is a product of non-empty faces of $\mathcal{P}$ and $\mathcal{P}'$. In particular, we have $\mathsf{dim}(F, F') = \mathsf{dim} F \times F' = \mathsf{dim} F + \mathsf{dim} F'$ and the facet incidences of the product are given by

$$I_{P \times P'}\big(F \times F'\big) = I_P(F) \uplus I_{P'}\big(F'\big).$$

The following definition distinguishes the faces of the product by their "type".

Definition 2.15. Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ with $\mathcal{P}_i$ a d_i-type on m_i facets for $i = 1, \ldots, r$. For a fixed $0 \leqslant k < d = d_1 + \cdots + d_r$ we call a composition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ with $0 \leqslant \lambda_i \leqslant d_i$ and $\lambda_1 + \cdots + \lambda_r = k$ a *face type* of dimension k. We denote by $\Lambda_k(\mathcal{P})$ the collection of k-dimensional face types for $\mathcal{P}$. For $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ we define the *cotype complex* of type $\boldsymbol{\lambda}$ as the join of coskeleton complexes

$$\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) := \Sigma_{\lambda_1}(\mathcal{P}_1) * \Sigma_{\lambda_2}(\mathcal{P}_2) * \cdots * \Sigma_{\lambda_r}(\mathcal{P}_r).$$

It is clear from the definition of the product that every face of $\mathcal{P}$ belongs to some type and this yields a partition of the coskeleton complex.

Proposition 2.16. *Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ and $0 \leqslant k < \mathsf{dim}\,\mathcal{P}$. Then*

$$\Sigma_k(\mathcal{P}) = \bigcup_{\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})} \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}). \qquad \square$$

The monotonicity of the embeddability dimension along subcomplexes yields our first bound for the projectability of products.

Corollary 2.17. *Let $\mathcal{P}$ be a product and $0 \leqslant k < \mathsf{dim}\,\mathcal{P}$. If there is a face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$ such that*

$$e < \mathsf{e\text{-}dim}(\Sigma_{\boldsymbol{\lambda}}) + d - m + 2$$

then there is no realization of $\mathcal{P}$ such that a projection to $\mathbb{R}^e$ retains the k-skeleton. $\square$

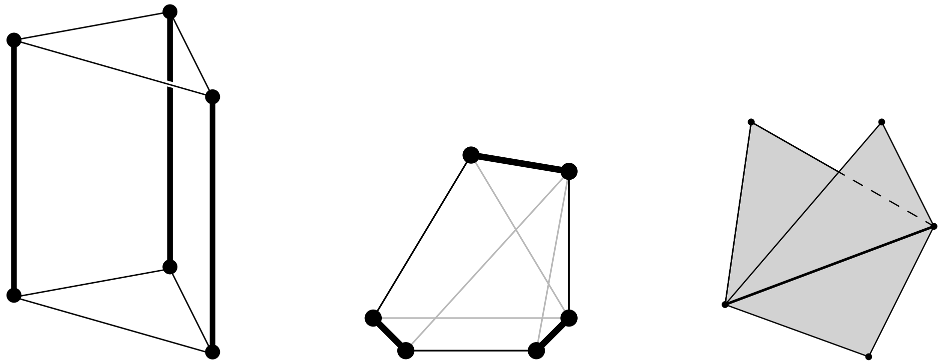

Fig. 1. The triangular prism to the left with bold vertical edges. An alleged projection in the middle with preserved vertical edges. And the associated cotype complex to the right.

Example 2.18. To illustrate the usefulness of the cotype complex, consider the following question: Is there a realization of a prism over a triangle $\mathcal{P} = \Delta_1 \times \Delta_2$ such that a projection to the plane preserves the three *vertical* edges (see Fig. 1). The ad-hoc negation of the question is that by *Desargues' theorem* (cf. [2, Sect. 14.3]) the three vertical edges in the prism meet in a common point (at infinity) and a linear projection retains this property. Using the developed machinery, the vertical edges represent the faces of type $\boldsymbol{\lambda} = (1, 0)$ and the cotype complex $\Sigma_{(1,0)}(\mathcal{P})$, shown in Fig. 1, consists of three triangles that share a common edge. Assuming such a projection exists, it necessarily preserves all vertices strictly and hence satisfies the conditions of Theorem 2.8. However, the cotype complex is visibly not planar and Corollary 2.17 yields the contradiction.

Remark 2.19. The definition of the cotype complex relies on properties of the product that are shared by other polytope constructions such as joins, direct sums, and wedge products (see Section 5). The common generalization is that of a *compound type* (cf. [15]) which is subject to further study.

2.3. Bounding the embeddability dimension

In general it is hard to decide the embeddability of a complex K into some $\mathbb{R}^e$. The following notions, taken and adapted from [8], show that in fortunate cases bounds on $\mathsf{e\text{-}dim}(\mathsf{K})$ can be obtained combinatorially.

For a simplicial complex $\mathsf{K} \subseteq 2^{[m]}$ we denote by $\mathcal{F}(\mathsf{K})$ the set of *minimal non-faces*, i.e. the inclusion-minimal sets in $2^{[m]} \setminus \mathsf{K}$. The *Kneser graph* $\mathsf{KG}(\mathcal{F})$ on a set system $\mathcal{F} \subseteq 2^{[m]}$ has the elements of $\mathcal{F}$ as vertices and $F, G \in \mathcal{F}$ share an edge iff F and G are disjoint. Furthermore, for a graph G we denote by $\chi(G)$ the *chromatic number* of G.

Definition 2.20. Let K be a simplicial complex on m vertices and $\mathcal{F} = \mathcal{F}(\mathsf{K})$ the collection of minimal non-faces. The *Sarkaria index* of K is

$$\mathsf{ind}_{\mathsf{SK}}\,\mathsf{K} := m - \chi\big(\mathsf{KG}(\mathcal{F})\big) - 1.$$

Theorem 2.21 *(Sarkaria's coloring/embedding theorem). (See [8, Sect. 5.8].) Let* K *be a simplicial complex. Then*

$$\text{e-dim}(\mathsf{K}) \geqslant \operatorname{ind}_{\mathsf{SK}} \mathsf{K}.$$

Every embedding of a simplicial complex K into a d-sphere gives rise to a $\mathbb{Z}_2$-equivariant map of the *deleted join* K_Δ^{*2} to a d-sphere. The $\mathbb{Z}_2$*-index* of K_Δ^{*2} is the smallest such d for which an equivariant map exists. In its original form in [8] the above theorem bounds from below the $\mathbb{Z}_2$-index of K_Δ^{*2} and thus also bounds from below the embeddability dimension of K.

The next observation reduces the calculation of the Sarkaria index of a product to its factors.

Proposition 2.22. *(See [16, Prop. 3.10].) Let* K *and* L *be simplicial complexes. Then*

$$\operatorname{ind}_{\mathsf{SK}}(\mathsf{K} * \mathsf{L}) = \operatorname{ind}_{\mathsf{SK}} \mathsf{K} + \operatorname{ind}_{\mathsf{SK}} \mathsf{L} + 1.$$

Thus it follows directly from Definition 2.15 that the Sarkaria index of a cotype complex is determined by its factors.

Corollary 2.23. *Let* $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_r$ *and let* $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P})$*. Then*

$$\operatorname{ind}_{\mathsf{SK}} \Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) = \sum_{i=1}^{r} \operatorname{ind}_{\mathsf{SK}} \Sigma_{\lambda_i}(\mathcal{P}_i) + r - 1. \qquad \square$$

We determine the exact embeddability dimensions and Sarkaria indices for two coskeleton complexes of an arbitrary combinatorial type. The result depends only on the number of facets.

Proposition 2.24. *Let* $\mathcal{P}$ *be a* d*-type on* m *facets. Then* $\Sigma_d(\mathcal{P}) = \Delta_{m-1}$ *is homeomorphic to an* $(m-1)$*-ball and*

$$m - 1 = \text{e-dim}\big(\Sigma_d(\mathcal{P})\big) = \operatorname{ind}_{\mathsf{SK}} \Sigma_d(\mathcal{P}).$$

For the $(d-1)$*-skeleton we have that* $\Sigma_{d-1}(\mathcal{P}) = \partial \Delta_{m-1} \cong S^{m-2}$ *and*

$$m - 2 = \text{e-dim}\big(\Sigma_{d-1}(\mathcal{P})\big) = \operatorname{ind}_{\mathsf{SK}} \Sigma_{d-1}(\mathcal{P}).$$

Proof. The first claim follows from the definition of the skeleton complex. Thus the embeddability dimensions are $m-1$ and $m-2$, respectively. For the Sarkaria index we get in the former case that the Kneser graph of the minimal non-faces of $\Sigma_d(\mathcal{P})$ has no vertices, whereas in the latter case the graph has no edges. $\square$

In the special case that we have an r-fold product $\mathcal{P}^r = \mathcal{P} \times \mathcal{P} \times \cdots \times \mathcal{P}$ of the same combinatorial type $\mathcal{P}$, bounds on the embeddability dimension of $\Sigma_k(\mathcal{P}^r)$ can be obtained by solving a *knapsack-type* problem.

Lemma 2.25. *Let* $\mathcal{P}$ *be a* d*-type and let* $r \geqslant 1$ *and* $0 \leqslant k \leqslant rd - 1$*. For* $i = 0, \ldots, d$ *set* $s_i = \operatorname{ind}_{\mathsf{SK}} \Sigma_i(\mathcal{P})$ *and let* s^* *be the optimal value of the integer linear program*

$$\begin{aligned} \max\ & s_0\,\mu_0 + s_1\,\mu_1 + \cdots + s_d\,\mu_d \\ \text{s.t.}\quad & 0\,\mu_0 + \ 1\,\mu_1 + \cdots + \ d\,\mu_d \ = \ k \\ & \quad\ \mu_0 + \quad \mu_1 + \cdots + \quad \mu_d \ = \ r \end{aligned}$$

with $\mu_0, \ldots, \mu_d \in \mathbb{Z}_{\geqslant 0}$. *Then* $\text{e-dim}(\Sigma_k(\mathcal{P}^r)) \geqslant s^* + r - 1$.

Proof. To a face type $\boldsymbol{\lambda} \in \Lambda_k(\mathcal{P}^r)$ associate the non-negative numbers $(\mu_0, \mu_1, \ldots, \mu_d)$ with

$$\mu_i = \#\{j \in [r]:\ \lambda_j = i\}.$$

They satisfy

$$\begin{aligned} 0\,\mu_0 + 1\,\mu_1 + \cdots + d\,\mu_d \ &= \ k \quad \text{and} \\ \mu_0 + \ \ \mu_1 + \cdots + \ \ \mu_d \ &= \ r \end{aligned}$$

since $\boldsymbol{\lambda}$ is a partition of k in r parts and the Sarkaria index of $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}^r)$ is given by $\sum_i s_i\,\mu_i + r - 1$. Vice versa, every such non-negative collection of numbers μ_i that satisfies the conditions of the integer program gives rise to a valid face type. □

3. Products of polygons

Denote by $\mathcal{P}_m$ the combinatorial type of an m-gon, that is, a 2-dimensional combinatorial type on $m \geqslant 3$ facets labeled in cyclic order. In this section we determine necessary conditions for the existence of a realizations of $\mathcal{P} = \mathcal{P}_{m_1} \times \mathcal{P}_{m_2} \times \cdots \times \mathcal{P}_{m_r}$, a product of polygons, that retain the k-skeleton under a suitable projection. To that end, we need to determine (bounds on) the embeddability dimension of $\Sigma_k(\mathcal{P})$ for $0 \leqslant k < 2r = \dim \mathcal{P}$.

For a single polygon, Proposition 2.24 leaves us to determine the Sarkaria index for the 0-th coskeleton complex of an m-gon.

Lemma 3.1. *Let $m \geqslant 3$ and $\mathcal{P}_m$ the combinatorial type of an m-gon. The Sarkaria index for the 0-th coskeleton complex is given by*

$$\text{ind}_{\text{SK}}\, \Sigma_0(\mathcal{P}_m) = \begin{cases} m - 3, & \text{if } m \text{ is even}, \\ m - 2, & \text{if } m \text{ is odd}. \end{cases}$$

Proof. We show that the Kneser graph of minimal non-faces of $\Sigma_0(\mathcal{P}_m)$ has chromatic number 2 and 1, respectively. For that let us determine the minimal non-faces of $\Sigma_0(\mathcal{P}_m)$: A subset $\sigma \subseteq [m]$ of the facets of $\mathcal{P}_m$ is a non-face of $\Sigma_0(\mathcal{P}_m)$ if and only if every vertex of $\mathcal{P}_m$ is incident to at least one facet F_i of $\mathcal{P}_m$ with $i \in \sigma$. If a vertex of $\mathcal{P}_m$ is covered twice by σ then every other minimal non-face intersects σ and thus σ is an isolated vertex in the Kneser graph. If σ covers every vertex exactly once, then $[m] \setminus \sigma$ is again a minimal non-face. It follows that for odd m the Kneser graph consists of isolated vertices alone while for even m there is exactly one edge. □

Example 3.2. Let us consider $\Sigma_0(\mathcal{P}_5)$, the 0-th coskeleton complex of the pentagon. The figure shows the triangles of the 0-th coskeleton complex of the pentagon which form a Möbius strip. Hence $\Sigma_0(\mathcal{P}_5)$ is not embeddable in the 2-sphere.

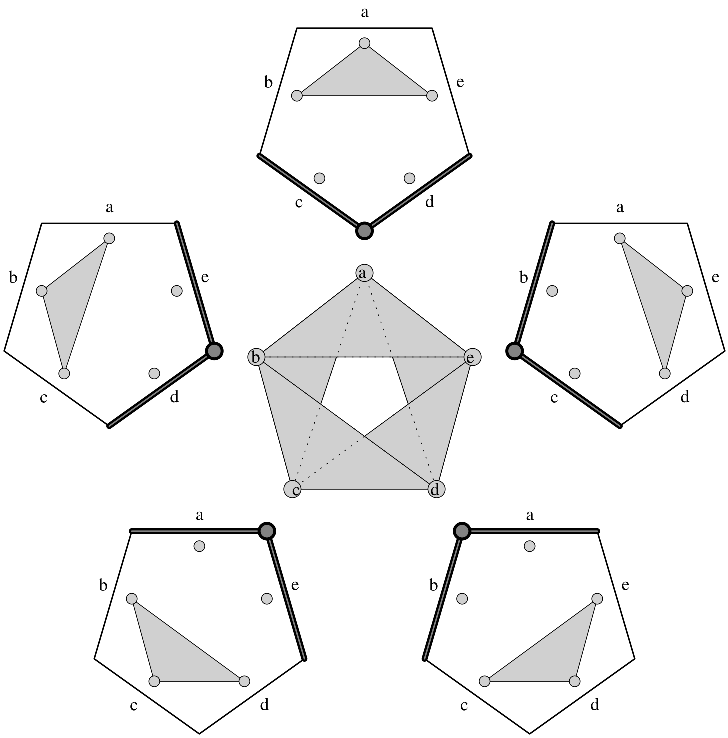

Fig. 2. The five triangles of the 0-th coskeleton complex of a pentagon fit together to form a Möbius strip.

The example shows that the 0-th coskeleton complex of an *odd* polygon has a certain twist to it that obstructs the embeddability into $(m-3)$-dimensional space.

Lemma 3.1 implies that projectability bounds for products of polygons arising from Corollary 2.23 will only depend on the total number of facets and the number of odd and even polygons. Thus, it suffices to consider the generic product

$$\mathcal{P} = \mathcal{P}_{\mathsf{even}}^{r_e} \times \mathcal{P}_{\mathsf{odd}}^{r_o},$$

of r_e *even* and r_o *odd* polygons. We denote by m the total number of facets and by $r = r_e + r_o$ the number of factors. For a product of polygons, we utilize the *knapsack-type* integer program introduced in Lemma 2.25.

Theorem 3.3. *Let $\mathcal{P}$ be a product of polygons with $r = r_o + r_e$ factors and m facets. For $0 \leqslant k \leqslant 2r$ the embeddability dimension of the k-th coskeleton complex is bounded by*

$$\mathsf{e\text{-}dim}\big(\varSigma_k(\mathcal{P})\big) \geqslant m - 1 - r + \left\lfloor \frac{k}{2} \right\rfloor + \mathsf{min}\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}.$$

Proof. In the spirit of Lemma 2.25 consider the following integer linear program

$$\begin{array}{llllllll}
\min & 2\mu_0^{\mathsf{even}} + & \mu_0^{\mathsf{odd}} + & \mu_1 & & & & \\
\text{s.t.} & & & \mu_1 & + 2\mu_2 & = & k \\
& \mu_0^{\mathsf{even}} + & \mu_0^{\mathsf{odd}} + & \mu_1 + & \mu_2 & = & r \\
& \mu_0^{\mathsf{even}} & & & & \leqslant & r_e \\
& & \mu_0^{\mathsf{odd}} & & & \leqslant & r_o
\end{array}$$

with $\mu_0^{\mathsf{even}}, \mu_0^{\mathsf{odd}}, \mu_1, \mu_2 \in \mathbb{Z}_{\geqslant 0}$. Every face type $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\mathcal{P})$ gives rise to a feasible solution by the association

$$\begin{array}{llll}
\mu_2 & := \#\{i\colon \lambda_i = 2\} & & \text{(polygons)} \\
\mu_1 & := \#\{i\colon \lambda_i = 1\} & & \text{(edges)} \\
\mu_0^{\mathsf{odd}} & := \#\{i\colon \lambda_i = 0,\ 1 \leqslant i \leqslant r_o\} & & \text{(odd vertices)} \\
\mu_0^{\mathsf{even}} & := \#\{i\colon \lambda_i = 0,\ r_o < i \leqslant r\} & & \text{(even vertices)}
\end{array}$$

and, vice versa, every feasible solution yields a face type. The integer program reduces to a problem in essentially two variables and the optimal solution is easily seen to be

$$\mu^* = r - \left\lfloor \frac{k}{2} \right\rfloor + \mathsf{max}\left\{0, r_e - \left\lceil \frac{k}{2} \right\rceil\right\}.$$

The result then follows from the fact that $\mathsf{e\text{-}dim}(\Sigma_k(\mathcal{P})) \geqslant m - 1 - \mu^*$. □

In order to put the above result in perspective, let us calculate upper bounds on the embeddability dimension.

Proposition 3.4. *Let $\mathcal{P}$ be a product of polygons with $r = r_o + r_e$ factors and let m be the number of facets. For $0 \leqslant k < 2r$ the embeddability dimension is bounded by*

$$\mathsf{e\text{-}dim}\big(\Sigma_k(\mathcal{P})\big) \leqslant \begin{cases} m - r - r_e - 1, & \text{if } k = 0, \\ m - r - 1, & \text{if } k = 1, \\ m - 1, & \text{otherwise}. \end{cases}$$

Proof. Let $\mathcal{P} = \mathcal{P}_{m_e}^{r_e} \times \mathcal{P}_{m_o}^{r_o}$ and for $\ell = \mathsf{min}\{k, 2\}$ define

$$\hat{\Sigma} = \Sigma_\ell(\mathcal{P}_{m_e})^{*r_e} * \Sigma_\ell(\mathcal{P}_{m_o})^{*r_o}.$$

We claim that $\hat{\Sigma}$ contains $\Sigma = \Sigma_k(\mathcal{P})$ as a subcomplex. By construction, $\hat{\Sigma}$ and Σ have identical vertex sets. For every admissible face type $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda_k(\mathcal{P})$ we have $\lambda_i \leqslant \ell$ for $i = 1, \ldots, r$ and, by Observation 2.10 and the relation of subcomplexes among joins, this shows $\Sigma_{\boldsymbol{\lambda}}(\mathcal{P}) \subseteq \hat{\Sigma}$. Since Σ is the union of all cotype complexes, this proves the claim. We will therefore bound the embeddability dimension of $\hat{\Sigma}$ from above.

For $\ell = 2$, we have by Proposition 2.24 that $\Sigma_2(\mathcal{P}_n) = \Delta_{n-1} \hookrightarrow \partial\Delta_n$ and thus $\hat{\Sigma}$ embeds into the boundary of $\Delta_{m_e}^{\oplus r_e} \oplus \Delta_{m_o}^{\oplus r_o}$, a simplicial sphere of dimension $r_e m_e + r_o m_o - 1 = m - 1$.

For $\ell = 1$, we again make use of Proposition 2.24 to get $\Sigma_1(\mathcal{P}_n) = \partial\Delta_{n-1}$ and therefore $\hat{\Sigma} \hookrightarrow \partial(\Delta_{m_e-1}^{\oplus r_e} \oplus \Delta_{m_o-1}^{\oplus r_o})$, which is a simplicial sphere of dimension $r_e(m_e - 1) + r_o(m_o - 1) - 1 = m - r - 1$.

For $\ell = 0$, the 0-th coskeleton complex of $\mathcal{P}_n$ may be embedded into the boundary of an $(n-1)$-simplex. However, for even $n = 2t$ we can do better: Consider the $(n-2)$-dimensional polytope $Q_t = \Delta_{t-1} \oplus \Delta_{t-1}$ and the mapping from the vertices of $\Sigma_0(\mathcal{P}_n)$ that maps the i-th vertex to the $\lfloor \frac{i}{2} \rfloor$-th vertex of the first summand if i is even and of the second otherwise. We claim that this gives an embedding. Every vertex v of $\mathcal{P}_n$ is the intersection of an odd and an even edge. Thus the corresponding facet $[n] \setminus I(v)$ is the disjoint union of $t-1$ odd and $t-1$ even vertices. These sets correspond to facets of Q_t. Thus $\Sigma_0(\mathcal{P}) = \hat{\Sigma}$ embeds into the boundary of $Q_t^{\oplus r_e} \oplus \Delta_{m_o-1}^{\oplus r_o}$ with $t = \frac{m_e}{2}$. $\square$

Combining the bounds on the embeddability dimensions of the coskeleton complexes of Theorem 3.3 with Corollary 2.12 we obtain the following obstructions to projectability of products of polygons.

Theorem 3.5. *Let $r = r_o + r_e$ and $0 \leqslant k < 2r$. There is no realization of a product of r_o odd and r_e even polygons such that a projection to e-dimensional space strictly preserves the k-skeleton if*

$$e < r + 1 + \left\lfloor \frac{k}{2} \right\rfloor + \min\left\{0, \left\lceil \frac{k}{2} \right\rceil - r_e\right\}. \qquad \square$$

In [17], e-dimensional polytopes with the $\lfloor \frac{e-2}{2} \rfloor$-skeleton of the r-fold product of *even* polygons are projections of suitable products of even polygons. The following corollary shows that this construction technique does not generalize to products of odd polygons.

Corollary 3.6. *There is no realization of an r_o-fold product of* odd *polygons such that the k-skeleton is strictly preserved under projection to $\mathbb{R}^e$ if*

$$e < r_o + 1 + \left\lfloor \frac{k}{2} \right\rfloor.$$

In the special case of $r_o = 2$ and $k = 0$ the result reduces to the well-known fact that a product of two odd m-gons does not project to an m^2-gon.

Another case of interest is $k = \lfloor \frac{e}{2} \rfloor - 1$. In case such a realization and projection exists, the resulting polytope is called neighborly, in analogy to the simplicial neighborly polytopes.

Corollary 3.7. *Let $r = r_e + r_o$ and $e \geqslant 1$. If*

$$\begin{cases} \lceil \frac{3e-2}{4} \rceil < r & \text{for } r_e < \lfloor \frac{e}{4} \rfloor, \\ \lceil \frac{e}{2} \rceil < r_o & \text{for } r_e \geqslant \lfloor \frac{e}{4} \rfloor, \end{cases}$$

then there is no realization of a product of r_e even and r_o odd polytopes such that a projection to e-space is neighborly.

Paraphrasing the situation for products of odd polygons, the result puts an upper bound of $\lceil \frac{3e-2}{4} \rceil$ on the number of odd polygons for a “neighborly” projection to $\mathbb{R}^e$.

4. Products of simplices

In this section we investigate obstructions to skeleta-preserving projections of products of simplices. Appealing to the results from Section 2, we bound the embeddability dimension for the respective coskeleton complexes. We denote by $\Delta_{n-1} = 2^{[n]}$ the combinatorial type of an $(n-1)$-simplex. The key to determining the embeddability dimension and the Sarkaria index of $\Sigma_k(\Delta_{n-1})$ is the following observation.

Observation. For $n \geqslant 1$ and $0 \leqslant k \leqslant n-1$ the k-th coskeleton complex $\Sigma_k(\Delta_{n-1})$ of the $(n-1)$-simplex is isomorphic to the k-skeleton of Δ_{n-1}.

Thus $\Sigma_k(\Delta_{n-1})$ is a *well known* complex and the calculation of the Sarkaria index involves the *classical* Kneser graphs $\mathsf{KG}_{n,\ell} = \mathsf{KG}\binom{[n]}{\ell}$ for $0 \leqslant \ell \leqslant n$, that is, the Kneser graphs on the collection of ℓ-sets of an n-set. Their chromatic numbers are a celebrated result in topological combinatorics.

Theorem 4.1. *(See Lovász [7].) For $1 \leqslant \ell \leqslant n$ the chromatic number of $\mathsf{KG}_{n,\ell}$ is given by*

$$\chi(\mathsf{KG}_{n,\ell}) = \begin{cases} n - 2\ell + 2 & \text{if } \ell \leqslant \frac{n+1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

This result immediately implies the Sarkaria index of the k-th skeleton complex $\Sigma_k(\Delta_{n-1})$.

Lemma 4.2. *For $n \geqslant 2$ and $0 \leqslant k \leqslant n-1$ the Sarkaria index of the k-th coskeleton complex $\Sigma_k = \Sigma_k(\Delta_{n-1})$ of the $(n-1)$-simplex is*

$$\mathsf{ind}_{\mathsf{SK}}\, \Sigma_k = \begin{cases} 2k+1, & \text{if } 0 \leqslant k \leqslant \frac{n-3}{2}, \\ n-2, & \text{if } \frac{n-3}{2} < k \leqslant n-2, \\ n-1, & \text{if } k = n-1. \end{cases}$$

Proof. By the above observation, we have $\mathsf{KG}(\mathcal{F}(\Sigma_k)) = \mathsf{KG}_{n,k+2}$. The first two cases follow directly from Theorem 4.1. The last case follows from Proposition 2.24. $\square$

In combination with Proposition 2.13 we obtain the following corollary.

Corollary 4.3. *Let $\Sigma_k = \Sigma_k(\Delta_{n-1})$ be the k-th coskeleton complex of an $(n-1)$-simplex for $n \geqslant 2$. Then the embeddability dimension satisfies*

$$\mathsf{e\text{-}dim}(\Sigma_k) = \begin{cases} 2k+1, & \text{if } 0 \leqslant k \leqslant \frac{n-3}{2}, \\ n-2, & \text{if } \frac{n-3}{2} < k \leqslant n-2, \\ n-1, & \text{otherwise.} \end{cases} \qquad \square$$

In the following we denote by

$$\Delta^r_{n-1} = \underbrace{\Delta_{n-1} \times \Delta_{n-1} \times \cdots \times \Delta_{n-1}}_{r}$$

an r-fold product of $(n-1)$-simplices.

Theorem 4.4. *Let $n \geqslant 2$, $r \geqslant 1$ and $0 \leqslant k < r(n-1)$. The embeddability dimension of the k-th coskeleton complex $\Sigma_k = \Sigma_k(\Delta_{n-1}^r)$ of the product of simplices Δ_{n-1}^r satisfies*

$$\mathsf{e\text{-}dim}(\Sigma_k) \geqslant \begin{cases} 2r+2k-1, & \text{if } 0 \leqslant k \leqslant r\lfloor\frac{n-3}{2}\rfloor, \\ \frac{1}{2}rn+k-1, & \text{if } r\lfloor\frac{n-3}{2}\rfloor < k \leqslant r\lfloor\frac{n-2}{2}\rfloor, \\ r(n-1)+\alpha-1, & \text{if } r\lfloor\frac{n-2}{2}\rfloor < k < r(n-1), \end{cases}$$

and

$$\alpha = \left\lfloor \frac{k - r\lfloor\frac{n-2}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor} \right\rfloor.$$

Proof. We use the knowledge gained from Lemma 4.2 to set up the integer linear program as in Lemma 2.25. Set $c = \lfloor\frac{n-3}{2}\rfloor$ and let $0 \leqslant k < r(n-1)$. The program is

$$\begin{array}{lll} \max & \sum_{j=0}^{c}(2j+1)\mu_j + (n-2)\sum_{j=c+1}^{n-2}\mu_j + (n-1)\mu_{n-1} & \\ \text{s.t.} & \mu_0 + \mu_1 + \cdots + \mu_{n-1} & = r \\ & 0\mu_0 + 1\mu_1 + \cdots + (n-1)\mu_{n-1} & = k \end{array}$$

and subject to the condition that the μ_i are non-negative and integral. Any feasible solution with value s gives the bound $\mathsf{e\text{-}dim}(\Sigma_k) \geqslant r-1+s$.

Using the two above constraints we rewrite the objective function

$$r + 2k - \min\left(\sum_{j=c+1}^{n-2}(2j-n+3)\mu_j + n\mu_{n-1}\right). \tag{1}$$

Note that all coefficients are non-negative and thus the minimum is at least 0.

For $0 < k \leqslant r\lfloor\frac{n-2}{2}\rfloor$ set $\ell = \lceil\frac{k}{r}\rceil \leqslant c+1$. Define $\mu = (\mu_0, \ldots, \mu_{n-1}) \in \mathbb{Z}^n$ by

$$\begin{pmatrix} \mu_{\ell-1} \\ \mu_\ell \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \ell-1 & \ell \end{pmatrix}^{-1} \begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} r\ell - k \\ k - r(\ell-1) \end{pmatrix}$$

and $\mu_j = 0$ otherwise. For n odd we have $\ell \leqslant \lfloor\frac{n-2}{2}\rfloor = c$ and μ gives a feasible solution with value 0 in the minimization above. If n is even and $\ell = c+1$ the feasible solution yields for the objective function (1) a value of $r + 2k - (k - r(\ell-1)) = k + \frac{1}{2}rn - r$. The embeddability dimension is therefore bounded by $\frac{1}{2}rn + k - 1$. Note that the second case is vacuous for n odd.

For $r\lfloor\frac{n-2}{2}\rfloor < k$, let $h = k - r\lfloor\frac{n-2}{2}\rfloor - \alpha\lfloor\frac{n+1}{2}\rfloor$ set

$$\mu_{n-1} = \alpha, \qquad \mu_c = r - \alpha - 1, \qquad \mu_{c+h} = 1$$

for n odd and

$$\mu_{n-1} = \alpha, \qquad \mu_{c+1} = r - \alpha - 1, \qquad \mu_{c+h+1} = 1$$

for n even and $\mu_j = 0$ for all other j. □

As can be seen in the proof, the feasible solution for $\ell \leqslant \lfloor \frac{n-3}{2} \rfloor$ is given by a basic solution to the linear program relaxation and it can be checked that this indeed gives the optimal solution. However, the coefficient for μ_{n-1} keeps this circumstance from being true for $\ell > \lfloor \frac{n-3}{2} \rfloor$.

In conjunction with Corollary 2.12 this gives the following definitive result concerning the non-projectability of skeleta of Δ^r_{n-1}.

Theorem 4.5. *Let $n \geqslant 2$ and $r \geqslant 1$ and set $\alpha = \lfloor \frac{k - r\lfloor \frac{n-2}{2} \rfloor}{\lfloor \frac{n+1}{2} \rfloor} \rfloor$. If*

$$e < \begin{cases} r + 2k + 1, & \text{for } 0 \leqslant k \leqslant r\lfloor \frac{n-3}{2} \rfloor, \\ \frac{1}{2}r(n-2) + k + 1, & \text{for } r\lfloor \frac{n-3}{2} \rfloor < k \leqslant r\lfloor \frac{n-2}{2} \rfloor, \\ r(n-2) + \alpha + 1, & \text{for } r\lfloor \frac{n-2}{2} \rfloor < k < r(n-1), \end{cases}$$

then there exists no realization of the r-fold product Δ^r_{n-1} of $(n-1)$-simplices such that a projection to $\mathbb{R}^e$ retains the k-skeleton. □

For $r = 1$ Theorem 4.5 states that there is no affine projection of the $(2k+2)$-simplex to $\mathbb{R}^{(2k+1)}$ which preserves the k-skeleton. This is exactly the linear Van Kampen–Flores theorem. Thus, in some sense Theorem 4.5 is a generalization of the Van Kampen–Flores theorem from simplices to products of simplices. As a special case it gives yet another proof that no product of two triangles maps linearly to a 9-gon.

Again, let us view the statement of Theorem 4.5 in comparison with upper bounds on the embeddability dimension of the complexes $\Sigma_k(\Delta^r_{n-1})$.

Proposition 4.6. *Let $\Sigma_k = \Sigma_k(\Delta^r_{n-1})$ be the k-th coskeleton complex of the r-fold product of $(n-1)$-simplices with $n \geqslant 2$ and $0 \leqslant k < r(n-1)$. Then*

$$\mathsf{e\text{-}dim}(\Sigma_k) \leqslant \min\{2k + 2r - 1, rn - 1\}.$$

Proof. We work along the same lines as in the proof of Proposition 3.4 and we use the fact that

$$\Sigma_\ell(\Delta_{n-1}) \cong \binom{[n]}{\leqslant \ell + 1} \hookrightarrow \partial \Delta_n$$

for all $0 \leqslant \ell \leqslant n-1$. Therefrom it follows that $\Sigma_k \hookrightarrow \partial \Delta_n^{\oplus r} = \partial(\Delta_{n^r})^\triangle$ and thus $\mathsf{e\text{-}dim}(\Sigma_k)$ is at most $rn - 1$. However, since $\dim \Sigma_k = r + k - 1$ the bound given by Proposition 2.13 is better for $k \leqslant \frac{1}{2}r(n-2)$. □

Combining the upper bounds with the lower bounds from Theorem 4.4 yields that the result of Theorem 4.5 is sharp for $k \leqslant r\lfloor \frac{n-3}{2} \rfloor$. On the geometric side, this is complemented in the work of Matschke, Pfeifle, and Pilaud [9] on *prodsimplicial-neighborly polytopes*. The constructions given in [9] yield products of simplices for which the projections retain the k-skeleta for

$k \leqslant r\lfloor \frac{n-3}{2} \rfloor$. Their constructions also include products of simplices of different dimensions and they generalize the topological obstructions to give bounds in the mixed case.

5. Wedge products

The *wedge product* $P \lhd Q$ of two polytopes P and Q is a geometric degeneration of the product Q^m that bears very interesting combinatorial properties. It corresponds to an iterated subdirect product in the sense of McMullen [10] and is dual to a *wreath product* as studied by Joswig and Lutz [5].

Our motivation for studying wedge products stems from the work of Rörig and Ziegler [13] on questions concerning the realizability of equivelar surfaces. In short, an equivelar surface is a 2-dimensional polytopal surface that satisfies certain regularity conditions. It is both combinatorially and geometrically challenging to construct equivelar surfaces as they exhibit extremal combinatorial behavior. For example, unlike triangulated surfaces, equivelar surfaces need not possesses a geometric realization with flat and convex faces (cf. Betke and Gritzmann [1]).

In [12] it is shown that a certain family of wedge products $\mathcal{W}_{r,n-1}$ contains equivelar surfaces in their 2-skeleta. Furthermore, for all $r \geqslant 3$ the surface contained in $\mathcal{W}_{r,1}$ possesses a straight-line realization in 3-space. The approach is to give a geometric realization of $\mathcal{W}_{r,1}$ such that a projection to $\mathbb{R}^4$ strictly preserves the surface and the resulting polytope carries the surface in its lower hull.

In this section we prove non-projectability results regarding skeleta of wedge products and, in particular, we show that for $r \geqslant 4$ and $n \geqslant 3$ there is no realization of the wedge product $\mathcal{W}_{r,n-1}$ such that a projection to $\mathbb{R}^4$ strictly preserves the equivelar surface.

5.1. Wedge products and products

Wedge products of polytopes were introduced in [12] from several perspectives such as an iteration of a generalized wedge construction and in terms of interior and exterior presentations. In this paper, we will only need the description in terms of facet-defining halfspaces.

Definition 5.1. (See [12, Def. 4.10].) For polytopes $P = \{y \in \mathbb{R}^d\colon\ a_i \cdot y \leqslant 1 \text{ for all } i = 1, \dots, m\}$ and $Q = \{x \in \mathbb{R}^{d'}\colon\ Bx \leqslant \mathbf{1}\}$ the *wedge product* of P and Q is the polytope

$$P \lhd Q := \left\{(x_1, \dots, x_m, y) \in \left(\mathbb{R}^{d'}\right)^m \times \mathbb{R}^d\colon\ Bx_i \leqslant (1 - a_i \cdot y)\mathbf{1} \text{ for all } i = 1, \dots, m\right\}.$$

The geometry and the combinatorics of wedge products are studied in [12]. For our purposes it is sufficient to know the combinatorial type of $P \lhd Q$ in the form of intersections of facets.

Theorem 5.2. *Let P and Q be polytopes with facets indexed by $[m]$ and $[n]$, respectively. The face lattice of $P \lhd Q$ is given by the collection of tuples $(H_1, \dots, H_m)$ with $H_1, \dots, H_m \subseteq [n]$ such that*

(1) *$H_i = I_Q(F_i)$ for some face $F_i \subseteq Q$ for all i, and*
(2) *$\{j \in [m]\colon\ H_j = [n]\} = I_P(G)$ for some face $G \subseteq P$.*

The order relation is given by componentwise reverse inclusion. The dimension of the face $(H_1, \dots, H_m)$ is given by $\sum_i \dim F_i + \dim G + |I_P(G)|$.

Proof. It follows from the lattice structure of $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ that the stated poset is a atomic and coatomic lattice. It is known that two atomic–coatomic lattices are isomorphic if and only if they have isomorphic atom–coatom incidences. The bijection on the collection of facets is clear and the vertices are determined by Theorem 4.13 in [12] and correspond to admissible tuples $(H_1, H_2, \dots, H_m)$ with F_i of dimension at most 0 and G a vertex. The dimension formula follows from the facet description above. □

An alternative approach to wedge products and Theorem 5.2 appears in [15, Thm. 2.21]. The following observation links the wedge product to the usual product.

Proposition 5.3. *(See [12, Prop. 4.12].) The intersection of the wedge product $P \lhd Q$ with the linear space $L = (\mathbb{R}^{d'})^m \times \{0\} \subset (\mathbb{R}^{d'})^m \times \mathbb{R}^d$ is affinely isomorphic to Q^m. In particular, the intersection is given by the faces $(H_1, \dots, H_m)$ with $H_i \neq [n]$.*

It follows from Proposition 5.2 that every k-face for $k \geqslant 0$ of the product Q^m is the unique intersection of L with a face of dimension $k + \operatorname{dim} P$ of $P \lhd Q$.

Lemma 5.4. *Let P and Q be polytopes with m being the number of facets of P. Then for any $0 \leqslant k \leqslant m \operatorname{dim} Q$ we have*

$$\Sigma_k\big(Q^m\big) \hookrightarrow \Sigma_{k+\operatorname{dim} P}(P \lhd Q).$$

We call the image of the k-skeleton of the product Q^m in $P \lhd Q$ the *special* $(k + \operatorname{dim} P)$-faces of the wedge product. These special faces cover all vertices of the wedge product by Theorem 5.2. The bottom line is that we can re-use the bounds obtained in Section 4 to handle projections of wedge products of polygons and simplices.

5.2. Projections of wedge products of polygons and simplices

In the following we restrict ourselves to the wedge product $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$ of an r-gon $\mathcal{P}_r$ and an $(n-1)$-simplex Δ_{n-1}. It follows from Theorem 5.2 that $\mathcal{W}_{r,n-1}$ is an $(r(n-1)+2)$-dimensional polytope with rn facets. Using the correspondence established in Lemma 5.4 we apply the result of Section 4 to the projectability of the k-skeleta of wedge products.

Proposition 5.5. *There exists no realization of the wedge product $\mathcal{W}_{r,n-1}$ of r-gon and $(n-1)$-simplex with $r \geqslant 4$ and $n \geqslant 2$ such that the projection to $\mathbb{R}^e$ preserves its special k-faces for $k \geqslant 2$ if*

$$e < \begin{cases} r + 2k - 1 & \text{if } 2 \leqslant k \leqslant r\lfloor \frac{n-3}{2} \rfloor + 2, \\ \frac{1}{2} r(n-2) + k + 1 & \text{if } r\lfloor \frac{n-3}{2} \rfloor + 2 < k \leqslant r\lfloor \frac{n-2}{2} \rfloor + 2, \\ r(n-2) + \alpha + 3 & \text{if } r\lfloor \frac{n-2}{2} \rfloor + 2 < k < r(n-1) + 2, \end{cases}$$

and

$$\alpha = \left\lfloor \frac{k - 2 - \lfloor \frac{n-2}{2} \rfloor}{\lfloor \frac{n+1}{2} \rfloor} \right\rfloor.$$

Proof. We are able to apply Theorem 2.8 since the special faces cover all vertices and every face of a strictly preserved face is also strictly preserved. The strict projection complex of a projection strictly preserving the special k-faces of the wedge product contains the $(k-2)$-nd coskeleton complex of the product Δ^r_{n-1} (see Lemma 5.4). Hence the embeddability dimension of the special k-faces of the wedge product is equal to the embeddability dimension of $\Sigma_{k-2}(\Delta^r_{n-1})$ given by Theorem 4.4. Plugging these bounds into Corollary 2.12 we obtain:

$$e < \mathsf{e\text{-}dim}\big(\Sigma_{k-2}\big(\Delta^r_{n-1}\big)\big) - r + 4 \leqslant \mathsf{e\text{-}dim}\big(\Sigma_k(\mathcal{W}_{r,n-1})\big) + \big(r(n-1)+2\big) - rn + 2. \qquad \square$$

Since the k-skeleton of the wedge product obviously contains the special k-faces we obtain the following result for the projectability of skeleta of the wedge product $\mathcal{W}_{r,n-1}$.

Theorem 5.6. *There exists no realization of the wedge product $\mathcal{W}_{r,n-1}$ of r-gon and $(n-1)$-simplex with $r \geqslant 4$ and $n \geqslant 2$ such that the projection to $\mathbb{R}^e$ preserves the k-skeleton for $k \geqslant 0$ if*

$$e < \begin{cases} r+2k-1 & \text{if } 2 \leqslant k \leqslant r\lfloor\frac{n-3}{2}\rfloor + 2, \\ \frac{1}{2}r(n-2)+k+1 & \text{if } r\lfloor\frac{n-3}{2}\rfloor + 2 < k \leqslant r\lfloor\frac{n-2}{2}\rfloor + 2, \\ r(n-2)+\alpha+3 & \text{if } r\lfloor\frac{n-2}{2}\rfloor + 2 < k < r(n-1)+2, \end{cases}$$

and

$$\alpha = \left\lfloor \frac{k-2-\lfloor\frac{n-2}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor} \right\rfloor.$$

Proof. The vertices of the wedge product correspond to the vectors $\mathcal{H}_V$ given by:

$$\mathcal{H}_V = \big\{(H_1,\dots,H_r) \in \mathcal{W}_{r,n-1}\colon\ H_i \neq [n] \Rightarrow |H_i| = n-1\big\}. \tag{2}$$

We pick a subfamily of vertices corresponding to the vectors $([n],[n],H_3,\dots,H_r)$ with $|H_i| = n-1$ for $i = 3,\dots,r$. Considering only the last $r-2$ components of the vector we obtain the following inclusion of coskeleton complexes:

$$\Sigma_0\big(\Delta^{r-2}_{n-1}\big) \hookrightarrow \Sigma_0(\mathcal{W}_{r,n-1}).$$

The embeddability dimension of $\Sigma_0(\Delta^{r-2}_{n-1})$ is $2r-5$ by Theorem 4.4. Thus we obtain the following bound on e with Corollary 2.12:

$$\mathsf{e\text{-}dim}\big(\Sigma_0(\mathcal{W}_{r,n-1})\big) + r(n-1) + 2 - rn + 2 \geqslant \mathsf{e\text{-}dim}\big(\Sigma_0\big(\Delta^{r-2}_{n-1}\big)\big) - r + 4 = r - 1 > e.$$

The 1-skeleton of the wedge product contains a subfamily of edges corresponding to the vectors $([n], H_2,\dots,H_r)$. As for the vertices we obtain an inclusion of coskeleton complexes:

$$\Sigma_0\big(\Delta^{r-1}_{n-1}\big) \hookrightarrow \Sigma_1(\mathcal{W}_{r,n-1}).$$

Since by Theorem 4.4 the embeddability dimension of $\Sigma_0(\Delta_{n-1}^{r-1})$ is $2r-3$ we obtain the following bound for the dimension projected onto using Corollary 2.12:

$$\mathsf{e}\text{-}\mathsf{dim}\big(\Sigma_1(\mathcal{W}_{r,n-1})\big) - r + 4 \geqslant \mathsf{e}\text{-}\mathsf{dim}\big(\Sigma_0\big(\Delta_{n-1}^{r-1}\big)\big) - r + 4 = r + 1 > e.$$

For $k \geqslant 2$ we simply use Proposition 5.5. □

5.3. *Equivelar surfaces in wedge products*

It is shown in [12] that the wedge product $\mathcal{W}_{r,n-1} = \mathcal{P}_r \lhd \Delta_{n-1}$ of an r-gon and an $(n-1)$-simplex carries a very interesting equivelar surface $\mathcal{S}_{r,2n}$ in its 2-skeleton. The main result of [12] is that in some cases this combinatorial embedding can be used to obtain a geometric embedding in 3-space. Using the machinery developed in Section 2 and the results of Section 4 we complement the above result about projections of equivelar surfaces.

The 2-skeleton of the wedge product $\mathcal{W}_{r,n-1}$ is a fertile ground for embedding equivelar surfaces. Consider the special 2-faces of Lemma 5.4. They correspond to:

$$\mathcal{H}_R = \big\{(H_1, \ldots, H_r) \in \mathcal{W}_{r,n-1}\colon |H_i| = n-1 \text{ for all } i \in [r]\big\}.$$

So for every choice $j_1, \ldots, j_r \in [n]$ the tuple

$$H = \big([n] \setminus j_1, [n] \setminus j_2, \ldots, [n] \setminus j_r\big)$$

represents a special 2-face of $\mathcal{W}_{r,n-1}$ and each such face is isomorphic to an r-gon. Indeed, for every $i \in [r]$ the tuple

$$H^i = \big([n] \setminus j_1, \ldots, [n] \setminus j_{i-1}, [n], [n] \setminus j_{i+1}, \ldots, [n] \setminus j_r\big)$$

corresponds to an edge of H by Theorem 5.2 and hence H is a 2-dimensional face with r edges. We denote the collection of these r-gon edges by $\mathcal{H}_E$: They correspond to tuples $(H_1, \ldots, H_r)$ with $|H_i| = n-1$ for all but a unique $i_0 \in [r]$ with $H_{i_0} = [n]$.

In [12] the following subcomplex of the wedge product $\mathcal{W}_{r,n-1}$ is discussed: For $r \geqslant 3$ and $n \geqslant 2$ consider the subcomplex $\mathcal{S}_{r,2n}$ generated by the following collection of r-gons of the wedge product $\mathcal{W}_{r,n-1}$:

$$\left\{\big([n] \setminus j_1, \ldots, [n] \setminus j_r\big)\colon \sum_{k=1}^{r} j_k \equiv 0, 1 \bmod n\right\} \subseteq \mathcal{H}_R.$$

The subcomplex $\mathcal{S}_{r,2n}$ contains all the vertices and all the edges $\mathcal{H}_E$ of $\mathcal{W}_{r,n-1}$. It is shown in [12] that $\mathcal{S}_{r,2n}$ is a regular (polyhedral) surface $\mathcal{S}_{r,2n}$ of type $\{r, 2n\}$, i.e. an (orientable) polyhedral 2-manifold that is

- *equivelar*: all faces are r-gons and every vertex is incident to $2n$ faces, and even
- *regular*: the automorphism group acts transitively on the flags of the surface.

For the special case $n = 2$ there are deformed realizations of the wedge products $\mathcal{W}_{r,1}$ and projections that yield embeddings of the surfaces $\mathcal{S}_{r,4}$ in $\mathbb{R}^3$.

Theorem 5.7. *(See [12, Thm. 4.26].) The wedge product $\mathcal{W}_{r,1}$ has a realization such that all the faces corresponding to the surface $\mathcal{S}_{r,4} \subset \mathcal{W}_{r,1}$ are preserved by the projection to $\mathbb{R}^4$. Hence there is a realization of $\mathcal{S}_{r,4}$ in $\mathbb{R}^3$.*

So there was hope that some realizations of the wedge products for other parameters r and n would yield realizations in $\mathbb{R}^3$ as well. But with the techniques developed in this article we obtain the following negative result.

Theorem 5.8. *There is no realization of the wedge product $\mathcal{W}_{r,n-1}$, with $n \geqslant 3$ and $r \geqslant 4$, such that all the faces corresponding to the surface $\mathcal{S}_{r,2n}$ are strictly preserved by the projection $\pi : \mathbb{R}^{2+r(n-1)} \to \mathbb{R}^e$ for $e < r+1$.*

Proof. We prove the theorem by contradiction. So assume that there exists a realization of $\mathcal{W}_{r,n-1}$ such that the surface $\mathcal{S}_{r,2n}$ is strictly preserved by the projection to $\mathbb{R}^e$ with $e < r+1$. By Theorem 2.8 the embeddability dimension of the strict projection complex $\mathsf{K} = \mathsf{K}(\mathcal{W}_{r,n-1}, \pi)$ is then

$$\text{e-dim}(\mathsf{K}) \leqslant rn - \bigl(r(n-1)+2\bigr) + e - 2 = r + e - 4 < 2r - 3. \tag{WP}$$

Since the polygons of the wedge product surface $\mathcal{S}_{r,2n}$ are strictly preserved by the projection π the simplicial complex K contains a subcomplex Σ corresponding to the polygons of $\mathcal{S}_{r,2n}$. The strict projection complex of all special r-gons is $\Sigma_0(\Delta^r_{n-1})$ by Lemma 5.4. Hence

$$\Sigma = \left\{ (j_1, \dots, j_r) \;\middle|\; \sum_{k=1}^{r} j_k \equiv 0, 1 \bmod n \right\} \subset \Sigma_0\bigl(\Delta^r_{n-1}\bigr).$$

We remove the asymmetry from Σ by only considering the edges $([n], [n] \setminus j_2, \dots, [n] \setminus j_r)$ for $j_i \in [n]$ of the wedge product. The strict projection complex of these edges is $\Sigma_0(\Delta^{r-1}_{n-1})$. By Theorem 4.4 the embeddability dimension of $\Sigma_0(\Delta^{r-1}_{n-1})$ is $2r-3$. Hence the embeddability dimension of K is at least $2r-3$ because $\Sigma_0(\Delta^{r-1}_{n-1}) \subseteq \Sigma \subseteq \mathsf{K}(\mathcal{W}_{r,n-1}, \pi)$. This is a contradiction to Eq. (WP). So there exists no realization of $\mathcal{W}_{r,n-1}$ such that the surface $\mathcal{S}_{r,2n}$ is strictly preserved by the projection to $\mathbb{R}^e$. $\square$

Theorem 5.8 does not obstruct straight-line realizations of the surfaces $\mathcal{S}_{r,2n}$ in $\mathbb{R}^3$ in general. It only exhibits the limitations of the approach taken in [12].

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