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Structure of some noetherian SI rings

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Abstract

We describe the structure of rings over which every cyclic (or finitely generated) right module is a direct sum of a projective module and an injective module.

1. Introduction

A ring R is called a right (left) SI ring if every singular right (left) R-module is injective. SI rings were initially introduced and investigated by Goodearl [9], and the structure of these rings was described as follows (cf. [9, Theorem 3.11]):

A ring R is right SI if and only if R is right nonsingular, and $R = K \oplus R_1 \oplus \cdots \oplus R_m$ (a ring-direct sum) where $K/\operatorname{Soc}(K)$ is semisimple and each R_i is Morita equivalent to a simple right noetherian domain D_i such that for every nonzero right ideal $C_i \subseteq D_i$, D_i/C_i is semisimple.

Concerning SI rings, Smith [19] introduced right (left) CDPI rings, i.e., rings each of whose cyclic right (left) modules is a direct sum of a projective module and an injective module. The question, if every right CDPI ring is right SI, remained open for several years (1979–1991). Finally in [17], as an application of their major theorem on finiteness of uniform dimension of certain cyclic extending

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modules, Osofsky and Smith have shown that a right CDPI ring is indeed right noetherian and right SI (cf. [17, Proposition 2]). On the other hand, Smith [19, Example 4.12] has proved that a right artinian right SI ring need not be right CDPI.

From these results it is natural to ask the question: When is a right SI ring right CDPI? In this note we will provide an answer to this question (Theorem 6).

Rings over which every finitely generated right module is a direct sum of a projective module and an injective module were initially investigated also by Smith in [18]. He called these rings right FGPI rings. In [12, Theorem 6] it was shown that if $\mathrm{Soc}(R_R) = 0$, then R is right FGPI iff R is left FGPI iff R is right and left SI. In this note we will describe the structure of all right FGPI rings (Theorem 8).

Following Faith [7], a ring R is called a right PCI ring if every cyclic right R-module is either isomorphic to R_R or injective. A right PCI ring is either semisimple artinian or a simple right noetherian right hereditary domain such that every singular right module is semisimple and injective (see Faith [7] and Damiano [4]). Hence every right PCI domain is right SI. On the other hand, by a result of Osofsky–Smith [17], for a ring R, if all cyclic singular right modules are injective, then all singular right modules are injective. This implies that a right SI domain is right PCI. Thus for domains, the two concepts of SI and PCI are equivalent. A right PCI domain (= right SI domain), which is not a division ring, was constructed by Cozzens in [3].

2. The results

Throughout this note we consider associative rings with identity and unitary modules. For a ring R and an R-module M we write M_R to indicate that M is a right R-module. The socle and the Jacobson radical of M are denoted respectively by $\mathrm{Soc}(M)$ and $J(M)$. The injective hull and the uniform dimension of M are denoted by $E(M)$ and u-dim(M), respectively.

A submodule C of a module M is called a closed submodule of M if C is itself a maximal essential extension of C in M. The module M is called an extending module (or a CS module) if every closed submodule of M is a direct summand. A ring R is called a right extending ring if R_R is an extending module. Clearly, every (quasi-) injective module is extending. For basic properties of injective and extending modules we refer to [1,6,8,15].

If a module M has finite composition length, we will denote its length by $l(M)$.

We first consider the artinian case. Namely, let A be a right artinian right CDPI ring. We write A in the form

$$A_A = A_1 \oplus \cdots \oplus A_n,$$

where each A_i is an indecomposable right ideal of A. In particular each A_i is a local right A-module. We define the following direct summands of A_A as follows:

- $B = B_1 \oplus \cdots \oplus B_m$ such that $B_i \in \{A_1, \ldots, A_n\}$ and the following properties hold:
  (B1) Each B_i is uniform with $l(B_i) \leqslant 2$.
  (B2) If $l(E(B_i)) \leqslant 2$, then $E(B_i)$ is projective.
  (B3) If $3 \leqslant l(E(B_i)) < \infty$ and $l(B_i) = 2$, then
    (b_3) there is no $B_{i'} \in \{A_1, \ldots, A_n\}$ $(i' \neq i)$ with $l(B_{i'}) = 2$ and $\mathrm{Soc}(B_{i'}) \cong \mathrm{Soc}(B_i)$, and
    (b'_3) a simple $A_h \in \{A_1, \ldots, A_n\}$ with $3 \leqslant l(E(A_h)) < \infty$ belongs to $\{B_1, \ldots, B_m\}$ iff A_h is isomorphic to the socle of such a B_i.
- $C = C_1 \oplus \cdots \oplus C_k$ such that $C_j \in \{A_1, \ldots, A_n\}$ and the following properties hold:
  (C1) Each C_j is uniform with $l(C_j) \leqslant 2$.
  (C2) For each C_j, $3 \leqslant l(E(C_j)) < \infty$.
  (C3) If for a C_j, $l(C_j) = 2$, then
    (c_3) there exists $C_{j'}$, $j' \neq j$, such that $l(C_{j'}) = 2$ and $\mathrm{Soc}(C_j) \cong \mathrm{Soc}(C_{j'})$, and
    (c'_3) a uniform $A_h \in \{A_1, \ldots, A_n\}$ belongs to $\{C_1, \ldots, C_k\}$ iff $\mathrm{Soc}(A_h)$ is isomorphic to the socle of such a C_j.
- $D = D_1 \oplus \cdots \oplus D_t$ such that $D_l \in \{A_1, \ldots, A_n\}$ and the following properties hold:
  (D1) Each D_l is either simple or u-dim$(D_l) \geqslant 2$.
  (D2) If D_l is simple, then $l(E(D_l)) < \infty$ and D_l is not embedded in either B or C.

Note that, by the definition of B, $\{B_1, \ldots, B_m\}$ also contains all uniform A_k from $\{A_1, \ldots, A_n\}$ with $l(E(A_k)) = \infty$. For the existence of such A_k's see Example 3.6 in the next section.

We conclude that $A_A = B \oplus C \oplus D$. By [17, Proposition 2], A is right SI. Hence A is right hereditary (cf. [9, Proposition 3.3]). In particular, A is right nonsingular. From this and the properties of B, C, D, there is no nonzero A-homomorphism between them. Hence $BC = CB = CD = DC = DB = BD = 0$, i.e., the following lemma holds.

Lemma 1. *$A = B \oplus C \oplus D$ is a ring-direct sum.*

Lemma 2. *B is a right extending ring.*

Proof. We can write $B = Q_1 \oplus Q_2 \oplus Q_3$ where Q_1 is the direct sum of all B_i that satisfy $l(E(B_i)) \leqslant 2$; Q_2 is the direct sum of all B_j with $l(E(B_j)) = \infty$, and

Q_3 is the direct sum of the remainder B_t, i.e., $3 \leqslant l(E(B_t)) < \infty$. As B is right nonsingular, it is easy to check that Q_i, $i = 1, 2, 3$, are ideals of B. Each Q_i is a right CDPI ring. By [6, Lemma 8.14], Q_1 is a right extending ring.

For Q_2, let U be a closed right ideal of Q_2. Then Q_2/U is a cyclic nonsingular module. Hence every minimal submodule S of Q_2/U embeds in Q_2. Whence $l(E(S)) = \infty$. Since $E(S)/S$ is semisimple, it follows that $E(S)$ can not be cyclic. Therefore, Q_2/U does not contain nonzero injective submodules. Thus Q_2/U must be projective. Whence U splits in Q_2, proving that Q_2 is right extending.

For Q_3, consider a $B_t \subseteq Q_3$, and let $[B_t]$ denote the direct sum of all B_k such that $E(B_k) \cong E(B_t)$. Clearly $[B_t]$ is a ring-direct summand of Q_3. $[B_t]$ is not a semisimple ring, because otherwise, as a semisimple ring-direct summand of B, $[B_t]$ is injective, a contradiction to the definition of Q_3. Hence there is a $B_k \subseteq [B_t]$ with $l(B_k) = 2$. By (B3), $[B_t] = B_k \oplus T$ where $T_{[B_t]}$ is a semisimple module. Now let V be a closed right ideal of $[B_t]$. If $B_k \cap V \neq 0$, then $B_k \subseteq V$. By modularity, and since $T_{[B_t]}$ is semisimple, we conclude that V is a direct summand of $[B_t]$. If $B_k \cap V = 0$, then by the same way we obtain that $B_k \oplus V$ is a direct summand of $[B_t]$ which implies that V is a direct summand of $[B_t]$. This shows that $[B_t]$ is a right extending ring. Since Q_3 is obviously a ring-direct sum of finitely many rings which are constructed in a similar way as $[B_t]$, it follows that Q_3 is a right extending ring. Thus B is right extending, as desired. □

Lemma 3. *For each simple submodule $S \subseteq C$, $l(E(S)) = 3$.*

Proof. We denote by $[C_1]$ the direct sum of all such C_i with $E(C_i) \cong E(C_1)$. Then $[C_1]$ is a ring-direct summand of C. $[C_1]$ is not a semisimple ring, because otherwise, every minimal right ideal of $[C_1]$ would be injective, a contradiction to the definition of C. Hence there is a uniform direct summand of $[C_1]$ that has length 2 (cf. (C1)). We may assume, without loss of generality, that $l(C_1) = 2$. By (C3) there exists a $C_{i_1} \in \{C_1, \ldots, C_k\}$ with $l(C_{i_1}) = 2$, and $\mathrm{Soc}(C_{i_1}) \cong \mathrm{Soc}(C_1)$, $i_1 \neq 1$. We write $[C_1] = C_1 \oplus C_{i_1} \oplus \cdots \oplus C_{i_p} \oplus V$ where V is semisimple, and $l(C_1) = l(C_{i_1}) = \cdots = l(C_{i_p}) = 2$, and all minimal submodules of $[C_1]$ are isomorphic to each other.

Suppose that for each t $(1 \leqslant t \leqslant p)$, $C_1 \oplus C_{i_t}$ does not have closed minimal submodules. Let S be an arbitrary minimal submodule of $C_1 \oplus C_{i_t}$. Then the closure S' of S in $C_1 \oplus C_{i_t}$ has length at least 2. Therefore, either $C_1 \oplus C_{i_t} = S' \oplus C_{i_t}$ or $C_1 \oplus C_{i_t} = C_1 \oplus S'$, and so $C_1 \oplus C_{i_t}$ is an extending module. Then by [6, Lemma 7.3(ii)] C_1 is C_{i_t}-injective for each $1 \leqslant t \leqslant p$. By [1, Proposition 16.13(2)], and since each minimal submodule of V is isomorphic to $\mathrm{Soc}(C_{i_t})$, C_1 is $(C_{i_1} \oplus \cdots \oplus C_{i_p} \oplus V)$-injective. Now, let f be an isomorphism $\mathrm{Soc}(C_1) \to \mathrm{Soc}(C_{i_t})$. Then $T = \{x + f(x) \mid x \in \mathrm{Soc}(C_1)\}$ is a minimal submodule of $C_1 \oplus C_{i_t}$. By assumption, the closure T' of T in $C_1 \oplus C_{i_t}$ has length at least 2. Because $T' \cap C_1 = T' \cap C_{i_t} = 0$, we have $C_1 \oplus C_{i_t} = T' \oplus C_1 = T' \oplus C_{i_t}$. This implies that $C_1 \cong C_{i_t}$. Thus C_1 is C_1-injective. This together with $(C_{i_1} \oplus \cdots \oplus C_{i_p} \oplus V)$-injectivity of C_1 implies that C_1 is injective, a contradiction to (C2).

Therefore, there is $t \in (1, \ldots, p\}$, say $t = 1$, such that $C_1 \oplus C_{i_1}$ contains a closed minimal submodule U. Notice that $C_1 \oplus C_{i_1}$ is cyclic. Hence the module $(C_1 \oplus C_{i_1})/U$ is cyclic, nonsingular, uniform (cf. for example, [6, 5.10(1)]), and of length 3. Moreover, by the Krull–Schmidt theorem (cf. [1, 12.9]), U can not be a direct summand of $C_1 \oplus C_{i_1}$. Whence $(C_1 \oplus C_{i_1})/U$ must be injective. As $C_1 \cap U = 0$, this implies that $E(C_1) \cong (C_1 \oplus C_{i_1})/U$, and so $l(E(C_1)) = 3$.

We can renumber the direct summands C_i so that $\{C_1, C_2, \ldots, C_q\}$ is a maximal set of $\{C_1, \ldots, C_k\}$ with $l(C_i) = 2$, $\mathrm{Soc}(C_i) \ncong \mathrm{Soc}(C_j)$ for $i \neq j$. Then $C = [C_1] \oplus [C_2] \oplus \cdots \oplus [C_q] \oplus G$ (a ring-direct sum), where each $[C_i]$ is constructed in a similar way as $[C_1]$, and G is semisimple. If $G \neq 0$, then each minimal submodule of G is injective, a contradiction to the definition of C. Hence $G = 0$, and so $C = [C_1] \oplus [C_2] \oplus \cdots \oplus [C_q]$. This shows that $l(E(C_j)) = 3$ for all $j = 1, \ldots, k$, proving Lemma 3. □

Lemma 4. *For each D_l, $l(D_l) = 1$ or 3. If $S \subseteq D_D$ is a minimal submodule, then $l(E(S)) = 2$, and $E(S)$ is not projective.*

Proof. All D_j are local modules. Assume that $l(D_j) \neq 1$, i.e., D_j is not simple. By (D1), $l(D_j) \geqslant 3$. From the structure theorem of (artinian) right SI rings (cf. [9, Theorem 3.11]), and the fact that D_j is local, it follows that $D_j/\mathrm{Soc}(D_j)$ is simple. Hence $l(D_j) = l(\mathrm{Soc}(D_j)) + 1$. It is enough to show that $l(\mathrm{Soc}(D_j)) = 2$. Now let S be a minimal submodule of $\mathrm{Soc}(D_j)$. Then D_j/S is a cyclic local right D-module which can not be projective. Since D is right CDPI, D_j/S must be injective. As D_j/S is indecomposable, $\mathrm{Soc}(D_j/S)$ is simple. This proves that $l(\mathrm{Soc}(D_j)) = 2$, and so $l(D_j) = 3$.

Next, let $\mathrm{Soc}(D_j) = S \oplus T$, where S, T are minimal submodules of D_j. As D_j is local, D_j/T must be injective and not projective. Whence $E(S) \cong D_j/T$, and $l(E(S)) = l(D_j/T) = 2$. We rewrite D in the form

$$D = S_1 \oplus \cdots \oplus S_k \oplus V_1 \oplus \cdots \oplus V_h,$$

where $l(S_1) = \cdots = l(S_k) = 1$, $l(V_1) = \cdots = l(V_h) = 3$, and S_i, $V_j \in \{D_1, \ldots, D_l\}$. Set $\mathrm{Soc}(V_i) = W_1 \oplus W_2$ with simple W_1 and W_2. As observed before, V_i/W_1 and V_i/W_2 are injective. It follows that the injective hull of each V_i $(i = 1, \ldots, h)$ is a direct sum of two indecomposable submodules $V_{i_1}^*$ and $V_{i_2}^*$ with $l(V_{i_1}^*) = l(V_{i_2}^*) = 2$. Now we consider S_i. If S_i is not embedable in $V_1 \oplus \cdots \oplus V_h$, then $(V_1 \oplus \cdots \oplus V_h)S_i = 0$. We may assume that $S_1, \ldots, S_t$ are not embedable in $V_1 \oplus \cdots \oplus V_h$, but $S_{t+1}, \ldots, S_k$ are. Then $(S_{t+1} \oplus \cdots \oplus S_k \oplus V_1 \oplus \cdots \oplus V_h)(S_1 \oplus \cdots \oplus S_t) = 0$. From this it follows that $S_1 \oplus \cdots \oplus S_t$ is an ideal of D. Since D is right nonsingular, we can similarly show that $(S_1 \oplus \cdots \oplus S_t)(S_{t+1} \oplus \cdots \oplus S_k \oplus V_1 \oplus \cdots \oplus V_h) = 0$, and so $S_{t+1} \oplus \cdots \oplus S_k \oplus V_1 \oplus \cdots \oplus V_h$ is also an ideal, and hence a ring-direct summand of D. This shows that $S_1, \ldots, S_t$ are injective right D-modules, which is a contradiction to the definition of D. Hence each S_i is embedable in some V_j, so the injective hull of each S_i has composition length 2, and is not projective. This completes the proof of Lemma 4. □

Lemma 5. *Let T be a right artinian right SI ring with $T_T = T_1 \oplus \cdots \oplus T_m \oplus T'$ ($m \geqslant 2$) where each T_i is uniform, $l(T_i) = 2$, and T'_T is semisimple. Assume further that for each minimal right ideal $S \subseteq T$, $l(E(S_T)) = 3$. If $V_1 \oplus \cdots \oplus V_q$ ($q \geqslant 2$) is a direct summand of T_T such that each V_i is uniform of length 2, then for any closed minimal submodule U of $V_1 \oplus \cdots \oplus V_q$, the factor module $(V_1 \oplus \cdots \oplus V_q)/U$ contains a nonzero injective submodule.*

Proof. We prove this statement by induction on q. For $q = 2$, $(V_1 \oplus V_2)/U$ is nonsingular uniform (see, for example, [6, 5.10(1)]) and of length 3. It follows that the minimal submodule of $(V_1 \oplus V_2)/U$ embeds in T_T. Hence $(V_1 \oplus V_2)/U$ is injective. From here we can follow the second step of the induction proof of [11, Claim 1, p. 146]. □

Now we can state the main theorem of this paper.

Theorem 6. *For a ring R the following conditions are equivalent*:

(a) *Every cyclic right R-module is a direct sum of a projective module and an injective module, i.e., R is a right CDPI ring.*

(b) *R has a ring-direct decomposition*

$$R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5,$$

where each R_i is a right SI ring. Furthermore:

- (i) *R_1 is right extending and right artinian.*
- (ii) *R_2 is right artinian with the following properties*:
  - (ii_1) *For each primitive idempotent $e \in R_2$, $l(eR_2) \leqslant 2$.*
  - (ii_2) *For any minimal right ideal S of R_2, $l(E(S)) = 3$.*
  - (ii_3) *If $e \in R_2$ is a primitive idempotent with $l(eR_2) = 2$, then there exists at least one other primitive idempotent $f \in R_2$ such that $l(fR_2) = 2$, $ef = fe = 0$, and $\mathrm{Soc}(eR_2) \cong \mathrm{Soc}(fR_2)$.*
- (iii) *R_3 is right artinian with the following properties*:
  - (iii_1) *For each primitive idempotent $e \in R_3$, $l(eR_3) = 1$ or $l(eR_3) = 3$.*
  - (iii_2) *For any minimal right ideal S of R_3, $l(E(S)) = 2$ and $E(S)$ is not projective.*
- (iv) *R_4 is a ring-direct sum of finitely many simple right and left SI rings with zero right socle and each with right uniform dimension $\geqslant 2$.*
- (v) *R_5 is a ring-direct sum of finitely many right SI domains which are not division rings.*

Proof. (a) $\Rightarrow$ (b). By [17, Proposition 2], R is right noetherian right SI. Hence by [9, Theorem 3.11] (see the Goodearl's theorem mentioned in the introduction), R has the ring-direct decomposition

$$R = A \oplus T,$$

where A is a right artinian right CDPI ring and T is a right CDPI ring with $\mathrm{Soc}(T) = 0$. By Lemma 1, $A = B \oplus C \oplus D$ (a ring-direct sum), where B, C, and D are defined as before Lemma 1. Set $R_1 = B$, $R_2 = C$, and $R_3 = D$. By Lemmas 2–4 we have $R_1 \in$ (i), $R_2 \in$ (ii), $R_3 \in$ (iii) of Theorem 6.

We have $R = R_1 \oplus R_2 \oplus R_3 \oplus T$. Again by [9, Theorem 3.11], $T = T_1 \oplus \cdots \oplus T_k$ where each T_i is a simple right noetherian right SI ring which is Morita equivalent to a right SI domain (and for each T_i, $\mathrm{Soc}(T_i) = 0$). Let U be a closed right ideal of T_i. Then T_i/U is a nonsingular right T_i-module. If T_i/U contains a nonzero injective submodule, then it is cyclic and isomorphic to the injective hull of some right ideal of T_i. As T_i is simple and right noetherian, we conclude that the injective hull of $T_{i_{T_i}}$ is finitely generated which implies that T_i is semisimple artinian (cf. [2, Corollary 1.29]), a contradiction. Hence T_i/U does not contain nonzero injective submodules. By (a) T_i/U is projective, so $T_i = U \oplus L$ for some right ideal $L \subseteq T_i$. This shows that each T_i is right extending.

If for some T_i, u-dim$(T_i) \geqslant 2$, then by [13], T_i is left Goldie and left extending. T_i is Morita equivalent to a right SI domain, say D_i. Hence D_i is left Goldie. In other words, D_i is a left Ore, right PCI domain. By [7, Theorem 22], D_i is left noetherian, and hence left PCI (see also [14, Corollary 4.3]). Equivalently, D_i is left SI. Thus T_i is left SI.

Now we renumber the direct summands T_i so that $T = T_1 \oplus \cdots \oplus T_m \oplus T_{m+1} \oplus \cdots \oplus T_k$ where all $T_1, \ldots, T_m$ have uniform dimension $\geqslant 2$, and u-dim$(T_{m+1}) = \cdots =$ u-dim$(T_k) = 1$. Set $R_4 = T_1 \oplus \cdots \oplus T_m$, $R_5 = T_{m+1} \oplus \cdots \oplus T_k$. Then we have a ring-direct decomposition $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5$ as desired.

(b) $\Rightarrow$ (a). It is clear that R_1 and R_5 have property (a).

We now consider R_2. By (ii), $R_2 = R_{21} \oplus \cdots \oplus R_{2k}$ where each R_{2i} is uniform, nonsingular, $l(R_{2i}) \leqslant 2$, and $l(E(R_{2i})) = 3$. Let X be a cyclic right R_2-module. Then $X = Y \oplus I$ where I is a maximal injective submodule of X. Since R_2 is right SI, I contains the singular submodule of X. Therefore Y is nonsingular and Y does not contain nonzero injective submodules. Hence there is a closed submodule U of R_2 such that $Y \cong R_2/U$. Our aim is to prove that Y_{R_2} is projective, implying that R_2 satisfies (a). We can write R_2 in the form $R_2 = V_1 \oplus \cdots \oplus V_h \oplus W$ where each $V_i \in \{R_{21}, \ldots, R_{2k}\}$ with $l(V_i) = 2$ and W is semisimple. Then $R_2/U = [(V_1 \oplus \cdots \oplus V_h) + U]/U + (W + U)/U$. Since $(W + U)/U$ is semisimple, $(W + U)/U = [((V_1 \oplus \cdots \oplus V_h) + U)/U \cap (W + U)/U] \oplus W'$ where W' is a submodule of $(W + U)/U$, that is projective and semisimple. Therefore $R_2/U = [(V_1 \oplus \cdots \oplus V_h) + U]/U \oplus W'$. Let $H = (V_1 \oplus \cdots \oplus V_h) \cap U$. Then $[(V_1 \oplus \cdots \oplus V_h) + U]/U \cong (V_1 \oplus \cdots \oplus V_h)/H$. Hence the projectivity of Y_{R_2} will follow from the following statement.

Claim. *Let $V_1 \oplus \cdots \oplus V_t$ ($t \geqslant 1$) be a direct summand of $(R_2)_{R_2}$ such that every V_i is a uniform right ideal with $l(V_i) = 2$, and let H be a closed submodule of $V_1 \oplus \cdots \oplus V_t$. If $(V_1 \oplus \cdots \oplus V_t)/H$ contains no nonzero injective submodules, then $(V_1 \oplus \cdots \oplus V_t)/H$ is projective.*

We prove this by induction on t. For $t = 1$ it is clear, because in this case either $H = 0$, or $H = V_1$.

Assume that the Claim is true for some $t \geqslant 1$. Let H be a closed submodule of $V_1 \oplus \cdots \oplus V_{t+1}$ such that $(V_1 \oplus \cdots \oplus V_{t+1})/H$ does not contain nonzero injective submodules. If $H = 0$ or $(V_1 \oplus \cdots \oplus V_{t+1})/H = 0$, we are done. Hence we consider only the case that $H \neq 0$, and $(V_1 \oplus \cdots \oplus V_{t+1})/H \neq 0$. Since $V_1 \oplus \cdots \oplus V_t \oplus V_{t+1}$ is nonsingular, either $H \supseteq V_{t+1}$ or $H \cap V_{t+1} = 0$.

If $H \supseteq V_{t+1}$, then $(V_1 \oplus \cdots \oplus V_h \oplus V_{t+1})/H \cong (V_1 \oplus \cdots \oplus V_t)/H'$ where $H' = (V_1 \oplus \cdots \oplus V_t) \cap H$ is a closed submodule of $V_1 \oplus \cdots \oplus V_t$. By the induction hypothesis, $(V_1 \oplus \cdots \oplus V_t)/H'$ is projective. We are done in this case.

For $H \cap V_{t+1} = 0$, we have two cases: Either $(V_1 \oplus \cdots \oplus V_t) \cap H = 0$ or $(V_1 \oplus \cdots \oplus V_t) \cap H \neq 0$.

(1) If $(V_1 \oplus \cdots \oplus V_t) \cap H = 0$, then H is uniform of length 1 or 2. If $l(H) = 1$, i.e., H is simple, then by Lemma 5, $(V_1 \oplus \cdots \oplus V_t \oplus V_{t+1})/H$ contains a nonzero injective submodule, a contradiction. Hence $l(H) = 2$. Then $(V_1 \oplus \cdots \oplus V_t) \oplus H = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1}$. This shows that $(V_1 \oplus \cdots \oplus V_t \oplus V_{t+1})/H$ is projective.

(2) For $(V_1 \oplus \cdots \oplus V_t) \cap H \neq 0$, set $K = (V_1 \oplus \cdots \oplus V_t) \cap H$. Hence K is closed in $V_1 \oplus \cdots \oplus V_t$. Since $(V_1 \oplus \cdots \oplus V_t)/K$ embeds in $(V_1 \oplus \cdots \oplus V_{t+1})/H$, $(V_1 \oplus \cdots \oplus V_t)/K$ does not contain nonzero injective submodules. Hence by the induction hypothesis, $(V_1 \oplus \cdots \oplus V_t)/K$ is projective. Therefore, K splits in $V_1 \oplus \cdots \oplus V_t$. By Krull–Schmidt Theorem, or by applying [1, 28.15], we have (after renumbering the summands if necessary) $V_1 \oplus \cdots \oplus V_t = V_1 \oplus \cdots \oplus V_l \oplus K$. It is clear that $l \leqslant t - 1$, and $V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} = (K \oplus V_1 \oplus \cdots \oplus V_l) \oplus V_{t+1}$. Hence $(V_1 \oplus \cdots \oplus V_t \oplus V_{t+1}/H \cong (V_1 \oplus \cdots \oplus V_l \oplus V_{t+1})/H'$ where $H' = (V_1 \oplus \cdots \oplus V_l \oplus V_{t+1}) \cap H$. Since $l \leqslant t - 1$, we can use the induction hypothesis to get that $(V_1 \oplus \cdots \oplus V_l \oplus V_{t+1})/H'$ is projective. This completes the proof of the claim, and therefore R_2 satisfies (a).

For R_3 we see that R_3 has all properties of C in [11, Theorem 7]. Moreover, R_3 is right nonsingular, hence by [11, Corollary 14], even every right R_3-module is a direct sum of a projective module and an injective module.

Considering R_4 we may assume that $V = R_4$ is a simple left and right SI ring (with zero socle). Then the classical right quotient ring Q of V is also the classical left quotient ring of V. Let X_V be any cyclic right V-module. Then $X = Y \oplus W$ where W is the maximal injective submodule of X. Since every singular right V-module is injective, Y is a nonsingular cyclic module. Hence Y is embedable in Q_V, i.e., $Y \cong yV$ for some $y \in Q$. Since Q is also the classical left quotient ring of V, $y = a^{-1}b$ ($a, b \in V$, with a regular). Hence $Y \cong bV \subseteq V$. This together with the fact that V is right (and left) hereditary (cf. [9, Proposition 3.3]) shows that Y is projective. Hence R_4 satisfies (a). □

Notice that by the above proof we can state (ii) of Theorem 6 as follows:

(ii′) R_2 is a finite ring-direct sum of indecomposable right artinian rings each of which is not right extending and has the following properties:
- (ii$_1$) For each primitive idempotent $e \in R_2$, $l(eR_2) \leqslant 2$.
- (ii$_2$) For any minimal right ideal S of R_2, $l(E(S)) = 3$.

The following is an immediate consequence of [9, Theorem 3.11] and Theorem 6.

Corollary 7. *Every right and left SI ring R with* $\mathrm{Soc}(R_R) = 0$ *is right and left CDPI, right and left extending.*

We remark that in [12, pp. 45–46] we also mentioned the question of describing the structure of a right CDPI ring R and expected to show that the (maximal) artinian direct summand A of R is right extending. This expectation was wrong, and it took us a long time to establish that in fact A contains a right extending direct summand B (cf. Lemma 2). The other summand is, in general, nonzero and not right extending (see the existence of such a ring in Section 3.2).

Theorem 8. *For a ring R the following conditions are equivalent*:

(a) *Every finitely generated right R-module is a direct sum of a projective module and an injective module, i.e., R is a right FGPI ring.*
(b) *R has a ring-direct decomposition $R = A_1 \oplus A_2 \oplus A_3 \oplus T$, where each A_i is a right SI ring. Moreover*:
  - (i) *A_1 is a right and left serial, right and left artinian ring with $J(A_1)^2 = 0$.*
  - (ii) *A_2 is a right artinian ring such that for each primitive idempotent $e \in A_2$, eA_2 is uniform, $l(eA_2) \leqslant 2$, and $l(E(eA_2)) = 3$.*
  - (iii) *A_3 is right artinian. For each primitive idempotent $e \in A_3$, either eA_3 is simple or $l(eA_3) = 3$. Moreover, if S is minimal right ideal of A_3, then $l(E(S)) = 2$ and $E(S)$ is not projective.*
  - (iv) *T is a right and left SI ring with* $\mathrm{Soc}(T_T) = 0$.

Proof. (a) ⇒ (b). Let R be a ring such that every finitely generated right R-module is a direct sum of a projective module and an injective module. By Theorem 6, $R = A \oplus T$ (a ring-direct sum) where A is right artinian right SI, T is right SI with $\mathrm{Soc}(T_T) = 0$. Hence by [12, Theorem 6], T is left SI. By [11, Corollary 14], $A = A_1 \oplus A_2 \oplus A_3$ (a ring-direct sum), where the A_1, A_2, and A_3 satisfy (i), (ii), and (iii) in Theorem 6, respectively.

(b) ⇒ (a) follows from [12, Theorem 6] and [11, Corollary 14]. □

Notice that by [11, Corollary 14], every right A_i-module $(i = 1, 2, 3)$ is a direct sum of a projective module and an injective module. But not all infinitely generated right (or left) modules over T have this decomposition property (cf. [11, Theorem 5]). On the other hand, by a result of [5] (see also [6, 13.5]), every right (or left) A_1-module is extending.

3. Examples

3.1. Let

$$T = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix},$$

where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. Then T is right and left artinian, right and left hereditary, right and left SI. Moreover, we can easily check that T is right extending. Hence this ring T is an example for the ring R_1 in Theorem 6. Another example for the ring R_1 of Theorem 6, that has an infinitely generated right injective hull, is given in Section 3.6 below.

3.2. Let U be the ring

$$\begin{bmatrix} \mathbb{C} & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{bmatrix}.$$

Then U is a right (and left) SI ring. Write U in the form $U = e_{11}U \oplus e_{22}U \oplus e_{33}U$, where

$$e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad e_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is clear that $e_{33}U \cong \operatorname{Soc}(e_{11}U) \cong \operatorname{Soc}(e_{22}U)$, and $l(e_{11}U) = l(e_{22}U) = 2$. Moreover,

$$E(e_{11}U) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and hence $l(E(e_{11}U)) = 3$. This shows that U is an example of the ring R_2 of Theorem 6. We can further show that U is not right extending. Namely, suppose on the contrary that U is right extending, then $e_{11}U \oplus e_{22}U$ is an extending right U-module. Hence by [6, 7.3(ii)], $e_{11}U$ is $e_{22}U$-injective. Let

$$L = \left\{ \begin{bmatrix} 0 & 0 & r \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \,\middle|\, r \in \mathbb{C} \right\}.$$

Then L is a minimal submodule of $e_{11}U \oplus e_{22}U$. There are two possibilities:

3.2.1. L is closed in $e_{11}U \oplus e_{22}U$. Hence L is a direct summand of $e_{11}U \oplus E_{22}U$. This is impossible by Krull–Schmidt Theorem (cf. [1, 12.9]).

3.2.2. L is not closed in $e_{11}U \oplus e_{22}U$. Then the closure L' of L in $e_{11}U \oplus e_{22}U$ has length at least 2. Therefore $e_{11}U \oplus e_{22}U = L' \oplus e_{22}U = e_{11}U \oplus L'$. This implies that $e_{11}U \cong e_{22}U$. Thus by [1, 16.13(2)], $e_{11}U$ is $(e_{11}U \oplus e_{22}U \oplus e_{33}U = U)$-injective, a contradiction.

3.3. If we take V to be the ring

$$\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix},$$

then V is left and right artinian, and nonsingular. However, V is not right extending. Write $V = e_{11}V \oplus e_{22}V$ where $e_{11}V$ is a local right V-module with u-dim$(e_{11}V) = 2$, $l(e_{11}V) = 3$ and $e_{22}V$ is simple. Since $V/J(V)$ is commutative, and V is left serial, every uniform right V-module is uniserial (cf. [10, Theorem 3.2]). Let S be a nonsingular simple right V-module. As V is a (right and left) SI-ring, $E(S)/S$ is semisimple (clearly, $l(E(S)) \geqslant 2$). Since $E(S)$ is uniserial, $E(S)/S$ is simple. Hence $l(E(S)) = 2$. Thus, V is an example for the ring R_3 of Theorem 6. Note that V is a left CDPI ring.

The above argument for V can be applied to show that the ring T in Section 3.1 is left CDPI.

3.4. Let C be any PCI domain (= SI domain) constructed in [3], and let $M_n(C)$ be the full $(n \times n)$-matrix ring over C. Then $M_n(C)$ is right and left hereditary, right and left noetherian. Hence by [6, 12.18], $M_n(C)$ is right and left extending. Thus, for $n > 1$, $M_n(C)$ is an example of the ring R_4 in Theorem 5.

3.5. The right (and left) SI domain constructed in [3] is an example for the ring R_5 of Theorem 6. However, it is unknown if there is a right SI domain which is not left SI.

3.6. Let

$$W = \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix},$$

where $\mathbb{Q}$ is the field of rational numbers. Then W is a right extending, right CDPI ring, in particular it is also an example of the ring R_1 in Theorem 6. Furthermore, by [9, Proposition 3.1], W is left SI. However, W (with an essential left socle) is not left artinian. Hence by Theorem 6, W is not a left CDPI ring.

Unlike the rings T in Section 3.1, U in Section 3.2 and V in Section 3.3, the right injective hull of the ring W in Section 3.6 is an infinitely generated right W-module. In particular, the ring W provides an example of a right artinian ring which has a uniform infinitely generated right module.

Moreover, though W is a right extending ring, $W \oplus W$ is not an extending right W-module, because, otherwise W would be a right co-H ring. By [16], W must be left artinian, but this is impossible.

3.7. Let $\mathbb{H}$ be the algebra of quaternions over $\mathbb{R}$. Then the ring

$$Y = \begin{bmatrix} \mathbb{R} & \mathbb{H} \\ 0 & \mathbb{R} \end{bmatrix}$$

is right and left artinian, right and left SI. However, since the indecomposable direct summand $e_{11}Y$ of Y_Y and the indecomposable direct summand Ye_{22} of ${}_YY$ both have composition length 5, Y is neither right nor left CDPI (cf. Theorem 6).

Another example for a right and left artinian right and left SI ring, that is neither right nor left CDPI, is the ring

$$\begin{bmatrix} \mathbb{R} & 0 & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{bmatrix}.$$

This ring is a subring of the right CDPI ring U in Section 3.2.

Examples in Sections 3.6 and 3.7 suggest the following question.

Question. Let R be a right and left artinian, right and left SI ring. Is R necessarily left CDPI if R is right CDPI?

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