Shanks's Transformation Revisited

B. Beckermann, A. Neuber, and G. Mühlbach
Institut für Angewandte Mathematik
Universität Hannover
Welfengarten 1
D-3000 Hannover 1, Germany

Submitted by Richard A. Brualdi

ABSTRACT

A unified and self-contained approach to the block structure of Shanks's table and its cross rules is presented. Wynn's regular and Cordellier's singular cross rules are derived by the Schur-complement method in a unified manner without appealing to Padé approximation. Moreover, by extending the definition of Shanks's transformation to certain biinfinite sequences and by introducing a parameter it is possible to get more consistency with respect to Möbius transformations. It is well known that Padé approximants in general don't have this property.

---

0. NOTATION

By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ we denote the sets of integers and of positive and nonnegative integers, respectively. $\mathbb{K}$ denotes the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers, respectively. $\mathbb{K}^{m\times k}$ is the set of all matrices with entries from $\mathbb{K}$ having m rows and k columns. $\overline{\mathbb{K}} := \mathbb{K} \cup \{\infty\}$ is the one-point compactification of $\mathbb{K}$, where we adopt the conventions

$$\frac{a}{\infty} := 0 \qquad \text{for} \quad a \in \mathbb{K},$$

$$a + \infty = \infty + a := \infty \qquad \text{for} \quad a \in \mathbb{K},$$

$$\frac{a}{0} := \infty \qquad \text{for} \quad a \in \overline{\mathbb{K}}, \quad a \neq 0.$$

LINEAR ALGEBRA AND ITS APPLICATIONS 173:191–219 (1992)

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

For a biinfinite sequence $\underline{c} = (c_n)_{n\in\mathbb{Z}}$,

$$H_k^{(n)} := H_k^{(n)}(\underline{c}) := \begin{pmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+k} \\ \vdots & \vdots & & \vdots \\ c_{n+k-1} & c_{n+k} & \cdots & c_{n+2k-2} \end{pmatrix}$$

is the $k \times k$ *Hankel matrix* of $\underline{c}$ with *starting element* c_n. By

$$\overline{H}_k^{(n)} := \overline{H}_k^{(n)}(\underline{c}) := \begin{pmatrix} c_n & \cdots & c_{n+k-1} & 1 \\ \vdots & & \vdots & \vdots \\ c_{n+k-1} & \cdots & c_{n+2k-2} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$$

we denote the corresponding *bordered Hankel matrix*. If the sequence $\underline{c}$ is fixed, we will drop the argument. If $\beta \in \mathbb{K}$, by $\underline{\beta}$ we denote the constant sequence $(\beta_n)_{n\in\mathbb{Z}}$ with $\beta_n = \beta$ for all $n \in \mathbb{Z}$. With regard to Hankel matrices the rows and columns of any matrix $A = (a_{ij}) \in \mathbb{K}^{m\times k}$ are listed as

$$A = A\begin{pmatrix} 0,\ldots,k-1 \\ 0,\ldots,m-1 \end{pmatrix} = \begin{pmatrix} a_{0,0} & \cdots & a_{0,k-1} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,k-1} \end{pmatrix}$$

if nothing is said to the contrary. $|H| :=\det H$ denotes the determinant of a square matrix H. By convention, $|H| = 1$ if H is the empty matrix.

1. SCHUR COMPLEMENTS

To be self-contained, we give a brief introduction to Schur complements and their properties (for proofs and further details see, e.g., [4]). If A is a rectangular matrix partitioned as

$$A = \begin{pmatrix} P & F \\ G & H \end{pmatrix},$$

where P is a nonsingular square *pivot* matrix, then the matrix

$$A/P := H - GP^{-1}F$$

is called the *Schur complement* of P in A. It is obtained by one Gauss block elimination step eliminating G or F by

row elimination:

$$\begin{pmatrix} I & 0 \\ -GP^{-1} & I \end{pmatrix} \begin{pmatrix} P & F \\ G & H \end{pmatrix} = \begin{pmatrix} P & F \\ 0 & A/P \end{pmatrix}$$

or by
column elimination:

$$\begin{pmatrix} P & F \\ G & H \end{pmatrix} \begin{pmatrix} I & -P^{-1}F \\ 0 & I \end{pmatrix} = \begin{pmatrix} P & 0 \\ G & A/P \end{pmatrix}.$$

Here I and 0 denote unit and zero matrices of suitable sizes. More generally, if

$$A = \begin{pmatrix} P & F & J \\ G & H & K \\ M & R & Q \end{pmatrix}$$

and if H is a nonsingular principal minor of A, then

$$A/H := \begin{pmatrix} P - FH^{-1}G & J - FH^{-1}K \\ M - RH^{-1}G & Q - RH^{-1}K \end{pmatrix}$$

is the *Schur complement* of H in A.

By definition, we get the *partition property*

$$\begin{pmatrix} P & F & J \\ G & H & K \\ M & R & Q \end{pmatrix} \Bigg/ P = \begin{pmatrix} \begin{pmatrix} P & F \\ G & H \end{pmatrix}/P & \begin{pmatrix} P & J \\ G & K \end{pmatrix}/P \\ \begin{pmatrix} P & F \\ M & R \end{pmatrix}/P & \begin{pmatrix} P & J \\ M & Q \end{pmatrix}/P \end{pmatrix}.$$

For square matrices A, *Schur's identity*

$$|A/P| = \frac{|A|}{|P|}$$

is immediate from the above elimination equations. Together with the partition property it implies the following *determinantal representation* of the entries of A/P $(0 < k < n)$: if

$$B\begin{pmatrix} 1,\dots,k \\ 1,\dots,k \end{pmatrix} := A\begin{pmatrix} 1,\dots,n \\ 1,\dots,n \end{pmatrix} \Big/ A\begin{pmatrix} k+1,\dots,n \\ k+1,\dots,n \end{pmatrix}$$

then

$$b_{i,j} = \frac{\left| A\begin{pmatrix} j, k+1,\dots,n \\ i, k+1,\dots,n \end{pmatrix} \right|}{\left| A\begin{pmatrix} k+1,\dots,n \\ k+1,\dots,n \end{pmatrix} \right|}.$$

Finally, the *quotient property* of Schur complements is easily proved [10]: if A is partitioned as above, then

$$A/\tilde{P} = \{A/P\}/\{\tilde{P}/P\}, \qquad \text{where} \quad \tilde{P} := \begin{pmatrix} P & F \\ G & H \end{pmatrix},$$

if P and $\tilde{P}$ (containing P) are nonsingular square submatrices of A.

These are all properties of Schur complements we are going to use.

2. INTRODUCTION TO THE PROBLEM AND MAIN RESULTS

Shanks's transformation [12] is known as a nonlinear sequence-to-sequence transformation mapping a sequence $\underline{c} = (c_n)_{n\in\mathbb{N}_0}$ into a family of sequences $(e_k(c_n))_{n\in\mathbb{N}_0}$, $k \in \mathbb{N}_0$. Here $e_k(c_n)$ is determined by the requirement that if $\underline{c}$ satisfies a linear inhomogeneous difference equation of order k

$$\sum_{i=0}^{k} v_i \cdot (c_{n+i} - \gamma) = 0, \qquad n \in \mathbb{N}_0,$$

with constant coefficients v_i such that $v_0 v_k \neq 0$, $\sum_{i=0}^{k} v_i \neq 0$, and with γ a constant, then $e_k(c_n) = \gamma$ for $n \in \mathbb{N}_0$. It is easily verified (see [2] or [14]) that

$$e_k(c_n) = \frac{\left|H_{k+1}^{(n)}(\underline{c})\right|}{\left|H_k^{(n)}\left(\underline{\Delta^2 c}\right)\right|} = -\frac{\left|H_{k+1}^{(n)}(\underline{c})\right|}{\left|\overline{H}_{k+1}^{(n)}(\underline{c})\right|} \tag{1}$$

if the denominator is different from zero. Here $\Delta^2 c$ denotes the sequence of second differences $\Delta^2 c_n := c_{n+2} - 2c_{n+1} + c_n$ of the sequence $\underline{c}$. In general, (1) is taken as definition of *Shanks's transformation* of order k,

$$\underline{c} \to \left(e_k(c_n)\right)_{n \in \mathbb{N}_0}.$$

The second equation of (1) allows a nice geometric interpretation (see [14, p. 136]). As an immediate consequence of the first representation under (1), Shanks's transformation is *homogeneous* and *translative*:

$$e_k(\alpha c_n + \beta) = \alpha e_k(c_n) + \beta$$

$$\text{for all constants } \alpha, \beta \in \mathbb{K} \text{ and all } k, n \in \mathbb{N}_0.$$

Also, it is known [1] that $e_k(c_n) = [n + k/k]_f(1)$, where the right-hand side denotes the value at the point $z = 1$ of the Padé approximant $r = P/Q$ of numerator degree $\partial P = n + k$ and of denominator degree $\partial Q = k$ of the formal power series $f(z) = c_0 + \sum_{n=1}^{\infty}(c_n - c_{n-1})z^n$, if it exists.

It is our aim to give a self-contained discussion of Shanks's transformation, of its block structure, and of computation methods, where no use is made of results from Padé approximation. In order to do this we slightly extend the definition of Shanks's transformation. It is natural to consider biinfinite sequences $\underline{c} = (c_n)_{n \in \mathbb{Z}}$ which are initially constant, i.e., $c_\kappa \neq c_{\kappa-1} = c_{\kappa-2} = c_{\kappa-3} = \cdots$ for some $\kappa \in \mathbb{Z}$, where with no restriction of generality $\kappa = 0$. Hence as the domain of Shanks's transformation we take the following set of sequences:

$$\mathscr{C} := \Big\{\underline{c} = \left(c_n\right)_{n \in \mathbb{Z}} : c_n \in \mathbb{K};$$

$$\text{there exists } \gamma \in \mathbb{K} \text{ such that } c_n = \gamma \text{ for } n < 0;\ c_0 \neq \gamma\Big\}.$$

Sometimes it will be convenient to consider the subset

$$\mathscr{C}_0 := \left\{\underline{c} = \left(c_n\right)_{n \in \mathbb{Z}} : c_n \in \mathbb{K};\ c_n = 0 \text{ for } n < 0;\ c_0 \neq 0\right\}.$$

It is the second quotient in (1), expressing $e_k(c_n)$ as a negative inverse Schur complement (cf. Section 1), that we use as our starting point. Given any sequence $\underline{c} = (c_n)_{n \in \mathbb{Z}} \in \mathscr{C}$, we take as definition of Shanks's transformation of order $k \geqslant -1$

$$\underline{c} \to \left(e_k(c_n)\right)_{n \in \mathbb{Z}},$$

where

$$e_k(c_n) := \begin{cases} U, \text{ undefined}, & \text{if } \left| H_{k+1}^{(n)}(\underline{c}) \right| = \left| \bar{H}_{k+1}^{(n)}(\underline{c}) \right| \\ & \qquad = 0, \\ -\dfrac{\left| H_{k+1}^{(n)}(\underline{c}) \right|}{\left| \bar{H}_{k+1}^{(n)}(\underline{c}) \right|} = \dfrac{-1}{\bar{H}_{k+1}^{(n)}(\underline{c}) / H_{k+1}^{(n)}(\underline{c})} & \text{else.} \end{cases} \tag{2}$$

The sequence $(e_k(c_n))_{n \in \mathbb{Z}}$ forms column k of *Shanks's table of the sequence* $\underline{c}$, whose entry in position (j, k) is $e_k(c_{j-k})$. According to the conventions adopted above, Shanks's table has in column $k = -1$ only entries ∞. In column $k = 0$ the given sequence $\underline{c}$ is reproduced. If $\underline{c} \in \mathscr{C}$, then in row -1, starting with column 0, we get entries γ, and in position $j \leqslant -2$, $k \geqslant 1$ an infinite block of U's occurs, as shown in Figure 1.

Note that only biinfinite sequences from $\mathscr{C}$ lead to a well-defined and constant initial row -1 in Shanks's table. It is easily checked that also under the more general definition (2) Shanks's transformation remains homogeneous and translative: if $\underline{c} \in \mathscr{C}$, then $\alpha \underline{c} + \beta \in \mathscr{C}$ and

$$e_k(\alpha c_n + \beta) = \alpha e_k(c_n) + \beta$$

$$\text{for all constants } \alpha, \beta \in \mathbb{K}, \quad k \geqslant -1, \quad n \in \mathbb{Z} \tag{3}$$

in the sense that either both sides of (3) are well defined and equal or both are undefined.

	-1	0	1		k-1	k	k+1	
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	
-3	∞	γ	U	$\cdots$	U	U	U	$\cdots$
-2	∞	γ	U	$\cdots$	U	U	U	$\cdots$
-1	∞	γ	γ	$\cdots$	γ	γ	γ	$\cdots$
0	∞	c_0	$e_1(c_{-1})$	$\cdots$	$e_{k-1}(c_{-k+1})$	$e_k(c_{-k})$	$e_{k+1}(c_{-k-1})$	$\cdots$
1	∞	c_1	$e_1(c_0)$	$\cdots$	$e_{k-1}(c_{-k+2})$	$e_k(c_{-k-1})$	$e_{k+1}(c_{-k})$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	
$n+k-1$	∞	c_{n+k-1}	$e_1(c_{n+k-2})$	$\cdots$	$e_{k-1}(c_n)$	$e_k(c_{n-1})$	$e_{k+1}(c_{n-2})$	$\cdots$
$n+k$	∞	c_{n+k}	$e_1(c_{n+k-1})$	$\cdots$	$e_{k-1}(c_{n+1})$	$e_k(c_n)$	$e_{k+1}(c_{n-1})$	$\cdots$
$n+k+1$	∞	c_{n+k+1}	$e_1(c_{n+k})$	$\cdots$	$e_{k-1}(c_{n+2})$	$e_k(c_{n+1})$	$e_{k+1}(c_n)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	
j								

FIG. 1. Shanks's table of a sequence $\underline{c} \in \mathscr{C}$.

It is well known that Shanks's table can be computed completely by Wynn's cross rules [15, 16] and their extensions due to Cordellier [6] if a block of U's occurs. Moreover, from the connections with Padé approximation one knows that there exist only square blocks of undefined values (cf. [6]). It is our aim to prove these results entirely in the narrow frame of Shanks's transformation based only upon the definition (2). This will be done making use of Schur complements and their properties, following ideas of Tempelmeier [13], who first proved Wynn's identity by the Schur-complement method. See also [3], where Schur complements are used to derive various algorithms of numerical analysis.

Let us briefly explain the idea. Suppose that there holds a functional relation

$$F\left(e_j\left(c_i\right),\dots,e_s\left(c_r\right)\right)=0,$$

where F is an unknown function of certain Shanks transforms of a sequence $\underline{c}$. How to find F? By the definition (2) the arguments of F are negative inverse Schur complements

$$F\left(\frac{-1}{\overline{H}_{j+1}^{(i)}(\underline{c})/H_{j+1}^{(i)}(\underline{c})},\dots,\frac{-1}{\overline{H}_{s+1}^{(r)}(\underline{c})/H_{s+1}^{(r)}(\underline{c})}\right)=0.$$

In order to find a suitable F, look for a "big matrix" A such that all matrices involved in these Schur complements are submatrices of A and all contain a nonsingular submatrix P, which should be chosen as large as possible. Then by the quotient property of Schur complements (cf. Section 1) also

$$F\left(\frac{-1}{\left\{\overline{H}_{j+1}^{(i)}/P\right\}/\left\{H_{j+1}^{(i)}/P\right\}},\dots,\frac{-1}{\left\{\overline{H}_{s+1}^{(r)}/P\right\}/\left\{H_{s+1}^{(r)}/P\right\}}\right)=0.$$

Now all Schur complements with respect to P are submatrices of A/P. Since this matrix has smaller dimensions than A, in general it will be easier to find a function F satisfying the last equation. For the cross rule (d) of Theorem 1 the corresponding F is found in Equation (9) of Lemma 2.

THEOREM 1 (Block structure and cross rules). *Let* $k\in\mathbb{N}_0$, $n\in\mathbb{Z}$, $n\geqslant -k$, $p\in\mathbb{N}$, *and let* $\underline{c}=(c_n)_{n\in\mathbb{Z}}\in\mathscr{C}$ *be given. For* $1\leqslant l\leqslant p$, *by*

$$N_l:=e_{k+l-1}\left(c_{n-l}\right),$$

$$W_l:=e_{k-1}\left(c_{n+l}\right),\qquad C_l:=e_k\left(c_{n+l-1}\right),\qquad E_l:=e_{k+p}\left(c_{n-l}\right),$$

$$S_l:=e_{k+p-l}\left(c_{n+l}\right)$$

we denote the particular elements of Shanks's table as displayed in Figure 2.

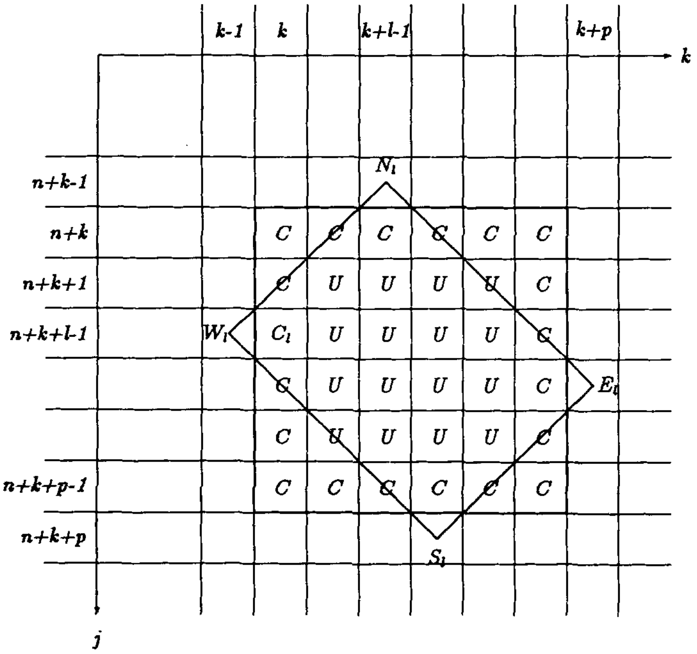

FIG. 2. A block of size $p = 6$ in Shanks's table of $\underline{c}$ with upper left corner $e_k(c_n) = C$, showing Cordellier's rectangle W_l, S_l, E_l, N_l for $l = 3$.

Under the assumptions

(i) *for* $l = 1, \ldots, p$, $C_l = C$,
(ii) *for* $l = 1, \ldots, p$, $W_l \neq C$,
(iii) *for at least one* $l_N \in \{1, \ldots, p\}$, $N_{l_N} \neq C$,
(iv) *for at least one* $l_S \in \{1, \ldots, p\}$, $S_{l_S} \neq C$,

there hold

(a) *for* $l = 1, \ldots, p$, N_l, S_l, E_l *are well defined and distinct from* C,

(b) *for* $l = 1, \ldots, p$ *the entries* $e_{k+p-l}(c_{n+l-1})$, $e_{k+l-1}(c_{n-l+1})$, $e_{k+p-1}(c_{n-l+1})$ *are well defined and equal to* C,

(c) *for* $l = 1, \ldots, p-2$ *and* $l < \lambda < p$ *the entries* $e_{k+\lambda-l}(c_{n+l-1})$, $e_{k+\lambda-1}(c_{n-l+1})$ *are undefined,*

and the equations (cross rules)

(d) $C \neq \infty$:

$$\frac{1}{N_l - C} + \frac{1}{S_l - C} = \frac{1}{W_l - C} + \frac{1}{E_l - C},$$

(e) $C \neq 0$:

$$\frac{1}{1/N_l - 1/C} + \frac{1}{1/S_l - 1/C} = \frac{1}{1/W_l - 1/C} + \frac{1}{1/E_l - 1/C},$$

(f) $C = \infty$:

$$N_l + S_l = W_l + E_l.$$

REMARKS.

1. By parts (a)–(c) of Theorem 1 the block structure of Shanks's table is described. Undefined entries occur only in square blocks bordered by equal entries distinct from their neighbors.

2. Notice that $p = 1$ is not excluded. In this case (d) and (f) are the "regular" cross rules due to Wynn [15, 16]. Cordellier [6] has proved (d) and (f) only for $p \geqslant 2$, where his proof departs from the known block structure of the Padé table. Formula (e) for $p = 1$ was recognized by Brezinski (cf. [1, part I, p. 90]). It is a particular case of the more general relation (4) below.

3. By the cross rules (d)–(e) the complete Shanks's table can be computed, proceeding either from left to right or from top to bottom. In case $p = 1$, by assumption the three known corners are different from the center C; hence also the computed corner is different from C. When a cross rule with $p = 1$ fails then a block is detected: If $W_1 \neq C = N_1 \neq S_1$ or $W_1 \neq C = S_1 \neq N_1$ then $E_1 = C$; if $W_1 \neq C = N_1 = S_1$, then $E_1 = U$ is undefined. Proceeding this way the size of a block is found. Then the singular cross rules of Theorem 1 can be used to compute the east border.

4. Cuyt [7] has identified certain multivariate Padé approximants of order $(n + k, k)$ as particular Shanks transforms $e_k(c_n)$. Here our Theorem 1 directly applies enlightening the structure of the corresponding "nonnormal multivariate Padé-approximation table". In fact, the structure theorem as well as the singular rules given by Cuyt [8] coincide with our results, though her proofs running in the Padé frame are different. A similar remark holds for the univariate two-point Padé table as defined in [5].

Another property of Shanks's table, called *reciprocal covariance* [9] or *duality* [1], is known only from its connection with Padé approximation. We are going to derive it directly from the definition (2) without appealing to Padé approximation.

THEOREM 2 (Reciprocal covariance or duality). *If* $\underline{c} = (c_n) \in \mathscr{C}$, $c_{-1} = \gamma$, *then* $\underline{\tilde{c}} = (\tilde{c}_n) \in \mathscr{C}_0$, *where*

$$\tilde{c}_n := \begin{cases} 0 & n < 0, \\ \dfrac{1}{e_n(c_{-n} - \gamma)} & n \geqslant 0, \end{cases}$$

is well defined. Moreover, for all $n, k \in \mathbb{N}_0$, $e_n(\tilde{c}_{k-n})$ *is well defined if and only if* $e_k(c_{n-k} - \gamma)$ *is. In this case*

$$e_k\left(c_{n-k} - \gamma\right) = \frac{1}{e_n\left(\tilde{c}_{k-n}\right)},$$

or equivalently

$$\frac{1}{e_k\left(c_{n-k}\right)} = \frac{e_n\left(\tilde{c}_{k-n}\right)}{1 + \gamma e_n\left(\tilde{c}_{k-n}\right)}.$$

REMARK. The last equation for $\underline{c} = (c_n)_{n \in \mathbb{Z}} \in \mathscr{C}_0$ means that Shanks's table of the sequence $\tilde{\underline{c}}$ is obtained by transposing Shanks's table of $\underline{c}$ and inverting its entries, and vice versa.

Observe that assumptions (i)–(iv) of Theorem 1 are not invariant under duality. They correspond to the natural calculation direction proceeding from left to right in Shanks's table. By a duality argument it is easy to show that under similar assumptions Shanks's table can be computed also proceeding from top to bottom. This can be used to handle the case $C = \infty$ in the proof of Theorem 1. We will not follow this line, which of course is influenced by the Padé background. Instead let us discuss the following questions, which are meaningful in the theory of Shanks's transformation based upon the definition (2) and having no simple interpretation in terms of Padé approximation.

Given Shanks's table of a sequence $\underline{c} \in \mathscr{C}$, consider the table of its inverses

$$t_{n,k} := \begin{cases} 1/e_k\left(c_{n-k}\right) & \text{if } e_k\left(c_{n-k}\right) \in \overline{\mathbb{K}} \text{ is well defined,} \\ U & \text{if } e_k\left(c_{n-k}\right) = U \text{ is undefined} \end{cases} \qquad \left(n \in \mathbb{Z}, \quad k \geqslant -1\right).$$

Is it possible to find a sequence $\underline{a} \in \mathscr{C}$ such that its Shanks's table does contain the table of the $t_{n,k}$ as an infinite block?

The answer is definitely no, for Shanks's table of $\underline{a}$ would contain a zero column $1/e_{-1}(c_n) = 0$ $(n \in \mathbb{Z})$, which in view of Theorem 1 would induce an infinite block bordered by zeros. But if we weaken the condition to require only that a finite part, say a triangle

$$t_{n,k} \qquad \left(n, k \geqslant 0, \quad n + k < \kappa\right)$$

should be part of Shanks's table of $\underline{a}$, then we get a positive answer. Since $1/e_k(c_{-k-1}) = 1/c_{-1}$, we will assume $\underline{c} \in \mathscr{C} \setminus \mathscr{C}_0$. In this case, with no loss of generality we may take $c_{-n} = \gamma = -1$ for $n \in \mathbb{N}$.

THEOREM 3 (Inverse Shanks's table). *Let* $\underline{c} \in \mathscr{C} \setminus \mathscr{C}_0$, $c_{-1} = -1$, *and* $\kappa \in \mathbb{N}$. *Then the sequence* $\underline{a} = (a_n)$ *with*

$$a_n = \begin{cases} -1, & n < 0, \\ 0, & 0 \leqslant n < \kappa, \\ \dfrac{1}{e_{n-\kappa}\left(c_{\kappa-n}\right) + 1}, & n \geqslant \kappa, \end{cases}$$

is well defined. Moreover:

(a) *For* $0 \leqslant n < \kappa$,

$$e_{\kappa-1}\left(a_{n+1-\kappa}\right) = 0 = \frac{1}{e_{-1}\left(c_{n+1}\right)}.$$

(b) *For* $k \in \mathbb{N}_0$,

$$e_{\kappa+k}\left(a_{-\kappa-k-1}\right) = -1 = \frac{1}{e_k\left(c_{-k-1}\right)}.$$

(c) *For* $n, k \geqslant 0$, $n + k < \kappa$, $e_{\kappa+k}(a_{n-\kappa-k})$ *is well defined if and only if* $e_k(c_{n-k})$ *is. In this case*

$$e_{\kappa+k}\left(a_{n-\kappa-k}\right) = \frac{1}{e_k\left(c_{n-k}\right)}.$$

Notice that Theorem 3 also gives a positive answer to the following question: Is it possible to find a sequence $\underline{a}$ where a finite part of the κth column of its Shanks's table is prescribed?

We will conclude this report with a discussion of some properties of Shanks's table with respect to Möbius transformations

$$\overline{\mathbb{K}} \ni z \to T(z) = \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{K}, \quad \Delta := ad - bc \neq 0.$$

It is easily checked that the cross rule (d) of Theorem 1 is invariant under such transformations:

$$\frac{1}{T(N_l) - T(C)} + \frac{1}{T(S_l) - T(C)} = \frac{1}{T(W_l) - T(C)} + \frac{1}{T(E_l) - T(C)}. \tag{4}$$

This follows at once from the fact that (d) is equivalent to

$$\mathcal{D}(C, E_l, N_l, S_l) = -\mathcal{D}(C, N_l, E_l, W_l),$$

where

$$\mathcal{D}(A, B, C, D) := \frac{A - C}{A - D} : \frac{B - C}{B - D}$$

denotes the cross ratio of four distinct elements of $\overline{\mathbb{K}}$, which is invariant under Möbius transformations. Notice that the transforms $T(e_k(c_n))$ in general are no longer Shanks transforms of some sequence $\underline{a}$, because $T(e_{-1}(c_n)) = T(\infty) = a/c$.

It is well known in Padé approximation [1] that homographic invariance of the values of Padé approximants is generally valid only for diagonal approximants. Nevertheless, by slightly generalizing the definition (2) we can get much more consistency of Shanks's table with respect to Möbius transformations.

Consider the family

$$T_\varepsilon(z) := \frac{z}{1 - \varepsilon z}, \qquad \varepsilon \in \mathbb{K}$$

of Möbius transformations, where ε is a parameter and $T_{-\varepsilon}$ is the inverse Möbius transformation of T_ε. We replace the definition (2) by

$$e_k^\varepsilon(c_n) := \begin{cases} U & \text{iff } e_k(c_n) \\ & \quad = U \text{ is undefined,} \\ T_\varepsilon(e_k(c_n)) = \dfrac{-\left|H_{k+1}^{(n)}(\underline{c})\right|}{\left|\overline{H}_{k+1}^{(n)}(\underline{c})\right| + \varepsilon\left|H_{k+1}^{(n)}(\underline{c})\right|} & \text{else,} \end{cases} \tag{2'}$$

i.e., when defined, *Shanks's transformation with parameter* ε, $e_k^{\varepsilon}(c_n)$, is a negative inverse Schur complement similar to $e_k^0(c_n) = e_k(c_n)$ where only the zero in the lower right corner of $\overline{H}_{k+1}^{(n)}(\underline{c})$ is to be replaced by ε. Clearly, any sequence $\underline{c} \in \mathscr{C}$ has a (generalized) Shanks's table with respect to the transformation (2′), where now column -1 consists of the constant sequence $\underline{\beta} = (\beta)$ with $\beta = e_{-1}^{\varepsilon}(c_n) = -1/\varepsilon$. Furthermore, the whole discussion of Shanks's transformation and table given above, which was based entirely upon the definition (2), remains valid with obvious changes if (2) is replaced by (2′).

Now in view of (4) we are able to express $T(e_k^{\varepsilon}(c_n))$ as a (generalized) Shanks transform.

Corollary 1. *Let* $\varepsilon \in \mathbb{K}$ *be given. Suppose that*

$$T(z) = \frac{az + b}{cz + d} \qquad \textit{with} \quad \Delta = ad - bc \neq 0$$

is a nonconstant Möbius transformation such that $T(-1/\varepsilon) \neq 0$. *Then for each sequence* $\underline{c} = (c_n) \in \mathscr{C}$, *for all* n, k,

$$T\left(e_k^{\varepsilon}(c_n)\right) = e_k^{\eta}(\tilde{c}_n),$$

where η *and* $\tilde{\underline{c}} = (\tilde{c}_n) \in \mathscr{C}$ *are defined by this equation, setting* $k = -1$: $(e_{-1}^{\eta}(\tilde{c}_n)) = T(-1/\varepsilon))$ *and* $k = 0$: $(e_0^{\eta}(\tilde{c}_n)) = T(c_n))$; *hence* $\eta = -1/T(-1/\varepsilon)$, *and for* $n \in \mathbb{Z}$

$$\tilde{c}_n = \left(T_{-\eta} \circ T \circ T_{\varepsilon}\right)(c_n) = \frac{(a - b\varepsilon)^2}{\Delta}\left(c_n + \frac{b}{a - b\varepsilon}\right).$$

Remark. For $\varepsilon = 0$, $c = 0$, $d = 1$ the assertion of Corollary 1 reduces to (3).

Note that the case $T(-1/\varepsilon) = (T \circ T_{\varepsilon})(\infty) = 0$ is discussed in Theorem 2 (by inverting and transposing Shanks's table) and in Theorem 3 (by inverting a finite part of Shanks's table). We will conclude this section by giving a modified form of these theorems with regard to generalized Shanks transforms $e_k^{\varepsilon}(c_n)$. Notice that for any nonconstant Möbius transformation S with inverse S^{-1} such that $S(\infty) \neq \infty$,

$$\delta_S := \left[S(z) - S(\infty)\right]\left[z - S^{-1}(\infty)\right] \in \mathbb{K}$$

does not depend on z. We will apply this to $S: = T_{-\eta} \circ T \circ T_{\varepsilon}$, with ε and η fixed to be chosen below.

Corollary 2. *For each sequence* $\underline{c} = (c_n) \in \mathscr{C}$, $c_{-1} = \gamma$ *with* $(T \circ T_{\varepsilon})(\gamma) \neq \infty$ *for all* $n, k \geqslant -1$,

$$T\left(e_k^{\varepsilon}\left(c_{n-k}\right)\right) = e_n^{\eta}\left(\tilde{c}_{k-n}\right),$$

where η *and* $\tilde{\underline{c}} = (\tilde{c}_n) \in \mathscr{C}$ *are defined by this equation setting* $n = -1$ *and* $n = 0$, *respectively, i.e.* $\eta = -1/T(T_{\varepsilon}(\gamma))$ *(which implies* $S(\gamma) = \infty \neq S(\infty)$*), and for* $k \geqslant -1$,

$$\tilde{c}_k = S\left(e_k\left(c_{-k}\right)\right) = S\left(\infty\right) + \delta_S T_{\gamma}\left(\frac{1}{e_k\left(c_{-k}\right)}\right).$$

Remark. For $\varepsilon = 0$, $T(z) = 1/z$ we obtain $\eta = -\gamma$, $S(\infty) = T_{-\eta}(0) = 0$, $S(0) = T_{-\eta}(\infty) = -1/\gamma$, and $\delta_S = [S(0) - S(\infty)][0 - S^{-1}(\infty)] = 1$; hence $\tilde{c}_k = T_{\gamma}(1/e_k(c_{-k}))$, as shown in Theorem 2.

Corollary 3. *Let* $\eta \in \mathbb{K}$ *be such that* $S(\infty) \neq \infty$ *(i.e.* $-1/\eta \neq T(-1/\varepsilon)$*); further let* $\kappa \in \mathbb{N}$. *For each sequence* $\underline{c} = (c_n) \in \mathscr{C}$, $c_{-1} = \gamma$ *with* $S(\gamma) \neq \infty$ *there exists a sequence* $\underline{a} = (a_n) \in \mathscr{C}$ *such that for* $-1 \leqslant n < \kappa$, $-1 \leqslant k \leqslant \kappa$, $n + k < \kappa$

$$T\left(e_k^{\varepsilon}\left(c_{n-k}\right)\right) = e_{\kappa+k}^{\eta}\left(a_{n-k-\kappa}\right), T\left(e_k^{\varepsilon}\left(c_{n-k} - \gamma + S^{-1}\left(\infty\right)\right)\right) = e_n^{\eta}\left(a_{k+\kappa-n}\right),$$

where, for $k < \kappa$, $a_{\kappa+k}$ *is uniquely defined by the last equation with* $n = 0$, *i.e.* $a_{-1} = S(\gamma)$, $a_0 = \cdots = a_{\kappa-1}$, *and for* $-1 \leqslant k < \kappa$

$$a_{\kappa+k} = S\left(e_k\left(c_{-k}\right) - \gamma + S^{-1}\left(\infty\right)\right) = S\left(\infty\right) + \delta_S T_{\gamma}\left(\frac{1}{e_k\left(c_{-k}\right)}\right).$$

Remark. For $\eta = \varepsilon = 0$, $T(z) = 1/z$, and therefore $S(z) = 1/z$, $\delta_S = 1$, $S(\infty) = 0$, the assertion of Corollary 3 coincides with Theorem 3. Note that the sequences $\underline{a}$ and $\tilde{\underline{c}}$ of the above corollaries are connected via $a_{\kappa+k} = \tilde{c}_k$, $-1 \leqslant k < \kappa$.

	-1	0	1		$\kappa-2$	$\kappa-1$	κ	$\kappa+1$		$2\kappa-2$	$2\kappa-1$
	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮
-2	$\tilde{a}_{-1}$	γ	U	⋯	U	U	U	U	⋯	U	U
-1	$\tilde{a}_{-1}$	γ	γ	⋯	γ	γ	γ	γ	⋯	γ	γ
0	$\tilde{a}_{-1}$	∞	∞	⋯	∞	∞	c_0	$e_1(c_{-1})$	⋯	$e_{\kappa-2}(c_{-\kappa+2})$	$e_{\kappa-1}(c_{-\kappa+1})$
1	$\tilde{a}_{-1}$	∞	U	⋯	U	∞	c_1	$e_1(c_0)$	⋯	$e_{\kappa-2}(c_{-\kappa+3})$	
	⋮	⋮	⋮		⋮	⋮	⋮	⋮			
$\kappa-2$	$\tilde{a}_{-1}$	∞	U	⋯	U	∞	$c_{\kappa-2}$	$e_1(c_{\kappa-3})$			
$\kappa-1$	$\tilde{a}_{-1}$	∞	∞	⋯	∞	∞	$c_{\kappa-1}$				
κ	$\tilde{a}_{-1}$	$\tilde{a}_0$	$\tilde{a}_1$	⋯	$\tilde{a}_{\kappa-2}$	$\tilde{a}_{\kappa-1}$					
	⋮	⋮	⋮								
$2\kappa-2$	$\tilde{a}_{-1}$	$e_{\kappa-2}(\tilde{a}_{2-\kappa})$									
$2\kappa-1$	$\tilde{a}_{-1}$	$e_{\kappa-1}(\tilde{a}_{1-\kappa})$									
2κ	$\tilde{a}_{-1}$										

FIG. 3. Table of quantities $S^{-1}(e_k(a_{n-k}))$, $\underline{a}$ from Corollary 3.

In view of (4), Corollary 3 can be illustrated by Figure 3, showing the quantities $S^{-1}(e_k(a_{n-k}))$. If the columns κ and $\kappa-1$ are prescribed as indicated by the frame in Figure 3, then by Theorem 1 the triangle to the right of the frame is uniquely determined by the cross rules (d), (f), whereas to the left a block of U's bordered by elements ∞ occurs. The elements $\tilde{a}_k = c_k - \gamma + S^{-1}(\infty)$, $0 \leqslant k < \kappa$, are uniquely determined by Theorem 1(f). Because $\tilde{a}_{-1} = S^{-1}(\infty)$, starting from the elements $\tilde{a}_k$ proceeding from top to the bottom the lower triangle is obtained—in particular in column 0, containing

$$\text{for} \quad -1 \leqslant k < \kappa: \qquad S^{-1}(a_{k+\kappa}) = e_k(\tilde{a}_{-k}) = e_k(c_{-k}) - \gamma + S^{-1}(\infty).$$

3. PROOFS

3.1. *Proof of Theorem 2*

In view of (3) we can assume without loss of generality that $c_{-1} = \gamma = 0$. Since

$$\left|H_{n+1}^{(-n)}(\underline{c})\right| = (-1)^{\binom{n+1}{2}} c_0^{n+1} \neq 0$$

by the definition (2),

$$e_n\left(c_{-n}\right) = -\frac{\left|H_{n+1}^{(-n)}(\underline{c})\right|}{\left|\overline{H}_{n+1}^{(-n)}(\underline{c})\right|} \neq 0.$$

Hence $\tilde{c}_n$ is well defined. Consider now the symmetric matrix

$$B\begin{pmatrix} -1,0,\ldots,n+2k+1 \\ -1,0,\ldots,n+2k+1 \end{pmatrix}$$

$$:= \begin{array}{ccccccccccccc}
\overbrace{\quad}^{1} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{k+1}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{k}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{n+1}} & \\
0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \} \, 1 \\
1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \\
1 & 0 & 0 & \cdots & 0 & 0 & \ddots & \vdots & 1 & \cdots & 1 & \\
\vdots & \vdots & & & \vdots & \vdots & \ddots & 1 & \vdots & & \vdots & \} \, k+1 \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \\
0 & 1 & 0 & \cdots & 0 & c_{-n-k} & \cdots & c_{-n-1} & c_{-n} & \cdots & c_0 & \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \} \, k \\
0 & 1 & \cdots & 1 & 0 & c_{-n-1} & \cdots & c_{k-n-2} & c_{k-n-1} & \cdots & c_{k-1} & \\
0 & 1 & 1 & \cdots & 1 & c_{-n} & \cdots & c_{k-n-1} & c_{k-n} & \cdots & c_k & \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \} \, n+1 \\
0 & 1 & 1 & \cdots & 1 & c_0 & \cdots & c_{k-1} & c_k & \cdots & c_{n+k} &
\end{array} .$$

It is easily checked that

$$\left|B\begin{pmatrix} k+1,\ldots,n+2k+1 \\ k+1,\ldots,n+2k+1 \end{pmatrix}\right| = \left|H_{n+k+1}^{(-n-k)}(\underline{c})\right| \neq 0,$$

$$\left|B\begin{pmatrix} 0,\ldots,n+2k+1 \\ 0,\ldots,n+2k+1 \end{pmatrix}\right| = (-1)^k \left|\overline{H}_{n+1}^{(k-n)}(\underline{c})\right|,$$

$$\left|B\begin{pmatrix} -1,0,\ldots,n+2k+1 \\ -1,0,\ldots,n+2k+1 \end{pmatrix}\right| = (-1)^{k+1} \left|H_{n+1}^{(k-n)}(\underline{c})\right|.$$

Moreover, there are no difficulties in computing the Schur complements

$$\beta_{i,j} := \frac{\left| B\begin{pmatrix} j, k+1, \dots, n+2k+1 \\ i, k+1, \dots, n+2k+1 \end{pmatrix} \right|}{\left| B\begin{pmatrix} k+1, \dots, n+2k+1 \\ k+1, \dots, n+2k+1 \end{pmatrix} \right|}$$

$$= \begin{cases} 0, & i = j = -1, \\ 1, & i = -1, \quad j \geqslant 0, \\ 1, & i \geqslant 0, \quad j = -1, \\ 0 = -\tilde{c}_{n+k-i-j}, & i, j \geqslant 0, \quad i + j > n + k, \\ \dfrac{\left| \overline{H}_{n+k+1-i-j}^{(-n-k+i+j)}(\underline{c}) \right|}{\left| H_{n+k+1-i-j}^{(-n-k+i+j)}(\underline{c}) \right|} = -\tilde{c}_{n+k-i-j} & \text{else}. \end{cases}$$

From Sylvester's determinantal identity [4] (which, in view of the determinantal representation of the Schur complement entries, is equivalent to Schur's identity) we know that

$$\det\left(\beta_{i,j}\right)_{i=0,\dots,k}^{j=0,\dots,k} = \frac{\left| B\begin{pmatrix} 0, \dots, n+2k+1 \\ 0, \dots, n+2k+1 \end{pmatrix} \right|}{\left| B\begin{pmatrix} k+1, \dots, n+2k+1 \\ k+1, \dots, n+2k+1 \end{pmatrix} \right|}.$$

As a consequence we have

$$\left| H_{k+1}^{(n-k)}(\tilde{\underline{c}}) \right| = (-1)^{k+1} \left| H_{k+1}^{(n-k)}(-\tilde{\underline{c}}) \right| = (-1)^{k+1} \det\left(\beta_{ij}\right)_{i=0,\dots,k}^{j=0,\dots,k}$$

$$= -\frac{\left| \overline{H}_{n+1}^{(k-n)}(\underline{c}) \right|}{\left| H_{n+k+1}^{(-n-k)}(\underline{c}) \right|}.$$

Similarly,

$$\left| \overline{H}_{k+1}^{(n-k)}(\tilde{\underline{c}}) \right| = (-1)^{k} \left| \overline{H}_{k+1}^{(n-k)}(-\tilde{\underline{c}}) \right| = (-1)^{k} \det\left(\beta_{ij}\right)_{i=-1,\dots,k}^{j=-1,\dots,k}$$

$$= (-1)^k \frac{\left| B\begin{pmatrix} -1, \dots, n+2k+1 \\ -1, \dots, n+2k+1 \end{pmatrix} \right|}{\left| B\begin{pmatrix} k+1, \dots, n+2k+1 \\ k+1, \dots, n+2k+1 \end{pmatrix} \right|} = -\frac{\left| H_{n+1}^{(k-n)}(\underline{c}) \right|}{\left| H_{n+k+1}^{(-n-k)}(\underline{c}) \right|}.$$

From these equations Theorem 2 is obvious.

3.2. *Proof of Theorem 3*

Notice that $a_n = \tilde{c}_{n-\kappa}$ for $n \geqslant 0$ where the sequence $\underline{\tilde{c}}$ is defined in Theorem 2. By elementary computations assertions (a) and (b) can be derived.

In order to prove (c) we will show that for $n, k \geqslant 0$, $n + k < \kappa$,

(i) $\left|H_{\kappa+k+1}^{(n-k-\kappa)}(\underline{a})\right| = \varepsilon\left|H_{n+1}^{(k-n)}(\underline{\tilde{c}})\right|$ and

(ii) $\left|\bar{H}_{\kappa+k+1}^{(n-k-\kappa)}(\underline{a})\right| = \varepsilon\left\{\left|H_{n+1}^{(k-n)}(\underline{\tilde{c}})\right| + \left|\bar{H}_{n+1}^{(k-n)}(\underline{\tilde{c}})\right|\right\}$

with $\varepsilon \in \{-1, 1\}$. In view of the definition (2), then (c) is a direct consequence of Theorem 2 with $\gamma = -1$.

After performing some elementary operations we can expand the determinant of the matrix $H_{\kappa+k-1}^{(n-k-\kappa)}(\underline{a})$: by subtracting the $(i + 1)$st column from the ith column $(i = 0, \ldots, \kappa - n - 1)$ and by subtracting the $(i + 1)$st row from the ith row $(i = 0, \ldots, k - 1)$ we get

$$\left|H_{\kappa+k+1}^{(n-k-\kappa)}(\underline{a})\right|$$

$$\begin{array}{ccc}
 & \begin{array}{cccccccccccc} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{\kappa-n-1}} & \overbrace{\quad}^{1} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{k}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{n+1}} \end{array} & \\
= \det & \left(\begin{array}{cccccccccccc}
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \iddots & -1 & 0 & * & * & \cdots & * & * & \cdots & * \\
\vdots & 0 & \iddots & 0 & 0 & * & * & \cdots & * & * & \cdots & * \\
0 & -1 & \iddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-1 & 0 & \cdots & 0 & 0 & * & * & \cdots & * & * & \cdots & * \\
0 & 0 & \cdots & 0 & 0 & * & * & \cdots & * & \tilde{c}_{k-n} & \cdots & \tilde{c}_k \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & * & * & \cdots & * & \tilde{c}_k & \cdots & \tilde{c}_{k+n}
\end{array}\right) & \begin{array}{l}
 \\
 \\
\left.\right\} k \\
 \\
\left.\right\} 1 \\
 \\
 \\
\left.\right\} k - n - \\
 \\
 \\
\left.\right\} n + 1 \\

\end{array}
\end{array}$$

where * denotes a possibly nonzero element. By suitable expansions with respect to the elements $a_{-1} = -1$ we obtain (i), where the *-entries are irrelevant.

The determinant $\left|\bar{H}_{\kappa+k+1}^{(n-k-\kappa)}(\underline{a})\right|$ can be treated similarly. By adding its first row to the last one we get the sum $u^T = (0, \ldots, 0, 1, \ldots, 1, 1)$ with $\kappa - n + k$ zeros and $n + 1$ entries 1. By adding the first column of the matrix thus obtained to its last one, the sum u results. Therefore $\left|\bar{H}_{\kappa+k+1}^{(n-k-\kappa)}(\underline{a})\right|$ is the determinant of the matrix on the right-hand side of the last equation bordered below by u^T and on the right by u. From this equation (ii) is obvious.

3.3. *Preliminaries to the Proof of Theorem 1*

To prove Theorem 1 we need some auxiliary results. The first one is a lemma due to Cordellier [6], but we will prove it by a different technique. This lemma can also be derived from general results on Toeplitz systems [11]. For completeness we give its simple proof. In the considerations that follow $\underline{c} \in \mathscr{C}$ is a fixed sequence. Therefore we will use the shorthand notation $H_k^{(n)} := H_k^{(n)}(\underline{c})$, $\bar{H}_k^{(n)} := \bar{H}_k^{(n)}(\underline{c})$. Always $p \in \mathbb{N}$ is assumed.

Lemma 1. *If for $l = 1, \ldots, p$ one has $|H_k^{(n+l)}| \neq 0$ and $|H_{k+1}^{(n+l-1)}| = 0$, then there exist coefficients $v_0, \ldots, v_k \in \mathbb{K}$ such that $v_0 v_k \neq 0$ and*

$$\delta_n = \delta_{n+1} = \cdots = \delta_{n+k+p-1} = 0, \tag{5}$$

where

$$\delta_i := \sum_{j=0}^{k} v_j c_{i+j} \qquad (i \in \mathbb{Z}). \tag{6}$$

Moreover, if for some $l \in \{1, \ldots, p\}$ one has $|\bar{H}_{k+1}^{(n+l-1)}| \neq 0$, then

$$\delta := \sum_{j=0}^{k} v_j \neq 0. \tag{7}$$

Proof. (5) is a system of $k + p$ homogeneous linear equations for $k + 1$ unknown components of the vector $\underline{v} = (v_0, \ldots, v_k)^T$. Consider the singular subsystem

$$H_{k+1}^{(n)} \underline{v} = \underline{0}. \tag{8}$$

It has a one-parameter solution $\underline{v}$ satisfying $v_0 v_k \neq 0$, for its matrix has $H_k^{(n+1)}$ as a nonsingular submatrix in the lower left and in the upper right corner. Hence we can fix a particular solution of (8) by the normalization condition $v_k = 1$. When $p = 1$, the proof is finished. Otherwise we prove (5) by induction on l. Assume that $\delta_n = \delta_{n+1} = \cdots = \delta_{n+k+l-1} = 0$ has been proved for $1 \leqslant l < p$, where the δ's are defined by (6) with respect to the solution $\underline{v}$ of (8) normalized by $v_k = 1$. To show $\delta_{n+k+l} = 0$ consider the

Schur complement $S = H_{k+1}^{(n+l)}/H_k^{(n+l)} = |H_{k+1}^{(n+l)}|/|H_k^{(n+l)}| = 0$ which is zero by assumption. On the other side S can be computed directly by elimination. By multiplying row j of $H_{k+1}^{(n+l)}$ by v_j $(j = 0, \ldots, k-1)$ and adding the products to row $k+1$ we get the row $(\delta_{n+l}, \ldots, \delta_{n+k+l})$, where $\delta_{n+l} \cdots = \delta_{n+l+k-1} = 0$ by the induction hypothesis and therefore $\delta_{n+l+k} = S = 0$.

(7) is proved indirectly. If $\delta = 0$ then $\underline{v} = (v_0, \ldots, v_k, 0)^T$ would be a nontrivial solution of $\bar{H}_{k+1}^{(n+l-1)}\underline{v} = \underline{0}$, contradicting $|\bar{H}_{k+1}^{(n+l-1)}| \neq 0$. This completes the proof of Lemma 1. ■

Our second lemma is a determinantal relation between Schur complements. Although it is the key to the cross rules, its proof is almost trivial. Cordellier [6] has given an extended form of it, and his proof is much more elaborate.

LEMMA 2. *Let N, S be nonsingular square matrices, $\xi \in \mathbb{K}$, and let U_1, U_2, V_1^T, V_2^T be suitable column vectors. Then*

$$\frac{\begin{vmatrix} N & 0 & U_1 \\ 0 & S & U_2 \\ V_1 & V_2 & \xi \end{vmatrix}}{\begin{vmatrix} N & 0 \\ 0 & S \end{vmatrix}} + \xi = \frac{\begin{vmatrix} N & U_1 \\ V_1 & \xi \end{vmatrix}}{|N|} + \frac{\begin{vmatrix} S & U_2 \\ V_2 & \xi \end{vmatrix}}{|S|}. \tag{9}$$

Here 0 *denotes a zero matrix of suitable size.*

Proof. Using the splitting

$$\begin{pmatrix} U_1 \\ U_2 \\ \xi \end{pmatrix} = \begin{pmatrix} U_1 \\ 0 \\ \xi \end{pmatrix} + \begin{pmatrix} 0 \\ U_2 \\ \xi \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix},$$

Equation (9) is easily obtained by suitable expansions of the resulting determinants. ■

In our proof of Theorem 1, for a given particular matrix A_0 we will determine a matrix T_0 such that $\tilde{A}_0 := T_0 A_0$ has many entries zero. Our final lemma shows that from such a representation, which is similar to the elimination equations of Section 1, Schur complements of A_0 can be computed explicitly.

LEMMA 3. *Let* A_0, T_0, *and* $\tilde{A}_0 := T_0 A_0$ *be partitioned consistently as*

$$A_0 = \begin{pmatrix} P & F & J \\ G & H & K \\ M & R & Q \end{pmatrix}, \qquad T_0 = \begin{pmatrix} U & B_1 & 0 \\ 0 & I & 0 \\ 0 & B_2 & L \end{pmatrix}, \qquad \tilde{A}_0 = \begin{pmatrix} \tilde{N} & 0 & 0 \\ G & H & K \\ 0 & 0 & \tilde{S} \end{pmatrix},$$

where U, L, *and* H *are nonsingular square matrices, and* I *and* 0 *denote unit and zero matrices of suitable sizes. Then*

$$A_0/H = \begin{pmatrix} U^{-1} & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} \tilde{N} & 0 \\ 0 & \tilde{S} \end{pmatrix} = \begin{pmatrix} U^{-1}\tilde{N} & 0 \\ 0 & L^{-1}\tilde{S} \end{pmatrix}.$$

Proof. Left to the reader. ■

3.4. *Proof of Theorem 1*

The proof will be separated into several parts:

Cases	Assumptions
I	$C = 0$
II	$C \neq 0, \infty$
III	$C = \infty$

Case I. By assumptions (i)–(iv), Lemma 1 applies, giving coefficients $v_0, \ldots, v_k$ with $v_0 v_k \neq 0$ such that (5) and (7) hold. Consider the matrix

$$T = T\begin{pmatrix} 0, \ldots, k+p+1 \\ \\ 0, \ldots, k+p+1 \end{pmatrix}$$

partitioned as shown on page 212, where I is the k-dimensional unit matrix and 0 denote suitable zero matrices. To prove assertions (a)–(d) we compute the product

$$\tilde{A} := T \cdot A, \qquad \text{where} \quad A := A\begin{pmatrix} 0, \ldots, k+p+1 \\ \\ 0, \ldots, k+p+1 \end{pmatrix} = \overline{H}_{k+p+1}^{(n-l)}.$$

Observe that the matrix A occurring in the denominator of E_l does contain all other matrices involved, by the definition (2), in the Shanks transforms W_l, S_l, E_l, N_l, C_l, and all these matrices do contain $H := H_k^{(n+l)}$ as a submatrix. By (5)–(7) we get the equation shown on page 213. Since $v_0 v_k \neq 0$ from the

$$
T := \begin{array}{ccccccccccccccccc|c}
\multicolumn{4}{c}{l} & \multicolumn{8}{c}{k} & \multicolumn{4}{c}{p-l+1} & l & \\
\hline
v_0 & v_1 & \cdots & v_{l-2} & v_{l-1} & v_l & \cdots & v_k & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & l \\
0 & \cdot & \cdot & & & & & & \cdot & \ddots & & \vdots & \cdot & & & \cdot & \cdot & \\
\vdots & & \cdot & \cdot & & & & & & \cdot & 0 & 0 & \cdot & & \cdot & \cdot & \cdot & \\
0 & & & \cdot & \cdot & & & & & & \cdot & 0 & \cdot & & & \cdot & \cdot & \\
0 & \cdots & 0 & 0 & v_0 & v_1 & \cdot & \cdot & \cdot & \cdot & \cdot & v_k & 0 & 0 & \cdots & 0 & 0 & \\
0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & \cdots & 0 & 0 & k \\
\cdot & & & & \cdot & 0 & \cdot & \cdot & & & & \cdot & \cdot & & & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & & & & \cdot & \cdot & 0 & \cdot & & & \cdot & \cdot & \\
0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \\
0 & \cdot & \cdot & \cdot & 0 & v_0 & \cdot & \cdot & \cdot & \cdot & \cdot & v_{k-1} & v_k & 0 & \cdots & 0 & 0 & p-l+1 \\
\cdot & & & & \cdot & 0 & v_0 & \cdot & \cdot & \cdot & \cdot & v_{k-2} & v_{k-1} & v_k & \cdot & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
\cdot & & & & \cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdot & & \cdot & 0 & \cdot & \\
0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & \cdots & 0 & v_0 & \cdots & v_{k-p+l-1} & v_{k-p+l} & \cdot & \cdots & v_k & 0 & \\
0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdots & 0 & 1 & 1
\end{array}
$$

$$
=: \begin{array}{cccc|c}
l & k & p-l+1 & 1 & \\
\hline
U & B_1 & 0 & 0 & l \\
0 & I & 0 & 0 & k \\
0 & B_2 & L & 0 & p-l+1 \\
0 & 0 & 0 & 1 & 1
\end{array}
$$

$$\tilde{A} = \begin{array}{cccccccccccccc}
& \multicolumn{4}{c}{\overbrace{\hphantom{\delta_{n-l}\ \cdots\ \cdots\ \delta_{n-1}}}^{l}} & \multicolumn{3}{c}{\overbrace{\hphantom{c_{n+l}\ \cdots\ c_{n+k+l-1}}}^{k}} & \multicolumn{4}{c}{\overbrace{\hphantom{c_{n+k+l}\ \cdots\ \cdots\ c_{n+k+p}}}^{p-l+1}} & \overbrace{\hphantom{\delta}}^{1} & \\
\multirow{12}{*}{$\left(\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right.$} & \delta_{n-l} & \cdots & \cdots & \delta_{n-1} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \delta & \\
& \vdots & & \iddots & 0 & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \\
& \vdots & \iddots & & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \Big\} \, l \\
& \delta_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \delta & \\
& c_n & \cdots & \cdots & c_{n+l-1} & c_{n+l} & \cdots & c_{n+k+l-1} & c_{n+k+l} & \cdots & \cdots & c_{n+k+p} & 1 & \\
& \vdots & & & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \Big\} \, k \\
& c_{n+k-1} & \cdots & \cdots & c_{n+k+l-2} & c_{n+k+l-1} & \cdots & c_{n+2k+l-2} & c_{n+2k+l-1} & \cdots & \cdots & c_{n+2k+p-1} & 1 & \\
& 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \delta_{n+k+p} & \delta & \\
& \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \iddots & \vdots & \vdots & \\
& \vdots & & & \vdots & \vdots & & \vdots & 0 & \iddots & & \vdots & \vdots & \Big\} \, p-l+1 \\
& 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \delta_{n+k+p} & \cdots & \cdots & \delta_{n+k+2p-l} & \delta & \\
& 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \big\} \, 1
\end{array}$$

particular structure of T, we infer that for $0 \leqslant r_1, s_1 \leqslant l$ and $k - l + 1 \leqslant r_2, s_2 \leqslant k + p$

$$\operatorname{rank} A \begin{pmatrix} s_1, s_1 + 1, \dots, s_2 \\ r_1, r_1 + 1, \dots, r_2 \end{pmatrix} = \operatorname{rank} \tilde{A} \begin{pmatrix} s_1, s_1 + 1, \dots, s_2 \\ r_1, r_1 + 1, \dots, r_2 \end{pmatrix}, \tag{10}$$

$$\operatorname{rank} A \begin{pmatrix} s_1, s_1 + 1, \dots, s_2, k + p + 1 \\ r_1, r_1 + 1, \dots, r_2, k + p + 1 \end{pmatrix}$$

$$= \operatorname{rank} \tilde{A} \begin{pmatrix} s_1, s_1 + 1, \dots, s_2, k + p + 1 \\ r_1, r_1 + 1, \dots, r_2, k + p + 1 \end{pmatrix}. \tag{11}$$

In order to prove the structure assertions (a)–(c) we have to check the rank of several submatrices of A of the form occurring on the left-hand side of (10), (11) with $r_1 - s_1 = r_2 - s_2 \in \{0, 1\}$. This is facilitated by the explicit representation of $\tilde{A}$. Accordingly, the following equivalences hold:

N_l is well defined and $N_l \neq C = 0$

$$\Leftrightarrow \quad |H_{k+l}^{(n-l)}| \neq 0$$

$$\Leftrightarrow \quad \operatorname{rank} A \begin{pmatrix} 0, \dots, k + l - 1 \\ 0, \dots, k + l - 1 \end{pmatrix} = k + l$$

$$\Leftrightarrow \quad \operatorname{rank} \tilde{A} \begin{pmatrix} 0, \dots, k + l - 1 \\ 0, \dots, k + l - 1 \end{pmatrix} = k + l$$

$$\Leftrightarrow \quad \delta_{n-1} \neq 0 \text{ and } |H_k^{(n+l)}| \neq 0$$

$$\Leftrightarrow \quad \delta_{n-1} \neq 0 \text{ and } W_l \neq C = 0.$$

Similarly, S_l is well defined and $S_l \neq C = 0 \Leftrightarrow \delta_{n+k+p} \neq 0$ and $W_l \neq C = 0$. Finally, E_l is well defined and $E_l \neq C = 0 \Leftrightarrow \delta_{n-1} \neq 0$, $\delta_{n+k+p} \neq 0$, and $W_l \neq C = 0$. From this it is evident that in case I, from assumptions (i)–(iv) assertion (a) follows.

Moreover, it is easily verified that for $1 \leqslant l \leqslant p$, $l \leqslant \lambda \leqslant p$

$$\operatorname{rank} H_{k+\lambda}^{(n-l+1)} = \operatorname{rank} \tilde{A}\begin{pmatrix} 0,\ldots,k+\lambda-1 \\ 1,\ldots,k+\lambda \end{pmatrix}$$

$$= k - 1 + \max\{l, 2\lambda - p\} < k + \lambda,$$

$$\operatorname{rank} \overline{H}_{k+\lambda}^{(n-l+1)} = \operatorname{rank} \tilde{A}\begin{pmatrix} 0,\ldots,k+\lambda-1, k+p+1 \\ 1,\ldots,k+\lambda, k+p+1 \end{pmatrix}$$

$$= k + 1 + \max\{l, 2\lambda - p\},$$

$$\operatorname{rank} H_{k+\lambda-l+1}^{(n+l-1)} = \operatorname{rank} \tilde{A}\begin{pmatrix} l-1,\ldots,k+\lambda-1 \\ l,\ldots,k+\lambda \end{pmatrix}$$

$$= k + \max\{0, 2\lambda - p - l\} < k + \lambda - l + 1,$$

$$\operatorname{rank} \overline{H}_{k+\lambda-l+1}^{(n+l-1)} = \operatorname{rank} \tilde{A}\begin{pmatrix} l-1,\ldots,k+\lambda-1, k+p+1 \\ l,\ldots,k+\lambda, k+p+1 \end{pmatrix}$$

$$= k + 2 + \max\{0, 2\lambda - p - l\}.$$

Consequently, for $l = 1, \ldots, p$

$$|H_{k+l}^{(n-l+1)}| = |H_{k+p}^{(n-l+1)}| = |H_{k+p-l+1}^{(n+l-1)}| = 0$$

and

$$|\overline{H}_{k+l}^{(n-l+1)}| \neq 0, \qquad |\overline{H}_{k+p}^{(n-l+1)}| \neq 0, \qquad \overline{H}_{k+p-l+1}^{(n+l-1)}| \neq 0,$$

which proves (b).

In the same way, for $l = 1, \ldots, p-2$ and $l < \lambda < p$ we get

$$|H_{k+\lambda}^{(n-l+1)}| = |H_{k+\lambda-l+1}^{(n+l-1)}| = 0 \quad \text{and} \quad |\overline{H}_{k+\lambda}^{(n-l+1)}| = |\overline{H}_{k+\lambda-l+1}^{(n+l-1)}| = 0,$$

which proves (c).

In order to prove (d) let us remark first that also certain Schur complements of submatrices of A directly can be obtained from corresponding submatrices of $\tilde{A}$. Let

$$A_0 := A\begin{pmatrix} 0,\ldots,k+p \\ 0,\ldots,k+p \end{pmatrix}, \qquad T_0 := T\begin{pmatrix} 0,\ldots,k+p \\ 0,\ldots,k+p \end{pmatrix},$$

$$\tilde{A}_0 := T_0 A_0 = \tilde{A}\begin{pmatrix} 0,\ldots,k+p \\ 0,\ldots,k+p \end{pmatrix},$$

and

$$H = H_k^{(n+l)} = A\begin{pmatrix} l,\ldots,k+l-1 \\ l,\ldots,k+l-1 \end{pmatrix}.$$

Then A_0, T_0, and $\tilde{A}$ can be partitioned consistently as in Lemma 3. Here it is clear that H is regular, and the regularity of U and L follows from $v_0 v_k \neq 0$. Moreover, the Schur complement A_0/H has the form

$$A_0/H = \begin{pmatrix} N & 0 \\ 0 & S \end{pmatrix}$$

with

$$N = \left[T\begin{pmatrix} 0,\ldots,l-1 \\ 0,\ldots,l-1 \end{pmatrix}\right]^{-1} \tilde{A}\begin{pmatrix} 0,\ldots,l-1 \\ 0,\ldots,l-1 \end{pmatrix}$$

$$= \begin{pmatrix} v_0 & \cdots & v_{l-1} \\ & \ddots & \vdots \\ 0 & & v_0 \end{pmatrix}^{-1} \begin{pmatrix} \delta_{n-l} & \cdots & \delta_{n-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ \delta_{n-1} & & 0 \end{pmatrix}$$

and

$$S = \left[T\begin{pmatrix} k+l,\ldots,k+p \\ k+l,\ldots,k+p \end{pmatrix}\right]^{-1} \tilde{A}\begin{pmatrix} k+l,\ldots,k+p \\ k+l,\ldots,k+p \end{pmatrix}$$

$$= \begin{pmatrix} v_k & & 0 \\ \vdots & \ddots & \\ v_{k-p+l} & \cdots & v_k \end{pmatrix}^{-1} \begin{pmatrix} 0 & & \delta_{n+k+p} \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ \delta_{n+k+p} & \cdots & \delta_{n+k+2p-l} \end{pmatrix}$$

where N and S both are nonsingular. Similarly, by the partition property of Schur complements A/H takes the form

$$A/H = \overline{H}_{k+p+1}^{(n-l)}/H_k^{(n+l)} = \begin{array}{cc} \begin{pmatrix} N & 0 & U_1 \\ 0 & S & U_2 \\ V_1 & V_2 & \xi \end{pmatrix} & \begin{array}{c} l \\ p-l+1 \\ 1 \end{array} \end{array}$$

with

$$N = H_{k+l}^{(n-l)}/H_k^{(n+l)}, \quad S = H_{k+p-l+1}^{(n+l)}/H_k^{(n+l)}, \quad \xi = \overline{H}_k^{(n+l)}/H_k^{(n+l)} = -\frac{1}{W_l},$$

$$\begin{pmatrix} N & U_1 \\ V_1 & \xi \end{pmatrix} = \overline{H}_{k+l}^{(n-l)}/H_k^{(n+l)}, \quad \text{and} \quad \begin{pmatrix} S & U_2 \\ V_2 & \xi \end{pmatrix} = \overline{H}_{k+p-l+1}^{(n+l)}/H_k^{(n+l)}.$$

From Schur's identity (or equivalently, from the quotient property of Schur complements) it follows that

$$\frac{\begin{vmatrix} N & U_1 \\ V_1 & \xi \end{vmatrix}}{|N|} = -\frac{1}{N_l}, \qquad \frac{\begin{vmatrix} S & U_2 \\ V_2 & \xi \end{vmatrix}}{|S|} = -\frac{1}{S_l}, \qquad \frac{\begin{vmatrix} N & 0 & U_1 \\ 0 & S & U_2 \\ V_1 & V_2 & \xi \end{vmatrix}}{\begin{vmatrix} N & 0 \\ 0 & S \end{vmatrix}} = -\frac{1}{E_l},$$

and (d) is a consequence of Lemma 2.

Case II. Consider the sequence $\underline{c}'$ defined by

$$c_n' := c_n - C, \qquad n \in \mathbb{Z}_0.$$

By the translativity of Shanks's transformation we know

$$e_k(c_n') = e_k(c_n) - C, \qquad k \geqslant -1, \quad n \in \mathbb{Z}.$$

Applying case I to Shanks's table of the sequence $\underline{c}'$ and going back to Shanks's table of $\underline{c}$ yields assertions (a)–(d).

Case III. If for $\underline{c} = (c_n)_{n \in \mathbb{Z}} \in \mathscr{C}$ assumptions (i)–(iv) hold with $C = \infty$, then by the translativity of Shanks's transformation they also hold for the sequence $\underline{c}' = (c_n')_{n \in \mathbb{Z}}$, where $c_n' := c_n - c_{-1} - 1$ $(n \in \mathbb{Z})$. Observe that $c_n' = -1$ for $n < 0$. Chose κ such that $2k + n + 2p < \kappa$. Then the indices of all

Shanks transforms involved in Theorem 1 lie in the triangle with vertices $(-1, -1)$, $(-1, \kappa - 1)$, $(\kappa - 1, -1)$ as described in Theorem 3. This theorem when applied to $\underline{c}'$ gives a sequence $\underline{a}$. Its Shanks's table also has a block of size p of constant values depending on the sequence $C(\underline{a}) = 1/C(\underline{c}') = 1/C(\underline{c}) = 1/\infty = 0$ with the upper left corner $(n + k, k + \kappa)$. Moreover, for $\underline{a}$ also assumptions (i)-(iv) of Theorem 1 prevail now with $C(\underline{a}) = 0$, as we have seen. Hence, from case I we infer that assertions (a)-(d) hold for Shanks's table of $\underline{a}$. Going back to the sequence $\underline{c}$ we get (a)-(c) and (f) for $\underline{c}$.

Finally (e) is obtained directly from (d). Multiplying (d) by C^2 and adding $2C$ on both sides, in view of

$$\frac{C^2}{X - C} + C = \frac{C^2 + XC - C^2}{X - C} = -\frac{X}{1 - X/C},$$

yields (e). This completes the proof of Theorem 1.

Remark. For the proof of case I of Theorem 1 we do not need the precise connections between the Schur complements $\tilde{A}/H$ and A/H as given above. It would be sufficient to know the block structure of A/H, i.e., that certain of its submatrices are zero, which is a simple consequence of Lemma 1 (cf. [6]). If in addition the structure assertions (a), (b), (c) of Theorem 1 are proved otherwise (for instance via Padé approximation; cf. [6]), then Lemma 3 and the first part of the proof of Theorem 1 concerning the matrices T and $\tilde{A}$ can be dropped.

REFERENCES

1 G. A. Baker and P. Graves-Morris, *Padé Approximants*, Parts I, II, Addison-Wesley, 1981.

2 Cl. Brezinski, *Accélération de la Convergence en Analyse Numérique*, Lecture Notes in Math. 584, Springer-Verlag, 1977.

3 Cl. Brezinski, Other manifestations of the Schur complement, *Linear Algebra Appl.* 111:231-247 (1988).

4 R. A. Brualdi and H. Schneider, Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley, *Linear Algebra Appl.* 52/53:769-791 (1983).

5 S. C. Cooper, A. Magnus, and J. H. McCabe, On the non-normal two-point Padé table, *J. Comput. Appl. Math.* 16:371-380 (1986).

6 Fl. Cordellier, Démonstration algébrique de l'extension de l'identité de Wynn aux tables de Padé non normales, in *Padé Approximation and Its*

Applications, Proceedings Antwerp 1979 (L. Wuytack, Ed.), Lecture Notes in Math. 765, pp. 36–60.

7 A. Cuyt, Regularity and normality of abstract Padé approximants, projection property and product property, *J. Approx. Theory* 35(1):1–11 (1982).

8 A. Cuyt, Singular rules for the calculation of nonnormal multivariate Padé approximants, *J. Comput. Appl. Math.* 14:289–301 (1986).

9 A. Cuyt and L. Wuytack, *Nonlinear Methods in Numerical Analysis*, Elsevier, 1987.

10 D. V. Ouellette, Schur complements and statistics, *Linear Algebra Appl.* 36:187–295 (1981).

11 G. Heinig and K. Rost, *Algebraic Methods for Toeplitz-like Matrices and Operators*, Akademie Verlag, Berlin, 1984.

12 D. Shanks, Nonlinear transformations of divergent and slowly convergent series, *J. Math. Phys.* 34:1–42 (1955).

13 U. Tempelmeier, A new proof of the cross-rule for the ε-algorithm based on Schur complements, *J. Comput. Appl. Math.* 21:55–61 (1988).

14 J. Wimp, *Sequence Transformations and Their Applications*, Academic Press, 1981.

15 P. Wynn, Upon systems of recursions which obtain among the quotients of the Padé table, *Numer. Math.* 8:264–269 (1966).

16 P. Wynn, Singular rules for certain non-linear algorithms, *BIT* 3:175–195 (1963).

Received 25 February 1991; final manuscript accepted 25 August 1991