Journal of Approximation Theory **119**, 18–40 (2002)
doi:10.1006/jath.2002.3708

Approximation of *Weak-to-Norm Continuous Mappings

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Communicated by Zeev Ditzian

Received October 30, 2000; accepted in revised form June 26, 2002

The purpose of this paper is to study the approximation of vector-valued mappings defined on a subset of a normed space. We investigate Korovkin-type conditions useful to recognize if a given sequence of linear operators is a so-called approximation process. First, we give a sufficient condition for this sequence to approximate the class of bounded, uniformly continuous functions. Then we present some sufficient and necessary conditions guaranteeing the approximation within the class of unbounded, *weak-to-norm continuous mappings. We also derive some estimates of the rate of convergence. We apply concrete approximation processes to derive representation formulae for semigroups of bounded linear operators. © 2002 Elsevier Science (USA)

Key Words: Korovkin test; linear approximation process; vector-valued mappings; representation formula.

1. INTRODUCTION

Korovkin's well-known result [2] states that if $(L_n)_{n\geqslant 1}$ is a sequence of positive linear operators on $\mathscr{C}([\![a,b]\!])$ then $||L_n(f)-f||_\infty \to 0$ for every $f\in\mathscr{C}([\![a,b]\!])$, provided the same is true for the following *test functions*: $f(u)=1,u,u^2$. Shisha and Mond [8] present a quantitative version of Korovkin's theorem, containing some estimates of the rate of convergence of $||L_n(f)-f||$ in terms of the corresponding rate of convergence computed for the test functions. Many authors have contributed to understanding the possible enlargement of the domain of approximation operators, in particular to include classes of unbounded functions. Ditzian [1] deals with continuous real-valued functions, defined on a closed and unbounded subset of the real line, which satisfy the growth condition $|f(u)|\leqslant M_f(1+u^2)\mu(u)$ with $\mu\geqslant 1$. He estimates the rate of the approximation in terms of the rate of

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0021-9045/02 $35.00

convergence for the test functions $1, u, u^2$ and $(u-t)^2\mu(u)$. Shaw and Yeh [7] study the case of functions defined on an open interval $]\!]a, b[\![$ of $\mathbb{R}$ and satisfying $|f(u)| = O(g_a(u))(u \to a^+)$ and $|f(u)| = O(g_b(u))(u \to b^-)$ (for some suitable convex functions g_a and g_b). The test functions determining the convergence rates are now the following: $1, u, u^2, g_a$ and g_b. Shaw [6] considers continuous functions on $\mathbb{R}^m$ with a prescribed growth at infinity. More precisely, he treats operators L_n defined by means of measures: $L_n(f)(t) = \int f(u)\, \mathrm{d}\mu_{n,t}(u)$, and the following classes of functions f. The first class consists of those real-valued functions whose growth is controlled by a convex function g. The second admissible class contains functions of the form $T(u)x$, where x belongs to a Banach space E, and $T(u)$ is a linear continuous operator from E into itself such that $T(u)$ is bounded on bounded subsets of $\mathbb{R}^m$ and $||T(u)|| \leqslant Mg(u)$. Many authors have also studied the case of vector-valued mappings defined on a compact Hausdorff space X see, e.g. [3–5]. The former studies the convergence of a net of quasi-positive linear operators to an operator T, that can be the identity on $\mathscr{C}(X; E)$. Actually, in [4] the value space E is a Dedekind complete normed vector lattice with normal unit order and, in [3] E is a normed linear space. Always in the setting of compactness of X, Prolla studies the approximation processes for the identity on $\mathscr{C}(X; E)$ by monotonically regular operators (that is the operators that are S-regular with S positive, see Section 2). Moreover, he gives a rate of approximation when X is a compact subset of a normed space and the process is made of dominated operators.

The purpose of this article is to give a generalization of the above results for classes of mappings defined on a convex subset of a vector space taking their values into a normed space. The paper is organized as follows.

In Section 2 we introduce the notation and definitions used in the sequel.

Replacing the previous assumption on the positivity of the operators L_n by the concept of the so-called dominated operators we proceed to find Korovkin-type conditions, as described in Section 3. We also derive there a Korovkin-type theorem on the approximation process within the class of bounded and uniformly continuous functions defined on a convex set, and find an estimate of the rate of convergence. In the end of the section we deduce a Korovkin-type theorem for *weak-to-norm continuous maps on bounded sets.

Section 4 deals with the case of unbounded functions. With X being a *weakly closed or open convex subset of a dual space $Y = Z'$, we present a Korovkin-type theorem for *weak-to-norm continuous maps on X, whose growth is controlled by a convex function. Under the additional assumption of the dimension of Y to be finite, we establish some new estimates of the rate of convergence. Theorems 4.1 and 4.2 generalize the corresponding results in [6, 7]. The main result in [1] is an easy consequence of our Theorem 4.2 under the additional requirement that the control function $(1+t^2)\mu(t)$ is strictly convex. Moreover, Theorem 4.2 extends the results of [6], providing them with estimates of the corresponding rate of convergence.

In the last section we apply developed theorems to same approximation process and improve a result of Shaw [6] on representation of semigroups.

2. NOTATION AND PRELIMINARY DEFINITIONS

In this work Y, Z and E will denote real or complex normed spaces, with their norms denoted by same symbol $||\cdot||$. As usual, Z' stands for the dual space of Z and $\pi(Z)$ stands for the dual space of Z' with *weak topology $\sigma(Z', Z)$, so Z is reflexive if and only if $\pi(Z) = Z''$. If $\phi \in \pi(Z)$, and X is a nonempty subset of Z', then by $\phi_{|X}$ we mean the restriction of ϕ to X.

We will often address to the following two functional spaces: $\mathscr{F}(X; E)$ and $\mathscr{B}(X; E)$ that are, respectively, the vector space of all mappings $F : X \to E$ and its subspace containing only the bounded mappings. The latter space is normed by the uniform norm $||\cdot||_X$

$$||F||_X := \sup_{u \in X} ||F(u)||.$$

For F belonging to the former space, $||F|| : X \to \mathbb{R}$ denotes the real-valued function $||F||(u) := ||F(u)||$.

With the usual symbol $\mathscr{C}(X; E)$ we denote the subspace of $\mathscr{F}(X; E)$ consisting of all continuous mappings.

Fix $g : X \to \mathbb{R}$ a strictly positive function. Then $\mathscr{C}(X; E, g)$ denotes the subspace of all mappings $F \in \mathscr{C}(X; E)$ such that $||F(u)|| \leqslant M_g(u)$ for every $u \in X$ and some constant $M > 0$, depending only on F. Finally, $UCB(X; E)$ is the subspace of all mappings of $\mathscr{C}(X; E)$ which are uniformly continuous and bounded.

In case $E = \mathbb{R}$ we abbreviate the above notation, writing $\mathscr{C}(X, g)$ instead of $\mathscr{C}(X; \mathbb{R}, g), \mathscr{F}(X)$ instead of $\mathscr{F}(X; \mathbb{R})$ and so on.

We also adopt the following notation: if $c \in E$, then, we shall denote again by c the constant mapping $F(u) = c$ $(u \in X)$.

If $f \in \mathscr{F}(X)$ and $x \in E, f \otimes x$ denotes the mapping of $\mathscr{F}(X; E)$ defined by $(f \otimes x)(u) := f(u)x$ $(u \in X)$.

For $t \in Y$, define $\psi_t : X \to \mathbb{R}$ by the formula $\psi_t(u) := ||u - t||$. Observe that if $\psi_{t_0}^2 \in \mathscr{C}(X, g)$, for some $t_0 \in Y$, then the same holds for every $t \in Y$.

Definition 2.1. Let Z be normed space, Y its dual space and $X \subset Y = Z'$. We say that $F : X \to E$ is **weak-to-norm continuous* if it is continuous from X equipped with the *weak topology $\sigma(Y, Z)$ in Y, into E with the norm topology. By $\mathscr{K}(X; E)$ we denote the space of all

*weak-to-norm continuous mappings from X into E. We set $\mathscr{K}(X;E,g) := \mathscr{K}(X;E) \cap \mathscr{C}(X;E,g)$.

We remark that every *weak-to-norm continuous mapping is in particular continuous and maps *weakly closed and bounded subsets of X in compact subsets of E. Moreover, if the dimension of Y is finite, then obviously $\mathscr{K}(X;E) = \mathscr{C}(X;E)$.

For $F \in UCB(X;E)$, as usual, we denote with $\omega(F,\cdot)$ its modulus of continuity,

$$\omega(F,h) := \sup\{||F(u) - F(t)|| \mid t,u \in X, ||t-u|| \leqslant h\} \qquad (h > 0).$$

The following definitions are based on the analogous ones in [5].

Definition 2.2. Let $L: D(L) \to \mathscr{F}(X;E)$ and $S: D(S) \to \mathscr{F}(X)$ be linear operators defined on some subspace $D(L)$ and $D(S)$ of $\mathscr{C}(X;E)$ and $\mathscr{C}(X)$, respectively. We say that

(a) L is *dominated* by S if $||F|| \in D(S)$, and

$$||L(F)(t)|| \leqslant S(||F||)(t)$$

for all $F \in D(L)$ and $t \in X$;

(b) L is *S-regular* if $f \otimes x \in D(L)$ and

$$L(f \otimes x) = S(f) \otimes x$$

for all $f \in D(S)$ and $x \in E$;

(c) L *preserves the constants* if $c \in D(L)$ and $L(c)(t) = c$, for all $c \in E$ and $t \in X$.

Below we present some examples of dominated and regular operators.

Example 2.1(Interpolation Operators). Let $\mathscr{L}(E)$ be the Banach algebra of the continuous linear operators on E and I be an index set. For every $i \in I$ fix a point $t_i \in X$ and an application $\Phi_i \in \mathscr{C}(X;\mathscr{L}(E))$, and set $\phi_i := ||\Phi_i||_{\mathscr{L}(E)} \in \mathscr{C}(X)$. We consider the operators $L: D(L) \to \mathscr{F}(X;E)$ and $S: D(S) \to \mathscr{F}(X)$, defined by

$$L(F)(t) := \sum_{i \in I} \Phi_i(t)(F(t_i)) \qquad \text{for any } F \in D(L),$$

$$S(f)(t) := \sum_{i \in I} \phi_i(t) f(t_i) \qquad \text{for any } f \in D(S)$$

for all $t \in X$. The domain $D(S)$ is the space of those functions $f \in \mathscr{C}(X)$ for which the family $(\phi_i(t)f(t_i))_{i \in I}$ is summable for all $t \in X$. The domain $D(L)$

is the space of the maps $F \in \mathscr{C}(X;E)$ such that $||F|| \in D(S)$. The inequality

$$||L(F)(t)|| \leqslant \sum_{i \in I} ||\Phi_i(t)||_{\mathscr{L}(E)} ||F(t_i)|| = S(||F||)(t)$$

implies that L is well defined on $D(L)$ and that L is dominated by S.

If for every $i \in I$, there exists $\psi_i \in \mathscr{C}(X)$ such that $\Phi_i(t)(v) = \psi_i(t)v$ $(t \in X, v \in E)$, then setting $\phi_i := \psi_i$, we have that L is S-regular. Moreover, if $\psi_i \geqslant 0$ then L is also dominated by S.

Example 2.2 (Integral Operators). Let $(E, ||\cdot||)$ be a Banach space and assume that for any $t \in X$, a positive finite measure $\mu_t : \mathscr{B}_X \to \mathbb{R}_+$ on the σ-algebra of all Borel subset of X is given. Define $D(L) := \mathscr{C}(X;E) \cap \bigcap_{t \in X} L^1(\mu_t;E)$, and $D(S) := \mathscr{C}(X) \cap \bigcap_{t \in X} L^1(\mu_t)$. Consider the operators $L : D(L) \to \mathscr{F}(X;E)$ and $S : D(S) \to \mathscr{F}(X)$ given by

$$L(F)(t) := \int_X F(u)\,\mathrm{d}\mu_t(u) \qquad \text{for any } F \in D(L),$$

$$S(g)(t) := \int_X g(u)\,\mathrm{d}\mu_t(u) \qquad \text{for any } g \in D(S)$$

for all $t \in X$. Trivially, L and S are linear and S is positive.

L is dominated *in natural way* by S:

$$||L(F)(t)|| = \left|\left|\int_X F(u)\,\mathrm{d}\mu_t(u)\right|\right| \leqslant \int_X ||F(u)||\,\mathrm{d}\mu_t(u) = S(||F||)(t).$$

Using the above estimate, we note that for an arbitrary $F \in \mathscr{C}(X;E)$, $S(||F||)$ is well defined provided $L(F)$ is defined.

By properties of the Bochner integral it is easy to verify that L is S-regular.

Moreover, we observe that L preserves the constants if and only if the measures μ_t have unit masses or, equivalently, $S(\mathbf{1})(t) = 1$ for all $t \in X$.

We will also make use of the following notation: if $\psi_t^2 \in D(S)$ then we write $\gamma^2(t) := S(\psi_t^2)(t)$.

3. A KOROVKIN-TYPE THEOREM FOR BOUNDED UNIFORMLY CONTINUOUS MAPPINGS BETWEEN NORMED SPACES

In this section we approximate vector valued, bounded and uniformly continuous mappings defined on a convex subset of a normed space.

THEOREM 3.1. *Let Y and E be normed spaces, X a convex subset of Y and $L_n : D(L_n) \to \mathscr{F}(X; E)$ a sequence of linear operator dominated by some positive linear operators $S_n : D(S_n) \to \mathscr{F}(X)$. We suppose that, for every $n \geqslant 1$ $UCB(X; E) \subset D(L_n)$, $UCB(X) \subset D(S_n)$* and *$\psi_t^2 \in D(S_n)$ for some (and hence for all) $t \in Y$. Then for each $F \in UCB(X; E)$, $t \in X$ and $\delta > 0$ one has*

$$||L_n(F)(t) - F(t)|| \leqslant ||L_n(F(t))(t) - F(t) + S_n(||F - F(t)||)(t)$$

$$\leqslant ||L_n(F(t))(t) - F(t)|| + \omega(F, \delta)[S_n(\mathbf{1})(t) + \delta^{-2}\gamma_n^2(t)], \quad (1)$$

where $\gamma_n^2(t) := S_n(\psi_t^2)(t)$.

Moreover, if L_n preserves the constants, then

$$||L_n(F)(t) - F(t)|| \leqslant \omega(F, \delta)[S_n(\mathbf{1})(t) + \delta^{-2}\gamma_n^2(t)].$$

In particular, taking $\delta = \gamma_n(t)$ we obtain

$$||L_n(F)(t) - F(t)|| \leqslant \omega(F, \gamma_n(t))[S_n(\mathbf{1})(t) + 1],$$

and if γ_n and $S_n(\mathbf{1})$ are bounded on $K \subset X$, then

$$||L_n(F) - F||_K \leqslant \omega(F, ||\gamma_n||_K)[||S_n(\mathbf{1})||_K + 1].$$

Proof. Fix $F \in UCB(X; E)$. For every $u \in X$ and $\delta > 0$, by the definition of $\omega(F, \cdot)$, we get the inequality

$$||F(u) - F(t)|| \leqslant \omega(F, ||t - u||) \leqslant (1 + \delta^{-2}||u - t||^2)\omega(F, \delta).$$

Applying the positive operator S_n we have

$$S_n(||F - F(t)||)(t) \leqslant \omega(F, \delta)(S_n(\mathbf{1})(t) + \delta^{-2}\gamma_n^2(t))$$

and

$$||L_n(F)(t) - F(t)|| \leqslant ||L_n(F - F(t))(t)|| + ||L_n(F(t))(t) - F(t)||$$

$$\leqslant S_n(||F - F(t)||)(t) + ||L_n(F(t))(t) - F(t)||,$$

as L_n is dominated by S_n. ∎

Note that the Theorem 3.1 yields the uniform convergence of $(L_n(F))_{n \geqslant 1}$ to F on those subsets of Y where the sequence $\gamma_n^2(t) = S_n(\psi_t^2)(t)$ converges to 0 uniformly.

When X is *weakly closed, convex and bounded subset of the dual space $Y = Z'$, then by Theorem 3.1 and the inclusions

$$\mathscr{K}(X;E) \subset UCB(X;E),$$

$$\psi_t^2 \in UCB(X) \subset D(S_n),$$

one obtains the following Korovkin-type theorem for *weak-to-norm continuous maps.

COROLLARY 3.1. *Let Z and E be normed spaces, Y the dual space of Z, X a *weakly closed, convex and bounded subset of the dual space $Y = Z'$, and $L_n : D(L_n) \to \mathscr{F}(X;E)$ a sequence of linear operator dominated by some positive linear operators $S_n : D(S_n) \to \mathscr{F}(X)$. We suppose that, for every $n \geqslant 1$ $UCB(X;E) \subset D(L_n)$, $UCB(X) \subset D(S_n)$ and set $\gamma_n^2(t) := S_n(\psi_t^2)(t)$. If for every $c \in E$ the following convergences hold:*

$$L_n(c) \to c \qquad \text{[resp. uniformly in } c \in E],$$

$$\gamma_n(t) \to 0 \qquad \text{[resp. uniformly in } t \in X],$$

then for each $F \in \mathscr{K}(X;E)$

$$L_n(F)(t) \to F(t) \qquad \text{[resp. uniformly on } X]$$

and moreover the inequalities of the Theorem 3.1 *hold.*

REMARK 3.1. In the setting of Corollary 3.1, X results to be a compact space with *weak topology and, in order to study the approximation process of the identity on $\mathscr{K}(X;E)$, the above result is slightly different from the analogue in [5, Theorem 1; 3, Corollary 5, Remark 4]. Prolla, dealing with dominated operators, requires that (X,d) is a metric space and the test functions depend on the metric d. In our case, of *weak-to-norm continuous mappings, this means to require the separability of Z and to use the metric d, given for every x, $y \in X$ by

$$d(x,y) := \sum_{n \geqslant 1} \frac{|\langle x-y, f_n\rangle|}{2^n},$$

where $f_n \in Z$, $\|f_n\| = 1$ and $(f_n)_{n \geqslant 1}$ is dense on the unitary sphere of Z. In Corollary 3.1 one does not need the separability of Z, and the test functions are based on the easier to use norm of the space. Nishishiraho tests the sequences of quasi-positive operators on a greater test set that in our context is

$$\{c\phi_{|X}^k | \phi \in \pi(Z), \quad k = 0,1,2 \text{ and } c \in E\}.$$

The cases of X closed and unbounded, or open are treated in the next section.

4. KOROVKIN-TYPE THEOREMS FOR UNBOUNDED MAPPINGS BETWEEN NORMED SPACES

As in the scalar case, where it is necessary to control the growth of the approximated functions (cf. [1]), for vector-valued mappings defined on subsets of Banach spaces we will have to assume appropriate conditions estimating the growth near the boundary of their domains of definition.

Since now we assume that $(Z, ||\cdot||)$ is a real normed space, Y its dual space, $(E, ||\cdot||)$ a normed space, and X a convex subset of $Y = Z'$, *weakly closed and unbounded or open. Fix $K \subset X$ *weakly closed and bounded and $g : X \to \mathbb{R}$ a function satisfying the following conditions:

(g_0) g is strictly positive, strictly convex, *weak-to-norm continuous on X and Fréchet differentiable on K such that $g' : K \to Y'$ is *weak-to-norm continuous and $g'(K) \subset \pi(Z)$.

We make the following growth hypothesis on g:

(g_1) for every $n \geqslant 1$ there exists a *weakly closed, convex and bounded subset B_n of X containing K such that for every $t \in X \setminus B_n$ one has $g(t) \geqslant n$ (or equivalently, for every $n \geqslant 1$ setting $B_n := g^{-1}([0, n])$ and requiring that $K \subset B_n$, B_n is bounded and $X \setminus B_n \neq \emptyset$). In case X is unbounded, we additionally require

$$\lim_{\substack{||t|| \to \infty \\ t \in X}} \frac{g(t)}{||t||} = +\infty. \tag{2}$$

Define the function $h : K \times X \to \mathbb{R}$ by setting

$$h(t, u) := g(u) - [g(t) + \langle g'(t), u - t \rangle]. \tag{3}$$

If hypothesis (g_0) holds, by the *weak-to-norm continuity of g' and the strict convexity of g, h is *weak-to-norm continuous and strictly positive for $u \neq t$.

In the remaining part of this section we state and prove two Korovkin-type theorems for *weak-to-norm continuous mappings with growth prescribed by g.

Theorem 4.1. *Let Z, Y, E, X, K, g and h be as above and there holds conditions* (g_0) *and* (g_1). *For each $n \geqslant 1$ let $L_n : D(L_n) \to \mathscr{F}(K; E)$ be a linear* *operator dominated by a linear positive operator* $S_n : D(S_n) \to \mathscr{F}(K)$, *with* $\mathscr{K}(X; E, g) \subset D(L_n)$ *and* $\mathscr{K}(X, g) \subset D(S_n)$.

Then for every $t \in K$ *the following statements are equivalent*:

(a) *For every* $c \in E$,

$$L_n(c)(t) \to c, \quad S_n(\mathbf{1})(t) \to 1 \quad \textit{and} \quad S_n(h(t, \cdot))(t) \to 0.$$

(b) *For every* $c \in E$, *and every continuous linear functional* $\phi \in \pi(Z)$,

$$L_n(c)(t) \to c, \quad S_n(\mathbf{1})(t) \to 1, \quad S_n(\phi_{|X}(t) \to \phi(t) \quad \textit{and} \quad S_n(g)(t) \to g(t).$$

(c) *For every* $F \in \mathscr{K}(X; E, g)$ *and* $f \in \mathscr{K}(X, g)$,

$$L_n(F)(t) \to F(t) \quad \textit{and} \quad S_n(f)(t) \to f(t).$$

If the convergences in (a) *are uniform with respect to* $t \in K$ *and with respect to* $c \in E$ *then* (c) *holds uniformly for* $t \in K$.

Moreover, *if the operators* L_n *are* S_n*-regular*, *then the above conditions are equivalent to one of the further statements*:

(d) *For every* $F \in \mathscr{K}(X; E, g)$,

$$L_n(F)(t) \to F(t).$$

(e) *For every* $F \in \mathscr{K}(X, g)$,

$$S_n(f)(t) \to f(t).$$

(f) *For every continuous linear functional* $\phi \in \pi(Z)$,

$$S_n(\mathbf{1})(t) \to 1, \quad S_n(\phi_{|X})(t) \to \phi(t) \quad \textit{and} \quad S_n(g)(t) \to g(t).$$

Remark 4.1. We remark that, if Y has finite dimension m, then denoting by $(pr_i)_{1 \leqslant i \leqslant m}$ the coordinate projections on Y, the above condition (b) reduces to the following one:

(b') *For every* $c \in E$, *and every* $i : 1 \dots m$,

$$L_n(c)(t) \to c, \quad S_n(\mathbf{1})(t) \to 1, \quad S_n(pr_i)(t) \to pr_i(t) \quad \textit{and} \quad S_n(g)(t) \to g(t),$$

and the convergences in (a) are uniform if and only if the same holds true for (b′).

This follows from the fact that $(pr_i)_{1 \leqslant i \leqslant m}$ forms a base of the space Y'.

Remark 4.2. If the space Z is reflexive, it is possible to simplify the hypotheses dropping the "$*$", substituting $\pi(Z)$ with Y' and forgetting of Z. So X will be a convex subset of the real reflexive Banach space Y, that is closed and unbounded or open; $K \subset X$ weakly closed and bounded;

$g : X \to \mathbb{R}$ strictly positive, strictly convex, weak-to-norm continuous on X and Fréchet differentiable on K such that $g' : K \to Y'$ is weak-to-norm continuous and satisfying the same growth hypotheses.

REMARK 4.3. Actually, as it is easy to check from the proof of the previous theorem, the hypothesis on g may be weakened. More precisely, if we substitute hypothesis (g_0) with the following:

(g_2) g is strictly positive, strictly convex, Fréchet differentiable on K, $g'(K) \subset \pi(Z), g'(K)$ is bounded in Y' and the function h, defined in (3), is lower semicontinuous with respect to *weak topology;

and leave the growth hypothesis (g_1), in the setting of Theorem 4.1, with further hypothesis that $g,\ h \in D(S_n)$, we obtain the implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c). Moreover if the operator L_n are S_n-regular, then we have the further implications (f) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Leftrightarrow$ (d).

THEOREM 4.2. *In the same setting of Theorem* 4.1 *assume in addition that* Y *has finite dimension and that* $\psi_t^2 \in \mathscr{C}(X, g)$ *for some* (*and hence for all*) $t \in Y$. *If* K *is convex and* $K_1 \subset \mathring{K}$ *is a closed subset*, *then for any* $F \in \mathscr{C}(X; E, g)$ *there exists a constant* $M > 0$ *depending only on* F, K, K_1 *and* g *such that the estimate*

$$||L_n(F)(t) - F(t)|| \leqslant ||L_n(F(t))(t) - F(t)|| + \omega(F, \delta)(S_n(\mathbf{1})(t) + \delta^{-2} S_n(\psi_t^2)(t)) + M S_n(h(t, \cdot))(t) \tag{4}$$

holds for all $\delta > 0$ *and* $t \in K_1$ (*here* $\omega(F, \cdot)$ *stands for the modulus of continuity of* F *on* K). *When* L_n *preserves the constants and* $S_n(\mathbf{1})(t) = 1$, *the above estimate becomes*

$$||L_n(F)(t) - F(t)|| \leqslant 2\omega(F, \gamma_n(t)) + M S_n(h(t, \cdot))(t). \tag{5}$$

Finally, *if* S_n *preserves the linear functionals*, *then*

$$||L_n(F)(t) - F(t)|| \leqslant 2\omega(F, \gamma_n(t)) + M(S_n(g)(t) - g(t)). \tag{6}$$

In case $dim(Y) = 1$, $X = [\![a, +\infty[\![$ $[resp. X =]\!] -\infty, b]\!]]$ *and* $K = [\![a, b]\!]$, *the previous estimates hold with* $K_1 = [\![a, b_1]\!]$ *for any* $b_1 < b$ $[$*resp.* $K_1 = [\![a_1, b]\!]$ *with* $a < a_1]$.

Before proving the theorems, we present two useful lemmas:

LEMMA 4.1. *Let* Z, Y, E, X, K, g *and* h *be as in the Theorem* 4.1 *and consider* $F \in \mathscr{K}(X; E, g)$. *Then there exist an integer* $v \geqslant 1$ *and a constant* $M > 0$ *such that*

$$\|F(t) - F(u)\| \leqslant Mh(t,u) \qquad \text{for any } t \in K \quad \text{and} \quad u \in X \setminus B_v. \tag{7}$$

Moreover, for any $\delta > 0$ *and any finite set* $\ell \subset Z$ *one gets*

$$\|F(t) - F(u)\| \leqslant \omega(F, K, X, I_{\ell,\delta}) + Mh(t,u) \qquad \text{for any } t \in K \quad \text{and } u \in X, \tag{8}$$

where $I_{\ell,\delta}$ *is the following neighborhood of* 0 *in the* **weak topology on* Y:

$$I_{\ell,\delta} := \{y \in Y \mid \forall \xi \in \ell: \ |y(\xi)| < \delta\}$$

and

$$\omega(F, K, X, I_{\ell,\delta}) := \sup\{\|F(t) - F(u)\| \mid t \in K, \quad u \in X, \ u \in t + I_{\ell,\delta}\}. \tag{9}$$

Proof (Estimate (7)). From the *weak-to-norm continuity of F, g, g' and the boundedness of K, it follows that there exists a positive constant $M_1 > 0$ such that for all $t \in K$ one has $\|F(t)\| \leqslant M_1$, $|g(t)| \leqslant M_1$, $\|g'(t)\|_{Y'} \leqslant M_1$ and $\|t\| \leqslant M_1$. Thus for $t \in K$ and $u \in X$ we get

$$\frac{\langle g'(t), u - t\rangle}{g(u)} \leqslant \frac{M_1\|u - t\|}{g(u)} \leqslant \frac{M_1}{g(u)}(\|u\| + M_1)$$

and then

$$\frac{h(t,u)}{g(u)} \geqslant 1 - M_1\frac{1 + \|u\| + M_1}{g(u)} \qquad \text{for all } t \in K \quad \text{and} \quad u \in X.$$

Hence, by the hypotheses on the growth of g, it follows that

$$0 < M_1\frac{1 + \|u\| + M_1}{g(u)} \leqslant \frac{M_1 + M_1^2}{n} + M_1\frac{\|u\|}{g(u)} \qquad \text{for any } u \in X \setminus B_n. \tag{10}$$

Fix $\varepsilon \in \left]0, 1\right[$. If X is bounded, that is $\|u\| \leqslant N$ for $u \in X$ and some constant N, then taking n greater than an appropriate integer v we obtain

$$M_1\frac{1 + \|u\| + M_1}{g(u)} \leqslant M_1\frac{1 + N + M_1}{g(u)} \leqslant M_1\frac{1 + N + M_1}{v} < \varepsilon$$

for all $n \geqslant v$ and $u \in X \setminus B_n$. If X is unbounded, then by (2), there exists $a > 0$ such that for any $\|u\| \geqslant a$ we have $M_1\frac{\|u\|}{g(u)} < \varepsilon/2$. Setting $v := 2M_1 \max\{a, 1 + M_1\}/\varepsilon$, we claim that $M_1\frac{\|u\|}{g(u)} < \varepsilon/2$ for any $n \geqslant v$ and $u \in X \setminus B_n$. Then one also has $M_1\frac{||u||}{g(u)} < M_1\frac{a}{n} \leqslant M_1\frac{a}{v} \leqslant \varepsilon/2$. Now, looking at (10) we obtain

$$M_1\,\frac{1+||u||+M_1}{g(u)} \leqslant \frac{M_1+M_1^2}{n} + M_1\,\frac{||u||}{g(u)} \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

for $n \geqslant v$ and $u \in X\backslash B_n$. Hence for $n \geqslant v$, $u \in X\backslash B_n$ and $t \in K$ we have

$$\begin{aligned}\frac{||F(t)-F(u)||}{h(t,u)} &= \frac{||F(t)-F(u)||}{g(u)}\,\frac{g(u)}{h(t,u)}\\ &\leqslant \frac{||F(u)||+M_1}{g(u)}\left(1-M_1\,\frac{1+||u||+M_1}{g(u)}\right)^{-1}\\ &\leqslant \frac{||F(u)||+M_1}{g(u)}\,(1-\varepsilon)^{-1}.\end{aligned}$$

The above inequality together with $||F(u)|| \leqslant Mg(u)$ and (g_1) accomplishes the proof of (7).

Estimate (8): Set

$$A := \{(t,u) | t \in K,\ \ u \in B_v \text{ and } u \notin t + I_{\ell,\delta}\}.$$

A is *weakly closed and bounded, because the same holds for K and B_v. Since h is *weak-to-norm continuous, then by Weierstrass' theorem, we deduce that h has a minimum m on A, and $m > 0$ because $h(t,u) = 0$ only for $u = t$. Moreover, since F is *weak-to-norm continuous, the same holds true for the function $||F||$, and, consequently, $||F||$ is bounded on the bounded set B_v. Hence we obtain

$$||F(t)-F(u)|| \leqslant 2||F||_{B_v}\,\frac{h(t,u)}{m} = M_2 h(t,u)$$

for every $t \in K$ and $u \in B_v\backslash(t + I_{\ell,\delta})$.

Recalling estimate (7) and definition (8), we conclude the proof of (8). ∎

The next lemma explains an important property of $\omega(F, K, I_{\ell,\delta})$, that will be used in the sequel.

Lemma 4.2. *Let X be a convex subset of $Y = Z'$, K a *weak closed and bounded subset of X, and let F be a **weak-*to-norm continuous mapping from X to E. Then for any $\varepsilon > 0$ there exist a finite set $\ell \subset Z$ and a constant $\delta > 0$ such that $\omega(F, K, X, I_{\ell,\delta}) \leqslant \varepsilon$.*

Proof. By the *weak-to-norm continuity of F, for a fixed $t \in K$ there exist a finite set $\ell_t \subset Z$ and $\delta_t > 0$ such that $||F(t) - F(u)|| < \varepsilon/2$ for $u \in t + I_{\ell_t,\delta_t}$. Trivially $K \subset \bigcup_{t\in K} t + I_{\ell_t,\delta_t/2}$. Since K is compact in the *weak topology, there are $t_1, t_2, \dots, t_n \in K$, such that

$$K \subset \bigcup_{i=1}^{n} t_i + I_{\ell_i,\delta_i/2}.$$

where we set $\ell_i := \ell_{t_i}$ and $\delta_i := \delta_{t_i}$. Let $\delta := \frac{1}{2}\min\{\delta_i,\ i = 1 \dots n\}$ and $\ell := \bigcup_{i=1}^{n} \ell_i$. We Prove that $I_{\ell,\delta}$ is the desired neighborhood of zero.

Fix $t \in K$ and $u \in (t + I_{\ell,\delta}) \cap X$. Let i be the index for which $t \in t_i + I_{\ell_i,\delta_i/2}$. For any $\xi \in \ell_i$ the inequality

$$|\xi(u - t_i)| \leqslant |\xi(u - t)| + |\xi(t - t_i)| < \delta + \delta_i/2 \leqslant \delta_i$$

holds, and thus $u \in t_i + I_{\ell_i,\delta_i}$. Therefore

$$||F(t) - F(u)|| \leqslant ||F(t) - F(t_i)|| + ||F(t_i) - F(u)|| < \varepsilon/2 + \varepsilon/2,$$

which yields the desired estimate for $\omega(F, K, X, I_{\ell,\delta})$. ∎

Now we prove our main results.

Proof of Theorem 4.1. First of all, observe that for given $F \in \mathscr{K}(X; E, g)$ and $t \in K$, applying S_n to both sides of (8) of Lemma 4.1, we obtain

$$S_n(||F - F(t)||)(t) \leqslant S_n(1)(t)\omega(F, K, I_{\ell,\delta}) + MS_n(h(t, \cdot))(t).$$

Consequently,

$$\begin{aligned} ||L_n(F)(t) - F(t)|| &\leqslant ||L_n(F(t))(t) - F(t)|| + S_n(||F - F(t)||)(t) \\ &\leqslant ||L_n(F(t))(t) - F(t)|| + S_n(\mathbf{1})(t)\omega(F, K, I_{\ell,\delta}) \\ &\quad + MS_n(h(t, \cdot))(t). \end{aligned} \tag{11}$$

We prove the implication (a) $\Rightarrow$ (c). Take $\varepsilon > 0$ and consider the zero neighborhood $I_{\ell,\delta}$ for which $\omega(F, K, I_{\ell,\delta}) \leqslant \varepsilon/6$. By Lemma 4.1, there exists a constant M such that relation (8) holds for $I_{\ell,\delta}$. In view of (a), for n sufficiently large we have $S_n(h(t, \cdot))(t) < \varepsilon/(3M)$, $S_n(\mathbf{1})(t) < 2$ and $||L_n(F(t))(t) - F(t)|| < \varepsilon/3$, and thus, using (11) we deduce

$$||L_n(F)(t) - F(t)|| \leqslant \varepsilon/3 + 2\varepsilon/6 + M\varepsilon/(3M) = \varepsilon,$$

that proves the convergence of $L_n(F)(t)$ to $F(t)$. It is clear that the convergence is uniform if the same holds for (a).

Fix $f \in \mathscr{K}(X, g)$. In order to prove the convergence of $S_n(f)(t)$ to $f(t)$ we proceed in the manner we made before substituting the norm $|| \cdot ||$ in E with the absolute value.

In order to prove the implication (c) $\Rightarrow$ (b), it is sufficient to observe that the constant functions are *weak-to-norm continuous, and the function g and all continuous functionals in $\pi(Z)$ belongs to $\mathscr{K}(X, g)$ (by (2)).

The implication (b) $\Rightarrow$ (a) follows directly from the identity

$$S_n(h(t,\cdot))(t) = S_n(g)(t) - g(t)S_n(\mathbf{1})(t) - S_n(\langle g'(t),\cdot\rangle)(t) + \langle g'(t), t\rangle S_n(\mathbf{1})(t). \tag{12}$$

Now we assume that L_n is S_n-regular. The implication (d) $\Rightarrow$ (c). Fix $f \in \mathscr{K}(X, g)$. Taking $x \in E$, by definition of S-regularity, we have

$$S_n(f)(t) \otimes x = L_n(f \otimes x)(t),$$

that converges to $f(t)x$. Since x is arbitrary we have the convergence of $S_n(f)(t)$ to $f(t)$. The implication (f) $\Rightarrow$ (b) follows from identity

$$L_n(c)(t) = L_n(\mathbf{1} \otimes c)(t) = S_n(\mathbf{1}) \otimes c$$

and the missing implication (e) $\Rightarrow$ (f) is immediate. The proof is complete. ∎

Proof of Theorem 4.2. Fix $F \in \mathscr{K}(X; E, g)$ and $\delta > 0$. By (7) for every $t \in K_1$ and $u \in X \backslash B_v$ we get

$$||F(t) - F(u)|| \leqslant M_1 h(t, u). \tag{13}$$

On the other hand, the inequality

$$||F(t) - F(u)|| \leqslant \omega(F, ||t-u||) \leqslant (1 + \delta^{-2}||t-u||^2)\omega(F, \delta) \tag{14}$$

holds for every $t \in K_1$ and $u \in K$ ($\omega(F, \delta)$ stands here for the modulus of continuity of F on K).

Now we discuss the case $t \in K_1$ and $u \in \overline{B_v \backslash K}$. Since Y is of finite dimension, K convex and $K_1 \subset \mathring{K}$, there exists a closed and convex set $K_\eta \subset \mathring{K}$ such that $K_1 \subset K_\eta$. From the convexity of B_v, K and K_η, it follows that

$$[a', a''] = [u, t] \cap \overline{K \backslash K_\eta}$$

for some $a' \in \bar{K} \backslash \mathring{K} = \partial K$ and $a'' \in \delta K_\eta$. Let $P : [\![0, 1]\!] \to [u, t]$ be the parametric representation of the segment, $P(s) := (1-s)u + st$ $(0 \leqslant s \leqslant 1)$, and $0 \leqslant s' < s'' \leqslant 1$ such that $P(s') = a'$ and $P(s'') = a''$. We set $\hat{g} := g \circ P : [\![0, 1]\!] \to [0, +\infty[$ and $\hat{h}(r, s) := \hat{g}(s) - [\hat{g}(r) + \hat{g}'(r)(s - r)]$. Note that $\hat{g}$ is strictly convex by the strict convexity of g. This yields $\hat{h}(s'', s') \leqslant \hat{h}(1, 0)$.

Observing that

$$\hat{g}'(r) = \langle g'(P(r)), P'(r) \rangle = \langle g'(P(r)), t - u \rangle,$$

and $P(s) - P(r) = (s - r)(t - u)$, we get

$$\hat{h}(r, s) = g(P(s)) - [g(P(r)) + \langle g'(P(r)), P(s) - P(r) \rangle] = h(P(r), P(s)).$$

Hence

$$0 < h(a'', a') = \hat{h}(s'', s') \leqslant \hat{h}(1, 0) = h(t, u),$$

and consequently

$$||F(t) - F(u)|| \leqslant ||F(t)|| + ||F(u)|| \leqslant 2||F||_{B_v} h(t, u)/h(a'', a'). \qquad (15)$$

Since $\partial K \cap \partial K_\eta = \emptyset$ and both ∂K and ∂K_η are compact, surely $\inf\{h(a'', a') | a' \in \partial K,\ a'' \in \partial K_\eta\} > 0$ and therefore

$$||F(t) - F(u)|| \leqslant M_2 h(t, u) \qquad \text{for any } t \in K_1, \quad u \in \overline{B_v \backslash K}. \qquad (16)$$

In case $dim(Y) = 1$, $X = [\![a, +\infty]\!]$, $K = [\![a, b]\!]$ and $K_1 = [\![a, b_1]\!]$ (with $b_1 < b$) relation (16) is established in the similar manner. One considers $K_\eta :$ $= [\![a, b_2]\!]$ with $b_1 < b_2 < b$ and finds $a' = b$ and $a'' = b_2$, which yield (16) in view of (15) and the inequality $0 < h(b_2, b) \leqslant h(t, u)$.

Combining inequalities (13), (14) and (16) we obtain

$$||F(t) - F(u)|| \leqslant (1 + \delta^{-2}||t - u||^2)\omega(F, \delta) + Mh(t, u)$$

for all $t \in K_1$ and $u \in X$. Now applying S_n and using the first inequality in (11) we obtain estimate (4). The last inequality (6) easily follows from relation (12). ∎

Remark 4.4. We stress the fact that the constant M in (4), (5) and (6) depends only on F, K, K_1 and g; in particular, it does *not* depend on the operators L_n or S_n.

Remark 4.5. From the previous theorems we deduce that an approximation process for real-valued functions S_n, defined by means of positive measures, yields another process L_n, for vector-valued functions. Note that the process L_n "inherits" the estimates valid for S_n.

5. EXAMPLES

The following examples deal with operators L_n that are S_n-regular. We shall use the same symbol to denote either of them.

In the sequel let $e_q(s) := s^q$. For $t \in \mathbb{R}^m$ and $q \geqslant 1$, $||t||_q$ will denote the norm $(\sum_{i=1}^m |t_i|^q)^{1/q}$. In $\mathbb{R}^m$ all the norms are equivalent, but we note that for $m > 1$ $||\cdot||_1^q$ is not strictly convex, thus both $1 + ||\cdot||_1^q$ and $\exp(||\cdot||_1)$ do not satisfy condition (g_0).

Example 5.1 (Bernstein–Chlodovsky). For every $n \geqslant 1$, let a_n be a positive real number. We define

$$C_n(F)(t) := \sum_{k=0}^{n} \binom{n}{k} F\left(\frac{a_n k}{n}\right) \left(\frac{t}{a_n}\right)^k \left(1 - \frac{t}{a_n}\right)^{n-k},$$

for every $F \in \mathscr{C}(\mathbb{R}_+; E)$ and $t \in [\![0, a_n]\!]$, and set

$$EXP := \bigcup_{w \geqslant 1} \mathscr{C}(\mathbb{R}_+; E, \exp(we_1)).$$

Theorem 5.1. *Assume $a_n \to +\infty$ and $a_n/n \to 0$ as $n \to \infty$. Then for every $F \in EXP$*

$$C_n(F) \to F$$

uniformly on compact subset of $\mathbb{R}_+$. Moreover, if $b \leqslant a_n$, there exists a constant $M > 0$ such that the estimate

$$||C_n(F)(t) - F(t)|| \leqslant 2\omega\left(F, \sqrt{t\frac{a_n - t}{n}}\right) + M\frac{a_n}{n}t,$$

holds for any $t \in [\![0, b]\!]$.

Proof. Fix $0 \leqslant t \leqslant b$, n and $F \in EXP$ such that $||F(t)|| \leqslant M \exp(wt)$ and $b \leqslant a_n$. In order to apply Theorem 4.2 with $g(t) = \exp(wt)$, we note that C_n preserves the constant and the linear functional, that is

$$C_n(\mathbf{1}) = \mathbf{1}, \qquad C_n(e_1)(t) = t.$$

Hence we will estimate γ_n and $S_n(g)(t) - g(t)$ of (6).

An easy computation shows that

$$\gamma_n^2(t) = C_n(e_2)(t) - t^2 = \frac{a_n t - t^2}{n},$$

$$\begin{aligned} C_n(e^{we_1})(t) &= \sum_{k=0}^{n} \binom{n}{k} e^{wka_n/n} \left(\frac{t}{a_n}\right)^k \left(1 - \frac{t}{a_n}\right)^{n-k} = \left[\frac{t}{a_n} e^{wa_n/n} + 1 - \frac{t}{a_n}\right]^n \\ &= \exp\left[n \ln\left(\frac{t}{a_n} e^{wa_n/n} - \frac{t}{a_n} + 1\right)\right] = \exp\left[\frac{n}{a_n} f_n\left(\frac{a_n}{n}\right)\right], \end{aligned}$$

where $f_n(s) \coloneqq a_n \ln[1 + t/a_n(e^{ws} - 1)]$. The application of mean value theorem to $f_n(s)$ in the interval $[\![0, a_n/n]\!]$ yields

$$C_n(e^{we_1})(t) = \exp\left[wt \frac{e^{w\xi_n}}{1 + t(e^{w\xi_n} - 1)/a_n}\right]$$

for some $0 \leqslant \xi_n \leqslant a_n/n$, hence

$$\begin{aligned} |C_n(e^{we_1})(t) - e^{wt}| &= e^{wt}\left\{\exp\left[wt \frac{e^{w\xi_n} - 1 - t(e^{w\xi_n} - 1)/a_n}{1 + t(e^{w\xi_n} - 1)/a_n}\right] - 1\right\} \\ &\leqslant e^{wt}\{\exp[wt(e^{w\xi_n} - 1)] - 1\}. \end{aligned}$$

From the last inequality and from

$$e^s - 1 \leqslant se^s \qquad \text{for any } s \in \mathbb{R}, \tag{17}$$

we obtain

$$|C_n(e^{we_1})(t) - e^{wt}| \leqslant \frac{a_n}{n} w^2 t \exp[(a_n/n + wte^{wa_n/n})],$$

that allows us to conclude. ∎

EXAMPLE 5.2. Our aim is to modify the operators of the previous example to approximate functions defined on $\mathbb{R}_+^m \coloneqq \{t = (t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_i \geqslant 0\}$.

For every $n \geqslant 1$, let a_n be a positive real number. We define

$$C_n^*(F)(t) := \sum_{k_1=0\ldots k_m=0} F\left(\frac{a_n k_1}{n}, \ldots, \frac{a_n k_m}{n}\right)$$
$$\binom{n}{k_1}\left(\frac{t_1}{a_n}\right)^{k_1}\left(1-\frac{t_1}{a_n}\right)^{n-k_1} \cdots \binom{n}{k_m}\left(\frac{t_m}{a_n}\right)^{k_m}\left(1-\frac{t_m}{a_n}\right)^{n-k_m}$$

for every $F \in \mathscr{C}(\mathbb{R}_+^m; E)$ and $t \in [\![0, a_n]\!]^m$.

Let P be the subspace of $\mathscr{C}(\mathbb{R}_+^m; E)$ of the functions that have a polynomial growth at infinity, that is

$$P := \{F \in \mathscr{C}(\mathbb{R}_+^m; E) | \exists M > 0,\ q > 1 : ||F(\cdot)|| \leqslant M(1 + ||\cdot||_2^q)\}.$$

Theorem 5.2. *Assume $a_n \to +\infty$ and $a_n/n \to 0$ as $n \to \infty$. For every $F \in P$*

$$C_n^*(F) \to F,$$

uniformly on compact subset of $\mathbb{R}_+^m$. Moreover, if $b \leqslant a_n$ then there exist two constants $M_1, M_2 > 0$ such that

$$||C_n^*(F) - F||_{[\![0,b]\!]^m} \leqslant 2\omega\left(F, \sqrt{b\frac{a_n}{n}m}\right) + M_1\sqrt{\frac{a_n}{n}} + M_2\frac{a_n}{n}.$$

Proof. Fix $b \geqslant 0$, $t \in [\![0, b]\!]^m$ and $F \in P$ and let n, $M > 0$, $q > 1$ be such that $||F(t)|| \leqslant M(1 + ||t||_2^q)$ and $b \leqslant a_n$. In order to apply Theorem 4.2, we consider $g(t) = 1 + ||t||_q^q$, that satisfies conditions $(g_{0,1})$ and, by equivalence of the norm on $\mathbb{R}^m$, $F \in \mathscr{C}(\mathbb{R}_+^m; E, 1 + ||\cdot||_q^q)$.

Noting that

$$C_n^*(pr_j^r)(t) = C_n(e_r)(t_j),$$

we obtain

$$C_n^*(\mathbf{1}) = \mathbf{1}, \qquad C_n^*(pr_j)(t) = t_j, \qquad C_n^*(pr_j^2)(t) = t_j^2 + \frac{a_n t_j - t_j^2}{n},$$

$$C_n^*(g)(t) - g(t) = \sum_{j=1}^m (C_n(e_q)(t_j) - t_j^q).$$

From Theorem 5.1 we have

$$|C_n(e_q)(t_j) - t_j^q| \leqslant 2\omega\left(e_q, \sqrt{t_j \frac{a_n - t_j}{n}}\right) + M\frac{a_n}{n}$$
$$\leqslant 2qb^{q-1}\sqrt{t_j \frac{a_n - t_j}{n}} + M\frac{a_n}{n}$$

for any $j = 1 \ldots m$ and $t_j \in [\![0, a_n]\!]$. It follows that for $t \in [\![0, b]\!]^m$

$$|C_n^*(g)(t) - g(t)| \leqslant 2mqb^{q-1}\sqrt{b\frac{a_n}{n}} + M\frac{a_n}{n}m.$$

Finally, from

$$\gamma_n^2(t) = C_n^*(\|\cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{a_n t_j - t_j^2}{n} \leqslant mb\frac{a_n}{n},$$

we conclude the proof. ∎

EXAMPLE 5.3. For every $n \geqslant 1$, let a_n be a positive real number. We define

$$V_n(F)(t) := \sum_{k=0}^{\infty} \binom{-n}{k} F\left(\frac{a_n k}{n}\right)\left(-\frac{t}{a_n}\right)^k \left(1 + \frac{t}{a_n}\right)^{-n-k},$$

$$\tilde{V}_n(F)(t) := \left(1 + \frac{1}{a_n}\right)^{-n} \sum_{k=0}^{\infty} \binom{-n}{k} F\left(\frac{a_n kt}{n}\right)\left(-\frac{1}{1 + a_n}\right)^k,$$

$$\hat{V}_n(F)(t) := 2^{-n} \sum_{k=0}^{\infty} \binom{-n}{k} F\left(\frac{kt}{n}\right)(-2)^{-k}$$

for every $F \in EXP$ and $t \in [\![0, +\infty[\![$. Note that for $a_n = 1$, V_n are the Baskakov operators and $\tilde{V}_n$ are obtained formally from V_n, replacing a_n with ta_n. When $a_n = 1$, from $\tilde{V}_n$ one gets the operators $\hat{V}_n$.

Since now we assume $a_n \geqslant \alpha > 0$, for some α and $a_n/n \to 0$ as $n \to \infty$.

THEOREM 5.3. *For every $F \in EXP$*

$$V_n(F)(t) \to F(t), \tag{18}$$

$$\tilde{V}_n(F)(t) \to F(t), \tag{19}$$

$$\hat{V}_n F(t) \to F(t), \tag{20}$$

uniformly on compact subset of $\mathbb{R}_+$. *Moreover, for every* $b > 0$, *there exists a constant* $M > 0$ *such that*

$$||V_n(F)(t) - F(t)|| \leqslant 2\omega\left(F, \sqrt{t\frac{a_n + t}{n}}\right) + M\frac{a_n}{n}t, \tag{21}$$

$$||\tilde{V}_n(F)(t) - F(t)|| \leqslant 2\omega\left(F, t\sqrt{\frac{1 + a_n}{n}}\right) + M\frac{a_n}{n}t^2, \tag{22}$$

$$||\hat{V}_n(F)(t) - F(t)|| \leqslant 2\omega\left(F, t\sqrt{\frac{2}{n}}\right) + M\frac{1}{n}t^2 \tag{23}$$

hold for any $t \in [\![0, b]\!]$.

Proof. Fix $0 \leqslant t \leqslant b$, n and $F \in EXP$ such that $||F(t)|| \leqslant Me^{wt}$. We proceed as in Theorem 5.1. It is easy to check

$$V_n(\mathbf{1}) = 1, \qquad V_n(e_1)(t) = t, \qquad \gamma_n^2(t) = V_n(e_2)(t) - t^2 = \frac{a_n t + t^2}{n}.$$

Using the mean value theorem, as in Theorem 5.1, we obtain

$$|V_n(e^{we_1})(t) - e^{wt}| = e^{wt}\left|\exp\left[wt\frac{(e^{w\xi_n} - 1)(1 + t/a_n)}{1 + t(\mathbf{1} - e^{w\xi_n})/a_n}\right] - 1\right|$$

for $0 \leqslant \xi_n \leqslant a_n/n$. Using twice (17), one gets

$$\begin{aligned}|V_n(e^{we_1})(t) - e^{wt}| \leqslant \frac{a_n}{n}w^2 te^{wt + w\xi_n}\frac{(1 + t/a_n)}{1 + t(1 - e^{w\xi_n})/a_n}\\ \exp\left[wt\frac{(e^{w\xi_n} - 1)(1 + t/a_n)}{1 + t(1 - e^{w\xi_n})/a_n}\right],\end{aligned}$$

which allows us to conclude the proofs of (18) and (21).

In order to prove the remaining, we note that for $t = 0$, the relations are obvious. For $t > 0$, we replace a_n with $a_n t$ in (21), and by means of Remark 4.4, we have (22) and (19). Setting $a_n = 1$ in (22), we obtain the remaining relations (23) and (20). ∎

It is possible to extend these operators in the same direction of Example 5.2. We omit this generalization for sake of brevity.

An immediate application of an approximation process for vector-valued mappings, is the representation of semigroups of operators.

THEOREM 5.4. *Let $T(\cdot): \mathbb{R}_+ \to \mathscr{L}(E)$ be a $\mathscr{C}_0$ one-parameter semigroup of bounded linear operators on E. Then the representation formulae*

$$\left[I - \frac{t}{a_n}\left(T\left(\frac{a_n}{n}\right) - I\right)\right]^{-n} x \to T(t)x, \tag{24}$$

$$\left[\left(1 + \frac{1}{a_n}\right)I - \frac{1}{a_n}T\left(\frac{a_n t}{n}\right)\right]^{-n} x \to T(t)x, \tag{25}$$

$$\left[2I - T\left(\frac{t}{n}\right)\right]^{-n} x \to T(t)x \tag{26}$$

hold for every $x \in E$ and uniformly on compact set of $\mathbb{R}_+$. Moreover, for every bounded set $J \subset \mathbb{R}_+$, there exists a constant $M > 0$, such that the estimates

$$\left\|\left[I - \frac{t}{a_n}\left(T\left(\frac{a_n}{n}\right) - I\right)\right]^{-n} x - T(t)x\right\| \leqslant 2\omega\left(T(\cdot)x, \sqrt{t\,\frac{a_n + t}{n}}\right) + M\frac{a_n}{n}t,$$

$$\left\|\left[\left(1 + \frac{1}{a_n}\right)I - T\left(\frac{a_n t}{n}\right)\right]^{-n} x - T(t)x\right\| \leqslant 2\omega\left(T(\cdot)x, t\sqrt{\frac{1 + a_n}{n}}\right) + M\frac{a_n}{n}t^2,$$

$$\left\|\left[2I - T\left(\frac{t}{n}\right)\right]^{-n} x - T(t)x\right\| \leqslant 2\omega\left(T(\cdot)x, t\sqrt{\frac{2}{n}}\right) + M\frac{t^2}{n}$$

hold for every $t \in J$.

Formulae (24) and (26) appear in [6]. There, Shaw proves (26) pointwise, and, by Chernoff's product formula, uniformly on compact sets only for contractive semigroups. Here, we have also an estimate of the rate of convergence. Formula (25) seems to be new as well as the operators $\tilde{V}_n$ and $\hat{V}_n$.

EXAMPLE 5.4. Set $G := \bigcup_{w \geqslant 1} \mathscr{C}(\mathbb{R}^m; E, \exp(we_2))$. We define

$$W_n(F)(t) := \left(\frac{n}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-\frac{n}{2}||u-t||^2} F(u)\, du$$

for any $F \in G$ ant $t \in \mathbb{R}^m$.

THEOREM 5.5. *For every* $F \in G$

$$W_n(F) \to F,$$

uniformly on compact subset of $\mathbb{R}^m$. *If* $||F(\cdot)|| \leqslant M \exp(we_2)$ *and if* $K \subset \mathbb{R}^m$ *is compact there exists a constant* $M > 0$ *such that*

$$||W_n(F) - F||_K \leqslant 2\omega\left(F, \sqrt{\frac{m}{n}}\right) + \frac{M}{n}$$

for $n \geqslant 4w$.

Proof. Fix $K \subset \mathbb{R}^m$ compact, $t \in K$, $F \in G$ and let n, w be such that $||F|| \leqslant M \exp(we_2)$, $n \geqslant 4w$ and set $g = \exp(we_2)$. In order to apply Theorem 4.2 it is sufficient to evaluate only γ_n and $W_n(g)(t) - g(t)$, because the relations

$$W_n(\mathbf{1}) = \mathbf{1}, \qquad W_n(pr_j)(t) = t_j$$

allow us to use inequality (6).

From $W_n(pr_j^2)(t) = t_j^2 + 1/n$ we obtain

$$\gamma_n^2(t) = \frac{m}{n}.$$

Setting $\xi_n := 1 - \frac{2w}{n}$, we can rewrite $W_n(g)$ as

$$\begin{aligned} W_n(g)(t) &= \prod_{j=1}^m \left(\frac{n}{2\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-\frac{n}{2}(u_j - t_j)^2} e^{wu_j^2}\, du_j \\ &= \prod_{j=1}^m \left(\frac{n}{2\pi}\right)^{1/2} \int_{\mathbb{R}} \exp\left(-\frac{n}{2}\xi_n (u_j - t_j/\xi_n)^2\right) \exp(wt_j^2/\xi_n)\, du_j \\ &= \prod_{j=1}^m \xi_n^{-1/2} \exp(wt_j^2/\xi_n) = \xi_n^{-m/2} \exp(w||t||^2/\xi_n). \end{aligned}$$

Then using (17) and the inequality

$$(1+s)^{\alpha} \leqslant 1 + \alpha s \qquad \text{for } 0<\alpha<1, \quad -1<s$$

with $\alpha = \frac{1}{2}$ and $1 + s = 1/\xi_n^m$, we obtain

$$\begin{aligned} |W_n(g)(t) - g(t)| &= e^{w||t||^2}\left\{\xi_n^{-m/2}\left[\exp\left(w||t||^2\left(\frac{1}{\xi_n}-1\right)\right)-1\right] + \xi_n^{-m/2} - 1\right\} \\ &\leqslant e^{w||t||^2}\left\{\xi_n^{-m/2} w||t||^2 \frac{1-\xi_n}{\xi_n}\exp\left(w||t||^2\frac{1-\xi_n}{\xi_n}\right) + \frac{1-\xi_n^m}{2\xi_n^m}\right\}. \end{aligned}$$

For $n \geqslant 4w$, there holds $\frac{1-\xi_n^m}{\xi_n^m} \leqslant m2^m(1-\xi_n)$. Hence, we have

$$|W_n(g)(t) - g(t)| \leqslant e^{w||t||^2}\left\{w||t||^2 \xi_n^{-(\frac{m}{2}+1)}\frac{w}{n}\exp\left(\frac{w^2||t||^2}{n\xi_n}\right) + 2^{m-1}m\frac{w}{n}\right\},$$

which concludes the proof. ∎

ACKNOWLEDGMENTS

The author gratefully acknowledges the helpful suggestions and encouragement of professor Francesco Altomare during the review of the paper, and thanks professor Sen-Yen Shaw for the careful reading of the manuscript.

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