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Mean Convergence of Hermite–Fejér Interpolation

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Weighted L^p convergence of Hermite–Fejér interpolation based on the zeros of generalized Jacobi polynomials is investigated. The main result of the paper gives necessary and sufficient conditions for such convergence for all continuous functions.

1. INTRODUCTION

The purpose of this paper is to investigate weighted L_p $(0 < p < \infty)$ convergence of Hermite–Fejér interpolating processes based on the roots of generalized Jacobi polynomials. If x_{kn}, $k = 1, 2,..., n$, are n distinct points and f is a bounded function, then the Hermite–Fejér interpolating polynomial $H_n(f)$ is defined to be the unique polynomial of degree at most $2n - 1$ which satisfies

$$H_n(f, x_{kn}) = f(x_{kn}), \qquad H_n'(f, x_{kn}) = 0; \qquad k = 1, 2,..., n. \tag{1}$$

Although, for any practical purpose, there are endlessly many papers in the literature dealing with convergence and divergence of Hermite–Fejér interpolation, most of these papers are based upon nice identities resulting from the specific choice of the interpolation nodes. Even when the interpolation nodes are chosen to be roots of ortogonal polynomials, most of the research has dealt with classical orthogonal polynomials and pointwise convergence

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and/or divergence. The only exception is given by three papers of G. Freud [3–5] where pointwise convergence of $H_n(f)$ is investigated when the interpolation nodes are zeros of general orthogonal polynomial systems. It is even more astonishing that though

$$\int H_n(f)\, d\alpha \to \int f\, d\alpha$$

when the nodes are zeros of orthogonal polynomials corresponding to $d\alpha$ where $d\alpha$ has a bounded support and f is Riemann–Stieltjes integrable with respect to $d\alpha$ [7, p. 89], L^p convergence of $H_n(f)$ to f has never been dealt with in the literature. This is in great contrast with Lagrange interpolation where sufficient attention seems to have been given to L^p convergence and/or divergence and its applications for general orthogonal polynomial systems. (See references in [1, 9–11, 15, 16].) The reason for the lack of the general theory appears to be the complicated structure of the explicit representation for the Hermite–Fejér interpolating polynomial which has been successfully overcome only by G. Freud. In this paper we present several theorems about weighted mean convergence and divergence of Hermite–Fejér interpolation processes, the most important being Theorem 5 where we give necessary and sufficient conditions for weighted mean convergence of $H_n(f)$ to f when the nodes are zeros of generalized Jacobi polynomials and f is continuous satisfying some prescribed growth condition.

2. Notations

General Notations. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and positive integers, respectively. The symbol "const" denotes some constant which is positive and independent of the variables and indices. Whenever "const" is used it will always be clear what variables and indices it is independent of. In each formula "const" may take a different value. The symbol "$\sim$" is used as follows. If A and B are two expressions depending on some variables and indices then

$$A \sim B \Leftrightarrow |AB^{-1}| \leqslant \text{const} \qquad \text{and} \qquad |A^{-1}B| \leqslant \text{const}.$$

Orthogonal polynomials. Let $d\alpha$ be a positive distribution on the real line such that the support of $d\alpha$ is infinite and all the moments of $d\alpha$ are finite. The corresponding set of orthogonal polynomials is denoted by $\{p_n(d\alpha)\}$:

$$p_n(d\alpha, x) = \gamma_n(d\alpha)\, x^n + \text{lower degree terms}, \qquad \gamma_n(d\alpha) > 0 \qquad \text{and}$$

$$\int_{\mathbb{R}} p_n(d\alpha)\, p_m(d\alpha)\, d\alpha = \delta_{nm}.$$

If $d\alpha$ is absolutely continuous, say $d\alpha = w\,dx$, then in the above and in all the subsequent notations $d\alpha$ is replaced by w. For example, we write $p_n(w)$, $\gamma_n(w)$, etc. The zeros of $p_n(d\alpha)$ are denoted by $x_{kn}(d\alpha)$ and they are indexed so that

$$x_{1n}(d\alpha) > x_{2n}(d\alpha) > \cdots > x_{nn}(d\alpha).$$

The reproducing kernel functions of the orthogonal system $\{p_n(d\alpha)\}$ are denoted by $K_n(d\alpha)$. Hence

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_k(d\alpha, t).$$

According to the Christoffel–Darboux formula [14, p. 43] $K_n(d\alpha)$ can be written as

$$K_n(d\alpha, x, t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} [p_n(d\alpha, x)\, p_{n-1}(d\alpha, t) - p_{n-1}(d\alpha, x)\, p_n(d\alpha, t)](x-t)^{-1}.$$

The Christoffel function $\lambda_n(d\alpha)$ is defined by

$$\lambda_n(d\alpha, x)^{-1} = K_n(d\alpha, x, x).$$

It is well known [7, p. 25] that

$$\lambda_n(d\alpha, x) = \min \int_{\mathbb{R}} |P(t)|^2\, d\alpha(t)$$

where the minimum is taken over all polynomials P of degree less than n such that $P(x) = 1$. The numbers $\lambda_{kn}(d\alpha)$ defined by

$$\lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn}(d\alpha))$$

are called the Cotes numbers. By the Gauss–Jacobi quadrature formula [14, p. 47]

$$\sum_{k=1}^{n} P(x_{kn}(d\alpha))\, \lambda_{kn}(d\alpha) = \int_{\mathbb{R}} P\, d\alpha$$

holds for every polynomial P of degree less than $2n$. If $\operatorname{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos\theta)$ is integrable over $[0, \pi]$ then by Szegö's theorem [14, p. 309]

$$0 < \lim_{n\to\infty} 2^{-n}\gamma_n(d\alpha) < \infty. \tag{2}$$

Lagrange interpolation. The Lagrange interpolating polynomials corresponding to the distribution $d\alpha$ and bounded function f are denoted by $L_n(d\alpha, f)$. They satisfy

$$L_n(d\alpha, f, x_{kn}(d\alpha)) = f(x_{kn}(d\alpha)),$$

$n \in \mathbb{N}$, $1 \leqslant k \leqslant n$. The polynomial $L_n(d\alpha, f)$ can conveniently be written in the form

$$L_n(d\alpha, f) = \sum_{k=1}^{n} f(x_{kn}(d\alpha))\, l_{kn}(d\alpha)$$

where the fundamental polynomials $l_{kn}(d\alpha)$ are defined by

$$l_{kn}(d\alpha, x) = \frac{p_n(d\alpha, x)}{p_n'(d\alpha, x_{kn}(d\alpha))(x - x_{kn}(d\alpha))}, \qquad 1 \leqslant k \leqslant n.$$

It is well known [14, p. 48] that

$$l_{kn}(d\alpha, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, \lambda_{kn}(d\alpha)\, p_{n-1}(d\alpha, x_{kn}(d\alpha))\, \frac{p_n(d\alpha, x)}{x - x_{kn}(d\alpha)}. \tag{3}$$

Hermite–Fejér interpolation. If the interpolation nodes $\{x_{kn}\}$ in (1) are taken to be the zeros $\{x_{kn}(d\alpha)\}$ of the orthogonal polynomials $p_n(d\alpha)$, then we denote the corresponding Hermite–Fejér interpolating polynomial by $H_n(d\alpha, x)$. Hence

$$H_n(d\alpha, f, x) = \sum_{k=1}^{n} f(x_{kn}(d\alpha)) \times \left[1 - \frac{p_n''(d\alpha, x_{kn}(d\alpha))}{p_n'(d\alpha, x_{kn}(d\alpha))}(x - x_{kn}(d\alpha))\right] l_{kn}(d\alpha, x)^2 \tag{4}$$

[14, p. 330]. G. Freud [3, p. 113] noticed that

$$-\frac{p_n''(d\alpha, x_{kn}(d\alpha))}{p_n'(d\alpha, x_{kn}(d\alpha))} = \frac{\lambda_n'(d\alpha, x_{kn}(d\alpha))}{\lambda_{kn}(d\alpha)}$$

so we can rewrite (4) as

$$H_n(d\alpha, f, x) = \sum_{k=1}^{n} f(x_{kn}(d\alpha)) \times \left[1 + \frac{\lambda_n'(d\alpha, x_{kn}(d\alpha))}{\lambda_{kn}(d\alpha)}(x - x_{kn}(d\alpha))\right] l_{kn}(d\alpha, x)^2. \tag{5}$$

If P is a polynomial of degree less than $2n$ then

$$P(x) = H_n(d\alpha, P, x) + \sum_{k=1}^{n} P'(x_{kn}(d\alpha))(x - x_{kn}(d\alpha))\, l_{kn}(d\alpha, x)^2$$

which is the Hermite interpolation formula [14, p. 331].

L^p and L_v^p spaces. If $0 < p \leqslant \infty$ then $f \in L^p$ on some interval Δ if $\|f\|_p < \infty$ where

$$\|f\|_p = \left[\int_\Delta |f(t)|^p\, dt\right]^{1/p}, \qquad 0 < p < \infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in \Delta} |f(t)|.$$

If $v \geqslant 0$ and $0 < p < \infty$ then $f \in L_v^p$ if $\|f\|_{v,p} < \infty$ where

$$\|f\|_{v,p} = \left[\int_{-\infty}^{\infty} |f(t)|^p\, v(t)\, dt\right]^{1/p}. \tag{6}$$

Naturally, when $0 < p < 1$, $\|\cdot\|_{v,p}$ and $\|\cdot\|_p$ are not norms, nevertheless we retain this notation for convenience. Also, in all of our L_v^p spaces v will have a bounded support so that the integration in (6) will actually be done only over some finite interval.

The function u is called a Jacobi weight function if u can be written as $u = w^{(a,b)}$ where

$$w^{(a,b)}(x) = (1-x)^a(1+x)^b, \qquad -1 \leqslant x \leqslant 1,$$

and $w^{(a,b)}(x) = 0$ if $|x| > 1$.

Generalized Jacobi weights. The function w is called a *GJ* weight function if $w \in L^1$ and $w = gu$ where $g > 0$ and u is a Jacobi weight function. Moreover,

$$w \in GJA \Leftrightarrow g^{\pm 1} \in L^\infty \qquad \text{on} \quad [-1, 1],$$

$$w \in GJB \Leftrightarrow g \in C \text{ and } m(t)\, t^{-1} \in L^1 \qquad \text{on} \quad [0, 1],$$

where m is the modulus of continuity of g, and

$$w \in GJC \Leftrightarrow g \in C^1 \text{ and } g' \in \text{Lip } 1 \qquad \text{on} \quad [-1, 1].$$

3. Auxiliary Propositions

Lemma 1. *Let Δ be a fixed interval. Let g be a positive continuous function in Δ such that g is differentiable on some set $D \subset \Delta$ and both* $\sup |g'(x)|$ *when* $x \in D$ *and* $\sup |g(x) - g(t) - g'(x)(x-t)|\,(x-t)^{-2}$ *when* $x \in D$ *and* $t \in \Delta$ *are finite. Let $d\alpha$ be supported in Δ and let $d\alpha_g$ be defined by $d\alpha_g = g\,d\alpha$. Then*

$$\left| g(x) \frac{d}{dx} K_n(d\alpha_g, x, x) - \frac{d}{dx} K_n(d\alpha, x, x) \right|$$
$$\leqslant \text{const} \cdot [|p_{n-2}(d\alpha, x)| + |p_{n-1}(d\alpha, x)| + |p_n(d\alpha, x)|]$$
$$\times [|p'_{n-1}(d\alpha, x)| + |p'_n(d\alpha, x)|]$$

uniformly for $x \in D$ and $n \in \mathbb{N}$.

Proof. For simplicity, we will use the notation $p_n(x) = p_n(d\alpha, x)$, $\gamma_n(d\alpha) = \gamma_n$, $K_n(d\alpha, x, t) = k_n(x, t)$, $k_n(x) = k_n(x, x)$, $P_n(x) = p_n(d\alpha_g, x)$, $\gamma_n(d\alpha_g) = \Gamma_n$, $K_n(d\alpha_g, x, t) = K_n(x, t)$ and $K_n(x) = K_n(x, x)$. Since K_n and k_n are reproducing kernels, we have for every number c

$$cK_n(x, y) - k_n(x, y) = \int_\Delta K_n(y, t)\, k_n(x, t)[c - g(t)]\, d\alpha(t).$$

Differentiating this identity with respect to x and substituting $y = x$ and $c = g(x)$ we obtain

$$g(x)\, K'_n(x) - k'_n(x) = 2 \int_\Delta K_n(x, t) \frac{\partial}{\partial x} k_n(x, t)[g(x) - g(t)]\, d\alpha(t)$$

which we rewrite in the form

$$g(x)\, K'_n(x) - k'_n(x) = 2g'(x) \int_\Delta K_n(x, t)(x - t) \frac{\partial}{\partial x} k_n(x, t)\, d\alpha(t)$$
$$+ 2 \int_\Delta K_n(x, t) \frac{\partial}{\partial x} k_n(x, t)[g(x) - g(t)$$
$$- g'(x)(x - t)]\, d\alpha(t) = 2g'(x)\, I_1 + 2I_2 \qquad (7)$$

for $x \in D$. Expression I_1 can be directly evaluated by noticing that $K_n(x, t)(x - t)$ is polynomial in t of degree n so that

$$I_1=\int_\Delta K_n(x,t)(x-t)\frac{\partial}{\partial x}k_{n+1}(x,t)\,d\alpha(t)$$

$$-p_n'(x)\int_\Delta K_n(x,t)(x-t)\,p_n(t)\,d\alpha(t)$$

$$=\frac{\partial}{\partial t}\left[K_n(x,t)(x-t)\right]|_{t=x}-p_n'(x)\int_\Delta K_n(x,t)(x-t)\,p_n(t)\,d\alpha(t).$$

Here the first term on the right-hand side equals $K_n(x)$. Applying the Christoffel–Darboux formula to $K_n(x,t)$ we obtain

$$\int_\Delta K_n(x,t)(x-t)\,p_n(t)\,d\alpha(t)=\frac{\Gamma_{n-1}}{\Gamma_n}P_n(x)\int_\Delta P_{n-1}(t)\,p_n(t)\,d\alpha(t)$$

$$-\frac{\Gamma_{n-1}}{\Gamma_n}P_{n-1}(x)\int_\Delta P_n(t)\,p_n(t)\,d\alpha(t)=-\frac{\Gamma_{n-1}}{\gamma_n}P_{n-1}(x).$$

Hence

$$I_1=K_n(x)+\frac{\Gamma_{n-1}}{\gamma_n}P_{n-1}(x)\,p_n'(x). \tag{8}$$

In order to estimate I_2 in (7), let us define M by

$$M=\sup_{x\in D,t\in\Delta}|g(x)-g(t)-g'(x)(x-t)|\,(x-t)^{-2}.$$

Then by Schwarz' inequality

$$(I_2)^2\leqslant M^2\int_\Delta K(x,t)^2(x-t)^2\,d\alpha(t)\cdot\int_\Delta\left[\frac{\partial}{\partial x}k_n(x,t)(x-t)\right]^2 d\alpha(t)$$

$$\leqslant\frac{M^2}{\min_{t\in\Delta}g(t)}\int_\Delta K_n(x,t)^2(x-t)^2\,d\alpha_g(t)$$

$$\cdot\int_\Delta\left[\frac{\partial}{\partial x}k_n(x,t)(x-t)\right]^2 d\alpha(t). \tag{9}$$

By the Christoffel–Darboux formula the first integral on the right-hand side of (9) equals

$$\frac{\Gamma_{n-1}^2}{\Gamma_n^2}\int_\Delta\left[P_n(x)\,P_{n-1}(t)-P_{n-1}(x)\,P_n(t)\right]^2 d\alpha_g(t)$$

$$=\frac{\Gamma_{n-1}^2}{\Gamma_n^2}\left[P_n(x)^2+P_{n-1}(x)^2\right] \tag{10}$$

whereas the second integral on the right-hand side of (9) can be evaluated by noticing that as a consequence of the Christoffel–Darboux formula we have

$$\frac{\partial}{\partial x} k_n(x, t)(x-t) = \frac{\gamma_{n-1}}{\gamma_n} [p_n'(x) p_{n-1}(t) - p_{n-1}'(x) p_n(t)] - k_n(x, t)$$

so that by orthogonality relations

$$\begin{aligned}\int_\Delta \left[\frac{\partial}{\partial x} k_n(x, t)(x-t)\right]^2 d\alpha(t) &= \frac{\gamma_{n-1}^2}{\gamma_n^2} p_n'(x)^2 + \frac{\gamma_{n-1}^2}{\gamma_n^2} p_{n-1}'(x)^2 + k_n(x)\\ &\quad - 2\frac{\gamma_{n-1}}{\gamma_n} p_n'(x) p_{n-1}(x) = \frac{\gamma_{n-1}^2}{\gamma_n^2} p_n'(x)^2 + \frac{\gamma_{n-1}^2}{\gamma_n^2} p_{n-1}'(x)^2\\ &\quad - \frac{\gamma_{n-1}}{\gamma_n} [p_n'(x) p_{n-1}(x) + p_n(x) p_{n-1}'(x)]\\ &\leqslant \left[\frac{\gamma_{n-1}^2}{\gamma_n^2} + \frac{1}{2}\right] [p_n'(x)^2 + p_{n-1}'(x)^2] + \frac{\gamma_{n-1}^2}{2\gamma_n^2} [p_n(x)^2 + p_{n-1}(x)^2].\end{aligned} \tag{11}$$

From (9), (10) and (11) we obtain

$$\begin{aligned} I_2^2 \leqslant &\frac{M^2}{\min_{t\in\Delta} g(t)} \frac{\Gamma_{n-1}^2}{\Gamma_n^2} [P_n(x)^2 + P_{n-1}(x)^2]\\ &\times \left\{ \left[\frac{\gamma_{n-1}^2}{\gamma_n^2} + \frac{1}{2}\right] [p_n'(x)^2 + p_{n-1}'(x)^2]\right.\\ &\left. + \frac{\gamma_{n-1}^2}{2\gamma_n^2} \left[p_n(x)^2 + p_{n-1}(x)^2\right]\right\}. \end{aligned} \tag{12}$$

Writing the recurrence formula for p_n as

$$\frac{\gamma_{n-1}}{\gamma_n} p_n(x) = (x - b_n) p_{n-1}(x) - \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2}(x)$$

where

$$b_n = \int_\Delta t p_{n-1}(t)^2 \, d\alpha(t),$$

we can see that

$$\frac{\gamma_{n-1}^2}{\gamma_n^2}[p_n(x)^2+p_{n-1}(x)^2] \leqslant \left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\}$$

$$\cdot\,[p_{n-1}(x)^2+p_{n-2}(x)^2]$$

$$\leqslant \left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\} k_n(x)$$

$$= \left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\} \frac{\gamma_{n-1}}{\gamma_n}$$

$$\times\,[p_n'(x)\,p_{n-1}(x)-p_{n-1}'(x)\,p_n(x)]$$

$$\leqslant \left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\} \frac{\gamma_{n-1}}{\gamma_n}$$

$$\times\,[p_n(x)^2+p_{n-1}(x)^2]^{1/2}[p_n'(x)^2+p_{n-1}'(x)^2]^{1/2}.$$

Since the zeros of p_n and p_{n-1} interlace, $p_n(x)^2+p_{n-1}(x)^2>0$, and we get

$$\frac{\gamma_{n-1}^2}{\gamma_n^2}[p_n(x)^2+p_{n-1}(x)^2]$$

$$\leqslant \left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\}^2 [p_n'(x)^2+p_{n-1}'(x)^2]$$

which substituted in (12) yields

$$|I_2| \leqslant \frac{M}{\sqrt{\min_{t\in\Delta} g(t)}}\frac{\Gamma_{n-1}}{\Gamma_n}$$

$$\cdot\left[\frac{\gamma_{n-1}^2}{\gamma_n^2}+\frac{1}{2}+\frac{1}{2}\left\{2\left[|x-b_n|+\frac{\gamma_{n-2}}{\gamma_{n-1}}\right]^2+\frac{\gamma_{n-1}^2}{\gamma_n^2}\right\}^2\right]^{1/2}$$

$$\cdot\,[|P_n(x)|+|P_{n-1}(x)|][|p_n'(x)|+|p_{n-1}'(x)|]. \tag{13}$$

Because g is positive on Δ, we have [7, p. 26]

$$K_n(x) \leqslant [\min_{t\in\Delta} g(t)]^{-1}\,k_n(x).$$

Thus by the Christoffel–Darboux formula

$$K_n(x) \leqslant [\min_{t\in\Delta} g(t)]^{-1} \frac{\gamma_{n-1}}{\gamma_n} [p_n'(x)\, p_{n-1}(x) - p_{n-1}'(x)\, p_n(x)]$$

$$\leqslant [\min_{t\in\Delta} g(t)]^{-1} \frac{\gamma_{n-1}}{\gamma_n} [|p_{n-1}(x)|$$

$$+ |p_n(x)|][|p_{n-1}'(x)| + p_n'(x)|]. \tag{14}$$

Similarly,

$$\Gamma_{n-1} \leqslant [\min_{t\in\Delta} g(t)]^{-1/2} \gamma_{n-1} \tag{15}$$

[14, p. 39]. Hence from (8), (14) and (15) we obtain

$$|I_1| \leqslant \frac{\gamma_{n-1}}{\gamma_n} \left\{ \frac{1}{\min_{t\in\Delta} g(t)} + \frac{1}{\sqrt{\min_{t\in\Delta} g(t)}} \right\}$$

$$\times [|P_{n-1}(x)| + |P_n(x)| + |p_{n-1}(x)|$$

$$+ |p_n(x)|][|p_{n-1}'(x)| + |p_n'(x)|]. \tag{16}$$

Substituting inequalities (13) and (16) in (7), and noticing that $\gamma_{n-1}/\gamma_n \leqslant |\Delta|/2$, $\Gamma_{n-1}/\Gamma_n \leqslant |\Delta|/2$, $|b_n| \leqslant |r| + |s|$ (if $\Delta = [r, s]$), we get

$$|g(x)\, K_n'(x) - k_n'(x)| \leqslant \text{const}[|P_{n-1}(x)| + |P_n(x)| + |p_{n-1}(x)|$$

$$+ |p_n(x)|][p_{n-1}'(x) + |p_n'(x)|]$$

uniformly for $x \in D$ and $n \in \mathbb{N}$. Now the lemma follows directly from Korous' theorem [14, p. 162] according to which if the support of $d\alpha$ is contained in Δ and

$$\sup_{\substack{x\in D\\ t\in D}} \left| \frac{g(x) - g(t)}{x - t} \right| < \infty, \qquad \min_{t\in\Delta} g(t) > 0$$

then

$$|P_n(x)| \leqslant \text{const}[|p_{n-1}(x)| + |p_n(x)|]$$

uniformly for $x \in D$ and $n \in \mathbb{N}$.

LEMMA 2. *Let* $w \in GJA$ *and let* $x_{kn}(w) = \cos \Theta_{kn}$ $(x_{0n} = 1,\ x_{n+1,n} = -1,$ $0 \leqslant \Theta_{kn} \leqslant \pi)$. *Then*

$$\Theta_{k+1,n} - \Theta_{kn} \sim \frac{1}{n} \tag{17}$$

uniformly for $0 \leqslant k \leqslant n$, $n \in \mathbb{N}$,

$$\lambda_n(w, x) \sim \begin{cases} \dfrac{1}{n} w(x) \sqrt{1-x^2}, & |x| \leqslant 1 - n^{-2}, \\ \dfrac{1}{n^2} w(1-n^{-2}), & 1 - n^{-2} \leqslant x \leqslant 1, \\ \dfrac{1}{n^2} w(-1+n^{-2}), & -1 \leqslant x \leqslant -1 + n^{-2}, \end{cases} \tag{18}$$

uniformly for $n \in \mathbb{N}$ *and*

$$\lambda_{kn}(w) \sim \frac{1}{n} w(x_{kn}(w)) \sqrt{1 - x_{kn}(w)^2} \tag{19}$$

uniformly for $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$. *If* $w \in GJB$ *then*

$$|p_n(w, x)| \leqslant \text{const} \begin{cases} [w(x)\sqrt{1-x^2}]^{-1/2}, & |x| \leqslant 1 - n^{-2}, \\ \sqrt{n}\, [w(1-n^{-2})]^{-1/2}, & 1 - n^{-2} \leqslant x \leqslant 1, \\ \sqrt{n}\, [w(-1+n^{-2})]^{-1/2}, & -1 \leqslant x \leqslant -1 + n^{-2}, \end{cases} \tag{20}$$

uniformly for $n \in \mathbb{N}$,

$$|p_n(w, x)| \sim \begin{cases} \sqrt{n}\, [w(1-n^{-2})]^{-1/2}, & 1 + x_{1n}(w) \leqslant 2x \leqslant 2, \\ \sqrt{n}\, [w(-1+n^{-2})]^{-1/2}, & -2 \leqslant 2x \leqslant -1 + x_{nn}(w), \end{cases} \tag{21}$$

uniformly for $n \in \mathbb{N}$ *and*

$$|p_{n-1}(w, x_{kn}(w))| \sim w(x_{kn}(w))^{-1/2} (1 - x_{kn}(w)^2)^{1/4} \tag{22}$$

uniformly for $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$. *If* $w \in GJC$ *then*

$$|\lambda_n'(w, x)| \leqslant \text{const} \begin{cases} \dfrac{1}{n} w(x)(1-x^2)^{-1/2}, & |x| \leqslant 1 - n^{-2}, \\ w(1-n^{-2}), & 1 - n^{-2} \leqslant x \leqslant 1, \\ w(-1+n^{-2}), & -1 \leqslant x \leqslant -1 + n^{-2}, \end{cases} \tag{23}$$

uniformly for $n \in \mathbb{N}$ *and*

$$|\lambda_n'(w, x_{kn}(w))| \leqslant \text{const}\, \frac{1}{n} w(x_{kn}(w))(1 - x_{kn}(w)^2)^{-1/2} \tag{24}$$

uniformly for $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$. *If* $w \in GJA$ *then*

$$|\lambda_n'(w, x_{kn}(w))| \leqslant \text{const } w(x_{kn}(w)) \tag{25}$$

uniformly for $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$.

Proof. Equation (17) was proved in [9, p. 367], (18) in [13, p. 336] and (19) follows directly from (17) and (18). Inequality (20) was proved in [2, p. 226] whereas (21) follows from Theorem 9.33 in [12, p. 171], (17) and (18). Equation (22) was proved in [12, p. 170]. The estimate (24) follows from (17) and (23). Now let us prove (23). Since $w \in GJC$ we can write w as $w\,dx = g\,d\alpha$ where $d\alpha$ is a Jacobi distribution and g is a positive continuous function in $[-1, 1]$ with $g' \in \text{Lip } 1$. Hence setting $\Delta = D = [-1, 1]$, we see that g satisfies the conditions of Lemma 1. Consequently,

$$|K_n'(w, x, x)| \leqslant \text{const} \left[|K_n'(d\alpha, x, x)| + \sum_{j=n-2}^{n} |p_j(d\alpha, x)| \cdot \sum_{j=n-1}^{n} |p_j'(d\alpha, x)| \right], \qquad |x| \leqslant 1. \tag{26}$$

The polynomials $p_j'(d\alpha, x)$ are themselves Jacobi polynomials with distribution $(1 - x^2)\,d\alpha$. Hence the two sums in (26) can be estimated by (20). By proceeding this way we obtain

$$\sum_{j=n-2}^{n} |p_j(d\alpha, x)| \sum_{j=n-1}^{n} |p_j'(d\alpha, x)| \leqslant \text{const} \begin{cases} nw(x)^{-1}(1 - x^2)^{-1}, & |x| \leqslant 1 - n^{-2}, \\ n^3 w(1 - n^{-2})^{-1}, & 1 - n^{-2} \leqslant x \leqslant 1, \\ n^3 w(-1 + n^{-2})^{-1}, & -1 \leqslant x \leqslant -1 + n^{-2}. \end{cases} \tag{27}$$

In order to estimate $K_n'(d\alpha, x, x)$ we notice that, in fact, it can be evaluated in a closed form as follows. By the Christoffel–Darboux formula

$$K_n(d\alpha, x, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \left[p_n'(d\alpha, x)\, p_{n-1}(d\alpha, x) - p_{n-1}'(d\alpha, x)\, p_n(d\alpha, x) \right]$$

so that

$$K_n'(d\alpha, x, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \left[p_n''(d\alpha, x)\, p_{n-1}(d\alpha, x) - p_{n-1}''(d\alpha, x)\, p_n(d\alpha, x) \right].$$

If a and b denote the parameters of the Jacobi distribution $d\alpha$ then $p_n(d\alpha)$ satisfies the differential equation [14, p. 60]

$$(1-x^2)\,Y'' = -\,n(n+a+b+1)\,Y + [a-b+(a+b+2)\,x]\,Y'.$$

Thus

$$K_n'(d\alpha, x, x) = \frac{a-b+(a+b+2)\,x}{1-x^2}\,K_n(d\alpha, x, x) - \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\,\frac{2n+a+b}{1-x^2}\,p_{n-1}(d\alpha, x)\,p_n(d\alpha, x).$$

Since $K_n = \lambda_n^{-1}$ we can apply (18) to estimate K_n, whereas p_n can be estimated by (20). By doing so we get

$$|K_n'(d\alpha, x, x)| \leqslant \text{const}\cdot n\cdot w(x)^{-1}(1-x^2)^{-3/2}, \qquad |x| \leqslant 1-n^{-2}. \tag{28}$$

When $1-n^{-2} \leqslant |x| \leqslant 1$ then we write

$$K_n'(d\alpha, x, x) = 2\sum_{j=0}^{n-1} p_j'(d\alpha, x)\,p_j(d\alpha, x)$$

and, since $p_j'(d\alpha)$ is also a Jacobi polynomial with distribution $(1-x^2)\,d\alpha$, we can use (20) to obtain

$$|K_n'(d\alpha, x, x)| \leqslant \text{const}\begin{cases} n^4 w(1-n^{-2})^{-1}, & 1-n^{-2} \leqslant x \leqslant 1,\\ n^4 w(-1+n^{-2})^{-1}, & -1 \leqslant x \leqslant -1+n^{-2}. \end{cases} \tag{29}$$

Comparing (27) and (28)–(29) we can conclude that the right-hand sides of (28) and (29) are larger than that of (27) so that (26) becomes

$$|K_n'(w, x, x)| \leqslant \text{const}\begin{cases} nw(x)^{-1}(1-x^2)^{-3/2}, & |x| \leqslant 1-n^{-2},\\ n^4 w(1-n^{-2})^{-1}, & 1-n^{-2} \leqslant x \leqslant 1,\\ n^4 w(-1+n^{-2})^{-1}, & -1 \leqslant x \leqslant -1+n^{-2}. \end{cases} \tag{30}$$

Observing that $K_n' = -\lambda_n'\lambda_n^{-2}$, (23) follows from (18) and (30). In order to prove (25), we write $w = gw^{(a,b)}$ so that by (18)

$$\lambda_n(w, x)^{-1} \leqslant \text{const}\; n(\sqrt{1-x}+n^{-1})^{-2a-1}(\sqrt{1+x}+n^{-1})^{-2b-1}, \qquad |x| \leqslant 1.$$

Applying a weighted version of Markov–Bernstein's inequality to the polynomial $\lambda_n(w)^{-1}$ [8, p. 51] (see also [12, p. 161]) we obtain

$$|\lambda_n'(w,x)|\,\lambda_n(w,x)^{-2} \leqslant \text{const} \cdot n^2(\sqrt{1-x}+n^{-1})^{-2a-2} \times (\sqrt{1+x}+n^{-1})^{-2b-2}, \qquad |x|\leqslant 1.$$

Substituting here $x = x_{kn}$ and using (17) and (19), inequality (25) follows immediately.

LEMMA 3. *Let $d\alpha$ be such that* $\operatorname{supp}(d\alpha)=[-1,1]$ *and* $\log \alpha'(\cos\theta) \in L^1$. *Then for every function f which is Riemann integrable in* $[-1,1]$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) f(x_{kn}(d\alpha)) \frac{p_{n-1}(d\alpha, x_{kn}(d\alpha))^2}{x - x_{kn}(d\alpha)} = \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{\frac{1+t}{1-t}}\,dt \tag{31}$$

uniformly for $1 + x_{1n}(d\alpha) \leqslant 2x \leqslant 2$ *and*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) f(x_{kn}(d\alpha)) \frac{p_{n-1}(d\alpha, x_{kn}(d\alpha))^2}{x - x_{kn}(d\alpha)} = -\frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{\frac{1+t}{1-t}}\,dt \tag{32}$$

uniformly for $-2 \leqslant 2x \leqslant -1 + x_{nn}(d\alpha)$.

Proof. In order to prove (31) we point out that for $x=1$ it has been proved in [12, p. 39]. For $1 + x_{1n}(d\alpha) \leqslant 2x \leqslant 2$ we will show that

$$\sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{1-x_{kn}} - \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{x-x_{kn}} \xrightarrow[n\to\infty]{} 0. \tag{33}$$

But

$$\left| \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{1-x_{kn}} - \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{x-x_{kn}} \right|$$
$$= (1-x) \left| \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{(1-x_{kn})(x-x_{kn})} \right|$$
$$\leqslant \sup_{|t|\leqslant 1} |f(t)| \cdot (1-x_{1n}) \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{(1-x_{kn})^2}, \qquad 1 + x_{1n} \leqslant 2x \leqslant 2.$$

Hence, if we prove that

$$\lim_{n\to\infty} (1-x_{1n}) \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{(1-x_{kn})^2} = 0 \tag{34}$$

then (33) holds and consequently so does (31). Let $\varepsilon \in (0, 1)$ be fixed. Then obviously

$$(1-x_{1n}) \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{(1-x_{kn})^2} \leqslant \frac{1-x_{1n}}{(1-\varepsilon)^2} + \sum_{x_{kn}>\varepsilon} \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{1-x_{kn}}$$

and applying (31) with $x = 1$ we obtain

$$\limsup_{n\to\infty} (1-x_{1n}) \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{(1-x_{kn})^2} \leqslant \frac{2}{\pi} \int_{\varepsilon}^{1} \sqrt{\frac{1+t}{1-t}}\, dt.$$

Letting here $\varepsilon \to 1$ we can see that (34) is satisfied and thus (31) holds uniformly for $1 + x_{1n} \leqslant 2x \leqslant 2$. The proof of (32) is analogous to the proof of (31).

LEMMA 4. *Let $w \in GJC$ and let $0 < \sigma < 1$. Then*

$$\sum_{k=1}^{n} l_{kn}(w, x)^2 \leqslant \text{const} \left[1 + \frac{\log n}{n} w(x)^{-1}(1-x^2)^{-1/2}\right] \tag{35}$$

and

$$\sum_{k=1}^{n} |x - x_{kn}(w)|\, l_{kn}(w, x)^2 \leqslant \text{const} \left[\frac{1}{\log n} + \frac{\log n}{n} w(x)^{-1}(1-x^2)^{-1/2}\right] \tag{36}$$

uniformly for $n \geqslant 2$ and $|x| \leqslant 1 - \sigma n^{-2}$.

Proof. Let $w = gw^{(a,b)}$ and let $0 \leqslant x \leqslant 1 - \sigma n^{-2}$. First we will consider the case when $a \geqslant -\frac{1}{2}$. We have

$$\sum_{k=1}^{n} l_{kn}(x)^2 = 1 - \sum_{k=1}^{n} (x - x_{kn}) \frac{\lambda_n'(x_{kn})}{\lambda_{kn}} l_{kn}(x)^2.$$

Thus by (19) and (24)

$$\sum_{k=1}^{n} l_{kn}(x)^2 \leqslant 1 + \text{const} \sum_{k=1}^{n} |x - x_k|\, (1-x_{kn}^2)^{-1}\, l_{kn}(x)^2. \tag{37}$$

If $x_{kn} \leqslant -\frac{1}{2}$ then by (3), (19) and (22)

$$|x - x_{kn}|\, (1-x_{kn}^2)^{-1}\, l_{kn}(x)^2 \leqslant \text{const} \frac{p_n(x)^2}{n} \lambda_{kn}$$

so that by the Gauss–Jacobi quadrature formula

$$\sum_{x_{kn} \leqslant -1/2} |x - x_{kn}| \, (1 - x_{kn}^2)^{-1} \, l_{kn}(x)^2 \leqslant \text{const} \frac{p_n(x)^2}{n}. \tag{38}$$

If $x_{kn} \geqslant -\frac{1}{2}$ then by (3), (19) and (22)

$$|x - x_{kn}| \, (1 - x_{kn}^2)^{-1} \, l_{kn}(x)^2 \leqslant \text{const} \frac{|p_n(x)|}{n} (1 - x_{kn})^{a/2 - 1/4} \, |l_{kn}(x)|.$$

Thus,

$$\begin{aligned} \sum_{x_{kn} \geqslant -1/2} & |x - x_{kn}| \, (1 - x_{kn}^2)^{-1} \, l_{kn}(x)^2 \\ & \leqslant \text{const} \frac{|p_n(x)|}{n} \sum_{k=1}^{n} (1 - x_{kn})^{a/2 - 1/4} \, |l_{kn}(x)| \\ & \leqslant \text{const} \frac{|p_n(x)|}{n} \left[\sum_{k=1}^{n} (1 - x_{kn})^{a - 1/2} \right]^{1/2} \left[\sum_{k=1}^{n} l_{kn}(x)^2 \right]^{1/2}. \end{aligned} \tag{39}$$

Combining (37), (38) and (39) we can conclude that

$$\begin{aligned} \sum_{k=1}^{n} l_{kn}(x)^2 \leqslant 1 &+ \text{const} \frac{p_n(x)^2}{n} \\ &+ \text{const} \frac{|p_n(x)|}{n} \left[\sum_{k=1}^{n} (1 - x_{kn})^{a - 1/2} \right]^{1/2} \cdot \left[\sum_{k=1}^{n} l_{kn}(x)^2 \right]^{1/2}. \end{aligned}$$

Solving this inequality for $\sum l_{kn}^2$ we obtain

$$\sum_{k=1}^{n} l_{kn}(x)^2 \leqslant \text{const} \left[1 + \frac{p_n(x)^2}{n} + \frac{p_n(x)^2}{n^2} \sum_{k=1}^{n} (1 - x_{kn})^{a - 1/2} \right]. \tag{40}$$

Applying (17) we see that

$$\sum_{k=1}^{n} (1 - x_{kn})^{a - 1/2} \sim \sum_{k=1}^{n} \left(\frac{k}{n} \right)^{2a - 1} \sim \begin{cases} n^{1 - 2a}, & -\frac{1}{2} \leqslant a < 0, \\ n \log n, & a = 0, \\ n, & a > 0. \end{cases} \tag{41}$$

If $a \geqslant 0$ then $0 \leqslant x \leqslant 1 - \sigma n^{-2}$ formula (35) follows directly from (20), (40) and (41). If $-\frac{1}{2} \leqslant a < 0$ then by (40) and (41)

$$\sum_{k=1}^{n} l_{kn}(x)^2 \leqslant \text{const} \left[1 + \frac{1}{n^{1 + 2a}} p_n(x)^2 \right]$$

so that by (20)

$$\max_{0\leqslant x\leqslant 1} \sum_{k=1}^{n} l_{kn}(x)^2 \leqslant \text{const} \tag{42}$$

which again implies (35) for $0 \leqslant x \leqslant 1 - \sigma n^{-2}$. Now let $-1 < a < -\frac{1}{2}$. Then by (19)

$$\begin{aligned}\sum_{x_{kn}\geqslant -1/2} l_{kn}(x)^2 &\leqslant \frac{\text{const}}{n} \sum_{x_{kn}\geqslant -1/2} w(x_{kn})\sqrt{1-x_{kn}^2}\,\frac{l_{kn}(x)^2}{\lambda_{kn}} \\ &\leqslant \frac{\text{const}}{n} \sum_{k=1}^{n} (1-x_{kn})^{a+1/2}\,\frac{l_{kn}(x)^2}{\lambda_{kn}}.\end{aligned} \tag{43}$$

Now we apply Hölder's inequality with $p = -(a+\frac{1}{2})^{-1}$ and $q = (a+\frac{3}{2})^{-1}$ to the right-hand side of (43). We get

$$\begin{aligned}\sum_{x_{kn}\geqslant -1/2} l_{kn}(x)^2 \leqslant \frac{\text{const}}{n} &\left[\sum_{k=1}^{n} (1-x_{kn})^{-1}\,\frac{l_{kn}(x)^2}{\lambda_{kn}}\right]^{-(a+1/2)} \\ &\cdot \left[\sum_{k=1}^{n} \frac{l_{kn}(x)^2}{\lambda_{kn}}\right]^{a+3/2}.\end{aligned}$$

By a result of G. Freud [6, p. 251]

$$\sum_{k=1}^{n} (1-x_{kn})^{-1}\,\frac{l_{kn}(x)^2}{\lambda_{kn}} = \lambda_n(\tilde{w}, x)^{-1}$$

where $\tilde{w}(x) = (1-x)\,w(x)$ so that $\tilde{w} \in GJC$. Also, we have

$$\sum_{k=1}^{n} \frac{l_{kn}(x)^2}{\lambda_{kn}} = \lambda_n(w, x)^{-1}$$

[7, p. 25]. Hence

$$\sum_{x_{kn}\geqslant -1/2} l_{kn}(x)^2 \leqslant \frac{\text{const}}{n}\,\lambda_n(\tilde{w}, x)^{a+1/2}\,\lambda_n(w, x)^{-(a+3/2)}$$

and by (18)

$$\sum_{x_{kn}\geqslant -1/2} l_{kn}(x)^2 \leqslant \text{const}, \qquad 0 \leqslant x \leqslant 1. \tag{44}$$

If $x_{kn} \leqslant -\frac{1}{2}$ then by (4), (19), (20) and (22),

$$l_{kn}(x)^2 \leqslant \text{const}\, n^{-2}$$

so that

$$\sum_{x_{kn} \leqslant -1/2} l_{kn}(x)^2 \leqslant \text{const}\,\frac{1}{n}, \qquad 0 \leqslant x \leqslant 1. \tag{45}$$

From (44) and (45) inequality (42) follows also for $-1 < a < -\frac{1}{2}$. Hence (35) holds for every $w \in GJC$ when $0 \leqslant x \leqslant 1 - \sigma n^{-2}$. Applying (35) with $w^*(x) = w(-x)$, we see that it also holds for $-1 + \sigma n^{-2} \leqslant x \leqslant 0$. In order to prove (36), we write

$$\begin{aligned}\sum_{k=1}^{n} |x - x_{kn}|\, l_{kn}(x)^2 &= \sum_{|x - x_{kn}| < 1/\log n} |x - x_{kn}|\, l_{kn}(x)^2 \\ &\quad + \sum_{|x - x_{kn}| \geqslant 1/\log n} l_{kn}(x)^2 \\ &\leqslant \frac{1}{\log n} \sum_{k=1}^{n} l_{kn}(x)^2 + \log n \sum_{k=1}^{n} (x - x_{kn})^2\, l_{kn}(x)^2.\end{aligned}$$

It follows from (3), (19) and (22) that

$$(x - x_{kn})^2\, l_{kn}(x)^2 \leqslant \text{const} \cdot \frac{p_n(x)^2}{n^2}.$$

Hence

$$\sum_{k=1}^{n} |x - x_{kn}|\, l_{kn}(x)^2 \leqslant \frac{1}{\log n} \sum_{k=1}^{n} l_{kn}(x)^2 + \text{const}\,\frac{\log n}{n}\, p_n(x)^2$$

and (36) follows from (20) and (35).

LEMMA 5. *Let $w \in GJA$ and let w be continuous in $(-1, 1)$. Then for every fixed nonnegative integer m there exist two polynomials R_1 and R_2 of the form $R_1(x) = (1 - x^2)^m\, \Pi_1(x)$ and $R_2(x) = (1 - x^2)^m\, \Pi_2(x)$ such that Π_1 and Π_2 are polynomials and*

$$\liminf_{n \to \infty} n p_n(w, x)^{-2}\, |R_1(x) - H_n(w, R_1, x) \geqslant 1 \tag{46}$$

uniformly for $1 + x_{1n}(w) \leqslant 2x \leqslant 2$ and

$$\liminf_{n \to \infty} n p_n(w, x)^{-2}\, |R_2(x) - H_n(w, R_2, x)| \geqslant 1 \tag{47}$$

uniformly for $-2 \leqslant 2x \leqslant -1 + x_{nn}(w)$.

Proof. If R is a polynomial and $2n > \deg R$ then

$$R(x) - H_n(w, R, x) = \sum_{k=1}^{n} R'(x_{kn}(w))(x - x_{kn}(w))\, l_{kn}(w, x)^2.$$

Omitting the unnecessary parameters, we can write this as

$$\begin{aligned} &np_n(x)^{-2}[R(x) - H_n(w, R, x)] \\ &\quad = \delta_n \sum_{k=1}^{n} (n\lambda_{kn})\, R'(x_{kn})\, \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \end{aligned} \tag{48}$$

for $1 + x_{1n} \leqslant 2x \leqslant 2$ where $\delta_n = \gamma_{n-1}^2/\gamma_n^2$. Let ρ be defined by

$$\rho = \sup_{\substack{n \in \mathbb{N} \\ 1 \leqslant j \leqslant n}} n\lambda_{jn} w(x_{jn})^{-1}(1 - x_{jn}^2)^{-1/2}.$$

Then by (19) ρ is finite. If ε fixed and $0 < \varepsilon < 1$ then by (48)

$$\begin{aligned} &np_n(x)^{-2}\, |R(x) - H_n(w, R, x)| \\ &\quad \geqslant \delta_n \left\{ \left| \sum_{|x_{kn}| \leqslant \varepsilon} (n\lambda_{kn})\, R'(x_{kn}) \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \right| \right. \\ &\qquad \left. - \rho \sum_{|x_{kn}| > \varepsilon} w(x_{kn}) \sqrt{1 - x_{kn}^2}\, |R'(x_{kn})| \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \right\} \end{aligned}$$

for $1 + x_{1n} \leqslant 2x \leqslant 2$. By Theorem 6.2.22 in [12, p. 85]

$$\lim_{n \to \infty} [n\lambda_{jn} - \pi w(x_{jn}) \sqrt{1 - x_{jn}^2}\,] = 0$$

uniformly for $|x_{jn}| \leqslant \varepsilon$. Thus, if $\delta > 0$ is fixed and $n \geqslant n_0(\delta)$, then

$$\begin{aligned} &np_n(x)^{-2}\, |R(x) - H_n(w, R, x)| \\ &\quad \geqslant \delta_n \left\{ \left| \sum_{|x_{kn}| \leqslant \varepsilon} \pi w(w_{kn}) \sqrt{1 - x_{kn}^2}\, R'(x_{kn})\, \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \right| \right. \\ &\qquad - \delta \sum_{|x_{kn}| \leqslant \varepsilon} |R'(x_{kn})|\, \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \\ &\qquad \left. - \rho \sum_{|x_{kn}| < \varepsilon} w(x_{kn}) \sqrt{1 - x_{kn}^2}\, |R'(x_{kn})|\, \lambda_{kn} \frac{p_{n-1}(x_{kn})^2}{x - x_{kn}} \right\} \end{aligned}$$

for $1 + x_{1n} \leqslant 2x \leqslant 2$. By (2), $\delta_n \to \frac{1}{4}$ as $n \to \infty$. Assuming without loss of generality that $m \geqslant 2$, we can apply Lemma 3 to conclude that all three sums inside the brackets converge uniformly for $1+x_{1n} \leqslant 2x \leqslant 2$ as $n \to \infty$ and passing to the limits we obtain

$$\begin{aligned}\liminf_{n\to\infty} np_n(x)^{-2}\,|R(x)-H_n(w,R,x)| \\ \geqslant \frac{1}{2}\left|\int_{-\varepsilon}^{\varepsilon} R'(t)\,w(t)(1+t)\,dt\right| - \frac{\delta}{2\pi}\int_{-\varepsilon}^{\varepsilon} |R'(t)|\sqrt{\frac{1+t}{1-t}}\,dt \\ -\frac{\rho}{2\pi}\int_{\varepsilon \leqslant |t| \leqslant 1} |R'(t)|\,w(t)(1+t)\,dt\end{aligned}$$

uniformly for $1+x_{1n} \leqslant 2x \leqslant 2$. Letting $\varepsilon \to 1$ and $\delta \to 0$ we get

$$\liminf_{n\to\infty} np_n(x)^{-2}\,|R(x)-H_n(w,R,x)| \geqslant \frac{1}{2}\left|\int_{-1}^{1} |R'(t)\,w(t)(1+t)\,dt\right| \quad (49)$$

uniformly for $1+x_{1n} \leqslant 2x \leqslant 2$. If we can show the existence of a polynomial R_1 of the form $R_1(x)=(1-x^2)^m\,\Pi_1(x)$ such that

$$\int_{-1}^{1} R_1'(t)\,w(t)(1+t)\,dt = 2 \quad (50)$$

then (46) will follow from (49). If (50) holds for no R_1 then for every $j=0, 1, 2,...$

$$\begin{aligned}0 &= \int_{-1}^{1}\left[\frac{t^{j+1}-1}{j+1}(1-t^2)^m\right]' w(t)(1+t)\,dt \\ &= \int_{-1}^{1} t^j(1-t^2)^m\,w(t)(1+t)\,dt \\ &\quad + \int_{-1}^{1}\frac{t^{j+1}-1}{j+1}\,[(1-t^2)^m]'\,w(t)(1+t)\,dt \\ &= \int_{-1}^{1} t^j\left\{(1-t^2)^m\,w(t)(1+t) - \int_{-1}^{t}[(1-s^2)^m]'\,w(s)(1+s)\,ds\right\}dt\end{aligned}$$

so that

$$(1-t^2)^m\,w(t)(1+t) = \int_{-1}^{t}[(1-s^2)^m]'\,w(s)(1+s)\,ds$$

for $-1<t<1$, and hence $w(t)(1+t)$ is absolutely continuous on every closed subinterval of $(-1,1)$ and also $[w(t)(1+t)]'=0$ for $-1<t<1$. Consequently, $w(t)=\text{const}(1+t)^{-1} \notin L^1$. This contradiction proves the existence of R_1 satisfying (50). The second part (47) of the lemma can be proved by analogous reasoning.

The following lemma is a partial case of a more general theorem proved in [11, Theorem 1].

LEMMA 6. *Let $w \in GJB$ and let ω be a Jacobi weight. If $1 < p < \infty$ and $(1-x^2)^{-1/4} w^{\pm 1/2} \in L^p_\omega$ then for every bounded function f in $[-1, 1]$*

$$\|L_n(w, F)\|_{\omega,p} \leqslant \text{const} \|f\|_\infty, \qquad n \in \mathbb{N},$$

where $F(x) = f(x)\sqrt{w(x)} \cdot (1-x^2)^{-1/4}$.

4. Main Results

THEOREM 1. *Let Δ be an interval. If $\operatorname{supp} w \subset \Delta$ then $\lim_{n\to\infty} H_n(w, R) = R$ in L^1_w for every polynomial R.*

Proof. If R is a polynomial and $\deg R < 2n$ then

$$\begin{aligned} R(x) - H_n(w, R, x) &= \sum_{k=1}^{n} R'(x_{kn}(w))(x - x_{kn}(w))\, l_{kn}(w, x)^2 \\ &= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_n(w, x) \sum_{k=1}^{n} R'(x_{kn}(w))\, p_{n-1}(w, x_{kn}(w))\, \lambda_{kn}(w)\, l_{kn}(w, x) \\ &= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_n(w, x)\, L_n(w, R' p_{n-1}(w)\, \lambda_n(w), x). \end{aligned}$$

Hence by Schwarz' inequality

$$\begin{aligned} \|R - H_n(w, R)\|_{w,1} &\leqslant \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \|p_n(w)\|_{w,2} \|L_n(w, R' p_{n-1}(w)\, \lambda_n(w))\|_{w,2} \\ &= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \left[\sum_{k=1}^{n} R'(x_{kn}(w))^2\, p_{n-1}(w, x_{kn}(w))^2\, \lambda_{kn}(w)^3 \right]^{1/2} \\ &\leqslant \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \|R'\|_\infty \|\lambda_n(w)\|_\infty \left[\sum_{k=1}^{n} p_{n-1}(w, x_{kn}(w))^2\, \lambda_{kn}(w) \right]^{1/2} \\ &= \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \|R'\|_\infty \|\lambda_n(w)\|_\infty \end{aligned} \tag{51}$$

where the ∞-norm is taken over Δ. Since $\operatorname{supp} w \subset \Delta$, $\gamma_{n-1}/\gamma_n \leqslant \frac{1}{2}|\Delta|$. Also, since w is an absolutely continuous weight distribution with compact support, $\lambda_n(w, x) \downarrow 0$ as $n \to \infty$ for every real x [7, p. 63] so that by Dini's theorem $\|\lambda_n(w)\|_\infty \to 0$ as $n \to \infty$. Thus, the theorem follows from (51).

THEOREM 2. *Let v be a Jacobi weight, not necessarily integrable. Suppose that f is Riemann integrable in $[-\delta, \delta]$ for every $0 < \delta < 1$ and*

$$|f(x)| \leqslant \text{const}\, v(x), \qquad -1 \leqslant x \leqslant 1. \tag{52}$$

Then $\lim_{n\to\infty} H_n(w,f) = f$ in L^1_w if either of the following two conditions are satisfied:

(i) *$w \in GJA$ and $wv\sqrt{1-x^2} \in L^\infty$ in $[-1, 1]$,*
(ii) *$w \in GJB$ and $wv/\sqrt[4]{1-x^2} \in L^1$ in $[-1, 1]$.*

Proof. Let $\varepsilon > 0$ be fixed. Let us choose a Jacobi weight u with positive integer parameters such that $u(x) \leqslant v(x)$ for $-1 \leqslant x \leqslant 1$ and $\omega \equiv w^2u^2\sqrt{1-x^2} \in L^1$ in $[-1, 1]$. Then $fu^{-1} \in L^2_\omega$ if either (i) or (ii) is satisfied. Thus we can pick a polynomial S (e.g., a partial sum of the orthogonal Fourier expansion of fu^{-1} in $\{p_n(\omega)\}$) such that

$$\|fu^{-1} - S\|_{\omega,2} \leqslant \varepsilon. \tag{53}$$

If $R = uS$ then R is a polynomial and by Theorem 1 $\lim_{n\to\infty} H_n(w, R) = R$ in L^1_w. Since by (53)

$$\begin{aligned}\|f-R\|_{w,1} &= \|f-uS\|_{w,1} = \|(fu^{-1}-S)\,u\|_{w,1}\\ &= \|(fu^{-1}-S)\sqrt{\omega}\,(1-x^2)^{-1/4}\|_1 \leqslant \|(fu^{-1}-S)\|_{\omega,2}\,\|(1-x^2)^{-1/4}\|_2\\ &= \sqrt{\pi}\,\|(fu^{-1}-S)\|_{\omega,2} \leqslant \sqrt{\pi}\,\varepsilon,\end{aligned} \tag{54}$$

we have

$$\begin{aligned}\limsup_{n\to\infty} \|f-H_n(w,f)\|_{w,1} &\leqslant \limsup_{n\to\infty} \|f-R\|_{w,1}\\ &\quad + \limsup_{n\to\infty} \|R-H_n(w,R)\|_{w,1} + \limsup_{n\to\infty} \|H_n(w,f-R)\|_{w,1}\\ &\leqslant \sqrt{\pi}\,\varepsilon + \limsup_{n\to\infty} \|H_n(w,f-R)\|_{w,1}\end{aligned} \tag{55}$$

if either (i) or (ii) is satisfied. Our next goal is to estimate the second term on the right side of (55). For this purpose let $\phi = f - R$. Then, because of the choice of u,

$$|\phi(x)| \leqslant cv(x), \qquad -1 \leqslant x \leqslant 1, \tag{56}$$

where the constant c depends on the constant in (52) and $\|S\|_\infty$ in $[-1, 1]$. Applying (5) and omitting the unnecessary parameters, we obtain

$$H_n(w,\phi,x) = \sum_{l=1}^{n} \phi(x_{kn})\, l_{kn}(x)^2 + \frac{\gamma_{n-1}}{\gamma_n} p_n(x)\, L_n(w, \phi\lambda'_n p_{n-1}, x)$$

so that by Schwarz' inequality

$$\|H_n(w,\phi)\|_{w,1} \leqslant \left\| \sum_{k=1}^{n} \phi(x_{kn})\, l_{kn}^2 \right\|_{w,1} + \|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2} \tag{57}$$

since $\gamma_{n-1}/\gamma_n \leqslant 1$ and $\|p_n\|_{w,2} = 1$. We will estimate the two terms on the right side of (57) separately. By the triangle inequality

$$\left\| \sum_{k=1}^{n} \phi(x_{kn})\, l_{kn}^2 \right\|_{w,1} \leqslant \sum_{k=1}^{n} |\phi(x_{kn})|\, \lambda_{kn}, \tag{58}$$

since $\|l_{kn}^2\|_{w,1} = \lambda_{kn}$. If $0 < \delta < 1$ is fixed then

$$\lim_{n\to\infty} \sum_{|x_{kn}| \leqslant \delta} |\phi(x_{kn})|\, \lambda_{kn} = \int_{-\delta}^{\delta} |\phi(t)|\, w(t)\, dt$$

because ϕ is Riemann integrable on $[-\delta, \delta]$ (see [7, p. 89]). Thus by (54)

$$\lim_{n\to\infty} \sum_{|x_{kn}| \leqslant \delta} |\phi(x_{kn})|\, \lambda_{kn} \leqslant \sqrt{\pi}\, \varepsilon. \tag{59}$$

Furthermore, by (56),

$$\sum_{|x_{kn}| > \delta} |\phi(x_{kn})|\, \lambda_{kn} \leqslant c \sum_{|x_{kn}| > \delta} v(x_{kn})\, \lambda_{kn}. \tag{60}$$

Let k_1 be the largest index k for which $x_{k+1,n} > \delta$, and let k_2 be the smallest index k for which $x_{k-1,n} < -\delta$. Applying (19) we obtain

$$v(x_{kn})\, \lambda_{kn} \sim \frac{1}{n}\, v(x_{kn})\, w(x_{kn}) \sqrt{1 - x_{kn}^2}, \qquad k = 1, k_1, k_2, n,$$

if either (i) or (ii) is satisfied. Also, under the same conditions, $v(x)\, w(x) \leqslant \text{const} \cdot (1 - x^2)^{-3/4}$. Hence by (17)

$$v(x_{kn})\, \lambda_{kn} \leqslant \text{const}\, n^{-1/2}, \qquad k = 1, k_1, k_2, n. \tag{61}$$

It follows from (17) that

$$v(t) \sim v(x_{kn}), \qquad x_{k+1,n} \leqslant t \leqslant x_{k-1,n}, \tag{62}$$

for $k = 2, 3, \ldots, n-1$. Another inequality we will need is the Markov–Stieltjes inequality [7, p. 29], according to which

$$\lambda_{kn} \leqslant \int_{x_{k+1,n}}^{x_{k-1,n}} w(t)\, dt, \qquad k = 2, 3, \ldots, n-1. \tag{63}$$

Combining (60), (61), (62) and (63) we obtain

$$\sum_{|x_{kn}|>\delta} |\phi(x_{kn})|\, \lambda_{kn} \leqslant \text{const} \left[\int_{\delta \leqslant |t| \leqslant 1} v(t)\, w(t)\, dt + n^{-1/2}\right]$$

so that

$$\limsup_{n\to\infty} \sum_{|x_{kn}|>\delta} |\phi(x_{kn})|\, \lambda_{kn} \leqslant \text{const} \int_{\delta \leqslant |t| \leqslant 1} v(t)\, w(t)\, dt. \tag{64}$$

If either (i) or (ii) holds then $vw \in L^1$ and then letting $\delta \to 1$ in (64), it follows from (58), (59) and (64) that

$$\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} \phi(x_{kn})\, l_{kn}^2 \right\|_{w,1} \leqslant \sqrt{\pi}\, \varepsilon. \tag{65}$$

Now we turn to estimating the second term on the right-hand side of (57). By the Gauss–Jacobi quadrature formula

$$\|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2}^2 = \sum_{k=1}^{n} \phi(x_{kn})^2\, \lambda_n'(x_{kn})^2\, p_{n-1}(x_{kn})^2\, \lambda_{kn}.$$

Noting that in both cases (i) and (ii) $w \in GJA$, we can apply (25) to obtain

$$\|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2}^2 \leqslant \text{const} \sum_{k=1}^{n} \phi(x_{kn})^2\, w(x_{kn})^2\, p_{n-1}(x_{kn})^2\, \lambda_{kn}$$

so that if we write $w = gw^{(a,b)}$ then

$$\|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2}^2 \leqslant \text{const} \sum_{k=1}^{n} \phi(x_{kn})^2\, w^{(2a,2b)}(x_{kn})^2\, p_{n-1}(x_{kn})^2\, \lambda_{kn}. \tag{66}$$

Now if (i) holds then we rewrite (66) as

$$\|L_n(w, \phi\lambda_n' p_{n-1}\|_{w,2}^2 \leqslant \text{const}\, \frac{1}{2} \sum_{k=1}^{n} [\phi(x_{kn})\, w^{(a,b)}(x_{kn}) \sqrt{1-x_{kn}^2}\,]^2 \times \left(\frac{1}{1+x_{kn}} + \frac{1}{1-x_{kn}}\right) p_{n-1}(x_{kn})^2\, \lambda_{kn}. \tag{67}$$

By the conditions of the theorem and (56) the function $\phi w^{(a,b)} \sqrt{1-x^2}$ is Riemann integrable in $[-1, 1]$. Hence by Lemma 3, the right side of (67) converges as $n \to \infty$, and evaluating its limit by (31) and (32) we obtain

$$\limsup_{n\to\infty} \|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2}^2 \leqslant \text{const}\, \frac{2}{\pi}\int_{-1}^{1} \phi(t)^2\, w^{(2a,2b)}(t)\sqrt{1-t^2}\, dt. \tag{68}$$

Since $\phi^2 w^{(2a,2b)}\sqrt{1-t^2} = g^{-1}(fu^{-1}-S)^2\,\omega$, we can conclude from (53) and (68) that

$$\limsup_{n\to\infty} \|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2} \leqslant \text{const}\,\varepsilon \tag{69}$$

if condition (i) of the theorem is satisfied. If condition (ii) holds then by (22) and (66)

$$\|L_n(w, \phi\lambda_n' p_{n-1})\|_{w,2}^2 \leqslant \text{const} \sum_{k=1}^{n} \phi(x_{kn})^2\, w^{(a,b)}(x_{kn})\sqrt{1-x_{kn}^2}\,\lambda_{kn}. \tag{70}$$

If $0<\delta<1$ is fixed then by the convergence of the Gauss–Jacobi quadrature process [7, p. 89] and by (53)

$$\lim_{n\to\infty} \sum_{|x_{kn}|\leqslant\delta} \phi(x_{kn})^2\, w^{(a,b)}(x_{kn})\sqrt{1-x_{kn}^2}\,\lambda_{kn} = \int_{-\delta}^{\delta} \phi(t)^2\, w^{(a,b)}(t)\sqrt{1-t^2}\, w(t)\, dt \leqslant \text{const}\,\varepsilon^2. \tag{71}$$

Taking (56) into consideration we obtain

$$\sum_{|x_{kn}|>\delta} \phi(x_{kn})^2\, w^{(a,b)}(x_{kn})\sqrt{1-x_{kn}^2}\,\lambda_{kn} \leqslant c^2 \sum_{|x_{kn}|>\delta} v(x_{kn})^2\, w^{(a,b)}(x_{kn})\sqrt{1-x_{kn}^2}\,\lambda_{kn}.$$

If (ii) is satisfied then $vw^{(a,b)}/\sqrt[4]{1-x^2}\in L^1$ so that there exists a number $\tau > -\frac{1}{4}$ such that

$$v(x)\, w^{(a,b)}(x)\sqrt{1-x^2} \leqslant (1-x^2)^{\tau}, \qquad -1\leqslant x\leqslant 1.$$

Hence

$$\sum_{|x_{kn}|>\delta} \phi(x_{kn})^2\, w^{(a,b)}(x_{kn})\sqrt{1-x_{kn}^2}\,\lambda_{kn} \leqslant c^2 \sum_{|x_{kn}|>\delta} v(x_{kn})(1-x_{kn}^2)^{\tau}\,\lambda_{kn},$$

$\tau > -\frac{1}{4}$. The right side of this inequality can be estimated in the same way as the right side of (60). By proceeding this way we obtain

$$\limsup_{n\to\infty} \sum_{|x_{kn}|>\delta} \phi(x_{kn})^2 w^{(a,b)}(x_{kn}) \sqrt{1-x_{kn}^2}\, \lambda_{kn}$$

$$\leqslant \text{const} \int_{\delta\leqslant|t|\leqslant 1} v(t)\, w(t)(1-t^2)^{-1/4}\, dt. \tag{72}$$

Combining (70), (71) and (72), and letting $\delta\to 1$, we see that (69) holds also if (ii) is satisfied. Finally, subsituting (57), (65) and (69) into (55) we see that

$$\limsup_{n\to\infty} \|f-H_n(w,f)\|_{w,1} \leqslant \text{const}\ \varepsilon$$

if either condition (i) or (ii) holds. Since $\varepsilon > 0$ is arbitrary, the theorem follows.

THEOREM 3. *Let u be a Jacobi weight function and let f be defined by $f(x)\equiv x$. Then $w^{-1}\in L_u^p$ if either of the following two conditions are satisfied*:

(i) $w\in GJA$, $1<p<\infty$ *and* $\lim_{n\to\infty} H_n(w,f)=f$ *in* L_u^p,

(ii) $w\in GJB$, $0<p<\infty$ *and* $\lim_{k\to\infty} H_{n_k}(w,f)=f$ *in* L_u^p *for some subsequence* $\{n_k\}$.

Proof. If $f(x)\equiv x$ then

$$f(x)-H_n(w,f,x)=\sum_{k=1}^{n} (x-x_{kn}(w))\, l_{kn}(w,x)^2$$

and, omitting the unnecessary parameters, we write it as

$$f(x)-H_n(w,f,x)=\frac{\gamma_{n-1}^2}{\gamma_n^2} p_n(x)^2 \sum_{k=1}^{n} \lambda_{kn}^2 \frac{p_{n-1}(x_{kn})^2}{x-x_{kn}}$$

so that by Cauchy's inequality

$$|f(x)-H_n(w,f,x)| \geqslant \frac{\gamma_{n-1}^2}{2\gamma_n^2} p_n(x)^2 \cdot \frac{1}{n}\left[\sum_{k=1}^{n} \lambda_{kn} |p_{n-1}(x_{kn})|\right]^2 \tag{73}$$

for $x_{1n}\leqslant x\leqslant 1$. Let T_n denote the Chebyshev polynomial of degree n. Then

$|T_n(t)| \leqslant 1$, $-1 \leqslant t \leqslant 1$ and $T_n(t) = 2^{n-1}t^n + \cdots$. Thus by the Christoffel–Darboux quadrature formula,

$$\sum_{k=1}^{n} \lambda_{kn} \,|\, p_{n-1}(x_{kn})| \geqslant \sum_{k=1}^{n} \lambda_{kn} T_{n-1}(x_{kn}) p_{n-1}(x_{kn})$$

$$= \int_{-1}^{1} T_{n-1}(t) p_{n-1}(t) \, w(t) \, dt = \frac{2^{n-2}}{\gamma_{n-1}}$$

and applying this inequality to (73), we obtain

$$|f(x) - H_n(w, f, x)| \geqslant \frac{2^{2n-5}}{\gamma_n^2} \frac{p_n(x)^2}{n}, \qquad x_{1n} \leqslant x \leqslant 1. \tag{74}$$

If condition (i) is satisfied then

$$\lim_{n \to \infty} \int_{(1+x_{1n})/2}^{1} |f(x) - H_n(w, f, x)|^p \, u(x) \, dx = 0$$

so that by (2) and (17)

$$\lim_{n \to \infty} \int_{(1+x_{1n})/2}^{1} |\, p_n(x)^2 \sqrt{1-x^2} \,|^p \, u(x) \, dx = 0. \tag{75}$$

But for $1 < p < \infty$

$$\left(\int_{(1+x_{1n})/2}^{1} \left| \frac{\sqrt{1-x^2}}{n\lambda_n(w, x)} \right|^p u(x) \, dx \right)^{1/p}$$

$$\leqslant \frac{1}{n} \sum_{k=0}^{n-1} \left(\int_{(1+x_{1n})/2}^{1} |\, p_k(x)^2 \sqrt{1-x^2} \,|^p \, u(x) \, dx \right)^{1/p}$$

$$\leqslant \frac{1}{n} \sum_{k=0}^{n-1} \left(\int_{(1+x_{1k})/2}^{1} |\, p_k(x)^2 \sqrt{1-x^2} \,|^p \, u(x) \, dx \right)^{1/p}$$

and, consequently,

$$\lim_{n \to \infty} \int_{(1+x_{1n})/2}^{1} \left| \frac{\sqrt{1-x^2}}{n\lambda_n(w, x)} \right|^p u(x) \, dx = 0. \tag{76}$$

If we write $w = gw^{(a,b)}$ and $u = w^{(c,d)}$ then by (17) and (18) formula (76) is equivalent to

$$\lim_{n \to \infty} n^{2(ap-c-1)} = 0$$

so that $c - ap > -1$ which means that $w^{-p}u$ is integrable in $[0, 1]$. The integrability of $w^{-p}u$ in $[-1,0]$ can be shown by similar arguments. If condition (ii) is satisfied then

$$\lim_{k\to\infty}\int_{(1+x_{1n_k})/2}^{1}|f(x)-H_{n_k}(w,f,x)|^p\,u(x)\,dx=0$$

and by (2) and (17)

$$\lim_{k\to\infty}\int_{(1+x_{1n_k})/2}^{1}\left|\frac{p_{n_k}(x)^2}{n_k}\right|^p u(x)\,dx=0$$

so that by (21)

$$\lim_{k\to\infty} w(1-n_k^{-2})^{-p}\int_{(1+x_{1n_k})/2}^{1}u(x)\,dx=0. \tag{77}$$

If we again write $w=gw^{(a,b)}$ and $u=w^{(c,d)}$ then by (17) formula (77) is equivalent to

$$\lim_{k\to\infty} n_k^{2(ap-c-1)}=0$$

and thus $w^{-p}u$ is integrable in $[0,1]$, whereas the integrability of $w^{-p}u$ in $[-1,0]$ may be proved using analogous arguments.

THEOREM 4. *Let $w\in GJB$, $p>0$, and let u and v be two Jacobi weight functions. We have* $\lim_{n\to\infty}H_n(w,R)=R$ *in* L_u^p *for every polynomial* R *satisfying the condition*

$$|R(x)|\leqslant \text{const}\, v(x),\qquad -1\leqslant x\leqslant 1, \tag{78}$$

if and only if $w^{-1}\in L_u^p$, *in particular, p is independent of v.*

Proof. If R is a polynomial and the degree of R is less than $2n$, then

$$R(x)-H_n(w,R,x)=\sum_{k=1}^{n}R'(x_{kn}(w))(x-x_{kn}(w))\,l_{kn}(w,x)^2.$$

By Theorem 6.3.14 in [12, p. 113] for every $0<p<\infty$ and Jacobi weight u there exists a constant $\sigma=\sigma(p,u)>0$ such that for every polynomial P of degree at most $2n$

$$\int_{-1}^{1}|P(t)|^p\,u(t)\,dt\leqslant 2\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}|P(t)|^p\,u(t)\,dt. \tag{79}$$

Thus

$$\int_{-1}^{1} |R(x)-H_n(w,R,x)|^p\, u(x)\,dx \leqslant 2\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |R(x)-H_n(w,R,x)|^p\, u(x)\,dx$$

$$\leqslant 2 \max_{|t|\leqslant 1} |R'(t)|^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left[\sum_{k=1}^{n} |x-x_{kn}|\, l_{kn}(w,x)^2\right]^p u(x)\,dx$$

so that by (36), if $n \geqslant 2$, then

$$\int_{-1}^{1} |R(x)-H_n(w,R,x)|^p\, u(x)\,dx \leqslant \text{const}\cdot \max_{|t|\leqslant 1} |R'(t)|^p$$

$$\times\left[(\log n)^{-p} + \left(\frac{\log n}{n}\right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (w(x)\sqrt{1-x^2})^{-p}\, u(x)\,dx\right]. \quad (80)$$

If $w \in GJB$, u is a Jacobi weight and $w^{-p}u \in L^1$ then there exists a number $q > 1$ such that $w^{-p}u \in L^q$ and $q^{-1} > 1 - p/2$. Applying Hölder's inequality with this q to the integral on the right side of (80) we obtain

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (1-x^2)^{-p/2}\, w(x)^{-p}\, u(x)\,dx$$

$$\leqslant \left[\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (1-x^2)^{-pq/2(q-1)}\,dx\right]^{(q-1)/q} \cdot \|w^{-p}u\|_q. \quad (81)$$

Simple computation shows that

$$\left[\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} (1-x2)^{-pq/2(q-1)}\,dx\right]^{(q-1)/q}$$

$$\leqslant \left[\frac{4(q-1)}{pq-2(q-1)}\right]^{(q-1)/q} \sigma^{(-p/2+(q-1)/q)} n^p n^{(1-q)/(2q)}$$

$$\leqslant \text{const}\; n^p n^{(1-q)/2q} \quad (82)$$

with the constant depending on q and σ. Combining estimates (80), (81) and (82) we get

$$\|R - H_n(w,R)\|_{u,p} \leqslant \text{const}\, \|R'\|_\infty [(\log n)^{-1} + \|w^{-p}u\|_q^{1/p} \log n \cdot n^{(1-q)/2pq}]$$

for $n \geqslant 2$. Since $q > 1$, we have $\lim_{n\to\infty} H_n(w,R) = R$ in L_u^p whenever $w^{-1} \in L_u^p$. If $\lim_{n\to\infty} H_n(w,R) = R$ in L_u^p for every polynomial R satisfying (78) then we choose $m \in \mathbb{N}$ so that $(1-x^2)^m \leqslant v(x)$ for $-1 \leqslant x \leqslant 1$, and applying Lemma 5 we pick a polynomial R_1 such that R_1 satisfies (78) with $R = R_1$ and (46) holds. Then

$$n^{-p} \int_{(1+x_{1n})/2}^{1} p_n(x)^{2p}\, u(x)\, dx \leqslant 2 \int_{(1+x_{1n})/2}^{1} |R_1(x) - H_n(w, R_1, x)|^p\, u(x)\, dx$$

for $n \geqslant n_1$ and thus

$$\lim_{n\to\infty} n^{-p} \int_{(1+x_{1n})/2}^{1} |p_n(x)|^{2p}\, u(x)\, dx = 0.$$

By (21) this is equivalent to

$$\lim_{n\to\infty} w(1-n^{-2})^{-p} \int_{(1+x_{1n})/2}^{1} u(x)\, dx = 0. \tag{83}$$

Writing $w = gw^{(a,b)}$ and $u = w^{(c,d)}$ and applying (17) we can see that (83) is equivalent to $c - ap > -1$ so that $w^{-p}u$ is integrable in $[0, 1]$. Applying the second part of Lemma 5, the integrability of $w^{-p}u$ in $[-1, 0]$ can be proved in a similar way.

THEOREM 5. *Let $w \in GJC$, $p > 0$, and let u and v be two Jacobi weight functions. Then* (i) $\lim_{n\to\infty} H_n(w,f) = f$ *in L_u^p for every function f which is continuous in $[-1, 1]$ and satisfies $|f(x)| \leqslant \text{const}\, v(x)$ for $-1 \leqslant x \leqslant 1$ if and only if* (ii) $w^{-1} \in L_u^p$. *In particular, p is independent of the rate at which f vanishes at ± 1.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 4. Also, if f is a polynomial then by Theorem 4(ii) $\Rightarrow \lim_{n\to\infty} H_n(w,f) = f$. Hence, it remains to show that if (ii) holds then

$$\|H_n(w,f)\|_{u,p} \leqslant \text{const}\, \|f\|_\infty \tag{84}$$

where the ∞-norm is taken over $[-1, 1]$. First let $p > 1$. Applying (5) and omitting the unnecessary parameters, we can write

$$H_n(w,f,x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x)^2 + \frac{\gamma_{n-1}}{\gamma_n} p_n(x)\, L_n(w, f\lambda_n' p_{n-1}, x) \tag{85}$$

Applying (79) with $P = H_n$ we get

$$\|H_n(w,f)\|_{u,p}^p \leqslant 2 \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |H_n(w,f,x)|^p\, u(x)\, dx \tag{86}$$

for some $\sigma = \sigma(p, u) > 0$. It follows from (35) that

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left| \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x)^2 \right|^p u(x)\, dx$$
$$\leqslant \text{const}\, \|f\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} [1 + n^{-1} \log nw(x)^{-1}(1-x^2)^{-1/2}]^p\, u(x)\, dx$$
$$\leqslant \text{const}\, \|f\|_\infty^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} [1 + w(x)^{-1}(1 + |\log(1-x^2)|)]^p\, u(x)\, dx.$$

Since $w \in GJC$ and u is a Jacobi weight, if $w^{-1} \in L_u^p$ then also $w^{-1} \log(1-x^2) \in L_u^p$. Consequently,

$$\int_{-1+\sigma^{-2}}^{1-\sigma n^{-2}} \left| \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x)^2 \right|^p u(x)\, dx \leqslant \text{const}\, \|f\|_\infty^p \times \|1 + (1 + |\log(1-x^2)|)\, w^{-1}\|_{u,p}. \tag{87}$$

Now we turn to estimating the second term on the right side of (85). By (22) and (24)

$$|\lambda_n'(x_{kn})\, p_{n-1}(x_{kn})| \leqslant \text{const}\, \frac{1}{n}\, w(x_{kn})^{1/2}(1-x_{kn}^2)^{-1/4}$$

uniformly for $1 \leqslant k \leqslant n$ and $n \in \mathbb{N}$. Hence, if $w = gw^{(a,b)}$, then we can write

$$f(x_{kn})\, \lambda_n'(x_{kn})\, p_{n-1}(x_{kn}) = f_n(x_{kn})\, n^{-1} w^{(a/2-1/4,\, b/2-1/4)}(x_{kn}) \tag{88}$$

where

$$\|f_n\|_\infty \leqslant \text{const}\, \|f\|_\infty. \tag{89}$$

Thus by (20) and (88)

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |p_n(x)\, L_n(w, f\lambda_n' p_{n-1}, x)|^p\, u(x)\, dx$$
$$\leqslant \text{const} \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |L_n(w, f_n w^{(a/2-1/4,\, b/2-1/4)}, x)|^p \times (1-x^2)^{p/4}\, w(x)^{-p/2}\, u(x)\, dx$$
$$\leqslant \text{const}\, \|L_n(w, f_n w^{(a/2-1/4,\, b/2-1/4)})\|_{\omega,p}^p$$

where $\omega = w^{(-pa/2+p/4,\, -pb/2+p/4)}\, u$. It is clear that $w^{-1} \in L_u^p$ is equivalent to

$(1-x^{2-1/4}w^{-1/2}\in L^p_\omega$, and also $u\in L^1$ means that $(1-x^2)^{-1/4}w^{1/2}\in L^p_\omega$. Hence, applying Lemma 6 and (89) we obtain

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} |p_n(x)\,L_n(w,f\lambda_n'\,p_{n-1},x)|^p\,u(x)\,dx \leqslant \text{const}\,\|f\|_\infty^p. \tag{90}$$

Since $\gamma_{n-1}/\gamma_n \leqslant 1$, inequality (84) follows from (85), (86), (87) and (90). If $0<p\leqslant 1$, then we proceed as follows. If $w=gw^{(a,b)}$ then we set $\alpha=\max(0,a)$, $\beta=\max(0,b)$ and define $\tilde{u}$ by $\tilde{u}=u(w^{(\alpha,\beta)})^{2-p}$. Then obviously $\tilde{u}\in L^1$ since $u\in L^1$ and $w^{-1}\in L^2_{\tilde{u}}$ since $w^{-1}\in L^p_u$. Thus, by (84),

$$\|H_n(w,f)\|_{\tilde{u},2} \leqslant \text{const}\,\|f\|_\infty. \tag{91}$$

But by Hölder's inequality

$$\begin{aligned}\|H_n(w,f)\|_{u,p} &= \|\,|H_n(w,f))|^p\,u\|_1^{1/p}\\ &= \|\,|H_n(w,f)\,\sqrt{\tilde{u}}\,|^p\,[w^{(-\alpha p,-\beta p)}u]^{(2-p)/2}\|_1^{1/p}\\ &\leqslant \|H_n(w,f)\|_{\tilde{u},2}\,\|w^{(-\alpha,-\beta)}\|_{u,p}^{(2-p)/2}\\ &\leqslant \text{const}\,\|H_n(w,f)\|_{\tilde{u},2}\,\|w^{-1}\|_{u,p}^{(2-p)/2}\end{aligned}$$

so that by (91), inequality (84) holds also for $0<p\leqslant 1$ whenever $w^{-1}\in L^p_u$. Hence, the proof of the theorem has been completed.

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