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THE UNDECIDABILITY OF THE LATTICE OF R.E. CLOSED SUBSETS OF AN EFFECTIVE TOPOLOGICAL SPACE

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The first-order theory of the lattice of recursively enumerable closed subsets of an effective topological space is proved undecidable using the undecidability of the first-order theory of the lattice of recursively enumerable sets. In particular, the first-order theory of the lattice of recursively enumerable closed subsets of Euclidean n-space, for all n, is undecidable. A more direct proof of the undecidability of the lattice of recursively enumerable closed subsets of Euclidean n-space, $n \geqslant 2$, is provided using the method of reduction and the recursive inseparability of the set of all formulae satisfiable in every model of the theory of SIBs and the set of all formulae refutable in some finite model of the theory of SIBs.

0. Introduction

The study of the lattices of r.e. substructures of a fixed recursively presented structure was introduced by Metakides and Nerode [13], and undecidability results for such lattices in the realm of recursive algebra have been produced for vector spaces [17], Boolean algebras [1], and algebraically closed fields [2].

The notion of a recursive presentation for a topological space differs somewhat from that for an algebraic structure. In recursive algebra, the objects that can be referred to in an effective manner are the elements of the structure under consideration, coded into ω. In recursive topology however, where most common examples are not countable, cardinality considerations suggest that the effective objects need not be the elements (points) of the space. Instead the objects coded into ω are the basic open sets. Given the notions of a fully effective topological space and an r.e. closed set which we define in the next section, the lattice of r.e. closed subsets of a fully effective space can be treated as a lattice of r.e. substructures in a manner analogous to the treatment of algebraic substructures.

We show that the first-order theory of the lattice of r.e. open subsets of an effective topological space is undecidable by producing a definable predicate which allows the reduction of the decidability of the theory of the lattice of r.e.

sets to the decidability of the theory of the lattice of r.e. open subsets of an effective topological space. It then follows from the undecidability of the theory of the lattice of r.e. sets [5, 7] that the theory of the lattice of r.e. open subsets (and hence r.e. closed subsets) of an effective topological space is undecidable answering a question raised by Kalantari in [9]. We thus obtain the undecidability of the theory of the lattice of r.e. closed subsets of Euclidean space in all dimensions. However, direct and interesting proofs are available coding less powerful results; we show that the first-order theory of the lattice of r.e. closed subsets of Euclidean n-space, $n \geqslant 2$, is undecidable. In this proof we apply the method of reduction to the first-order theory of the lattice of r.e. closed subsets of recursive points (an intermediate structure) and the first-order theory of a countable, symmetric, irreflexive binary relation (SIB) and then apply the fact that the set of all formulae satisfiable in every model of the theory of SIBs and the set of all formulae refutable in some finite model are recursively inseparable [12].

1. Notation and definitions

A topological space X with a countable basis Δ for the topology on X is *fully effective* if the basis elements are coded into ω, and the operations of union and inclusion on the basis elements are effective. The open subsets of X consisting of r.e. unions of basic open sets are *r.e. open* sets. An *r.e. closed* set is the complement in X of an r.e. open set. In this article, $\mathbb{R}^2$ is considered as a fully effective topological space with a basis consisting of open balls in $\mathbb{R}^2$ with rational centers and rational radii. The r.e. closed subsets of $\mathbb{R}^2$ form a lattice, denoted $\mathscr{L}(R)$, with the usual set operations $\cap$ and $\cup$.

In the course of proving our undecidability result, we introduce the lattice of r.e. closed subsets of recursive points of Euclidean 2-space, denoted $\mathscr{L}(RR)$. We require the following definitions.

x is a *recursive real* if x satisfies one of the following equivalent definitions:

(i) There is a recursive sequence of rational numbers $\{r_k\}$ which converges effectively to x, i.e., there is a total recursive function $f(n)$ such that $k \geqslant f(n) \Rightarrow |x - r_k| \leqslant 2^{-n}$.

(ii) The set of all rational numbers greater than x and the set of all rational numbers less than x are r.e. sets.

A *recursive point* in $\mathbb{R}^2$ is a point (x, y) such that x and y are recursive reals. An *r.e. open set of recursive points* is the set of recursive points in an r.e. open set. An *r.e. closed set of recursive points* is the set of recursive points in an r.e. closed set.

Finally, we refer to a symmetric, irreflexive binary relation as a *SIB*. In the context of this article, we assume that SIBs are defined on initial segments of the natural numbers.

2. The general result

We consider (X, Δ) a fully effective topological space, where Δ satisfies four topological axioms [9].

I. Δ is closed under finite intersections.

II. $\emptyset, X \in \Delta$.

III. Every basic open set δ is connected, i.e., δ cannot be written as a disjoint union of two open subsets of X.

IV. Every nonempty basic open set δ contains two nonempty basic open sets with disjoint closures and the closures contained in δ.

In addition, we require that (X, Δ) has an inclusion algorithm [9], i.e.,

(1) Given $\varepsilon, \delta \in \Delta$, there is a uniform effective procedure which computes $\varepsilon \cap \delta$.

(2) Given $\delta, \varepsilon_1, \ldots, \varepsilon_n \in \Delta$, there is a uniform effective procedure which determines whether $\delta \subseteq \varepsilon_1 \cup \cdots \cup \varepsilon_n$ and $\bar{\delta} \subseteq \varepsilon_1 \cup \cdots \cup \varepsilon_n$.

Theorem. *Let (X, Δ) be as above. The theory of the lattice of r.e. open subsets of (X, Δ) is undecidable.*

Proof. We construct a set U_1 and predicates $CC(A, U)$ and $S(W, U)$ in the first-order language of lattice theory. $\{W \mid S(W, U_1)\}$ will be a lattice under intersection and union which is isomorphic to $\mathscr{C}$, and so its theory will be undecidable.

Let $D = \delta_0, \delta_1, \ldots$ be an effective list of Δ. We construct a recursive sequence $\gamma_0, \gamma_1, \ldots$ of basic open sets such that for all i and j, $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, and we construct a set U_1 such that U_1 is the largest r.e. open set with the property that $U_1 \subseteq X - (\bigcup \bar{\gamma}_i)$. At each stage n in the construction below we shall specify $\gamma_1, \ldots, \gamma_n$ and β_n, where $\beta_n \neq \emptyset$ and $\bar{\gamma}_i \cap \bar{\beta}_n = \emptyset$ for all $i \leq n$.

We note that for any nonempty basic open set α, the four topological axioms satisfied by (X, Δ) and its inclusion algorithm allow us to effectively define two nonempty basic open sets α_1 and α_2 such that $\bar{\alpha}_1 \cap \bar{\alpha}_2 = \emptyset$ and $\alpha_1, \alpha_2 \subseteq \alpha$. When we say that α_1 and α_2 are the least basic open sets such that $\bar{\alpha}_1 \cap \bar{\alpha}_2 = \emptyset$ and $\alpha_1, \alpha_2 \subseteq \alpha$, we mean least with respect to our indexing of basic open sets.

Construction

Stage 0. Let γ_0 and β_0 be the least nonempty basic open sets such that $\bar{\gamma}_0 \cap \bar{\beta}_0 = \emptyset$ and $\gamma_0, \beta_0 \subseteq \delta_0 = X$. Declare $\delta_0 \nsubseteq U_1$.

Stage $s+1$. Assume we have defined nonempty basic open sets $\gamma_1, \ldots, \gamma_s, \beta_s$ such that $\#\gamma_1 < \#\gamma_2 < \cdots < \#\gamma_s$, where $\#\gamma_i$ is the Gödel number of γ_i, and $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, and $\bar{\gamma}_i \cap \bar{\beta}_s = \emptyset$ for all i and j.

Let γ_{s+1} and μ_{s+1} be the least basic open sets such that $\bar{\gamma}_{s+1} \cap \bar{\mu}_{s+1} = \emptyset$ and $\gamma_{s+1}, \mu_{s+1} \subseteq \beta_s$ and $\#\gamma_{s+1}$ is greater than $\#\gamma_s$. Put δ_{s+1} into U_1 if $\delta_{s+1} \cap (\gamma_1 \cup$

$\cdots \cup \gamma_{s+1} \cup \mu_{s+1}) = \emptyset$. Otherwise declare $\delta_{s+1} \nsubseteq U_1$. Finally, if $\delta_{s+1} \cap \mu_{s+1} \neq \emptyset$, then let β_{s+1} be the least nonempty basic open set in $\delta_{s+1} \cap \mu_{s+1}$, and let $\beta_{s+1} = \mu_{s+1}$ otherwise. (Note that if δ_{s+1} is not put into U_1 at stage s, either $\delta_{s+1} \cap \gamma_i \neq \emptyset$ for some $i \leq s+1$ or $\delta_{s+1} \supseteq \beta_{s+1}$ in which case we are guaranteed that $\delta_{s+1} \supseteq \gamma_{s+1}$.)

This completes the construction of U_1.

We now define a predicate $CC(A, U)$ in the first-order language of the theory of the lattice of open sets which says that A is a component of the interior of $X - U$. When we relativize this predicate to the lattice of r.e. open subsets of (X, Δ), $CC(A, U)$ says that A is in the complement of U, A cannot be disconnected into two disjoint r.e. open sets, and all r.e. open supersets of A in the complement of U can be disconnected into two disjoint r.e. open sets. ("A is a relativized component of the interior of $X - U$".)

For the U_1 constructed above, it is easy to check that the relativized version of $CC(A, U_1)$ holds if and only if $A = \gamma_i$, for some γ_i constructed above.

We define a predicate $S(W, U)$ which holds of two r.e. open sets W and U if and only if $W = U$ union some set of relativized components of the interior of $X - U$. In particular, $S(W, U)$ holds if and only if

(a) U is a subset of W.

(b) W contains all relativized components of the interior of $X - U$ which it intersects.

(c) If V is any r.e. open set which contains U and contains all the relativized components of the interior of $X - U$ which it intersects and if V intersects all the relativized components of the interior of $X - U$ that W intersects, then V contains W.

Thus $S(W, U_1)$ holds if and only if $W = U_1 \cup$ some r.e. set of γ_i, i.e., $W = U_1 \cup \{\gamma_i \mid i \in W_e\}$ for some r.e. set W_e.

It is easy to see that $\{W \mid S(W, U_1)\}$ is a lattice under intersection and union which is isomorphic to $\mathscr{E}$. It then follows from the Hermann [7] or Harrington [5] coding that the theory of the lattice of r.e. open subsets of (X, Δ) is undecidable.

In the following sections we supply a proof for the undecidability of the theory of the lattice of r.e. closed subsets of Euclidean n-space, $n \geq 2$, coding less powerful results.

3. The classical case for Euclidean 2-space

The proof of the undecidability of the theory of the lattice of r.e. closed subsets of Euclidean 2-space, $\mathscr{L}(R)$, is a relativization to $\mathscr{L}(RR)$ of a new proof of the undecidability of the lattice of closed subsets of Euclidean 2-space, denoted $\mathscr{L}(\mathscr{E}^2)$. The proof for $\mathscr{L}(\mathscr{E}^2)$ relies heavily on the fact that many standard topological notions are definable in the first-order language of the theory of

$\mathscr{L}(\mathscr{E}^2)$ [6]. We define formulae D and F in the first-order language of the theory of $\mathscr{L}(\mathscr{E}^2)$ to provide the effective translation from a formula in the language of the theory of SIBs to a formula in the language of the theory of $\mathscr{L}(\mathscr{E}^2)$ that will be needed to apply the method of reduction.

D and F have free variables x, y, and z and parameters a_1, a_2, and a_3 which vary over elements of $\mathscr{L}(\mathscr{E}^2)$. D can be viewed as describing the domain of a SIB, and F can be viewed as describing the relation of a SIB.

$D(z, a_1, a_2, a_3)$ is satisfied by z in $\mathscr{L}(\mathscr{E}^2)$ just in case a_2 is empty and z is a point, or z is a point in a_2.

The notion of an arc is central to our definition of F, and we use the classical topological definition—an arc is a metric continuum with exactly two non-cut points. This definition is first-order expressible in the language of the theory of $\mathscr{L}(\mathscr{E}^2)$ [6]. We also need the notion of a transverse crossing of two arcs A and B. We define a *transverse crossing* (*intersection*) of two arcs A and B to be an intersection of A and B in exactly one point such that there is an open connected set C' containing the intersection of A and B such that $C' - A$ is disconnected into two connected open sets which are both intersected by B, and such that if any connected open set containing the point of intersection of A and B is disconnected into two connected open sets by the removal of A, then B intersects both sets of this disconnection. It is a straightforward exercise to generate a formula expressing this notion in the first-order language of the theory of $\mathscr{L}(\mathscr{E}^2)$.

$F(x, y, a_1, a_2, a_3)$ is then satisfied by pairs of elements (x, y) in $\mathscr{L}(\mathscr{E}^2)$ just in case:

(i) x and y are elements of a_2, and a_1 is an arc containing a_2.

(ii) a_3 does not intersect $a_1 - a_2$.

(iii) a_3 has the property that each of its points is an element of an arc contained in a_3, and this arc has endpoints in a_2 but no other points of a_2 and intersects other arcs in a_3 transversally.

(iv) x and y are the endpoints of an arc in a_3 containing no other points of a_2, and this arc intersects other arcs in a_3 transversally.

It should be clear that such an F can be written as a first-order formula (although a very long one) in the language of the theory of $\mathscr{L}(\mathscr{E}^2)$ since the notions of arc and transverse intersection are themselves first-order expressible.

4. The relativization to $\mathscr{L}(RR)$

We now consider the lattice of r.e. closed subsets of recursive points of Euclidean 2-space $\mathscr{L}(RR)$ and the topology it inherits from $\mathscr{L}(R)$. We relativize the D and F described above to the language of the theory of $\mathscr{L}(RR)$.

In this setting, $D(z, a_1, a_2, a_3)$ expresses that either a_2 is empty and z is a recursive point, or z is a recursive point in a_2.

$F(x, y, a_1, a_2, a_3)$ expresses that:

(i) x and y are recursive points in a_2.

(ii) a_2 is a subset of a_1.

(iii) a_3 only intersects a_1 at points of a_2.

(iv) a_1 is a relativized arc: a_1 cannot be disconnected into two r.e. closed sets of recursive points, a_1 contains two recursive points such that if either of these points is removed from a_1, the resulting set cannot be disconnected into two r.e. closed sets of recursive points, and there is a linear ordering on a_1 such that the order topology determined by this linear ordering is the same as the given toplogy of r.e. closed sets of recursive points on a_1.

(v) Every point in a_3 is contained in a relativized arc A (as in (iv)) contained in a_3, A contains only two points of a_2 as endpoints, and A has the property that if it intersects another relativized arc B contained in a_3 with endpoints in a_2 in exactly one recursive point not in a_2, then at this point of intersection there is an r.e. open set of recursive points R such that the removal of A from R disconnects R into two r.e. open sets of recursive points which themselves cannot be disconnected into two r.e. open sets of recursive points, and B intersects both sets of the disconnection. Furthermore, if any r.e. open set of recursive points contains the point of intersection of A and B and the removal of A disconnects this set into two r.e. open sets of recursive points which themselves can not be disconnected into two r.e. open sets of recursive points, then B intersects both sets of the disconnection. ((v) is a relativization of the property of a transverse crossing.)

(vi) There exists a relativized arc contained in a_3 with endpoints x and y (the free variables in F), containing no other points of a_2, which intersects all other relativized arcs in a_3 with endpoints in a_2 transversally, as described in (v).

5. The main theorem for $\mathscr{L}(RR)$

Theorem. *The theory of the lattice of r.e. closed subsets of recursive points of Euclidean 2-space, $\mathscr{L}(RR)$, is undecidable.*

Lemma 1 (Lavrov [12]). *The set of all formulae satisfiable in every model of the theory of SIBs and the set of all formulae refutable in some finite model are recursively inseparable.*

In addition, we require the following two lemmas in order to prove the theorem.

Lemma 2. *Given values for a_1, a_2, a_3 in $\mathscr{L}(RR)$, there is a SIB $(\mathfrak{A}, R)$ such that $(\{z \mid D(z, a_1, a_2, a_3)$ is true in $\mathscr{L}(RR)\}$, $\{(x, y) \mid F(x, y, a_1, a_2, a_3)$ is true in $\mathscr{L}(RR)\})$ is isomorphic to $(\mathfrak{A}, R)$.*

Lemma 3. *Given a finite SIB $(\mathfrak{A}, R)$, we can find values of the parameters a_1, a_2, a_3 such that for x, y, z elements of $\mathscr{L}(RR)$, $(\mathfrak{A}, R)$ is isomorphic to $(\{z \mid D(z, a_1, a_2, a_3)$ is true in $\mathscr{L}(RR)\}$, $\{(x, y) \mid F(x, y, a_1, a_2, a_3)$ is true in $\mathscr{L}(RR)\})$.*

Proof of Lemma 2. We (non-effectively) enumerate the recursive points in $\mathbb{R}^2$ and define a mapping g from $A = \{z \mid D(z, a_1, a_2, a_3)$ is true in $\mathscr{L}(RR)\}$ into $\mathbb{N}$ as follows: Take the first point that appears in the enumeration which is also in A and map it to 0, take the second such point distinct from the first and map it to 1, and so on. $\mathfrak{A}$ is defined to be the range of g. If a_2 is empty, then R is defined to be empty, otherwise we (non-effectively) list all pairs of recursive points in $\mathbb{R}^2$ and add $(g(a), g(b))$ to R if and only if (a, b) satisfies $F(x, y, a_1, a_2, a_3)$ in $\mathscr{L}(RR)$.

To see that this $(\mathfrak{A}, R)$ is indeed a SIB, we observe that for a, b, a_1, a_2, a_3 to satisfy F, it must be the case that a and b are the distinct endpoints of a relativized arc so R is irreflexive. It is also the case that a, b, a_1, a_2, a_3 satisfy F if and only if b, a, a_1, a_2, a_3 satisfy F because the notion of relativized arc is symmetric in a and b; so R is symmetric.

Proof of Lemma 3. We choose $a_1 =$ all recursive reals in $[0, 1] \subseteq x$-axis, $a_2 = \{0, \frac{1}{2}, \frac{3}{4}, \ldots, (2^{n-1} - 1)/2^{n-1}\} \subseteq a_1$ where n is the number of elements in $\mathfrak{A}$, and $a_3 =$ the union of all recursive points in semicircles lying above the x-axis with endpoints $(2^i - 1)/2^i$ and $(2^j - 1)/2^j$ in a_2 for $(i, j) \in R$.

We claim that z satisfies $D(z_1, a_1, a_2, a_3)$ in $\mathscr{L}(RR)$ if and only if $z \in a_2$, and (x, y) satisfies $F(x, y, a_1, a_2, a_3)$ if and only if $(x, y) = ((2^i - 1)/2^i, (2^j - 1)/2^j)$ for $(i, j) \in R$. The isomorphism is then $i \mapsto (2^i - 1)/2^i$ for $i \in \mathfrak{A}$.

The claim is clear for D: all the elements of a_2 are recursive points, recursive points are r.e. closed (an easy exercise), thus a_2 is a finite union of r.e. closed sets so it is r.e. closed.

For F we must show that $(2^i - 1)/2^i$, $(2^j - 1)/2^j$, a_1, a_2, a_3 satisfy F if and only if $(i, j) \in R$. To verify that if $(i, j) \in R$, then $(2^i - 1)/2^i$, $(2^j - 1)/2^j$, a_1, a_2, a_3 satisfy F, we must show that:

(i) The recursive points in $[0, 1]$ form a relativized arc.

(ii) The recursive points in a semicircle with rational center and rational radius form a relativized arc.

(iii) Semicircles of recursive points with rational endpoints on the x-axis which intersect at exactly one point not on the x-axis do so transversally (relativized).

(iv) Semicircles in a_3 which intersect relativized arcs in a_3 with endpoints in a_2 at exactly one point not in a_2 do so transversally (relativized).

The checks for (i) and (ii) are essentially the same and involve trivial facts about r.e. closed sets including: the recursive reals in an interval cannot be disconnected by two r.e. closed sets, an r.e. closed subset of recursive points of the closed unit interval which cannot be disconnected into two r.e. closed sets cannot omit a recursive point between any two points in it, the union of an r.e.

closed set and a recursive point is r.e. closed, and r.e. closed sets are closed in the $\mathbb{R}^2$ topology.

For (iii) it is clear that the semicircles in a_3 intersect at a recursive point—just solve their defining equations simultaneously. The neighborhood C' we choose as a witness for a transverse intersection of two semicircles of recursive points A and B with rational endpoints on the x-axis is an open rational circle (rational center, rational radius) lying above the x-axis, and containing the point of intersection. C' cannot be disconnected into two r.e. closed sets because the recursive reals in an interval cannot be disconnected by two r.e. closed sets. The interior and exterior of the closed curve A^u, formed by the union of A and the span of A on $[0, 1]$, are r.e. open so $C' - A$ is disconnected into two r.e. open sets which themselves cannot be disconnected into two r.e. open sets. Now for A and B to intersect at a point above the x-axis, it must be the case that the interior and exterior of A^u each contain a point of B. This implies that both components of $C' - A$ contain points of B; otherwise there would be a disconnection of B into two r.e. closed sets of recursive points. The same proof shows that any open N' which cannot be disconnected into two r.e. closed sets and which contains the intersection point of A and B has the property that if $N' - A$ has two components, then B intersects them both.

(iv) requires, in addition to (iii), that given endpoints in a_2, no three distinct semicircles intersect at exactly one point. This is easily verified using analytic geometry.

To verify that if $(i, j) \notin R$, then $(2^i - 1)/2^i$, $(2^j - 1)/2^j$, a_1, a_2, a_3 do not satisfy F, observe that there is no semicircle in a_3 connecting such $(2^i - 1)/2^i$, $(2^j - 1)/2^j$; therefore, any arc A which connects them must be the union of segments from more than one semicircle in a_3. A simple analytic geometry proof shows that such an arc must have a non-transverse intersection, namely at a point where two segments in A from distinct semicircles meet.

Proof of the Theorem. We define an effective transformation ** from the language of the theory of SIBs based on R to the language of the theory of the lattice of $\mathscr{L}(RR)$ as follows:

$$
\begin{aligned}
(x = y)^{**} &= D(x, a_1, a_2, a_3) \wedge D(y, a_1, a_2, a_3) \wedge x = y,\\
(R(x, y))^{**} &= D(x, a_1, a_2, a_3) \wedge D(y, a_1, a_2, a_3) \wedge F(x, y, a_1, a_2, a_3),\\
(\neg C)^{**} &= \neg(C^{**}),\\
(C_1 \wedge C_2)^{**} &= C_1^{**} \wedge C_2^{**},\\
(\forall x\, C(x))^{**} &= \forall x\, (D(x, a_1, a_2, a_3) \Rightarrow (C(x))^{**}),\\
(\exists x\, C(x))^{**} &= \exists x\, (D(x, a_1, a_2, a_3) \wedge (C(x))^{**}).
\end{aligned}
$$

A straightforward induction on the length of formulae shows that for any formula φ in the language of the theory of SIBs and any given set of values of the parameters a_1, a_2, a_3, φ^{**} with those values is satisfiable in $\mathscr{L}(RR)$ if and only if φ is satisfiable in the SIB defined in Lemma 2 for those values.

Let

$$T = \{\varphi(R(x, y)) \mid \forall a_1, a_2, a_3\, (\varphi(R(x, y)))^{**} \text{ is satisfiable in } \mathscr{L}(RR)\}.$$

By Lemmas 2 and 3, T is a separation of the universally satisfiable and the finitely refutable formulae in the language of the theory of SIBs. If the theory of $\mathscr{L}(RR)$ were decidable, then T would be a recursive separation, contradicting Lemma 1.

6. The main theorem for $\mathscr{L}(R)$

Theorem. *The theory of the lattice of r.e. closed subsets of Euclidean 2-space, $\mathscr{L}(R)$, is undecidable.*

The key observation for the proof of this theorem is that in $\mathscr{L}(R)$, the recursive points are still distinguishable from arbitrary points.

Lemma. *The atoms of $\mathscr{L}(R)$ are recursive points.*

Proof of Lemma. Because $\mathbb{R}^2$ is Hausdorff, any r.e. closed set with more than one point has a non-trivial r.e. closed subset and thus cannot be an atom of the lattice. An r.e. closed set consisting of exactly one point is a recursive point because we can effectively determine the expansion of x to within $1/2^n$ for any n. The decidability of the theory of real closed fields is used to obtain a $1/2^n$-net covering some closed bounded ball containing x and then again to identify an element of this covering net which contains x.

Proof of Theorem. In the language of the theory of $\mathscr{L}(R)$ we can define a recursive point as an atom of the lattice:

$$\mathrm{At}(x) \equiv \neg x = \emptyset \wedge \forall y\, (y \subseteq x \Rightarrow (y = \emptyset \vee y = x)).$$

In this language we can thus define an equivalence relation $\equiv_e$ on the r.e. closed subsets of Euclidean 2-space:

$$C_1 \equiv_e C_2 \quad \text{if and only if} \quad \forall x\, (\mathrm{At}(x) \wedge x \subseteq C_1 \Rightarrow x \subseteq C_2) \wedge \forall y\, (\mathrm{At}(y) \wedge y \subseteq C_2 \Rightarrow y \subseteq C_1).$$

We also define an effective transformation from the language of the theory of

$\mathscr{L}(RR)$ to the language of the theory of $\mathscr{L}(R)$ as follows:

$$\begin{aligned}
(x=y)^* &= x \equiv_e y, \\
(x \cup y)^* &= x \cup y, \\
(x \cap y)^* &= x \cap y, \\
(\neg\sigma)^* &= \neg(\sigma^*), \\
(\sigma_1 \wedge \sigma_2)^* &= \sigma_1^* \wedge \sigma_2^*, \\
(\exists x\, \sigma(x))^* &= \exists x\, \sigma^*(x), \\
(\forall x\, \sigma(x))^* &= \forall x\, \sigma^*(x).
\end{aligned}$$

Suppose the theory of $\mathscr{L}(R)$ were decidable. Let φ be a formula in the language of the theory of $\mathscr{L}(RR)$. A straightforward induction shows that for any φ in the language of the theory of $\mathscr{L}(RR)$, $\mathscr{L}(R) \models \varphi^*$ if and only if $\mathscr{L}(RR) \models \varphi$, so a decision procedure for the theory of $\mathscr{L}(R)$ yields a decision procedure for the theory of $\mathscr{L}(RR)$, contradicting the previous theorem.

Corollary. *The lattice of r.e. closed subsets of Euclidean n-space, $n \geqslant 2$, is undecidable.*

Proof. The proof for $\mathbb{R}^2$ generalizes directly to the lattice of r.e. closed subsets of recursive points of Euclidean n-space, $n > 2$, with the only change being the values we choose for a_1, a_2, a_3 to obtain an isomorphic copy of a given finite SIB $(\mathfrak{A}, R)$. As before, $a_1 =$ all recursive reals in $[0,1]$, $a_2 = 0, \frac{1}{2}, \frac{3}{4}, \ldots, (2^{n-1}-1)/2^{n-1}$, where n is the number of elements in $\mathfrak{A}$. Now $a_3 =$ the union of all recursive points in *non-intersecting* semicircles, each of which is contained in a sphere of diameter equal to the distance between the points being connected, where if $(i,j) \in R$, we connect $(2^i-1)/2^i$ and $(2^j-1)/2^j$. Note that there are at most $n(n-1)/2$ arcs required. Take $A = 360/(n(n-1)/2)$. Number the elements in R (for each non-ordered pair). If $(i,j) \in R$ and (i,j) was given number m, then the semicircle connecting $(2^i-1)/2^i$ and $(2^j-1)/2^j$ is placed at an angle equal to $m \cdot A$ (with respect to the xy-plane in xyz-space).

This avoids the technical difficulties associated with intersecting arcs in $\mathbb{R}^2$.

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