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THREE PROBLEMS ON THE LENGTHS OF INCREASING RUNS

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Let $U_1, U_2, \ldots$ be a sequence of independent, uniform (0, 1) r.v.'s and let $R_1, R_2, \ldots$ be the lengths of increasing runs of $\{U_i\}$, i.e., $X_1 = R_1 = \inf\{i: U_{i+1} < U_i\}, \ldots, X_n = R_1 + R_2 + \cdots + R_n = \inf\{i: i > X_{n-1}, U_{i+1} < U_i\}$. The first theorem states that the sequence $(\frac{3}{2}n)^{1/2}(X_n - 2n)$ can be approximated by a Wiener process in strong sense.

Let $\tau(n)$ be the largest integer for which $R_1 + R_2 + \cdots + R_{\tau(n)} \leq n$, $R_n^* = n - (R_1 + R_2 + \cdots + R_{\tau(n)})$ and $M_n = \max\{R_1, R_2, \ldots, R_{\tau(n)}, R_n^*\}$. Here M_n is the length of the longest increasing block. A strong theorem is given to characterize the limit behaviour of M_n.

The limit distribution of the lengths of increasing runs is our third problem.

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Uniform distribution	increasing runs
Wiener process	strong invariance principle
law of iterated logarithm	limit theorems

1. Introduction

Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) r.v.'s, i.e.,

$$\mathbf{P}(U_i < x) = x \quad (0 \leq x \leq 1, i = 1, 2, \ldots)$$

and let $R_1, R_2, \ldots$ be the lengths of the increasing runs of $\{U_i\}$, i.e.,

$$X_1 = R_1 = \inf\{i: U_{i+1} < U_i\},$$
$$X_2 = R_1 + R_2 = \inf\{i: i > R_1, U_{i+1} < U_i\},$$
$$\vdots$$
$$X_n = R_1 + R_2 + \cdots + R_n = \inf\{i: i > X_{n-1}, U_{i+1} < U_i\},$$
$$\vdots$$

Pittel (1980) proved that the finite-dimensional distributions of the sequence

$$X_n^* = (\tfrac{2}{3}n)^{-1/2}(X_n - 2n)$$

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converge to the corresponding finite-dimensional distributions of the Wiener process. One of the aims of the present paper is to prove an analogous strong version of this theorem.

In fact we prove the following strong invariance principle.

Theorem 1.A. *One can construct*

(i) *a probability space* $\{\Omega, \mathcal{A}, P\}$,

(ii) *a sequence* $U_1, U_2, \ldots$ *of independent, uniform* $(0, 1)$ *r.v.'s and a Wiener process* $\{W(t); t \geq 0\}$ *(both defined on* Ω*) such that*

$$n^{-1/2+\varepsilon}|(\tfrac{3}{2})^{1/2}(X_n - 2n) - W(n)| \to 0 \text{ a.s.} \quad (n \to \infty) \tag{1.1}$$

for suitable $\varepsilon > 0$.

Remark 1. It is not hard to get a better rate in (1.1) but here we do not intend to find the best possible rate.

Eq. (1.1) implies that the limit properties of the process $n^{1/2}X_n^*$ are the same as those of $W(n)$. For example we have the following.

Consequence 1.A.1

$$\limsup_{n\to\infty} \frac{\sqrt{3}(X_n - 2n)}{2(n \log\log n)^{1/2}} = 1,$$

$$\liminf_{n\to\infty} \left(\frac{12}{\pi^2} \frac{\log\log n}{n}\right)^{1/2} \sup_{1\leq j\leq n} |X_j - 2j| = 1,$$

$$\mathbf{P}\left\{(\tfrac{3}{2})^{1/2} \sup_{1\leq j\leq n} (X_j - 2j) > n^{1/2}u\right\} \to \tfrac{1}{2}(1 - \Phi(u)) \quad \textit{as } n \to \infty$$

where

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2}\, dx.$$

The above definition of the lengths of increasing runs is taken from Pittel (1980); however, it does not seem to be a very natural one. In order to present a more natural definition introduce the following notations.

Let $\tau_n = \tau(n)$ be the largest integer for which

$$R_1 + R_2 + \cdots + R_{\tau(n)} \leq n;$$

further let

$$R_n^* = n-(R_1+R_2+\cdots+R_{\tau(n)}), \tag{1.2}$$

$$S_n = R_n - 1, \qquad Y_n = S_1+S_2+\cdots+S_n = X_n - n,$$

$$S_n^* = \begin{cases} R_n^* - 1 & \text{if } R_n^* > 0, \\ 0 & \text{if } R_n^* = 0, \end{cases}$$

$$\rho_1 = \rho(1) = \min\{i: S_i > 0\},$$

$$\vdots$$

$$\rho_{n+1} = \rho(n+1) = \min\{i: i > \rho_n, S_i > 0\},$$

$$\vdots$$

$$T_n = S_{\rho(n)}, \qquad Z_n = T_1+T_2+\cdots+T_n.$$

The r.v. Z_n/n seems to be closer to the natural concept of the average of the lengths of the increasing runs than the r.v. X_n/n. Let us give a concrete example. Suppose

$$U_1 < U_2 < U_3, \qquad U_3 > U_4 > U_5, \qquad U_5 < U_6, \qquad U_6 > U_7,$$

$$U_7 < U_8 < U_9 < U_{10}, \qquad U_{10} > U_{11} > U_{12} > U_{13} > U_{14}, \ldots .$$

In this case

$$X_1 = 3, \quad X_2 = 4, \quad X_3 = 6, \quad X_4 = 10, \quad X_5 = 11, \quad X_6 = 12, \quad X_7 = 13, \ldots,$$

$$R_1 = 3, \quad R_2 = 1, \quad R_3 = 2, \quad R_4 = 4, \quad R_5 = 1, \quad R_6 = 1, \ldots,$$

$$T_1 = 2, \quad T_2 = 1, \quad T_3 = 3, \ldots .$$

That is to say, in the above sequence the blocks

$$(U_1, U_2), \qquad (U_5), \qquad (U_7, U_8, U_9)$$

are considered as increasing runs and the blocks

$$(U_3, U_4), \qquad (U_6), \qquad (U_{10}, U_{11}, U_{12}, U_{13})$$

are the decreasing runs.

Investigating the sequence $\{Z_n\}$ instead of $\{X_n\}$ the following analogue result will be proved.

Theorem 1.B. *One can construct*

(i) *a probability space* $\{\Omega, \mathcal{A}, P\}$,

(ii) *a sequence* $U_1, U_2, \ldots$ *of independent, uniform* $(0,1)$ *r.v.'s and a Wiener process* $\{W(t); t \geq 0\}$ *(both defined on* Ω*) such that*

$$n^{-1/2+\varepsilon}|2(\tfrac{5}{11})^{1/2}(Z_n - \tfrac{3}{2}n) - W(n)| \to 0 \text{ a.s.} \quad (n\to\infty) \tag{1.3}$$

for a suitable $\varepsilon > 0$.

As an analogue of Consequence 1.A.1 we have the following.

Consequence 1.B.1

$$\limsup_{n\to\infty} \frac{10^{1/2}(Z_n - \frac{3}{2}n)}{(11n \log\log n)^{1/2}} = 1,$$

$$\liminf_{n\to\infty} \frac{4}{\pi}\left(\frac{10 \log\log n}{11n}\right)^{1/2} (Z_n - \tfrac{3}{2}n) = 1,$$

$$\mathbf{P}\left\{2(\tfrac{5}{11})^{1/2} \sup_{1\le j\le n} (Z_j - \tfrac{3}{2}j) > n^{1/2}u\right\} \to 2(1-\Phi(u)).$$

Our second problem is to investigate the length of the longest increasing block in the sequence $U_1, U_2, \ldots, U_n$. In order to formulate our result introduce the following notation,

$$M_n = \max\{R_1, R_2, \ldots, R_{\tau(n)}, R_n^*\}.$$

Clearly M_n is the length of the longest increasing block.

Then we have the following.

Theorem 2.A. *For any $\varepsilon > 0$ we have*

$$l_n(\varepsilon) = l_n < M_n < u_n = u_n(\varepsilon) \tag{1.4}$$

with probability 1 *if n is big enough where*

$$l_n(\varepsilon) = \begin{cases} [f(n)] - 3 & \text{if } \alpha(n) \le \varepsilon, \\ [f(n)] - 2 & \text{if } \alpha(n) > \varepsilon, \end{cases}$$

$$u_n(\varepsilon) = \begin{cases} [f(n)] + 2 & \text{if } \alpha(n) \le 1-\varepsilon, \\ [f(n)] + 3 & \text{if } \alpha(n) > 1-\varepsilon, \end{cases}$$

$$f(n) = (\log n)/b_n - \tfrac{1}{2}, \qquad \alpha(n) = f(n) - [f(n)]$$

and b_n is the solution of the equation

$$b_n \, \mathrm{e}^{b_n} = \mathrm{e}^{-1} \log n. \tag{1.5}$$

More formally speaking: For any $\varepsilon > 0$ and for almost all $\omega \in \Omega$ there exists a random integer $n_0 = n_0(\varepsilon, \omega)$ such that (1.4) *holds if $n \ge n_0$.*

Remark 2. From definition (1.5) it is easy to see that

$$b_n = \log\log n - \log\log\log n - 1 + \mathrm{o}(1).$$

Remark 3. Since the difference of the upper and lower estimators of (1.4) is $4 \le u_n - l_n \le 5$ and M_n is integer-valued, the r.v. M_n has three or four possible values only with probability 1 if n is big enough.

Remark 4. Theorem 2.A clearly implies

$$\lim_{n\to\infty} \frac{\log\log n}{\log n} M_n = 1 \quad \text{a.s.}$$

This result was proved recently by Pittel (1981).

Our following result shows that (1.4) nearly gives the best possible estimators.

Theorem 2.B. *For any $\varepsilon > 0$ there exist sequences $n_1 = n_1(\omega, \varepsilon) < n_2 = n_2(\omega, \varepsilon) < \cdots$ and $m_1 = m_1(\omega, \varepsilon) < m_2 = m_2(\omega, \varepsilon) < \cdots$ such that*

$$M_{n_i} \geq V(n_i) \quad \textit{and} \quad M_{m_i} \leq J(m_i)$$

where

$$V(n) = \begin{cases} [f(n)] & \textit{if } \alpha(n) \leq \varepsilon, \\ [f(n)] + 1 & \textit{if } \alpha(n) > \varepsilon, \end{cases}$$

$$J(n) = \begin{cases} [f(n)] - 1 & \textit{if } \alpha(n) \leq 1 - \varepsilon, \\ [f(n)] & \textit{if } \alpha(n) > 1 - \varepsilon, \end{cases}$$

with $f(n)$ and $\alpha(n)$ as defined in Theorem 2.A.

In order to formulate our third result introduce the following notations,

$$\delta(j, k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise}, \end{cases}$$

$\Psi_n = \Psi(n)$ is the largest integer for which $\rho(\Psi(n)) \leq \tau_n$,

$$\frac{3}{n} \sum_{i=1}^{\psi(n)} \delta(T_i, k) = \lambda(k, n).$$

Then we have the following.

Theorem 3

$$\lim_{n\to\infty} \sup_k \left| \lambda(k, n) - \frac{k^2 + 3k + 1}{(k+3)!} \right| = 0 \quad \text{a.s.}$$

The proof of this theorem does not require any new idea and it will be omitted.

2. Proofs of Theorems 1.A and 1.B

Let

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } x \geq y. \end{cases}$$

Then we clearly have the following.

Lemma 1

$$S_1+S_2+\cdots+S_{\tau_n}=R_1+R_2+\cdots+R_{\tau_n}-\tau_n-\tau_n=n-R_n^*-\tau_n, \tag{2.1}$$

$$\sum_{i=1}^{n-1}\alpha(U_i,U_{i+1})=S_1+S_2+\cdots+S_{\tau_n}+S_n^*, \tag{2.2}$$

$$|(S_1+S_2+\cdots+S_{\tau_n}+S_n^*-\tfrac{1}{2}n)-(\tfrac{1}{2}n-\tau_n)|\leq 1, \tag{2.3}$$

$$\mathbf{E}\,\alpha(U_i,U_{i+1})=\mathbf{E}\,\alpha^2(U_1,U_{i+1})=\tfrac{1}{2}, \tag{2.4}$$

$$\mathbf{E}\,\alpha(U_i,U_{i+1})\alpha(U_j,U_{j+1})=\begin{cases}\frac{1}{6} & \text{if } j=i+1,\\ \frac{1}{4} & \text{if } j>i+1,\end{cases} \tag{2.5}$$

$$\mathbf{E}\left(\sum_{i=1}^{n-1}(\alpha(U_i,U_{i+1})-\tfrac{1}{2})\right)^2=\tfrac{1}{12}(n+1) \tag{2.6}$$

and

$$\mathbf{E}\left(\sum_{i=1}^{n-1}(\alpha(U_i,U_{i+1})-\tfrac{1}{2})\right)^4=\mathrm{O}(n^2). \tag{2.7}$$

Especially the last formula follows from the fact that

$$\mathbf{E}(\alpha(U_i,U_{i+1})-\tfrac{1}{2})(\alpha(U_j,U_{j+1})-\tfrac{1}{2})(\alpha(U_k,U_{k+1})-\tfrac{1}{2})(\alpha(U_l,U_{l+1})-\tfrac{1}{2})=0$$

if $i\leq j\leq k\leq l$ and $\max\{j-i,l-k\}\geq 2$.

Eqs. (2.2), (2.3), (2.7), the Markov inequality and the Borel–Cantelli lemma imply the following lemma.

Lemma 2. *For any* $\varepsilon>0$ *we have*

$$n^{-3/4-\varepsilon}(\tau_n-\tfrac{1}{2}n)\to 0 \quad \text{a.s.} \quad (n\to\infty). \tag{2.8}$$

Our next lemma is a trivial consequence of Theorem 4 of Kuelbs and Philipp [2].

Lemma 3. *One can construct*

(i) *a probability space* $\{\Omega,\mathcal{A},P\}$,

(ii) *a sequence* $U_1,U_2,\ldots$ *of independent, uniform* (0, 1) *r.v.'s and a Wiener process* $\{W(t);t\geq 0\}$ *(both defined on* Ω*) such that*

$$n^{-1/2+\varepsilon}(12^{1/2}\sum_{i=1}^{n-1}(\alpha(U_i,U_{i+1})-\tfrac{1}{2})-W(n))\to 0 \quad \text{a.s.} \quad (n\to\infty) \tag{2.9}$$

for a suitable $\varepsilon>0$.

Eqs. (2.2), (2.3) and (2.9) imply the following lemma.

Lemma 4. *We have*

$$n^{-1/2+\varepsilon}(12^{1/2}(\tfrac{1}{2}n-\tau(n))-W(n))\to 0 \text{ a.s.} \quad (n\to\infty). \tag{2.10}$$

Let $\nu(n)$ be the smallest integer for which $\tau(\nu(n))=n$. Then (2.8) implies the following lemma.

Lemma 5

$$n^{-3/4-\varepsilon}(\nu(n)-2n)\to 0 \text{ a.s.} \quad (n\to\infty). \tag{2.11}$$

Since (2.11) and the continuity properties of the Wiener process imply

$$\lim_{n\to\infty} n^{-3/8-\varepsilon}(W(\nu(n))-W(2n))=0 \text{ a.s.},$$

replacing n by $\nu(n)$ in (2.10) we get the following lemma.

Lemma 6

$$\nu(n)^{-1/2+\varepsilon}(12^{1/2}(\tfrac{1}{2}\nu(n)-n)-W(\nu(n)))\to 0 \text{ a.s.} \quad (n\to\infty), \tag{2.12}$$

$$n^{-1/2+\varepsilon}(3^{1/2}(\nu(n)-2n)-W(2n))\to 0 \text{ a.s.} \quad (n\to\infty) \tag{2.13}$$

and

$$n^{-1/2+\varepsilon}((\tfrac{3}{2})^{1/2}(\nu(n)-2n)-W_1(n))\to 0 \text{ a.s.} \quad (n\to\infty) \tag{2.14}$$

where the Wiener process $W_1(n)$ is defined by $2^{1/2}W_1(n)=W(2n)$.

By (1.2) we have

$$R_1+R_2+\cdots+R_{\tau_n}-\tau_n=n-\tau_n-R_n^*$$

and replacing n by $\nu(n)$ we get

$$R_1+R_2+\cdots+R_n-n=\nu(n)-n.$$

Having (2.14) we get (1.1).

In order to prove (1.3) we introduce some further notations,

$$\beta(x,y,z)=\begin{cases}1 & \text{if } y<\min\{x,z\},\\ 0 & \text{otherwise,}\end{cases}$$

$$\gamma_i=\alpha(U_i,U_{i+1})-\tfrac{3}{2}\beta(U_i,U_{i+1},U_{i+2}).$$

Then we clearly have the following lemma.

Lemma 7

$$Z_{\psi_n}=S_1+S_2+\cdots+S_{\tau_n}=\sum_{i=1}^{n-1}\alpha(U_i,U_{i+1})-S_n^*, \tag{2.15}$$

$$\psi_n = \begin{cases} \sum_{i=1}^{n-2} \beta(U_i, U_{i+1}, U_{i+2}) & \text{if } S_n^* = 0, \\ \sum_{i=1}^{n-2} \beta(U_i, U_{i+1}, U_{i+2}) + 1 & \text{if } S_n^* > 0, \end{cases} \tag{2.16}$$

$$\mathbf{E}\,\beta(U_i, U_{i+1}, U_{i+2}) = \mathbf{E}\,\beta^2(U_i, U_{i+1}, U_{i+2}) = \tfrac{1}{3}, \tag{2.17}$$

$$\mathbf{E}\,\beta(U_i, U_{i+1}, U_{i+2})\beta(U_j, U_{j+1}, U_{j+2}) = \begin{cases} 0 & \text{if } j = i+1, \\ \tfrac{2}{15} & \text{if } j = i+2, \\ \tfrac{1}{9} & \text{if } j > i+2, \end{cases} \tag{2.18}$$

$$\mathbf{E}\,\alpha(U_i, U_{i+1})\beta(U_j, U_{j+1}, U_{j+2}) = \begin{cases} 0 & \text{if } i = j, \\ \tfrac{1}{3} & \text{if } i = j+1, \\ \tfrac{1}{8} & \text{if } i = j+2, \\ \tfrac{5}{24} & \text{if } i = j-1, \\ \tfrac{1}{6} & \text{otherwise}, \end{cases} \tag{2.19}$$

$$\mathbf{E}\,\gamma_i = 0, \tag{2.20}$$

$$\mathbf{E}\,\gamma_i\gamma_j = \begin{cases} \tfrac{5}{4} & \text{if } j = i, \\ -\tfrac{31}{48} & \text{if } j = i+1, \\ \tfrac{9}{80} & \text{if } j = i+2, \\ 0 & \text{otherwise}, \end{cases} \tag{2.21}$$

$$\mathbf{E}(\gamma_1 + \gamma_2 + \cdots + \gamma_n)^2 = \tfrac{11}{60}n + \tfrac{101}{120}, \tag{2.22}$$

$$n^{-3/4-\varepsilon}|\psi_n - \tfrac{1}{3}n| \to 0 \text{ a.s.} \quad (\varepsilon > 0). \tag{2.23}$$

Lemma 8. *For a suitable $\varepsilon > 0$ we have*

$$n^{-1/2+\varepsilon}\left|2(\tfrac{15}{11})^{1/2} \sum_{i=1}^{n-2} \gamma_i - W(n)\right| \to 0 \text{ a.s.}$$

and

$$n^{-1/2+\varepsilon}\left|2(\tfrac{15}{11})^{1/2}(Z_{\psi(n)} - \tfrac{3}{2}\psi(n)) - W(n)\right| \to 0 \text{ a.s.}$$

in the same sense as it was stated in Lemma 3.

Proof. Since $\{\gamma_i\}$ is an m-dependent sequence (with $m = 3$), our statement is a simple consequence of Theorem 4 of Kuelbs and Philipp [2] which states the validity of the strong invariance principle in case of mixing random variables.

Now, (1.3) can be obtained in the same way as we got (1.1).

3. Proofs of Theorems 2.A and 2.B

Proof of Theorem 2.A. Set

$$A_{ik} = \{\omega : U_{i+1} < U_{i+2} < \cdots < U_{i+k}\}.$$

Then clearly we have

$$\mathbf{P}(A_{ik}) = \frac{1}{k!} \geq (2\pi k)^{-1/2}\left(\frac{e}{k}\right)^k.$$

Further

$$\begin{aligned}\mathbf{P}(M_n < k) &= \mathbf{P}(\overline{A_{0k} + A_{1k} + \cdots + A_{n-k,k}}) \\ &= \mathbf{P}(\bar{A}_{0k}\bar{A}_{1k} \cdots \bar{A}_{n-k,k}) \\ &\leq \mathbf{P}(\bar{A}_{0k}\bar{A}_{kk} \cdots \bar{A}_{jk,k}) = (\mathbf{P}(\bar{A}_{0k}))^{j+1} = \left(1 - \frac{1}{k!}\right)^{j+1} \\ &\leq \exp\left\{-\frac{j+1}{k!}\right\}\end{aligned}$$

where j is the largest integer for which $(j+1)k \leq n$.

Let $k = l_n + 1$ and observe that

$$l_n + 1 \leq \frac{\log n}{b_n} - \tfrac{3}{2} - \varepsilon,$$

and (by definition of b_n)

$$\log \frac{\log n}{b_n} - 1 = b_n.$$

Then

$$\begin{aligned}\frac{j+1}{k!} &= \frac{(j+1)k}{k \cdot k!} \geq \frac{n-k}{k}\frac{1}{k!} \geq \frac{n-k}{k}(2\pi k)^{-1/2}\left(\frac{e}{k}\right)^k \\ &= \frac{n-k}{\sqrt{2\pi}} k^{-3/2} \exp\{-k(\log k - 1)\} \\ &\geq \frac{n-k}{\sqrt{2\pi}} k^{-3/2} \exp\left\{-(l_n + 1)\left(\log \frac{\log n}{b_n} - 1\right)\right\} \\ &= \frac{n-k}{\sqrt{2\pi}} k^{-3/2} \exp\{-(l_n + 1)b_n\} \\ &\geq \frac{n-k}{\sqrt{2\pi}} k^{-3/2} \exp\left\{-\left(\frac{\log n}{b_n} - \tfrac{3}{2} - \varepsilon\right)b_n\right\} \\ &\geq \mathrm{O}((\log n)^{\varepsilon/2}).\end{aligned}$$

Hence

$$P_n = \mathbf{P}(M_n < l_n + 1) \leq \exp\{-\mathrm{O}((\log n)^{\varepsilon/2})\}.$$

Since for any $\theta > 1$

$$\sum_{n=1}^{\infty} P_{[\theta^n]} < \infty,$$

by the Borel–Cantelli lemma we get

$$M_{[\theta^n]} > l_{[\theta^n]} \text{ a.s.}$$

(except finitely many n). Let $[\theta^n] \leq N < [\theta^{n+1}]$. Since $M_N \geq M_{[\theta^n]}$ and

$$l_N(\varepsilon) \leq l_{[\theta^{n+1}]}(\varepsilon) \leq l_{[\theta^n]}(\tfrac{1}{2}\varepsilon)$$

we have the lower estimation in (1.4).

In order to get the upper estimator of (1.4) observe that

$$\begin{aligned}\mathbf{P}\{M_n \geq k\} &= \mathbf{P}(A_{0k} + A_{1k} + \cdots + A_{n-k,k}) \\ &\leq (n-k)\mathbf{P}(A_{0k}) \\ &\leq \frac{n}{k!} \leq \frac{n}{\sqrt{k}}\left(\frac{\mathrm{e}}{k}\right)^k = \frac{n}{\sqrt{k}} \exp\{-k(\log(k-1)\}\end{aligned}$$

and

$$u_n \geq \frac{\log n}{b_n} + \tfrac{1}{2} + \varepsilon.$$

Hence we have

$$\begin{aligned}\mathbf{P}\{M_n \geq u_n\} &\leq n\sqrt{\frac{b_n}{\log n}} \exp\left\{-\left(\frac{\log n}{b_n} + \tfrac{1}{2} + \varepsilon\right) \log \frac{\log n}{\mathrm{e}^{b_n}}\right\} \\ &\leq \mathrm{O}((\log n)^{-1-\varepsilon/2}).\end{aligned}$$

Choosing $n = [\theta^N]$ we get the proof in the same way as above.

Proof of Theorem 2.B. At first we give two lemmas.

Lemma 9. *We have*

$$\mathbf{P}\{M_{2k-1} \geq k\} = \frac{k^2+1}{(k+1)!}.$$

Proof. The proof is trivial.

Lemma 10

$$\mathbf{P}(M_n \geq k) \geq 1 - \left(1 - \frac{k^2+1}{(k+1)!}\right)^{[n/(2k-1)]},$$

$$\mathbf{P}(M_n \leq k) \geq \left(1 - \frac{k^2+1}{(k+1)!}\right)^{2[n/(2k-1)]}.$$

Proof. The proof follows quite elementary by Lemma 9. It is essentially the same as the proof of Theorem 5 of Erdös–Révész [1]. The details will be omitted

Proof of Theorem 2.B (*continued*). The proof of Theorem 2.B can be easily obtained by applying Lemma 10 and the Borel–Cantelli lemma. It is again essentially the same as the proof of Lemma 4 of Erdös–Révész [1] and the details will be omitted.

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