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The multi-resolution method applied to the sideways heat equation ☆

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Abstract

We consider the sideways heat equation $u_{xx}(x,t) = u_t(x,t)$, $0 \leqslant x < 1$, $t \geqslant 0$. The solution $u(x,t)$ on the boundary $x = 1$ is a known function $g(t)$. This is an ill-posed problem, since the solution—if it exists—does not depend continuously on the boundary, i.e., small changes on the boundary may result in big changes in the solution. In this paper, we shall use the multi-resolution method based on the Shannon MRA to obtain a well-posed approximating problem and obtain an estimate for the difference between the exact solution and the solution of the approximating problem defined in V_j.

Keywords: The sideways heat equation; Multi-resolution analysis; Shannon multi-resolution solution

1. Introduction

When an aircraft or a ballistic rocket goes back into the atmosphere from the outer space, a lot of heat is generated due to the friction between the aircraft and the atmosphere. As a result, the temperature of the surface of the aircraft can be raised to as high as a few

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thousands degrees. The sideways heat equation may be utilized to estimate how the heat flow rate (HFR) varies as a function of time.

In fact, it is very difficult to measure the HFR at the surface of the aircraft. However, it is relatively easy to install some sensors somewhere inside the aircraft to measure the temperature at that position. To estimate the HFR from the data of the temperature, the sideways heat equation is established as follows:

$$\begin{cases} u_{xx}(x,t) = u_t(x,t), & 0 \leqslant x < 1,\ t \geqslant 0, \\ u(1,t) = g(t), & t \geqslant 0, \end{cases} \tag{1}$$

where $u(x,t)$ denotes the temperature and Eq. (1) corresponds to a situation in which the point $x = 0$ is inaccessible, but for which one can make a measurement at $x = 1$.

Let the function $g(t)$ and $u(x,t)$ be extended to the whole real t axis by defining $g(t)$ and $u(x,t)$ to be zero for $t < 0$.

For a function $h(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, its Fourier transform is given by

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t) e^{-i\xi t}\, dt.$$

Throughout the whole discussion, we will suppose the function $\hat{g}(\xi) \in C^2(\mathbb{R})$ and it has compact support. Then it is easy to see

Theorem 1.1.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(\xi) e^{(1-x)\sqrt{i\xi}} e^{i\xi t} d\xi$$

is the solution of Eq. (1)*, where*

$$\sqrt{i\xi} = \bigl(1 + i \cdot \operatorname{sign}(\xi)\bigr)\sqrt{|\xi|/2}.$$

Since $\hat{u}(x,\xi) = \hat{g}(\xi) e^{(1-x)\sqrt{i\xi}}$, where $\sqrt{i\xi}$ has non-negative real part and tends to infinity as $|\xi| \to \infty$, so the existence of a solution in $L^2(\mathbb{R})$ depends on a rapid decay of $\hat{g}(\xi)$ at the high frequencies. However, in practice, $\hat{g}_m(\xi)$ is the measured data, $\hat{g}_m(\xi)$ need not decay rapidly at the high frequencies, therefore the solution may not exist in $L^2(\mathbb{R})$. Furthermore, if it exists, it does not depend continuously on the initial condition, i.e., Eq. (1) is ill-posed. For example (this example is very similar to [2]), considering the following problem:

$$\begin{cases} u_{xx}(x,t) = u_t(x,t), & 0 \leqslant x < 1,\ t \geqslant 0, \\ u(1,t) = g_n(t), & t \geqslant 0, \end{cases} \tag{2}$$

where

$$g_n(t) = \begin{cases} \frac{\cos 2n^2 t}{n^2}, & 0 \leqslant t \leqslant t_0, \\ 0, & t > t_0. \end{cases}$$

The solution of Eq. (2) is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{+\infty} \frac{\cos(2n^2 t + j\frac{\pi}{2})}{n^2} \frac{[\sqrt{2}n(1-x)]^{2j}}{(2j)!}, & 0 \leqslant t \leqslant t_0, \\ 0, & t > t_0. \end{cases}$$

Note that the function $g_n(t)$ converges uniformly to zero as n tends to infinity, but the solution $u_n(x,t)$ does not tend to zero.

However, if we impose an a priori bound on the solution u with $\|u(0,\cdot)\|_{L^2} \leqslant M$ and allow some imprecision in the matching of the data, i.e., we consider the following problem:

$$\begin{cases} u_{xx}(x,t) = u_t(x,t), & 0 \leqslant x < 1,\ t \geqslant 0, \\ u(x,0) = 0, & 0 \leqslant x < 1, \\ \|u(1,\cdot) - g_m(\cdot)\| \leqslant \varepsilon, & \\ \|u(0,\cdot)\|_{L^2} \leqslant M, & \end{cases} \tag{3}$$

then we have the stability in the following sense: any two solutions of Eq. (3) satisfy [6,7]

$$\big\|u_1(x,t) - u_2(x,t)\big\| \leqslant 2M^{1-x}\varepsilon^x, \quad 0 \leqslant x \leqslant 1. \tag{4}$$

Furthermore, Levine [6] had proved that the inequality (4) is sharp (it is also implicit in the earlier paper by Carasso [7]). Therefore, we cannot expect to find a numerical method for the approximating solution that satisfies a better error estimate.

Multi-resolution techniques to solving Eq. (1) have been used by L. Elden and T. Reginska [1,3–5], etc. They described a multi-resolution Galerkin method which is based on the Meyer MRA. It was demonstrated that using the multi-resolution Galerkin method Eq. (1) can be solved efficiently; they also give a rule for choosing an appropriate multi-resolution space V_j. However, up to now, the theoretical results concerning the error estimates were unsatisfactory. In papers [1] and [3], the results about multi-resolution method were not better than about the Fourier method; in paper [1], although the authors imposed an additional assumption on the Meyer wavelet function, the error estimate was not optimal.

In this paper, we will get the optimal error bound and will not impose any additional assumption on the Meyer wavelet function. In this sense, we think we have less restrictive assumptions than in earlier papers and get better result.

The outline of this paper is as follows: in Section 2, we introduce the Shannon MRA and give some useful properties; in Section 3, we shall define a Shannon multi-resolution solution; in Section 4, we give some useful lemmas; finally, in Section 5, we obtain our main result.

2. The Shannon multi-resolution analysis

Definition 2.1 [8]. A multi-resolution analysis (MRA) of $L^2(\mathbb{R})$ is a set of increasing, closed linear subspaces $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$, called scaling spaces, satisfying

(a) $\bigcap_{-\infty}^{\infty} V_j = \{0\}$ and $\overline{\bigcup_{-\infty}^{\infty} V_j} = L^2(\mathbb{R})$;
(b) $f(\cdot) \in V_0$ if and only if $f(2^j\cdot) \in V_j$ for all $j \in \mathbb{Z}$;
(c) $f(\cdot) \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$;
(d) there exists a function $\phi(\cdot) \in V_0$ such that $\{\phi_{0,k}(t)\colon k \in \mathbb{Z}\}$ is an orthonormal basis in V_0, where $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$ for all $j,k \in \mathbb{Z}$. The function $\phi(\cdot)$ is called the scaling function of the multi-resolution analysis.

From Definition 2.1, we can see that

$$V_j = \overline{\operatorname{span}\{\phi_{j,k}(\cdot)\}}_{k\in\mathbb{Z}}, \qquad \phi_{j,k}(t) = 2^{j/2}\phi\big(2^j t - k\big), \quad j,k\in\mathbb{Z}.$$

Moreover, there exists a wavelet function $\psi(\cdot)\in L^2(\mathbb{R})$ determined by $\phi(\cdot)$ such that the set of functions $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ $(k\in\mathbb{Z})$ satisfy: for fixed j, $\{\psi_{j,k}(t)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j which is the orthogonal complement of the space V_j in V_{j+1} $(V_{j+1} = V_j \oplus W_j)$. $\{\psi_{j,k}(t)\}_{j,k\in\mathbb{Z}}$ is called an wavelet basis of $L^2(\mathbb{R})$.

Let the orthogonal projection of $L^2(\mathbb{R})$ onto V_j and $\hat{V}_j = \overline{\operatorname{span}\{\hat{\phi}_{j,k}(\cdot)\}}_{k\in\mathbb{Z}}$ be denoted by P_j and $\hat{P}_j$, respectively, i.e., $\forall f(\cdot)\in L^2(\mathbb{R})$,

$$P_j f(t) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk}\rangle \phi_{jk}(t), \qquad \hat{P}_j f(t) = \sum_{k\in\mathbb{Z}} \langle f, \hat{\phi}_{jk}\rangle \hat{\phi}_{jk}(t).$$

It is easy to see $\widehat{P_j f}(\xi) = \hat{P}_j \hat{f}(\xi)$.

In this paper the Shannon MRA will be applied. The Shannon scaling function is $\phi(t) = \frac{\sin \pi t}{\pi t}$ and its Fourier transform is

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| \leqslant \pi, \\ 0, & \text{o.w.} \end{cases}$$

The corresponding wavelet function $\psi(t)$ is given by

$$\hat{\psi}(\xi) = \begin{cases} e^{-i\frac{\xi}{2}}, & \pi \leqslant |\xi| \leqslant 2\pi, \\ 0, & \text{o.w.} \end{cases}$$

Then we have the following properties:

(1) For any $k\in\mathbb{Z}$,

$$\operatorname{supp} \hat{\phi}_{j,k}(\xi) = \big\{\xi\colon |\xi| \leqslant \pi 2^j\big\}, \qquad \operatorname{supp} \hat{\psi}_{j,k}(\xi) = \big\{\xi\colon \pi 2^j \leqslant |\xi| \leqslant \pi 2^{j+1}\big\},$$

i.e., P_j can be considered *a low pass filter*.

(2) $\forall f(\cdot)\in L^2(\mathbb{R})$,

$$f(t) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \phi_{j,k}(t) + \sum_{l\geqslant j}\sum_{k\in\mathbb{Z}} \langle f, \psi_{l,k}\rangle \psi_{l,k}(t).$$

(3) For any $j\in\mathbb{Z}$,

$$\widehat{P_j f}(\xi) = \hat{f}(\xi), \quad |\xi| \leqslant \pi 2^j.$$

3. A multi-resolution solution

In this section, we shall define a multi-resolution solution based on the Shannon MRA.

Let the infinite-dimensional matrix D_j is given by

$$\big\{(D_j)_{l,k}(x)\big\}_{l\in\mathbb{Z},\,k\in\mathbb{Z}} = \big\{\big\langle \phi'_{j,l}(t), \phi_{j,k}(t)\big\rangle\big\}_{l\in\mathbb{Z},\,k\in\mathbb{Z}},$$

where $\phi(\cdot)$ is the Shannon scaling function, $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$. Then we have

Theorem 3.1.

(a) $(D_j)_{l,k}(x) = -(D_j)_{k,l}(x)$, $(D_j)_{l,k}(x) = (D_j)_{l-k,0}(x)$, $l, k \in \mathbb{Z}$;
(b) $\|D_j\| \leqslant \pi^2 2^{j+1}$;
(c) *let* $f_1(\cdot)$ *and* $f_2(\cdot)$ *are the real continuous functions on* $[-\pi^2 2^{j+1}, \pi^2 2^{j+1}]$, $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *denotes the eigenprojection family of the Hermitian operator* $-iD_j$. *Define*

$$r(i\lambda) =: f_1(\lambda) + i f_2(\lambda),$$
$$r(D_j) =: \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} r(i\lambda)\, dE_\lambda = \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} \big[f_1(\lambda) + i f_2(\lambda)\big]\, dE_\lambda.$$

Then we have

$$\big\|r(D_j)\big\| \leqslant \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \big|r(i\lambda)\big|.$$

Proof. (a) and (b) follow closely the proof in [3].

(c) Define the matrix $A =: -iD_j$, it is easy to see A is a Hermitian matrix, i.e.,

$$\bar{A}^T = \overline{(-iD_j)^T} = iD_j^T = -iD_j = A.$$

By the spectral theorem [9], we have

$$A = \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} \lambda\, dE_\lambda,$$

where E_λ is a spectral family about A.

It follows that

$$D_j = \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} i\lambda\, dE_\lambda.$$

If $r(\cdot)$ is a continuous function and $r(D_j) =: \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} r(i\lambda)\, dE_\lambda$, then

$$\begin{aligned}
\left\|\sum_{j=1}^{n} r(i\lambda_j)\Delta E_{\lambda_j} x\right\|^2 &= \left\langle \sum_{j=1}^{n} r(i\lambda_j)\Delta E_{\lambda_j} x, \sum_{j=1}^{n} r(i\lambda_j)\Delta E_{\lambda_j} x \right\rangle \\
&= \sum_{j=1}^{n} \big|r(i\lambda_j)\big|^2 \langle \Delta E_{\lambda_j} x, \Delta E_{\lambda_j} x\rangle \quad (\Delta E_{\lambda_i}\Delta E_{\lambda_j} = 0,\ i \neq j) \\
&\leqslant \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \big|r(i\lambda_j)\big|^2 \sum_{j=1}^{n} \langle \Delta E_{\lambda_j} x, \Delta E_{\lambda_j} x\rangle \\
&= \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \big|r(i\lambda_j)\big|^2 \left\langle \sum_{j=1}^{n} \Delta E_{\lambda_j} x, \sum_{j=1}^{n} \Delta E_{\lambda_j} x \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&(\Delta E_{\lambda_i} \Delta E_{\lambda_j} = 0,\ i \neq j)\\
&= \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \left| r(i\lambda_j)\right|^2 \left\| \sum_{j=1}^{n} \Delta E_{\lambda_j} x \right\| \\
&= \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \left| r(i\lambda_j)\right|^2 \|Ix\|.
\end{aligned}
$$

Therefore

$$
\left\| r(D_j) \right\| \leqslant \max_{-\pi^2 2^{j+1} \leqslant \lambda \leqslant \pi^2 2^{j+1}} \left| r(i\lambda)\right|. \qquad \square
$$

Now we defined $u_j(x,t) \in V_j$ to be the Shannon multi-resolution solution of Eq. (1) for $g(t)$, i.e.,

$$
u_j(x,t) =: \sum_{k \in \mathbb{Z}} c_k(x) \phi_{j,k}(t),
$$

where the infinite-dimensional vector

$$
\begin{aligned}
c(x) &= \left\{ c_k(x) \right\}_{k \in \mathbb{Z}} =: \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda, \\
\gamma &= \left\{ \gamma(k) \right\}_{k \in \mathbb{Z}} =: \left\{ \left\langle g(t), \phi_{j,k}(t) \right\rangle \right\}_{k \in \mathbb{Z}};
\end{aligned}
$$

then we get

Theorem 3.2. *The infinite-dimensional vector $c(x)$ is the solution of the following system*:

$$
\begin{cases}
c_{xx} = D_j c, & 0 \leqslant x < 1, \\
c(1) = \gamma.
\end{cases}
$$

Proof. (i) Firstly, we prove

$$
c_x = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda.
$$

In fact,

$$
\begin{aligned}
&\left\| \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} \left[\frac{e^{(1-x-\Delta x)\sqrt{i\lambda}} - e^{(1-x)\sqrt{i\lambda}}}{\Delta x} \right] dE_\lambda - \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda \right\| \\
&\quad = \left\| \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} \left[\frac{e^{(1-x-\Delta x)\sqrt{i\lambda}} - e^{(1-x)\sqrt{i\lambda}}}{\Delta x} - (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \right] dE_\lambda \right\|
\end{aligned}
$$

$$\leqslant \|\gamma\| \left\| \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} \left[\frac{e^{(1-x-\Delta x)\sqrt{i\lambda}} - e^{(1-x)\sqrt{i\lambda}}}{\Delta x} - (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \right] dE_\lambda \right\|$$

$$\leqslant \|\gamma\| \max_{|\lambda| \leqslant \pi^2 2^{j+1}} \left| \frac{e^{(1-x-\Delta x)\sqrt{i\lambda}} - e^{(1-x)\sqrt{i\lambda}}}{\Delta x} - (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \right| \to 0$$

$$(\Delta x \to 0).$$

So

$$c_x = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} (-\sqrt{i\lambda}) e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda.$$

Similarly

$$c_{xx} = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} (i\lambda) e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda.$$

On the other hand, by using the spectral theorem [9], it is easy to see

$$D_j c = \left(\int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} i\lambda \, dE_\lambda \right) \left(\gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda \right)$$

$$= \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} (i\lambda) e^{(1-x)\sqrt{i\lambda}} \, dE_\lambda.$$

Therefore

$$c_{xx} = D_j c.$$

(ii) By the definition of the spectral family, we have

$$c(1) = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} e^{(1-1)\sqrt{i\lambda}} \, dE_\lambda = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} dE_\lambda = \gamma. \qquad \square$$

4. Auxiliary lemmas

In order to obtain our main result, in this section, we shall give some useful lemmas.

Lemma 4.1 [10]. *Let $f(t) \in L^2(\mathbb{R})$ have a Fourier transform $\hat{f}(\xi) \in L^1(\mathbb{R})$, then $\sum_{n\in Z} \langle f, \phi_{m,n} \rangle \phi_{m,n}(t) \to f(t)$ uniformly on $\mathbb{R}$.*

Lemma 4.2. *Let $u(x,t)$ be the exact solution of Eq.* (1), *then we have*

(a) $\widehat{(u_{xx})} = (\hat{u})_{xx}$;

(b) $(P_j u)_{xx} = P_j(u_{xx})$;
(c) $P_j u$ *satisfies the following equation*:

$$\begin{cases} (P_j u)_{xx} = P_j(P_j u)_t, & 0 \leqslant x < 1,\ t \geqslant 0, \\ P_j u(1,t) = P_j g(t), & t \geqslant 0. \end{cases}$$

Proof. (a) can be proved easily.

(b) can be obtained immediately by Lemma 4.1 if we let $f(t) = u_{xx}(x,t)$.

(c) By (b) and $u(x,t)$ satisfies the following equation:

$$\begin{cases} u_{xx}(x,t) = u_t(x,t), & 0 \leqslant x < 1,\ t \geqslant 0, \\ u(1,t) = g(t), & t \geqslant 0. \end{cases}$$

We know $P_j u(x,t)$ satisfies:

$$\begin{cases} (P_j u)_{xx} = P_j(P_j u)_t + P_j[(I - P_j)u]_t, & 0 \leqslant x < 1,\ t \geqslant 0, \\ P_j u(1,t) = P_j g(t), & t \geqslant 0. \end{cases}$$

Using $\widehat{P_j u}(x,\xi) = \hat{u}(x,\xi)$, $|\xi| \leqslant \pi 2^j$ and $\operatorname{supp} \hat{\phi}_{j,k}(\xi) = \{\xi\colon |\xi| \leqslant \pi 2^j\}$, we have

$$\begin{aligned} \big\| P_j\big[(I - P_j)u\big]_t \big\|_{L^2}^2 &= \big\| \widehat{P_j} i\xi(\hat{u} - \widehat{P_j u})_t \big\|_{L^2}^2 \\ &= \big\| \widehat{P_j} i\xi(\hat{u} - \widehat{P_j u}) \big\|_{|\xi| \leqslant \pi 2^j}^2 + \big\| \widehat{P_j} i\xi(\hat{u} - \widehat{P_j u}) \big\|_{|\xi| > \pi 2^j}^2 \\ &= 0. \end{aligned}$$

So $P_j u(x,t)$ satisfies:

$$\begin{cases} (P_j u)_{xx} = P_j(P_j u)_t, & 0 \leqslant x < 1,\ t \geqslant 0, \\ P_j u(1,t) = P_j g(t), & t \geqslant 0. \end{cases} \qquad \square$$

Since we suppose $\hat{g}(\xi)$ has compact support, so $\exists J \in \mathbb{N}$ such that $\operatorname{supp} \hat{u} \subseteq [-\pi 2^J, \pi 2^J]$. Then we have the following results:

Lemma 4.3. *Let* $\hat{g}(\xi) e^{(1-x)\sqrt{i\xi}} \in C^2(\mathbb{R})$, $\operatorname{supp} \hat{u} \subseteq [-\pi 2^J, \pi 2^J]$, $J \in \mathbb{N}$. *Then if* $j \geqslant J$, *we have* $\sum_{k \in Z} b_k(x) \phi'_{j,k}(t)$ *uniformly convergent on* $t \in \mathbb{R}$, *where* $b_k(x) = \langle u(x,t), \phi_{j,k}(t) \rangle$.

Proof. (i) Firstly we prove $\sum_{k \in \mathbb{Z}} |b_k(x)| < +\infty$.

Since $\hat{g}(\xi) e^{(1-x)\sqrt{i\xi}} \in C^2(\mathbb{R})$ and $\operatorname{supp} \hat{u} \subseteq [-\pi 2^J, \pi 2^J]$, $J \in \mathbb{N}$, then if $j \geqslant J$, we have

$$\begin{aligned} b_k(x) &= \big\langle u(x,t), \phi_{j,k}(t) \big\rangle = \big\langle \hat{u}(x,\xi), \hat{\phi}_{j,k}(\xi) \big\rangle \\ &= \int_{-\pi 2^j}^{\pi 2^j} \hat{u}(x,\xi) e^{-ik\xi/2^j}\, d\xi = 0 + \frac{2^j}{ik} \int_{-\pi 2^j}^{\pi 2^j} \hat{u}_\xi(x,\xi) e^{-ik\xi/2^j}\, d\xi \\ &= \frac{2^{2j}}{k^2} \big[\hat{u}_\xi(x,\xi) e^{-ik\xi/2^j}\big]_{-\pi 2^j}^{\pi 2^j} - \frac{2^{2j}}{k^2} \int_{-\pi 2^j}^{\pi 2^j} \hat{u}_{\xi\xi}(x,\xi) e^{-ik\xi/2^j}\, d\xi \\ &= 2^{2j} \frac{c_1(x)}{k^2}. \end{aligned}$$

Hence $\sum_{k\in\mathbb{Z}} |b_k(x)| < +\infty$.

(ii) Since

$$\begin{aligned}\sum_{k\in\mathbb{Z}} b_k(x)\phi'_{j,k}(t) &= 2^{\frac{3j}{2}}\frac{1}{\pi}\sum_{k\in\mathbb{Z}} b_k(x)\frac{\pi(2^jt+k)\cos\pi(2^jt+k)-\sin\pi(2^jt+k)}{(2^jt+k)^2}\\ &= 2^{\frac{3j}{2}}\Biggl(\frac{1}{\pi}\sum_{k=0}^{+\infty} b_k(x)\frac{\pi(2^jt+k)\cos\pi(2^jt+k)-\sin\pi(2^jt+k)}{(2^jt+k)^2}\\ &\quad+\frac{1}{\pi}\sum_{k=0}^{+\infty} b_k(x)\frac{\pi(2^jt-k)\cos\pi(2^jt-k)-\sin\pi(2^jt-k)}{(2^jt-k)^2}\Biggr)\\ &= 2^{\frac{3j}{2}}(I_1+I_2),\end{aligned}$$

then if $j \geqslant J$, we get I_1 uniformly convergent on $t \in \mathbb{R}$.

Now we consider I_2. For any ε_0, $\forall k \in \mathbb{Z}$: if $|2^jt-k| < \varepsilon_0$, then we have

$$\begin{aligned}&\frac{1}{\pi}\sum_{k=0}^{+\infty} b_k(x)\frac{\pi(2^jt-k)\cos\pi(2^jt-k)-\sin\pi(2^jt-k)}{(2^jt-k)^2}\\ &\quad= \frac{1}{\pi}\sum_{k=0}^{+\infty} b_k(x)\Biggl\{\frac{\pi(2^jt-k)[1-\frac{\pi^2(2^jt-k)^2}{2!}+\frac{(\cos\xi)\pi^4(2^jt-k)^4}{4!}]}{(2^jt-k)^2}\\ &\qquad-\frac{[\pi(2^jt-k)-\frac{(-\cos\xi)\pi^3(2^jt-k)^3}{3!}]}{(2^jt-k)^2}\Biggr\}\\ &\quad= \frac{1}{\pi}\sum_{k=0}^{+\infty} b_k(x)\bigl[c_1\bigl(2^jt-k\bigr)+c_2\bigl(2^jt-k\bigr)^3\bigr],\end{aligned}$$

therefore we obtain $\sum_{k=0}^{+\infty} b_k(x)\phi'_{j,k}(t)$ uniformly convergent on $|2^jt-k| \leqslant \varepsilon_0$. On the other hand, if $|2^jt-k| > \varepsilon_0$, we have

$$\begin{aligned}&\frac{1}{\pi}\sum_{k=0}^{+\infty}\left|b_k(x)\frac{\pi(2^jt-k)\cos\pi(2^jt-k)-\sin\pi(2^jt-k)}{(2^jt-k)^2}\right|\\ &\quad\leqslant \sum_{k=0}^{+\infty}\bigl|b_k(x)\bigr|\left|\frac{\cos\pi(2^jt-k)}{2^jt-k}\right|+\sum_{k=0}^{+\infty}\bigl|b_k(x)\bigr|\left|\frac{\sin\pi(2^jt-k)}{\pi(2^jt-k)^2}\right|\\ &\quad= 2\sum_{k=0}^{+\infty}\bigl|b_k(x)\bigr|\left|\frac{1}{2^jt-k}\right|\\ &\quad< \frac{2}{\varepsilon_0}\sum_{k=0}^{+\infty}\bigl|b_k(x)\bigr|,\end{aligned}$$

therefore we obtain $\sum_{k=0}^{+\infty} b_k(x)\phi'_{0,k}(t)$ uniformly convergent on $|2^jt-k| > \varepsilon_0$. $\square$

Lemma 4.4. *Let* $P_j u(x,t) =: \sum_{k\in\mathbb{Z}} b_k(x)\phi_{j,k}(t)$, $\operatorname{supp}\hat{u} \subseteq [-\pi 2^J, \pi 2^J]$, $J \in \mathbb{N}$. *Then if* $j \geqslant J$, *we have the infinite vector* $b(x)$ *satisfies*:

$$\begin{cases} b_{xx} = D_j b, & 0 \leqslant x < 1, \\ b(1) = \gamma, \end{cases}$$

where the infinity vector $b(x) = \{b_k(x)\}_{k\in\mathbb{Z}} =: \{\langle u(x,t), \phi_{j,k}(t)\rangle\}_{k\in\mathbb{Z}}$, $\gamma = \{\gamma(k)\}_{k\in\mathbb{Z}} =: \{\langle g(t), \phi_{j,k}(t)\rangle\}_{k\in\mathbb{Z}}$.

Proof. Note that $P_j u(1,t) = P_j g(t)$, we have

$$\sum_{k\in\mathbb{Z}} b_k(1)\phi_{j,k}(t) = \sum_{k\in\mathbb{Z}} \bigl\langle g(t), \phi_{j,k}(t)\bigr\rangle \phi_{j,k}(t)$$
$$\Longrightarrow \quad b_k(1) = \bigl\langle g(t), \phi_{j,k}(t)\bigr\rangle \quad \Longrightarrow \quad b(1) = \gamma.$$

By Theorem 4.1 (let $f(t) = u_{xx}(x,t)$) and Lemma 4.3, we obtain

$$\begin{aligned}
(P_j u)_{xx} &= \left(\sum_{k\in\mathbb{Z}} b_k(x)\phi_{j,k}(t)\right)_{xx} = \sum_{k\in\mathbb{Z}} b_k''(x)\phi_{j,k}(t), \\
P_j(P_j u)_t &= \sum_{k\in\mathbb{Z}} \bigl\langle (P_j u)_t, \phi_{j,k}(t)\bigr\rangle \phi_{j,k}(t) \\
&= \sum_{k\in\mathbb{Z}} \left\langle \left(\sum_{m\in\mathbb{Z}} b_m(x)\phi_{j,m}(t)\right)_t, \phi_{j,k}(t)\right\rangle \phi_{j,k}(t) \\
&= \sum_{k\in\mathbb{Z}} \left\langle \sum_{m\in\mathbb{Z}} b_m(x)\phi_{j,m}'(t), \phi_{j,k}(t)\right\rangle \phi_{j,k}(t) \\
&= \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} (D_j)_{mk} b_m(x)\right)\phi_{j,k}(t).
\end{aligned}$$

Therefore $b_k''(x) = \sum_{m\in\mathbb{Z}} (D_j)_{mk} b_m(x)$, i.e., $b_{xx} = D_j b$. $\square$

5. Main result

In this section, the main result will be given.

Let $u_j(x,t)$ be the Shannon multi-resolution solution for $g(t)$, $v_j(x,t)$ be the Shannon multi-resolution solution for the measured data $g_m(t)$, i.e.,

$$v_j(x,t) =: \sum_{k\in\mathbb{Z}} c_k^m(x)\phi_{j,k}(t),$$

where the infinite-dimensional vector

$$c_m(x) = \bigl\{c_m^k(x)\bigr\}_{k\in\mathbb{Z}} =: \gamma_m \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} e^{(1-x)\sqrt{i\lambda}}\, dE_\lambda,$$
$$\gamma_m = \bigl\{\gamma_m(k)\bigr\}_{k\in\mathbb{Z}} =: \bigl\{\bigl\langle g_m(t), \phi_{j,k}(t)\bigr\rangle\bigr\}_{k\in\mathbb{Z}}.$$

The following theorem indicates that the Shannon multi-resolution solution which we have defined is stability.

Theorem 5.1. *If $\|g(t)-g_m(t)\|_{L^2}\leqslant\varepsilon$ and $j=j(\varepsilon)$ is such that*

$$2^j=\frac{1}{\pi^2}\left(\log\frac{M}{\varepsilon}\right)^2,$$

then we have

$$\left\|u_j(x,t)-v_j(x,t)\right\|_{L^2}\leqslant M^{1-x}\varepsilon^x.$$

Proof. Since $u_j(x,t)$ and $v_j(x,t)$ are given by

$$u_j(x,t)=:\sum_{k\in\mathbb{Z}}c_k(x)\phi_{j,k}(t),\qquad v_j(x,t)=:\sum_{k\in\mathbb{Z}}c_k^{(m)}(x)\phi_{j,k}(t),$$

then we have

$$\begin{aligned}\left\|u_j(x,t)-v_j(x,t)\right\|_{L^2}&=\left\|c_m(x)-c(x)\right\|_{l^2}=\left\|\left(\gamma_m-\gamma\right)\int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}}e^{(1-x)\sqrt{i\lambda}}\,dE_\lambda\right\|_{l^2}\\&\leqslant\left\|\int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}}e^{(1-x)\sqrt{i\lambda}}\,dE_\lambda\right\|_{l^2}\left\|\gamma^m-\gamma\right\|_{L^2}\\&\leqslant\max_{|\lambda|\leqslant\pi^2 2^{j+1}}\left|e^{(1-x)\sqrt{i\lambda}}\right|\left\|P_jg_m(t)-P_jg(t)\right\|_{l^2}\\&\leqslant\max_{|\lambda|\leqslant\pi^2 2^{j+1}}\left|e^{(1-x)\sqrt{i\lambda}}\right|\varepsilon=\max_{|\lambda|\leqslant\pi^2 2^{j+1}}e^{(1-x)\sqrt{\frac{|\lambda|}{2}}}\varepsilon\\&\leqslant e^{(1-x)\pi 2^{\frac{j}{2}}}\varepsilon.\end{aligned}$$

Using

$$2^j=\frac{1}{\pi^2}\left(\log\frac{M}{\varepsilon}\right)^2,$$

we get

$$\left\|u_j(x,t)-v_j(x,t)\right\|_{L^2}\leqslant M^{1-x}\varepsilon^x.\qquad\square$$

Note. If $\|u_{j_1}(x,t)-v_{j_1}(x,t)\|_{L^2}\leqslant M^{1-x}\varepsilon^x$ and $j_2>j_1$, then $\|u_{j_2}(x,t)-v_{j_2}(x,t)\|_{L^2}\leqslant M^{1-x}\varepsilon^x$.

Theorem 5.2. *If $j=j(\varepsilon)$ is such that*

$$2^j=\frac{2}{\pi}\left(\log\frac{M}{\varepsilon}\right)^2,$$

then we have

$$\left\|u(x,t)-P_ju(x,t)\right\|_{L^2}\leqslant M^{1-x}\varepsilon^x.$$

Proof. Since P_j is an orthogonal projection and $\widehat{P_j u}(x,\xi) = \hat{u}(x,\xi)$ for $|\xi| \leqslant \pi 2^j$, then we have

$$\left\|u(x,t) - P_j u(x,t)\right\|_{L^2}^2 = \left\|\hat{u}(x,\xi) - \widehat{P_j u}(x,\xi)\right\|_{L^2}^2 = \left\|(I - \hat{P}_j)\chi_j^+(\xi)\hat{u}(x,\xi)\right\|_{L^2}^2 \leqslant \left\|\chi_j^+(\xi)\hat{u}(x,\xi)\right\|_{L^2}^2,$$

where $\chi_j^+(\xi)$ denotes the characteristic function of the interval $[\pi 2^j, +\infty)$.

Note that the Fourier transform of the exact solution u has the form $\hat{g}(\xi)e^{(1-x)\sqrt{i\xi}}$, then the above inequality reduces to

$$\begin{aligned}
&\left\|u(x,t) - P_j u(x,t)\right\|_{L^2}^2 \\
&\quad\leqslant \int\limits_{|\xi|\geqslant\pi 2^j} \left|\hat{g}(\xi)e^{(1-x)\sqrt{i\xi}}\right|^2 d\xi = \int\limits_{|\xi|\geqslant\pi 2^j} e^{-x\sqrt{2|\xi|}}\left|\hat{g}(\xi)e^{\sqrt{i\xi}}\right|^2 d\xi \\
&\quad\leqslant e^{-x\sqrt{\pi 2^{j+1}}} \int\limits_{\xi\in\mathbb{R}} \left|\hat{g}(\xi)e^{\sqrt{i\xi}}\right|^2 d\xi \leqslant e^{-x\sqrt{\pi 2^{j+1}}} M^2,
\end{aligned}$$

the last inequality is because $\hat{g}(\xi)$ has compact support.

Using

$$2^j = \frac{2}{\pi}\left(\log\frac{M}{\varepsilon}\right)^2,$$

we get

$$\left\|u(x,t) - P_j u(x,t)\right\|_{L^2} \leqslant M^{1-x}\varepsilon^x. \qquad \square$$

In this paper, we are interested in the norm estimation of the distance between the Shannon multi-resolution solution $v_j(x,t)$ for the measured data $g_m(t)$ and the unknown solution $u(x,t)$ of Eq. (1) for the exact data $g(t)$. Let $u_j(x,t)$ denote the Shannon multi-resolution solution for the exact data $g(t)$, we have

$$\left\|u(x,\cdot) - v_j(x,\cdot)\right\|_{L^2} \leqslant \left\|u(x,\cdot) - P_j u(x,\cdot)\right\|_{L^2} + \left\|P_j u(x,\cdot) - u_j(x,\cdot)\right\|_{L^2} + \left\|u_j(x,\cdot) - v_j(x,\cdot)\right\|_{L^2},$$

therefore it remains to estimate the second one, i.e., the norm of the function

$$w(x,t) =: P_j u(x,t) - u_j(x,t).$$

Theorem 5.3. *Let* $\operatorname{supp}\hat{u} \subseteq [-\pi 2^J, \pi 2^J]$, $J \in \mathbb{N}$. *Then if* $j \geqslant J$, *we have* $P_j u(x,t) = u_j(x,t)$.

Proof. Define $w(x,t) =: P_j u(x,t) - u_j(x,t)$, then

$$\begin{aligned}
w(x,t) &= \sum_{k\in\mathbb{Z}} b_k(x)\phi_{j,k}(t) - \sum_{k\in\mathbb{Z}} c_k(x)\phi_{j,k}(t) = \sum_{k\in\mathbb{Z}} \left[b_k(x) - c_k(x)\right]\phi_{j,k}(t) \\
&= \sum_{k\in\mathbb{Z}} w_k(x)\phi_{j,k}(t),
\end{aligned}$$

where $w_k(x) = b_k(x) - c_k(x)$, $k \in \mathbb{Z}$.

By Theorem 3.2 and Lemma 4.4, we obtain the infinite-dimensional vector $w(x) = \{w_k(x)\}_{k\in\mathbb{Z}}$ satisfies:

$$\begin{cases} w_{xx} = D_j w, & 0 \leqslant x < 1, \\ w(1) = 0. \end{cases}$$

As in the proof of Theorem 3.2, we have $w(x) = \gamma \int_{-\pi^2 2^{j+1}}^{\pi^2 2^{j+1}} e^{(1-x)\sqrt{i\lambda}}\, dE_\lambda$ and $\gamma = 0$. Hence $w = 0$, i.e., $P_j u = u_j$. $\square$

Theorem 5.4. *Let $\hat{g}(\xi)e^{(1-x)\sqrt{i\xi}} \in C^2(\mathbb{R})$, $\operatorname{supp} \hat{u} \subseteq [-\pi 2^J, \pi 2^J]$, $J \in \mathbb{N}$, $\|g - g_m\|_{L^2} \leqslant \varepsilon$. Then if $j = j(\varepsilon) \geqslant J$ is such that*

$$2^j = \frac{2}{\pi}\left(\log \frac{M}{\varepsilon}\right)^2,$$

we have

$$\left\| u(x,t) - v_j(x,t) \right\|_{L^2} \leqslant 2M^{1-x}\varepsilon^x.$$

Proof. By Theorems 5.1–5.3, the result is immediately obtained. $\square$

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