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Approximate approximations from scattered data

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Abstract

The aim of this paper is to extend the approximate quasi-interpolation on a uniform grid by dilated shifts of a smooth and rapidly decaying function to scattered data quasi-interpolation. It is shown that high order approximation of smooth functions up to some prescribed accuracy is possible, if the basis functions, which are centered at the scattered nodes, are multiplied by suitable polynomials such that their sum is an approximate partition of unity. For Gaussian functions we propose a method to construct the approximate partition of unity and describe an application of the new quasi-interpolation approach to the cubature of multi-dimensional integral operators.

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1. Introduction

The approximation of multivariate functions from scattered data is an important theme in numerical mathematics. One of the methods to attack this problem is quasi-interpolation. One takes values $u(\mathbf{x}_j)$ of a function u on a set of nodes $\{\mathbf{x}_j\}_{j \in J}$ and constructs an approximant of u

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by linear combinations

$$\sum_{j\in J} u(\mathbf{x}_j)\eta_j(\mathbf{x}),$$

where $\eta_j(\mathbf{x})$ is a set of basis functions. Using quasi-interpolation there is no need to solve large algebraic systems. The approximation properties of quasi-interpolants in the case that $\mathbf{x}_j$ are the nodes of a uniform grid are well-understood. For example, the quasi-interpolant

$$\sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\varphi\left(\frac{\mathbf{x}-h\mathbf{j}}{h}\right) \tag{1.1}$$

can be studied via the theory of principal shift-invariant spaces, which has been developed in several articles by de Boor et al. (see e.g. [3,4]). Here φ is supposed to be a compactly supported or rapidly decaying function. Based on the Strang-Fix condition for φ, which is equivalent to polynomial reproduction, convergence and approximation orders for several classes of basis functions were obtained (see also [22], [8]). Scattered data quasi-interpolation by functions, which reproduce polynomials, has been studied by Buhmann et al. [2] and Dyn and Ron [5] (see also [26] for further references). Other methods for scattered data approximation include moving least squares (see [6,10]), which among others have attracted attention in the context of approximate solutions of partial differential equations as so-called meshless methods (see [1] and the references therein). As a rule, the methods reproduce polynomials, at least locally, but the shape functions η_j are not available analytically in simple forms. The computation of the approximant requires solving a linear algebraic system for each point $\mathbf{x} \in \mathbb{R}^n$.

In order to extend the quasi-interpolation (1.1) to general classes of approximating functions with good analytical properties, another concept of approximation procedures, called *approximate approximations*, was proposed in [11,12]. These procedures have the common feature, that they are accurate without being convergent in a rigorous sense. Consider, for example, the quasi-interpolant on the uniform grid

$$\mathcal{M}\,u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\,\eta\left(\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{D}}\right), \tag{1.2}$$

where η is sufficiently smooth and of rapid decay, h and D are two positive parameters. It was shown that if $\mathcal{F}\eta - 1$ has a zero of order N at the origin ($\mathcal{F}\eta$ denotes the Fourier transform of η), then $\mathcal{M}\,u$ approximates u pointwise

$$|\mathcal{M}u(\mathbf{x}) - u(\mathbf{x})| \leqslant c_{N,\eta}\,(h\sqrt{D})^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \varepsilon\,|\nabla_{N-1}u(\mathbf{x})|, \tag{1.3}$$

with a constant $c_{N,\eta}$ not depending on u, h, and D, and the positive number ε can be made arbitrarily small if D is sufficiently large (see [14,15]). In general, there is no convergence of the *approximate quasi-interpolant* $\mathcal{M}u(\mathbf{x})$ to $u(\mathbf{x})$ as $h \to 0$. However, one can fix D such that up to any prescribed accuracy $\mathcal{M}u$ approximates u with order $O(h^N)$. The lack of convergence as $h \to 0$, which is not perceptible in numerical computations for appropriately chosen D, is compensated by a greater flexibility in the choice of approximating functions η. In applications, this flexibility enables one to obtain simple and accurate formulae for values of various integral and pseudo-differential operators of mathematical physics (see [14,16,18] and the review paper [23]) and to develop explicit semi-analytic time-marching algorithms for initial boundary value problems for linear and non linear evolution equations [13,9].

Up to now the approximate quasi-interpolation approach was extended to nonuniform grids in two directions. The case that the set of nodes is a smooth image of a uniform grid was studied in [17]. It was shown that formulae similar to (1.2) preserve the basic properties of approximate quasi-interpolation. A similar result for quasi-interpolation on piecewise uniform grids was obtained in [7]. It is the purpose of the present paper to generalize the method of approximate quasi-interpolation to functions with values given on a rather general grid $\{\mathbf{x}_j\}_{j\in J}$ by modifying the approximating functions. More precisely, we consider approximations of the form

$$Mu(\mathbf{x}) = \sum_{j\in J} \sum_{\mathbf{x}_k\in\overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k)\,\mathcal{P}_{j,k}(\mathbf{x})\,\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right), \tag{1.4}$$

where $\overline{\mathrm{st}}(\mathbf{x}_j)$ is some finite set of nodes near $\mathbf{x}_j$ (see Definition 3.1). The functions $\mathcal{P}_{j,k}$ are polynomials and h_j are scaling parameters. We show that one can achieve the approximation of u with arbitrary order N up to a small saturation error, as long as an "approximate partition of unity" $\left\{\widetilde{\mathcal{P}}_j(\mathbf{x})\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\right\}$ with other polynomials $\widetilde{\mathcal{P}}_j$ exists. Here we mean that for any $\varepsilon>0$ one can find polynomials such that

$$\sup_{\mathbb{R}^n}\left|\sum_{j\in J}\widetilde{\mathcal{P}}_j(\mathbf{x})\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)-1\right|<\varepsilon. \tag{1.5}$$

Then one can choose the polynomials $\mathcal{P}_{j,k}$ in (1.4) such that pointwise

$$|Mu(\mathbf{x})-u(\mathbf{x})|\leqslant C\,\sup_j h_j^N\,\|\nabla_N u\|_{L_\infty}+\varepsilon\,|u(\mathbf{x})|.$$

This estimate is valid as long as

$$\sum_{j\in J}\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\geqslant c>0$$

and η is sufficiently smooth and of rapid decay, but is not subjected to additional requirements as the Strang-Fix condition. Moreover, for the special case of scattered nodes close to a piecewise uniform grid we propose a method to construct polynomials $\mathcal{P}_j$ such that the sum

$$\sum_{j\in J}\mathcal{P}_j(\mathbf{x})\,\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_j^2 D)}$$

approximates the constant function 1 up to an arbitrary prescribed accuracy. This method does not require solving a large system of linear equations. Instead, in order to obtain locally an analytic representation of the partition of unity and consequently of the quasi-interpolant (1.4), one has to solve a small number of linear systems of moderate size.

We give a simple example of formula (1.4). Let $\{x_j\}$ be a sequence of points on $\mathbb{R}$ such that $0<x_{j+1}-x_j\leqslant 1$. Consider a sequence of functions ζ_j on $\mathbb{R}$ supported by a fixed neighborhood of the origin. Suppose that the sequence $\{\zeta_j(x-x_j)\}$ forms an approximate partition of unity on $\mathbb{R}$,

$$\left|1-\sum_j\zeta_j(x-x_j)\right|<\varepsilon.$$

One can easily see that the quasi-interpolant

$$M_h u(x) = \sum_j u(hx_j) \left(\frac{x_{j+1} - x/h}{x_{j+1} - x_j} \zeta_j \left(\frac{x}{h} - x_j \right) + \frac{x/h - x_{j-1}}{x_j - x_{j-1}} \zeta_{j-1} \left(\frac{x}{h} - x_{j-1} \right) \right)$$

satisfies

$$|M_h u(x) - u(x)| \leqslant c\, h^2 \, \|u''\|_{L_\infty(\mathbb{R})} + \varepsilon \, |u(x)|,$$

where the constant c depends on the functions ζ_j.

Note, that by a suitable choice of η it is possible to obtain explicit semi-analytic or other efficient approximation formulae for multi-dimensional integral and pseudo-differential operators which are based on the quasi-interpolant (1.4). So the cubature of those integrals, which is one of the applications of the approximate quasi-interpolation on uniform grids, can be carried over to the case when the integral operators are applied to functions given at scattered nodes.

The outline of the paper is as follows. In Section 2 we show that, under some mild restrictions on the scattered nodes, an approximate partition of unity can be obtained from a given system of rapidly decaying approximating functions if these functions are multiplied by polynomials. Using the approximate partition of unity, one can construct quasi-interpolants of high order approximation rate up to some prescribed saturation error. This will be shown in Section 3. Section 4 contains an application to the cubature of convolution integral operators. A construction of the approximate partition of unity for the case of Gaussians and some numerical examples are given in Section 5.

2. Approximate partition of unity

In this section we show that an approximate partition of unity of $\mathbb{R}^n$ can be obtained from a given system of approximating functions centered at scattered nodes $\{\mathbf{x}_j\}_{j\in J}$ if these functions are multiplied by polynomials. Here J denotes an infinite index set. We are mainly interested in rapidly decaying basis functions which are supported on the whole space. But we start with the simpler case of compactly supported basis functions.

2.1. Basis functions with compact support

Lemma 2.1. *Let $\{B(\mathbf{x}_j, h_j)\}_{j\in J}$ be an open locally finite covering of $\mathbb{R}^n$ by balls centered in $\mathbf{x}_j$ and radii h_j. Suppose that the multiplicity of this covering does not exceed a positive constant μ_n and that there are positive constants c_1 and c_2 satisfying*

$$c_1 h_m \leqslant h_j \leqslant c_2 h_m \tag{2.1}$$

provided the balls $B(\mathbf{x}_j, h_j)$ and $B(\mathbf{x}_m, h_m)$ have common points. Furthermore, let $\{\eta_j\}$ be a bounded sequence of continuous functions on $\mathbb{R}^n$ such that $\operatorname{supp}\eta_j \subset B(\mathbf{x}_j, h_j)$. *We assume that the functions $\mathbb{R}^n \ni \mathbf{y} \to \eta_j(h_j\,\mathbf{y})$ are uniformly continuous with respect to j and*

$$s(\mathbf{x}) := \sum_{j\in J} \eta_j(\mathbf{x}) \geqslant c \quad on \;\; \mathbb{R}^n, \tag{2.2}$$

where c is a positive constant. Then for any $\varepsilon > 0$ there exists a sequence of polynomials $\{\mathcal{P}_j\}$ with the following properties:

(i) *the function*

$$\Theta := \sum_{j\in J} \mathcal{P}_j \, \eta_j \tag{2.3}$$

satisfies

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \textit{for all}\, \mathbf{x} \in \mathbb{R}^n; \tag{2.4}$$

(ii) *the degrees of all $\mathcal{P}_j$ are bounded (they depend on the least majorant of the continuity moduli of η_j and the constants $\varepsilon, c, c_1, c_2, \mu_n$)*;

(iii) *there exists a constant c_0 such that $|\mathcal{P}_j(\mathbf{x})| < c_0$ for all j and $\mathbf{x} \in B(\mathbf{x}_j, h_j)$.*

Proof. Since the functions $B(\mathbf{x}_j, 1) \ni \mathbf{y} \to s(h_j\,\mathbf{y})$ are continuous uniformly with respect to j, for an arbitrary positive δ there exist polynomials $\mathcal{P}_j$ subject to

$$\left|\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right| < \delta \;\text{ on }\; B(\mathbf{x}_j, h_j),$$

and the degree of $\mathcal{P}_j$, deg $\mathcal{P}_j$, is independent of j. Letting $\delta = \varepsilon\,(\mu_n \|\eta\|_{L_\infty})^{-1}$ we obtain

$$\left|\eta_j(\mathbf{x}) \left(\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right)\right| \leqslant \frac{\varepsilon}{\mu_n}.$$

Then

$$\sup_{\mathbb{R}^n} \sum_{j\in J} \left|\eta_j(\mathbf{x}) \left(\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right)\right| \leqslant \varepsilon, \tag{2.5}$$

since at most μ_n terms of this sum are different from zero. But

$$\begin{aligned}\sum_{j\in J} \eta_j(\mathbf{x}) \left(\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right) &= \sum_{j\in J} \eta_j(\mathbf{x})\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \sum_{j\in J} \eta_j(\mathbf{x}) \\ &= \sum_{j\in J} \eta_j(\mathbf{x})\mathcal{P}_j(\mathbf{x}) - 1,\end{aligned}$$

which proves (2.4). □

Remark 2.2. Let the functions $\{\eta_j\}_{j\in J}$ in Lemma 2.1 satisfy the additional hypothesis $\eta_j \in C^k(\mathbb{R}^n)$. Then one can find a sequence of polynomials $\{\mathcal{P}_j\}$ of degrees L_j such that

$$\sup_{B(\mathbf{x}_j, h_j)} \left|\mathcal{P}_j(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right| \leqslant C(k)\, \frac{h_j^k}{L_j^k} \sup_{B(\mathbf{x}_j, h_j)} |\nabla_k s(\mathbf{x})|$$

(see, e.g., [19]). Here $\nabla_k s$ denotes the vector of partial derivatives $\{\partial^{\boldsymbol{\alpha}} s\}_{|\boldsymbol{\alpha}|=k}$. The estimate shows that it suffices to take polynomials $\mathcal{P}_j$ with deg $\mathcal{P}_j > c(k)\,\varepsilon^{-1/k}$ in order to achieve the error ε in (2.5).

2.2. Basis functions with noncompact support

Here we consider approximating functions supported on the whole $\mathbb{R}^n$. We suppose that the functions $\{\eta_j\}_{j\in J}$, are scaled translates

$$\eta_j(\mathbf{x}) = \eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)$$

of a sufficiently smooth function η with rapid decay.

First we formulate a result on weighted polynomial approximation which follows from [20, Theorem 4.2]. If we denote by $w_{\delta,p}$, $\delta > 1$, $p > 0$, the weight function

$$w_{\delta,p}(\mathbf{x}) = \exp\left(-p\sum_{k=1}^{n}|x_k|^{\delta}\right), \tag{2.6}$$

then for any $g \in W^r_\infty(\mathbb{R}^n)$ there exists a polynomial $\mathcal{P}$ of degree at most $2N-1$ in each variable $x_1,\ldots,x_n$, such that

$$\|w_{\delta,p}(g-\mathcal{P})\|_{L_\infty} \leqslant c\,N^{(1-\delta)r/\delta}\left(\|w_{\delta,p}g\|_{L_\infty} + \sum_{k=1}^{n}\|w_{\delta,p}\partial_k^r g\|_{L_\infty}\right), \tag{2.7}$$

with a constant c depending only on the weight function.

Lemma 2.3. *For given $\varepsilon > 0$ there exist a number L_ε and polynomials $\mathcal{P}_j$ of degree* $\deg \mathcal{P}_j \leqslant L_\varepsilon$ *such that the function Θ defined by* (2.3) *satisfies* (2.4) *under the following assumptions on η, the nodes $\{\mathbf{x}_j\}_{j\in J}$ and the scaling parameters $\{h_j\}$:*

1. *There exists $K > 0$ such that*

$$c_K := \left\|\sum_{j\in J}\left(1+h_j^{-1}|\cdot-\mathbf{x}_j|\right)^{-K}\right\|_{L_\infty} < \infty. \tag{2.8}$$

2. *There exist $\delta > 1$ and $p > 0$ such that*

$$\left\|\frac{(1+|\cdot|)^K}{w_{\delta,p}}\eta\right\|_{L_\infty},\ \left\|\frac{(1+|\cdot|)^K}{w_{\delta,p}}\nabla\eta\right\|_{L_\infty} \leqslant c_{\delta,p} < \infty, \tag{2.9}$$

with the weight function $w_{\delta,p}$ defined in (2.6).

3. *There exists $C > 0$ such that for all indices $j, m \in J$*

$$\frac{h_j}{h_m}\,w_{\delta,p}\left(\frac{\mathbf{x}_j-\mathbf{x}_m}{h_j+h_m}\right) \leqslant C. \tag{2.10}$$

4. (2.2) *is valid.*

Proof. From (2.8) and (2.9) the sum

$$\sum_{j\in J}\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) = s(\mathbf{x})$$

converges absolutely for any $\mathbf{x}$ to a positive, smooth and bounded function s. Suppose that we have shown that for any $\varepsilon > 0$ and all indices j there exist polynomials $\mathcal{P}_j$ such that

$$\left|\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\left(\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)-\frac{1}{s(\mathbf{x})}\right)\right| \leqslant \frac{\varepsilon}{c_K}\left(1+\frac{|\mathbf{x}-\mathbf{x}_j|}{h_j}\right)^{-K}, \tag{2.11}$$

(c_K is defined in (2.8)) and $\deg \mathcal{P}_j \leqslant L_\varepsilon$. Then

$$\sup_{\mathbb{R}^n}\sum_{j\in J}\left|\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\left(\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)-\frac{1}{s(\mathbf{x})}\right)\right| \leqslant \varepsilon,$$

and as in the proof of Lemma 2.1 we conclude

$$\sup_{\mathbb{R}^n}\left|\sum_{j\in J}\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)-1\right| \leqslant \varepsilon.$$

Let us fix an index j and make the change of variables $\mathbf{y} = {h_j}^{-1}(\mathbf{x}-\mathbf{x}_j)$. Then (2.11) is proved if we show that there exists a polynomial $\mathcal{P}_j$ such that for all $\mathbf{y} \in \mathbb{R}^n$

$$\left|\eta(\mathbf{y})\left(\mathcal{P}_j(\mathbf{y})-\frac{1}{\tilde{s}(\mathbf{y})}\right)\right| \leqslant \frac{\varepsilon}{c_K}(1+|\mathbf{y}|)^{-K}, \tag{2.12}$$

with $\tilde{s}(\mathbf{y}) = s(h_j\mathbf{y}+\mathbf{x}_j)$. Since $\tilde{s}^{-1} \in W^1_\infty(\mathbb{R}^n)$ according to the estimate (2.7) we can find a polynomial $\mathcal{P}_j$ satisfying

$$\sup_{\mathbb{R}^n}\left|\mathcal{P}_j(\mathbf{y})-\frac{1}{\tilde{s}(\mathbf{y})}\right| w_{\delta,p}(\mathbf{y}) < \frac{\varepsilon}{c_{\delta,p}\,c_K},$$

with the constant $c_{\delta,p}$ in the decay condition (2.9). Now (2.12) follows immediately from

$$|\eta(\mathbf{y})|(1+|\mathbf{y}|)^K \leqslant c_{\delta,p}\,w_{\delta,p}(\mathbf{y}).$$

By (2.7), the degree of the polynomial $\mathcal{P}_j$ depends on the weighted norm

$$\begin{aligned}\sup_{\mathbb{R}^n} w_{\delta,p}(\mathbf{y})\left|\nabla\frac{1}{\tilde{s}(\mathbf{y})}\right| &= h_j\sup_{\mathbb{R}^n} w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\left|\nabla\frac{1}{s(\mathbf{x})}\right| \\ &\leqslant \sup_{\mathbb{R}^n}\frac{1}{(s(\mathbf{x}))^2} w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\sum_{m\in J}\frac{h_j}{h_m}\left|\nabla\eta\left(\frac{\mathbf{x}-\mathbf{x}_m}{h_m}\right)\right|.\end{aligned} \tag{2.13}$$

Since by (2.9)

$$\left|\nabla\eta\left(\frac{\mathbf{x}-\mathbf{x}_m}{h_m}\right)\right| \leqslant c_{\delta,p}\,w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_m}{h_m}\right)\left(1+\frac{|\mathbf{x}-\mathbf{x}_m|}{h_m}\right)^{-K},$$

a uniform bound of (2.13) with respect to j can be established if the sums

$$\sum_{m\in J}\frac{h_j}{h_m}w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_m}{h_m}\right)\left(1+\frac{|\mathbf{x}-\mathbf{x}_m|}{h_m}\right)^{-K}$$

are uniformly bounded for all j. In view of

$$\left|\frac{x-a}{\sigma}\right|^\delta + \left|\frac{x-b}{\tau}\right|^\delta \geqslant \frac{|a-b|^\delta}{(\sigma^{\delta/(\delta-1)}+\tau^{\delta/(\delta-1)})^{\delta-1}} \geqslant \left|\frac{a-b}{\sigma+\tau}\right|^\delta$$

for any $x \in \mathbb{R}$ and $\delta > 1$ we obtain the inequality

$$w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_m}{h_m}\right) \leqslant w_{\delta,p}\left(\frac{\mathbf{x}_j-\mathbf{x}_m}{h_j+h_m}\right).$$

Therefore, the condition (2.10) on the nodes $\{\mathbf{x}_j\}$ and the corresponding parameters $\{h_j\}$ guarantees that the degree of the polynomials $\mathcal{P}_j$ can be chosen not depending on j. $\square$

3. Quasi-interpolants of a general form

In this section we study the approximation of sufficiently smooth functions by the quasi-interpolant (1.4). For simplicity we consider functions u of the class $W_\infty^N(\mathbb{R}^n)$ which have continuous bounded partial derivatives

$$\partial^{\boldsymbol{\alpha}} u = \frac{\partial^{\boldsymbol{\alpha}} u}{\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}}, \quad |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n,$$

up to the order $|\boldsymbol{\alpha}| < N$ and $\partial^{\boldsymbol{\alpha}} u \in L_\infty(\mathbb{R}^n)$ if $|\boldsymbol{\alpha}| = N$. We will show that within the class of generating functions of the form polynomial times compactly supported or rapidly decaying generating function it suffices to have an approximate partition of unity in order to construct approximate quasi-interpolants of high order accuracy up to some prescribed saturation error.

Definition 3.1. Let $\mathbf{x}_j \in \mathbf{X}$. A collection of $m_N = \dfrac{(N-1+n)!}{n!(N-1)!} - 1$ nodes $\mathbf{x}_k \in \mathbf{X}$ will be called *star* of $\mathbf{x}_j$ and denoted by $\mathrm{st}(\mathbf{x}_j)$ if the Vandermonde matrix

$$\left\{(\mathbf{x}_k-\mathbf{x}_j)^{\boldsymbol{\alpha}}\right\}, \ |\boldsymbol{\alpha}| = 1,\ldots,N-1, \ \mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j), \tag{3.1}$$

is not singular. The union of the node $\mathbf{x}_j$ and its star $\mathrm{st}(\mathbf{x}_j)$ is denoted by $\overline{\mathrm{st}}(\mathbf{x}_j) = \mathbf{x}_j \cup \mathrm{st}(\mathbf{x}_j)$.

Let us assume the following hypothesis concerning the grid $\{\mathbf{x}_j\}_{j\in J}$:

Condition 3.1. For any $\mathbf{x}_j$ there exists a ball $B(\mathbf{x}_j, h_j)$ which contains m_N nodes $\mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j)$ with

$$|\det V_{j,h_j}| = \left|\det\left\{\left(\frac{\mathbf{x}_k-\mathbf{x}_j}{h_j}\right)^{\boldsymbol{\alpha}}\right\}_{|\boldsymbol{\alpha}|=1,\mathbf{x}_k\in\mathrm{st}(\mathbf{x}_j)}^{N-1}\right| \geqslant c, \tag{3.2}$$

with $c > 0$ not depending on $\mathbf{x}_j$.

3.1. Compactly supported basis functions

Theorem 3.2. *Suppose that the function system* $\{\eta_j\}_{j\in J}$ *satisfies the conditions of Lemma* 2.1, *let* $u \in W_\infty^N(\mathbb{R}^n)$ *and* $\varepsilon > 0$ *arbitrary. There exist polynomials* $\mathcal{P}_{j,k}$, *independent of* u, *whose degrees* *are uniformly bounded, such that the quasi-interpolant*

$$Mu(\mathbf{x}) = \sum_{k\in J} u(\mathbf{x}_k) \sum_{\overline{\mathrm{st}}(\mathbf{x}_j)\ni\mathbf{x}_k} \mathcal{P}_{j,k}(\mathbf{x})\eta_j(\mathbf{x}) \tag{3.3}$$

satisfies the estimate

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C h_m^N \sup_{B(\mathbf{x}_m,\lambda h_m)} |\nabla_N u| + \varepsilon\, |u(\mathbf{x})|, \tag{3.4}$$

where $\mathbf{x}_m$ is an arbitrary node and $\mathbf{x}$ is any point of the ball $B(\mathbf{x}_m, h_m)$. By λ we denote a constant greater than 1 *which depends on c_1 and c_2 in* (2.1). *The constant C does not depend on h_m, m and ε.*

Proof. For given ε we choose polynomials $\mathcal{P}_j(\mathbf{x})$ such that the function (2.3) satisfies

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

and introduce the auxiliary quasi-interpolant

$$M^{(1)}u(\mathbf{x}) = \sum_{j\in J} \left(\sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)}{\boldsymbol{\alpha}!} (\mathbf{x} - \mathbf{x}_j)^{\boldsymbol{\alpha}} \right) \mathcal{P}_j(\mathbf{x})\eta_j(\mathbf{x}). \tag{3.5}$$

We use the Taylor expansion u around $\mathbf{y} \in \mathbb{R}^n$

$$u(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{y})}{\boldsymbol{\alpha}!} (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}} + R_N(\mathbf{y}, \mathbf{x}) \tag{3.6}$$

with the remainder satisfying

$$|R_N(\mathbf{y}, \mathbf{x})| \leqslant c_N |\mathbf{x} - \mathbf{y}|^N \sup_{B(\mathbf{y},|\mathbf{x}-\mathbf{y}|)} |\nabla_N u|. \tag{3.7}$$

Taking $\mathbf{y} = \mathbf{x}_j$ we write $M^{(1)}u(\mathbf{x})$ as

$$M^{(1)}u(\mathbf{x}) = u(\mathbf{x})\Theta(\mathbf{x}) - \sum_{j\in J} R_N(\mathbf{x}_j, \mathbf{x})\mathcal{P}_j(\mathbf{x})\eta_j(\mathbf{x}),$$

which gives

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant \sum_{j\in J} |R_N(\mathbf{x}_j, \mathbf{x})\mathcal{P}_j(\mathbf{x})\eta_j(\mathbf{x})| + |u(\mathbf{x})|\, |\Theta(\mathbf{x}) - 1|.$$

This, together with the estimate for the remainder (3.7), shows that for $\mathbf{x} \in B(\mathbf{x}_m, h_m)$

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant C_1 h_m^N \sup_{B(\mathbf{x}_m,\lambda h_m)} |\nabla_N u| + \varepsilon |u(\mathbf{x})|, \tag{3.8}$$

where the ball $B(\mathbf{x}_m, \lambda h_m)$ contains all balls $B(\mathbf{x}_j, h_j)$ such that $B(\mathbf{x}_j, h_j)$ and $B(\mathbf{x}_m, h_m)$ intersect.

As the next step we approximate in $M^{(1)}u$ the values of the derivatives $\partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)$ by a linear combination of $u(\mathbf{x}_k)$, where $\mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j)$. Let $\{a_{\boldsymbol{\alpha}}^{(j)}\}_{1 \leqslant |\boldsymbol{\alpha}| \leqslant N-1}$ be the unique solution of the linear system with m_N unknowns

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(j)}}{\boldsymbol{\alpha}!} (\mathbf{x}_k - \mathbf{x}_j)^{\boldsymbol{\alpha}} = u(\mathbf{x}_k) - u(\mathbf{x}_j), \quad \mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j). \tag{3.9}$$

Denoting by $\{b_{\boldsymbol{\alpha},k}^{(j)}\}$ the elements of the inverse of V_{j,h_j}, cf. (3.2), the solution of (3.9) is given by

$$a_{\boldsymbol{\alpha}}^{(j)} = \frac{\boldsymbol{\alpha}!}{h_j^{|\boldsymbol{\alpha}|}} \sum_{\mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j)} b_{\boldsymbol{\alpha},k}^{(j)} (u(\mathbf{x}_k) - u(\mathbf{x}_j)).$$

If the derivatives $\{\partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)\}$ in (3.5) are replaced by $\{a_{\boldsymbol{\alpha}}^{(j)}\}$, then we obtain the formula

$$\begin{aligned} Mu(\mathbf{x}) = \sum_{j \in J} \Bigg\{ & u(\mathbf{x}_j) \left(1 - \sum_{\mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j)} \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b_{\boldsymbol{\alpha},k}^{(j)} \left(\frac{\mathbf{x} - \mathbf{x}_j}{h_j} \right)^{\boldsymbol{\alpha}} \right) \\ & + \sum_{\mathbf{x}_k \in \mathrm{st}(\mathbf{x}_j)} u(\mathbf{x}_k) \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b_{\boldsymbol{\alpha},k}^{(j)} \left(\frac{\mathbf{x} - \mathbf{x}_j}{h_j} \right)^{\boldsymbol{\alpha}} \Bigg\} \mathcal{P}_j(\mathbf{x}) \eta_j(\mathbf{x}) \\ = \sum_{j \in J} & \sum_{\mathbf{x}_k \in \overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k) \, \mathcal{P}_{j,k}(\mathbf{x}) \, \eta_j(\mathbf{x}), \end{aligned}$$

which can be rewritten as the quasi-interpolant (3.3). From (3.6) and (3.9) follows that

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{h_j^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} (a_{\boldsymbol{\alpha}}^{(j)} - \partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)) \left(\frac{\mathbf{x}_k - \mathbf{x}_j}{h_j} \right)^{\boldsymbol{\alpha}} = R_N(\mathbf{x}_j, \mathbf{x}_k),$$

hence the boundedness of $\| V_{j,h_j}^{-1} \|$ from Condition 3.1 and the estimate of the remainder (3.7) imply

$$|a_{\boldsymbol{\alpha}}^{(j)} - \partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)| \leqslant \boldsymbol{\alpha}! \, C_2 \, h_j^{N-|\boldsymbol{\alpha}|} \sup_{B(\mathbf{x}_j, h_j)} |\nabla_N u|.$$

Therefore, we obtain the inequality

$$|Mu(\mathbf{x}) - M^{(1)} u(\mathbf{x})| \leqslant C_2 \sum_{j \in J} h_j^N \sup_{B(\mathbf{x}_j, h_j)} |\nabla_N u| \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \left| \frac{\mathbf{x} - \mathbf{x}_j}{h_j} \right|^{|\boldsymbol{\alpha}|} |\mathcal{P}_j(\mathbf{x}) \eta_j(\mathbf{x})|$$

and, for any $\mathbf{x} \in B(\mathbf{x}_m, h_m)$,

$$|Mu(\mathbf{x}) - M^{(1)} u(\mathbf{x})| \leqslant C_3 \, h_m^N \sup_{B(\mathbf{x}_m, \lambda h_m)} |\nabla_N u|.$$

This inequality and (3.8) lead to (3.4). □

3.2. Quasi-interpolants with noncompactly supported basis functions

Theorem 3.3. *Suppose that in addition to the conditions of Lemma* 2.3 *the inequality*

$$\Big\| \sum_{j\in J} \Big(1 + h_j^{-1}|\cdot - \mathbf{x}_j|\Big)^{N-K} \Big\|_{L_\infty} < \infty \tag{3.10}$$

is fulfilled, let $u \in W_\infty^N(\mathbb{R}^n)$ and $\varepsilon > 0$ arbitrary. There exist polynomials $\mathcal{P}_{j,k}$, independent of u, whose degrees are uniformly bounded, such that the quasi-interpolant

$$Mu(\mathbf{x}) = \sum_{k\in J} u(\mathbf{x}_k) \sum_{\overline{\mathrm{st}}(\mathbf{x}_j)\ni\mathbf{x}_k} \mathcal{P}_{j,k}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) \tag{3.11}$$

satisfies the estimate

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C \sup_{m\in J} h_m^N \, \|\nabla_N u\|_{L_\infty} + \varepsilon\, |u(\mathbf{x})|. \tag{3.12}$$

The constant C does not depend on u and ε.

Proof. Analogously to (3.5) we introduce the quasi-interpolant

$$M^{(1)}u(\mathbf{x}) = \sum_{j\in J} \left(\sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x}_j)}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{x}_j)^{\boldsymbol{\alpha}}\right)\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)$$

and obtain the estimate

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant \sum_{j\in J} \left| R_N(\mathbf{x},\mathbf{x}_j)\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\right| + |u(\mathbf{x})(\Theta(\mathbf{x})-1)|.$$

From (2.12) we have

$$\left|\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\right| \leqslant \frac{1}{c}\left|\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\right| + \frac{\varepsilon}{c_K}\left(1 + \frac{|\mathbf{x}-\mathbf{x}_j|}{h_j}\right)^{-K},$$

with the lower bound c of $s(\mathbf{x})$ (see (2.2)). Together with (2.9) and (3.7) this provides

$$\begin{aligned}&\left|R_N(\mathbf{x}_j,\mathbf{x})\mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\eta\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right)\right| \\ &\quad\leqslant c_N h_j^N \|\nabla_N u\|_{L_\infty} \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|^N \left(\frac{c_{\delta,p}}{c}\, w_{\delta,p}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) + \frac{\varepsilon}{c_K}\right)\left(1 + \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|\right)^{-K},\end{aligned}$$

resulting in

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant |u(\mathbf{x})|\,|\Theta(\mathbf{x}) - 1| + c_N \sup_{m\in J} h_m^N \,\|\nabla_N u\|_{L_\infty}$$
$$\times \left(\frac{c_{\delta,p}}{c} \left\| w_{\delta,p} |\mathbf{x}|^N \right\|_{L_\infty} \sum_{j\in J} \left(1 + \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|\right)^{-K} + \frac{\varepsilon}{c_K} \sum_{j\in J} \left(1 + \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|\right)^{N-K} \right).$$

Now we can proceed as in the proof of Theorem 3.2. □

Remark 3.4. Let the parameter $\kappa_{\mathbf{x}}$ be chosen for a fixed $\mathbf{x}$ so that

$$\sum_{|\mathbf{x}_j - \mathbf{x}| > \kappa_{\mathbf{x}}} w_{\delta,p} \left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|^N \left(1 + \left|\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right|\right)^{-K} < \varepsilon.$$

Then the estimate (3.12) can be sharpened to

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C \max_{|\mathbf{x}_j - \mathbf{x}| \leqslant \kappa_{\mathbf{x}}} h_j^N \sup_{B(\mathbf{x},\kappa_{\mathbf{x}})} |\nabla_N u| + \varepsilon \left(|u(\mathbf{x})| + \|\nabla_N u\|_{L_\infty} \right).$$

4. Application to the computation of integral operators

Here we discuss a direct application of the quasi-interpolation formula (3.11) for the important example $\eta(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2}$. Suppose that the density of the integral operator with radial kernel

$$\mathcal{K}u(\mathbf{x}) = \int_{\mathbb{R}^n} g(|\mathbf{x} - \mathbf{y}|) u(\mathbf{y})\, \mathrm{d}\mathbf{y} \tag{4.1}$$

is approximated by the quasi-interpolant

$$Mu(\mathbf{x}) = \sum_{j\in J} \sum_{\mathbf{x}_k \in \overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k)\, \mathcal{P}_{j,k}\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_j}\right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/h_j^2}. \tag{4.2}$$

Using the following lemma it is easy to derive a cubature formula for (4.1).

Lemma 4.1. *Any* $\mathcal{P}(\mathbf{x}) = \sum_{|\boldsymbol{\beta}|=0}^{L} c_{\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}$ *can be written as*

$$\mathcal{P}(\mathbf{x}) = \mathrm{e}^{|\mathbf{x}|^2} \sum_{|\boldsymbol{\beta}|=0}^{L} c_{\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}|^2},$$

with the polynomial $\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t})$ *being defined by*

$$\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t}) = \left(\frac{1}{2i}\right)^{|\boldsymbol{\beta}|} H_{\boldsymbol{\beta}}\left(\frac{\mathbf{t}}{2i}\right), \tag{4.3}$$

where $H_{\boldsymbol{\beta}}$ denotes the Hermite polynomial of n variables

$$H_{\boldsymbol{\beta}}(\mathbf{t}) = \mathrm{e}^{|\mathbf{t}|^2}(-\partial_{\mathbf{t}})^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{t}|^2}.$$

Proof. We are looking for the polynomial $\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t})$ defined by the relation

$$\mathbf{x}^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{x}|^2} = \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}|^2}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{4.4}$$

Taking the Fourier transforms

$$\mathcal{F}(\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}|^2})(\lambda) = \pi^{n/2}\mathrm{e}^{-\pi^2|\lambda|^2}\mathcal{S}_{\boldsymbol{\beta}}(2\pi i\,\lambda)$$

and

$$\mathcal{F}(\mathbf{x}^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{x}|^2})(\lambda) = \pi^{n/2}\left(-\frac{\partial_\lambda}{2\pi i}\right)^{\boldsymbol{\beta}}\mathrm{e}^{-\pi^2|\lambda|^2}$$

we obtain (4.3). □

In view of Lemma 4.1 we can write $\mathcal{P}_{j,k}(\mathbf{x})\,\mathrm{e}^{-|\mathbf{x}|^2} = \mathcal{T}_{j,k}(\partial_{\mathbf{x}})\,\mathrm{e}^{-|\mathbf{x}|^2}$ with some polynomials $\mathcal{T}_{j,k}(\mathbf{x})$. Then (4.2) can be rewritten as

$$Mu(\mathbf{x}) = \sum_{j\in J}\sum_{\mathbf{x}_k\in\overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k)\,\mathcal{T}_{j,k}(h_j\,\partial_{\mathbf{x}})\,\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/h_j^2}.$$

The cubature formula for the integral $\mathcal{K}u$ is obtained by replacing u by its quasi-interpolant Mu

$$\begin{aligned}\tilde{\mathcal{K}}u(\mathbf{x}) &= \mathcal{K}Mu(\mathbf{x})\\ &= \sum_{j\in J}\sum_{\mathbf{x}_k\in\overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k)\,\mathcal{T}_{j,k}(h_j\,\partial_{\mathbf{x}})\,h_j^n\int_{\mathbb{R}^n} g(h_j|\mathbf{z}|)\,\mathrm{e}^{-|\mathbf{z}-\mathbf{t}_j|^2}\,\mathrm{d}\mathbf{z},\end{aligned} \tag{4.5}$$

where $\mathbf{t}_j = (\mathbf{x}-\mathbf{x}_j)/h_j$. By introducing spherical coordinates in $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} g(h_j|\mathbf{z}|)\mathrm{e}^{-|\mathbf{z}-\mathbf{t}_j|^2}\,\mathrm{d}\mathbf{z} = \mathrm{e}^{-|\mathbf{t}_j|^2}\int_0^\infty \varrho^{n-1}g(h_j\varrho)\,\mathrm{e}^{-\varrho^2}\mathrm{d}\varrho\int_{S^{n-1}}\mathrm{e}^{2\varrho|\mathbf{t}_j|\cos(\omega_{\mathbf{t}_j},\omega)}\,\mathrm{d}\sigma_\omega,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The integral over S^{n-1} can be represented by means of the modified Bessel functions of the first kind I_n in the following way:

$$\begin{aligned}\int_{S^{n-1}}\mathrm{e}^{2\varrho|\mathbf{t}_j|\cos(\omega_{\mathbf{t}_j},\omega)}\,\mathrm{d}\sigma_\omega &= \frac{2\,\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int_0^\pi \mathrm{e}^{2\varrho|\mathbf{t}_j|\cos\vartheta}(\sin\vartheta)^{n-2}\,\mathrm{d}\vartheta\\ &= 2\pi^{n/2}(\varrho\,|\mathbf{t}_j|)^{1-n/2}I_{(n-2)/2}(2\varrho|\mathbf{t}_j|)\end{aligned}$$

(see [24, p.154] and [25, p.79]). Using the notation

$$\mathcal{L}_j(r) = 2\,\pi^{n/2}\,r^{1-n/2}\mathrm{e}^{-r^2}\int_0^\infty \varrho^{n/2}\,\mathrm{e}^{-\varrho^2}g(h_j\,\varrho)\,I_{(n-2)/2}(2\varrho\,r)\,\mathrm{d}\varrho,$$

relation (4.5) leads to the following cubature formula for the integral Ku

$$\tilde{\mathcal{K}}u(\mathbf{x}) = \sum_{j\in J}h_j^n\sum_{\mathbf{x}_k\in\overline{\mathrm{st}}(\mathbf{x}_j)} u(\mathbf{x}_k)\,\mathcal{T}_{j,k}(h_j\,\partial_{\mathbf{x}})\,\mathcal{L}_j\left(\frac{|\mathbf{x}-\mathbf{x}_j|}{h_j}\right).$$

5. Construction of the Θ-function with Gaussians

In this section we propose a method to construct the approximate partition of unity for the basis functions

$$\eta_j(\mathbf{x}) = (\pi D)^{-n/2} \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_j^2 D)}$$

if the set of nodes $\mathbf{X} = \{\mathbf{x}_j\}_{j\in J}$ satisfy the following condition piecewise with different grid sizes h_j.

Condition 5.1. There exist a domain Ω, $h > 0$ and $\kappa_1 > 0$ such that for any $\mathbf{j} \in \mathbb{Z}^n \cap \Omega$ the ball $B(h\mathbf{j}, h\kappa_1)$ centered at $h\mathbf{j}$ with radius $h\kappa_1$ contains nodes of $\mathbf{X}$.

5.1. Scattered nodes close to a piecewise uniform grid

Let us explain the assumption on the nodes: Suppose that the nodes are located in some domain $\Omega_1 \subset \mathbb{R}^n$ and satisfy Condition 5.1 with $h = h_1$. A subset of nodes $\mathbf{x}_k \in \mathbf{X}_2$ lie in a bounded subdomain $\Omega_2 \subset \Omega_1$ and satisfy Condition 5.1 with $h = h_2 = Hh_1$ for some small H. To keep good local properties of quasi-interpolants one wants to approximate the data at these nodes by functions of the form polynomial times $\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2/(h_2^2 D)}$, whereas outside Ω_2 quasi-interpolants with functions of the form polynomial times $\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_1^2 D)}$ should be used.

Our aim is to develop a simple method to construct polynomials $\mathcal{P}_j$ such that

$$\Theta(\mathbf{x}) = (\pi D)^{-n/2} \left\{ \sum_{\mathbf{x}_j\in\mathbf{X}_1} \mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_1\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_1^2 D)} + \sum_{\mathbf{x}_k\in\mathbf{X}_2} \mathcal{P}_k\left(\frac{\mathbf{x}-\mathbf{x}_k}{h_2\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2/(h_2^2 D)} \right\} \tag{5.1}$$

is almost the constant function 1 for $\mathbf{x} \in \Omega_1$. Here $\mathbf{X}_2$ denotes the set of nodes $\mathbf{x}_k \in \Omega_2$ and the $\mathbf{X}_1$ contains the remaining nodes $\mathbf{X} \setminus \mathbf{X}_2$ and possibly some auxiliary nodes outside the domain Ω_1.

First we derive a piecewise uniform grid on $\mathbb{R}^n$ which is associated to the splitting of the set of scattered nodes into $\mathbf{X}_1$ and $\mathbf{X}_2$. We start with Poisson's summation formula for Gaussians

$$(\pi D)^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D} = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathrm{e}^{-\pi^2 D|\mathbf{k}|^2} \mathrm{e}^{2\pi i(\mathbf{x},\mathbf{k})/h_1},$$

which shows that

$$\left| 1 - (\pi D)^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D} \right| \leqslant C_1\, \mathrm{e}^{-\pi^2 D},$$

with some constant C_1 depending only on the space dimension.

Obviously, for any $\varepsilon > 0$ there exists $D > 0$ and a subset $Z \in \mathbb{Z}^n$ such that the function system $\{\mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D}\}_{\mathbf{m}\in Z}$ forms an approximate partition of unity on the domain Ω_1 with accuracy ε. We can represent any of these functions very accurately by a linear combination of dilated

Gaussians due to the equation (see [16])

$$e^{-|\mathbf{x}|^2/D_1} = \left(\frac{D_1}{\pi D(D_1 - h^2 D)}\right)^{n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} e^{-h^2|\mathbf{m}|^2/(D_1-h^2D)} e^{-|\mathbf{x}-h\mathbf{m}|^2/h^2D}$$
$$-e^{-|\mathbf{x}|^2/D_1} \sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} e^{2\pi i(D_1-h^2D)(\mathbf{x},\mathbf{k})/hD_1} e^{-\pi^2 D(D_1-h^2D)|\mathbf{k}|^2/D_1}, \tag{5.2}$$

which is valid for any $D_1 > h^2 D > 0$. Applied to our setting with $h = h_2$ and $D_1 = h_1^2 D$ we obtain the approximate refinement relation

$$\left| e^{-|\mathbf{x}|^2/h_1^2D} - \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}}\, e^{-|\mathbf{x}-h_2\mathbf{k}|^2/h_2^2D} \right| \leqslant C_2\, e^{-|\mathbf{x}|^2/h_1^2D}\, e^{-\pi^2 D(1-H^2)} \tag{5.3}$$

(because by assumption $h_2 = Hh_1$) with the coefficients

$$a_{\mathbf{k}} = \frac{e^{-H^2|\mathbf{k}|^2/(1-H^2)D}}{(\pi D(1-H^2))^{n/2}}.$$

Again, the constant C_2 depends only on the space dimension. Define by $S \in \mathbb{Z}^n$ the minimal index set such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus S} a_{\mathbf{k}} < e^{-\pi^2 D(1-H^2)}.$$

Then it is clear from (5.3) that for any disjoint Z_1 and Z_2 with $Z_1 \cup Z_2 = Z$

$$\max_{\mathbf{x}\in\Omega_1} \left| 1 - (\pi D)^{-n/2} \left(\sum_{\mathbf{m}\in Z_1} e^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2D} + \sum_{\mathbf{m}\in Z_2} \sum_{\mathbf{k}\in S} a_{\mathbf{k}}\, e^{-|\mathbf{x}-h_1\mathbf{m}-h_2\mathbf{k}|^2/h_2^2D} \right) \right|$$
$$\leqslant C_3\, e^{-\pi^2 D(1-H^2)}. \tag{5.4}$$

Condition 5.2. Denote $Z_2 = \{\mathbf{m} \in \mathbb{Z}^n : h_1\mathbf{m} + h_2\mathbf{k} \in \Omega_2 \text{ for all } \mathbf{k} \in S\}$. The constant κ_1 of Condition 5.1 and the domain Ω_2 are such that for all nodes $\mathbf{x}_k \in \Omega_2$, i.e. the nodes belonging to $\mathbf{X}_2$, one can find $\mathbf{m} \in Z_2$, $\mathbf{k} \in S$ with $|\mathbf{x}_k - h_1\mathbf{m} - h_2\mathbf{k}| < \kappa_1 h_2$.

Setting $Z_1 = Z \setminus Z_2$ we connect the index sets Z_1, Z_2 with the splitting of the scattered nodes into $\mathbf{X}_1, \mathbf{X}_2$, where the set $\mathbf{X}_1$ is formed by the nodes $\mathbf{X}\setminus\mathbf{X}_2$ and, additionally, the nodes $h_1\mathbf{m} \notin \Omega_1$, $\mathbf{m} \in Z_1$.

By this way we construct an approximate partition of unity on the domain Ω_1 using Gaussians with the "large" scaling factor h_1 centered at the uniform grid $G_1 := \{h_1\mathbf{m}\}_{\mathbf{m}\in Z_1}$ outside Ω_2 and using Gaussians with scaling factor h_2 and the centers $G_2 := \{h_1\mathbf{m} + h_2\mathbf{k}\}_{\mathbf{m}\in Z_2,\mathbf{k}\in S}$ in Ω_2.

It is obvious, that the above definition of piecewise quasi-uniformly distributed scattered nodes and the construction of an associated approximate partition of unity on piecewise uniform grids can be extended to finitely many scaling factors h_ℓ. Since there will be no difference for the subsequent considerations we will restrict to the two-scale case.

From (5.4) we see that for any $\varepsilon > 0$, and given h_1 and h_2 there exists $D > 0$ such that the linear combination

$$(\pi D)^{-n/2} \left(\sum_{g_1 \in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/(h_1^2 D)} + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \, \mathrm{e}^{-|\mathbf{x}-g_2|^2/(h_2^2 D)} \right), \tag{5.5}$$

with $\tilde{a}_{g_2} = a_{\mathbf{k}}$ for $g_2 = h_1\mathbf{m} + h_2\mathbf{k}$, $\mathbf{m} \in Z_2, \mathbf{k} \in S$, approximates in Ω_1 the constant function 1 with an error less than $\varepsilon/2$. The idea of constructing the Θ-function (5.1) is to choose for each $g_1 \in G_1$ and $g_2 \in G_2$ finite sets of nodes $\Sigma(g_1) \subset \mathbf{X}_1$ and $\Sigma(g_2) \subset \mathbf{X}_2$, respectively, and to determine polynomials $\mathcal{P}_{j,g_\ell}$ such that

$$\sum_{\mathbf{x}_j \in \Sigma(g_\ell)} \mathcal{P}_{j,g_\ell} \left(\frac{\mathbf{x} - \mathbf{x}_j}{h_\ell \sqrt{D}} \right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_\ell^2 D)} \quad \text{approximate} \quad \mathrm{e}^{-|\mathbf{x}-g_\ell|^2/(h_\ell^2 D)}, \ \ \ell = 1, 2.$$

Note that in the case $g_1 \notin \Omega_1$ by construction $g_1 \in \mathbf{X}_1$, then trivially $\Sigma(g_1) = g_1$ with the corresponding polynomial $\mathcal{P}_{1,g_1} = 1$. Therefore in the following we always assume $g_\ell \in \Omega_1$.

If the L_∞-error of the sums over g_ℓ can be controlled, then we get

$$\sum_{g_1 \in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/(h_1^2 D)} \asymp \sum_{\mathbf{x}_j \in \mathbf{X}_1} \mathcal{P}_j \left(\frac{\mathbf{x} - \mathbf{x}_j}{h_1 \sqrt{D}} \right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/(h_1^2 D)},$$

with the polynomials

$$\mathcal{P}_j = \sum_{g_1 \in G(\mathbf{x}_j)} \mathcal{P}_{j,g_1} \tag{5.6}$$

and

$$\sum_{g_2 \in G_2} \tilde{a}_{g_2} \mathrm{e}^{-|\mathbf{x}-g_2|^2/(h_2^2 D)} \asymp \sum_{\mathbf{x}_k \in \mathbf{X}_2} \mathcal{P}_k \left(\frac{\mathbf{x} - \mathbf{x}_k}{h_2 \sqrt{D}} \right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2/(h_2^2 D)},$$

with the polynomials

$$\mathcal{P}_k = \sum_{g_2 \in G(\mathbf{x}_k)} \tilde{a}_{g_2} \mathcal{P}_{k,g_2}, \tag{5.7}$$

where we denote $G(\mathbf{x}_j) = \{g : \mathbf{x}_j \in \Sigma(g)\}$. Note that we have to choose the sets of nodes $\Sigma(g_\ell)$ such that $G(\mathbf{x}_j) \subset G_\ell$ are nonempty finite sets for any node $\mathbf{x}_j \in \mathbf{X}_\ell$. Additionally, one has to choose these sets such that for some $\kappa_1 > 0$ and any $g_\ell \in G_\ell$ the ball $B(g_\ell, \kappa_1 h_\ell)$ contains at least one node $\mathbf{x}_j \in \mathbf{X}_\ell$. This is always possible, since Conditions 5.1 resp. 5.2 are valid.

The proposed method for constructing an approximate partition of unity does not require solving a large algebraic system. Instead, to obtain the local representation of Θ one has to solve a small number of approximation problems, which are reduced in the next subsection to linear systems of moderate size.

After this preparation we write Θ as

$$\begin{aligned}\Theta(\mathbf{x}) = (\pi D)^{-n/2} \sum_{g_1 \in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/(h_1^2 D)} + \sum_{g_1 \in G_1} \omega_{g_1} \left(\frac{\mathbf{x}}{h_1} \right) \\ +(\pi D)^{-n/2} \sum_{g_2 \in G_2} \tilde{a}_{g_2} \, \mathrm{e}^{-|\mathbf{x}-g_2|^2/(h_2^2 D)} + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \omega_{g_2} \left(\frac{\mathbf{x}}{h_2} \right),\end{aligned}$$

where

$$\omega_{g_\ell}(\mathbf{y}) = (\pi D)^{-n/2} \left\{ \sum_{h_\ell \mathbf{y}_j \in \Sigma(g_\ell)} \mathcal{P}_{j,g_\ell}\left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} - \mathrm{e}^{-|\mathbf{y}-g_\ell/h_\ell|^2/D} \right\}, \qquad (5.8)$$

with $\mathbf{y}_j = \mathbf{x}_j/h_\ell$, $\mathbf{x}_j \in \mathbf{X}_\ell$. Hence for sufficiently large D and all $\mathbf{x} \in \Omega_1$

$$|\Theta(\mathbf{x}) - 1| < \frac{\varepsilon}{2} + \sum_{g_1 \in G_1} \left|\omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right)\right| + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \left|\omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right)\right|. \qquad (5.9)$$

5.2. Construction of polynomials

Let us introduce

$$\omega(\mathbf{y}) := (\pi D)^{-n/2} \left\{ \sum_{\mathbf{y}_j \in \Sigma} \mathcal{P}_j\left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} - \mathrm{e}^{-|\mathbf{y}|^2/D} \right\}, \qquad (5.10)$$

where Σ is some finite point set in $\mathbb{R}^n$. We will describe a method for constructing polynomials $\mathcal{P}_j$ such that $\mathrm{e}^{\rho|\mathbf{y}|^2}|\omega(\mathbf{y})|$ for some $\rho > 0$ becomes small. In what follows we use the representation

$$\mathcal{P}_j(\mathbf{x}) = \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}.$$

Hence by Lemma 4.1

$$\mathcal{P}_j\left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} = \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_{\mathbf{y}})\, \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D},$$

and ω can be written as

$$\omega(\mathbf{y}) = (\pi D)^{-n/2} \left(\sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_{\mathbf{y}})\, \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} - \mathrm{e}^{-|\mathbf{y}|^2/D} \right). \qquad (5.11)$$

To estimate the L_∞-norm of ω we represent this function as convolution.

Lemma 5.1. *Let $\mathcal{P}$ be a polynomial and let $0 < D_0 < D$. Then*

$$\mathcal{P}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/D} = c_1\, \mathrm{e}^{-|\mathbf{x}|^2/(D-D_0)} * \mathcal{P}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/D_0},$$

where $*$ stands for the convolution operator and

$$c_1 = \left(\frac{D}{\pi D_0 (D - D_0)}\right)^{n/2}.$$

Proof. From

$$\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/D} = c_1 \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{x}-\mathbf{t}|^2/(D-D_0)}\, \mathrm{e}^{-|\mathbf{t}-\mathbf{y}|^2/D_0}\, d\mathbf{t}$$

we obtain

$$\mathcal{P}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/D} = \mathcal{P}(-\partial_{\mathbf{y}})\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/D} = c_1 \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{x}-\mathbf{t}|^2/(D-D_0)}\, \mathcal{P}(\partial_{\mathbf{t}})\mathrm{e}^{-|\mathbf{t}-\mathbf{y}|^2/D_0}\, \mathrm{d}\mathbf{t}. \qquad \square$$

Using Lemma 5.1 and (5.11) we write ω as

$$\omega(\mathbf{y}) = \frac{c_1}{(\pi D)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{y}-\mathbf{t}|^2/(D-D_0)} G_{\mathbf{c}}(\mathbf{t})\, \mathrm{d}\mathbf{t}, \tag{5.12}$$

where we denote for $\mathbf{c} = \{c_{j,\boldsymbol{\beta}}\}$

$$G_{\mathbf{c}}(\mathbf{t}) = \sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_{\mathbf{t}})\mathrm{e}^{-|\mathbf{t}-\mathbf{y}_j|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0}. \tag{5.13}$$

Then, by Cauchy's inequality, we obtain

$$\|\omega\|_{L_\infty} \leqslant c_2\, \|G_{\mathbf{c}}\|_{L^2},$$

where

$$c_2 = (\pi D_0)^{-n/2}(2\pi(D-D_0))^{-n/4}.$$

An estimate for the sum of $|\omega_{g_\ell}|$ can be derived from

Lemma 5.2. *Let* $0 < D_0 < D$ *and denote* $\rho = \dfrac{D-D_0}{(D-D_0)^2 + DD_0}$. *Then the estimate*

$$\sup_{\mathbb{R}^n} |\omega(\mathbf{y})|\, \mathrm{e}^{\rho|\mathbf{y}|^2} \leqslant c_3\sqrt{Q(\mathbf{c})} \tag{5.14}$$

is valid, where for $\mathbf{c} = \{c_{j,\boldsymbol{\beta}}\}$ *the quadratic form* $Q(\mathbf{c})$ *is defined by*

$$Q(\mathbf{c}) = \int_{\mathbb{R}^n} \mathrm{e}^{2(D-D_0)|\mathbf{t}|^2/(DD_0)}\, (G_{\mathbf{c}}(\mathbf{t}))^2\, d\mathbf{t}, \tag{5.15}$$

with $G_{\mathbf{c}}$ *from* (5.13) *and*

$$c_3 = \frac{D^{n/4}}{(2\pi^3 D_0(D-D_0)((D-D_0)^2 + DD_0))^{n/4}}.$$

Proof. Starting with (5.12) and using

$$|\mathbf{x}-\mathbf{t}|^2 = \left|\sqrt{a}\mathbf{x} - \frac{\mathbf{t}}{\sqrt{a}}\right|^2 + (1-a)|\mathbf{x}|^2 + \frac{a-1}{a}|\mathbf{t}|^2$$

for $a > 0$, we derive the representation

$$\omega(\mathbf{y}) = c_1 \frac{\mathrm{e}^{-(1-a)|\mathbf{y}|^2/(D-D_0)}}{(\pi D)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}-a\mathbf{x}|^2/(a(D-D_0))}\, \mathrm{e}^{(1-a)|\mathbf{t}|^2/(a(D-D_0))} G_{\mathbf{c}}(\mathbf{t})\, \mathrm{d}\mathbf{t}.$$

Then Cauchy's inequality leads to

$$\left|\omega(\mathbf{y})\, \mathrm{e}^{(1-a)|\mathbf{y}|^2/(D-D_0)}\right| \leqslant c_3 \left(\int_{\mathbb{R}^n} \mathrm{e}^{2(1-a)|\mathbf{t}|^2/a(D-D_0)}\, (G_{\mathbf{c}}(\mathbf{t}))^2\, d\mathbf{t}\right)^{1/2}, \tag{5.16}$$

with

$$c_3 = (\pi D_0)^{-n/2} \left(\frac{a}{2\pi(D-D_0)}\right)^{n/4}.$$

If we choose the parameter a such that

$$\frac{(1-a)|\mathbf{t}|^2}{a(D-D_0)} - \frac{|\mathbf{t}|^2}{D_0} = -\frac{|\mathbf{t}|^2}{D}, \quad \text{i.e.} \quad a = \frac{D\, D_0}{(D-D_0)^2 + D\, D_0},$$

then the right-hand side of (5.16) takes the form (5.15). □

Next we find an explicit expression of the quadratic form $Q(\mathbf{c})$. Using (5.13), after elementary calculations one obtains

$$Q(\mathbf{c}) = \left(\frac{\pi D}{2}\right)^{n/2} \Bigg(1 - 2 \sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathcal{C}_{\boldsymbol{\beta},0}(\mathbf{y}_j, \mathbf{0}) + \sum_{\mathbf{y}_j,\mathbf{y}_k \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} \sum_{|\boldsymbol{\gamma}|=0}^{L_k} c_{j,\boldsymbol{\beta}}\, c_{k,\boldsymbol{\gamma}}\, \mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{y}_j, \mathbf{y}_k)\Bigg),$$

where the function $\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ is given by

$$\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}_{\boldsymbol{\beta}}(-\sqrt{D}\partial_{\mathbf{x}})\mathcal{S}_{\boldsymbol{\gamma}}(-\sqrt{D}\partial_{\mathbf{y}})\, \mathrm{e}^{-(D|\mathbf{x}-\mathbf{y}|^2/2-(D-D_0)(|\mathbf{x}|^2+|\mathbf{y}|^2))/D_0^2} \tag{5.17}$$

and the polynomials $\mathcal{S}_{\boldsymbol{\beta}}$ are defined in (4.3). Hence the minimum of $Q(\mathbf{c})$ is attained by the solution $\mathbf{c} = \{c_{j,\boldsymbol{\beta}}\}$ of the linear system

$$\sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}}\, \mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{y}_j, \mathbf{y}_k) = \mathcal{C}_{\mathbf{0},\boldsymbol{\gamma}}(\mathbf{0}, \mathbf{y}_k), \quad \mathbf{y}_k \in \Sigma,\ 0 \leqslant |\boldsymbol{\gamma}| \leqslant L_k. \tag{5.18}$$

Then by Lemma 5.2 the sum

$$\sum_{\mathbf{y}_j \in \Sigma} \mathcal{P}_j \left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} = \sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} c_{j,\boldsymbol{\beta}} \left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right)^{\boldsymbol{\beta}} \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} \tag{5.19}$$

approximates $\mathrm{e}^{-|\mathbf{y}|^2/D}$ with

$$(\pi D)^{-n/2} \left| \mathrm{e}^{-|\mathbf{y}|^2/D} - \sum_{\mathbf{y}_j \in \Sigma} \mathcal{P}_j \left(\frac{\mathbf{y}-\mathbf{y}_j}{\sqrt{D}}\right) \mathrm{e}^{-|\mathbf{y}-\mathbf{y}_j|^2/D} \right| \leqslant c_3\, \mathrm{e}^{-\rho|\mathbf{y}|^2} r^{1/2},$$

where ρ and c_3 are given in Lemma 5.2 and

$$r := \min_{\mathbf{c}}\ Q(\mathbf{c}). \tag{5.20}$$

In the next section we show that (5.18) has a unique solution and give an estimate of r.

5.3. Existence and estimates

Let us give another representation of $Q(\mathbf{c})$ defined by (5.15). We define polynomials $T_{\boldsymbol{\beta}}$ by

$$T_{\boldsymbol{\beta}}(\mathbf{x}) = \mathrm{e}^{|\mathbf{x}|^2/D_0} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}|^2/D_0} \tag{5.21}$$

and introduce the transformed points

$$\mathbf{t}_j = \frac{D}{D_0}\mathbf{y}_j, \quad \mathbf{y}_j \in \Sigma.$$

Then, because of

$$\frac{D-D_0}{D\,D_0}\,|\mathbf{t}|^2 - \frac{1}{D_0}\,|\mathbf{t}-\mathbf{y}_j|^2 = -\frac{1}{D}\,|\mathbf{t}-\mathbf{t}_j|^2 + \frac{D-D_0}{D_0^2}\,|\mathbf{y}_j|^2$$

and in view of (5.13) the quadratic form $Q(\mathbf{c})$ can be written as

$$Q(\mathbf{c}) = \int_{\mathbb{R}^n} \left(\mathrm{e}^{-|\mathbf{t}|^2/D} - \sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} \widetilde{c}_{j,\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y}_j)\, \mathrm{e}^{-|\mathbf{t}-\mathbf{t}_j|^2/D} \right)^2 \mathrm{d}\mathbf{t}, \tag{5.22}$$

with

$$\widetilde{c}_{j,\boldsymbol{\beta}} = \mathrm{e}^{(D-D_0)|\mathbf{y}_j|^2/D_0^2} c_{j,\boldsymbol{\beta}}.$$

Since $T_{\boldsymbol{\beta}}$ are polynomials of degree $|\boldsymbol{\beta}|$, the minimum problem for $Q(\mathbf{c})$ is equivalent to finding the best L_2 approximation

$$\min_{d_{j,\boldsymbol{\beta}}} \int_{\mathbb{R}^n} \left(\mathrm{e}^{-|\mathbf{t}|^2/D} - \sum_{\mathbf{y}_j \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_j} d_{j,\boldsymbol{\beta}} (\mathbf{t}-\mathbf{t}_j)^{\boldsymbol{\beta}}\, \mathrm{e}^{-|\mathbf{t}-\mathbf{t}_j|^2/D} \right)^2 \mathrm{d}\mathbf{t}.$$

Lemma 5.3. *Let $\{\mathbf{x}_j\}$ a finite collection of nodes. For all $L_j \geqslant 0$ the polynomials $\mathcal{P}_j$ of degree L_j, which minimize*

$$\left\| \mathrm{e}^{-|\cdot|^2} - \sum_j \mathcal{P}_j(\cdot - \mathbf{x}_j)\, \mathrm{e}^{-|\cdot - \mathbf{x}_j|^2} \right\|_{L_2}, \tag{5.23}$$

are uniquely determined.

Proof. The application of Lemma 4.1 gives for $\mathcal{P}_j(\mathbf{x}) = \sum_{|\boldsymbol{\beta}|=0}^{L_j} d_{j,\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}$

$$\left\| \mathrm{e}^{-|\cdot|^2} - \sum_j \mathcal{P}_j(\cdot - \mathbf{x}_j)\mathrm{e}^{-|\cdot - \mathbf{x}_j|^2} \right\|_{L_2}^2$$

$$= \int_{\mathbb{R}^n} \left(\mathrm{e}^{-|\mathbf{x}|^2} - \sum_j \sum_{|\boldsymbol{\beta}|=0}^{L_j} d_{j,\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2} \right)^2 d\mathbf{x}$$

$$= \left(\frac{\pi}{2}\right)^{n/2} \left(1 - 2\sum_j \sum_{|\boldsymbol{\beta}|=0}^{L_j} d_{j,\boldsymbol{\beta}} \mathcal{B}_{\boldsymbol{\beta},0}(\mathbf{x}_j, 0) + \sum_{j,k} \sum_{|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|=0}^{L_j,L_k} d_{j,\boldsymbol{\beta}} d_{k,\boldsymbol{\gamma}} \mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x}_j, \mathbf{x}_k) \right),$$

where we use the notation

$$\mathcal{B}_{\boldsymbol{\beta},\gamma}(\mathbf{x},\mathbf{y}) = \mathcal{S}_{\boldsymbol{\beta}}(-\partial_{\mathbf{x}})\,\mathcal{S}_{\gamma}(-\partial_{\mathbf{y}})\,\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/2}.$$

The coefficients $\{d_{j,\boldsymbol{\beta}}\}$ minimize (5.23) if they satisfy the system of linear equations

$$\sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} d_{j,\boldsymbol{\beta}}\,\mathcal{B}_{\boldsymbol{\beta},\gamma}(\mathbf{x}_j,\mathbf{x}_k) = \mathcal{B}_{0,\gamma}(0,\mathbf{x}_k). \tag{5.24}$$

Hence the uniqueness of the polynomials $\mathcal{P}_j$ is equivalent to the invertibility of the matrix $\|\mathcal{B}_{\boldsymbol{\beta},\gamma}(\mathbf{x}_j,\mathbf{x}_k)\|$ of the system (5.24). In the following we show that this matrix is positive definite. We use the representation

$$\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/2} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x})}\mathrm{e}^{-i(\mathbf{t},\mathbf{y})}\,d\mathbf{t},$$

which implies

$$\mathcal{B}_{\boldsymbol{\beta},\gamma}(\mathbf{x},\mathbf{y}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\overline{\mathcal{S}_{\gamma}(-i\mathbf{t})}\,\mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x})}\mathrm{e}^{-i(\mathbf{t},\mathbf{y})}\,d\mathbf{t}.$$

Let $\{v_{j,\boldsymbol{\beta}}\}$ be a constant vector and consider the sesquilinear form

$$\begin{aligned}
&\sum_{j,k}\sum_{|\boldsymbol{\beta}|,|\gamma|=0}^{L_j,L_k} \mathcal{B}_{\boldsymbol{\beta},\gamma}(\mathbf{x}_j,\mathbf{x}_k)\,v_{j,\boldsymbol{\beta}}\,\overline{v_{k,\gamma}}\\
&\quad= (2\pi)^{-n/2}\sum_{j,k}\sum_{|\boldsymbol{\beta}|,|\gamma|=0}^{L_j,L_k} v_{j,\boldsymbol{\beta}}\,\overline{v_{k,\gamma}}\int_{\mathbb{R}^n}\mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\overline{\mathcal{S}_{\gamma}(-i\mathbf{t})}\,\mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x}_j-\mathbf{x}_k)}\,d\mathbf{t}\\
&\quad= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\mathrm{e}^{-|\mathbf{t}|^2/2}\left|\sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} v_{j,\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_j)}\right|^2 d\mathbf{t}\geqslant 0.
\end{aligned}$$

The change of integration and summation is valid because the integrand is absolutely integrable and the sums are finite. We have to show that the inequality is strict when $\{v_{j,\boldsymbol{\beta}}\}\neq 0$. This is equivalent to showing that

$$\sigma(\mathbf{t}) = \sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} v_{j,\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_j)} = 0$$

identically only if $v_{j,\boldsymbol{\beta}} = 0$ for all j and $\boldsymbol{\beta}$. To this end similar to [21, Lemma 3.1] we introduce the function

$$\begin{aligned}
f_\varepsilon(\mathbf{x}) &:= \int_{\mathbb{R}^n}\mathrm{e}^{-\varepsilon^2|\mathbf{t}|^2/4}\,\sigma(\mathbf{t})\,\mathrm{e}^{-i(\mathbf{t},\mathbf{x})}\,d\mathbf{t} = \sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} v_{j,\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\int_{\mathbb{R}^n}\mathrm{e}^{-\varepsilon^2|\mathbf{t}|^2/4}\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_j-\mathbf{x})}\,d\mathbf{t}\\
&= \varepsilon^{-n}\sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} v_{j,\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\int_{\mathbb{R}^n}\mathrm{e}^{-|\mathbf{t}|^2/4}\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_j-\mathbf{x})/\varepsilon}\,d\mathbf{t}\\
&= \left(\frac{4\pi}{\varepsilon^2}\right)^{n/2}\sum_{j}\sum_{|\boldsymbol{\beta}|=0}^{L_j} v_{j,\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/\varepsilon^2}.
\end{aligned}$$

Let us fix an index k and consider the function $f_\varepsilon(\mathbf{x})$ on the ball $B(\mathbf{x}_k,\varepsilon)$ for sufficiently small $\varepsilon > 0$. If $\mathbf{x} \in B(\mathbf{x}_k,\varepsilon)$ and $\mathbf{x}_j \neq \mathbf{x}_k$, then obviously $\mathcal{S}_{\boldsymbol\beta}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/\varepsilon^2} \to 0$ as $\varepsilon \to 0$. Since $f_\varepsilon(\mathbf{x}) = 0$, for any $\delta > 0$ there exists ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ and $\mathbf{x} \in B(\mathbf{x}_k,\varepsilon)$

$$\left|\sum_{|\boldsymbol\beta|=0}^{L_k} v_{k,\boldsymbol\beta}\,\mathcal{S}_{\boldsymbol\beta}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2/\varepsilon^2}\right| < \delta. \tag{5.25}$$

Setting $\mathbf{t} = (\mathbf{x}-\mathbf{x}_k)/\varepsilon$ we obtain therefore from (5.25)

$$\left|\sum_{|\boldsymbol\beta|=0}^{L_k} v_{k,\boldsymbol\beta}\,\mathcal{S}_{\boldsymbol\beta}(\varepsilon^{-1}\partial_{\mathbf{t}})\mathrm{e}^{-|\mathbf{t}|^2}\right| = \mathrm{e}^{-|\mathbf{t}|^2}\left|\sum_{|\boldsymbol\beta|=0}^{L_k} \varepsilon^{-|\boldsymbol\beta|}\, p_{\boldsymbol\beta}(\mathbf{t})\right| < \delta$$

for all $|\mathbf{t}| \leqslant 1$ and $\varepsilon \in (0,\varepsilon_0)$, where $p_{\boldsymbol\beta}$ are certain polynomials of degree $|\boldsymbol\beta|$ not depending on ε. The inequality is valid for any $\delta > 0$ only if these polynomials vanish, which implies for $\varepsilon = 1$

$$\sum_{|\boldsymbol\beta|=0}^{L_k} v_{k,\boldsymbol\beta}\,\mathcal{S}_{\boldsymbol\beta}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2} = 0.$$

Since by (4.4)

$$\mathcal{S}_{\boldsymbol\beta}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2} = (\mathbf{x}-\mathbf{x}_k)^{\boldsymbol\beta}\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_k|^2},$$

we conclude $v_{k,\boldsymbol\beta} = 0$ for all $\boldsymbol\beta$. □

Let now for given Σ and degrees L_j the coefficient vector $\mathbf{c} = \{c_{j,\boldsymbol\beta}\}$ be a unique solution of the linear system (5.18). To estimate $r = Q(\mathbf{c})$ we denote by $\mathbf{y}_k \in \Sigma$ the point closest to $\mathbf{0}$ and by L_k the degree of the polynomial $\mathcal{P}_k$.

Lemma 5.4. *The minimal value* (5.20) *can be estimated by*

$$r \leqslant \left(\frac{\pi}{2}\right)^{n/2} \frac{D^{L_k+1+n/2}\,|\mathbf{y}_k|^{2(L_k+1)}}{D_0^{2(L_k+1)}(L_k+1)!}.$$

Proof. It follows from the representation (5.22) that

$$r = \int_{\mathbb{R}^n}\left(\sum_{\mathbf{y}_j\in\Sigma}\sum_{|\boldsymbol\beta|=0}^{L_j}\widetilde{c}_{j,\boldsymbol\beta}T_{\boldsymbol\beta}(\mathbf{t}-\mathbf{y}_j)\,\mathrm{e}^{-|\mathbf{t}-\mathbf{t}_j|^2/D} - \mathrm{e}^{-|\mathbf{t}|^2/D}\right)^2 \mathrm{d}\mathbf{t}$$

$$\leqslant \min_{\mathcal{P}\in\Pi_{L_k}} \int_{\mathbb{R}^n}\left(\mathcal{P}(\mathbf{t})\mathrm{e}^{-|\mathbf{t}-\mathbf{t}_k|^2/D} - \mathrm{e}^{-|\mathbf{t}|^2/D}\right)^2 \mathrm{d}\mathbf{t}$$

$$= \left(\frac{D}{2}\right)^{n/2} \min_{\mathcal{P}\in\Pi_{L_k}} \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2}\left(\mathcal{P}(\mathbf{t}) - \mathrm{e}^{-|\mathbf{z}_k|^2}\mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_k)}\right)^2 \mathrm{d}\mathbf{t},$$

with $\mathbf{t}_k = D\mathbf{y}_k/D_0$, $\mathbf{z}_k = \sqrt{D}\mathbf{y}_k/D_0$, and Π_{L_k} denotes the set of polynomials of degree L_k. The minimum is attained when

$$\mathcal{P}(\mathbf{t}) = \frac{1}{\sqrt{2^{|\boldsymbol{\beta}|}\boldsymbol{\beta}!\pi^{n/2}}} \sum_{|\boldsymbol{\beta}|=0}^{L_k} a_{\boldsymbol{\beta}} H_{\boldsymbol{\beta}}(\mathbf{t}),$$

with the coefficients

$$\begin{aligned} a_{\boldsymbol{\beta}} &= \frac{\mathrm{e}^{-|\mathbf{z}_k|^2}}{\sqrt{2^{|\boldsymbol{\beta}|}\,\boldsymbol{\beta}!\,\pi^{n/2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2} H_{\boldsymbol{\beta}}(\mathbf{t}) \mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_k)}\,d\mathbf{t} \\ &= \frac{\mathrm{e}^{-|\mathbf{z}_k|^2}}{\sqrt{2^{|\boldsymbol{\beta}|}\boldsymbol{\beta}!\pi^{n/2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_k)}(-\partial_{\mathbf{t}})^{\boldsymbol{\beta}} \mathrm{e}^{-|\mathbf{t}|^2}\,d\mathbf{t}. \end{aligned}$$

Integrating by parts, we obtain

$$a_{\boldsymbol{\beta}} = \pi^{n/4} \frac{(-1)^{|\boldsymbol{\beta}|}\mathbf{z}_k^{\boldsymbol{\beta}}}{\sqrt{\boldsymbol{\beta}!}}\, \mathrm{e}^{-|\mathbf{z}_k|^2/2},$$

which together with

$$\sum_{|\boldsymbol{\beta}|=L_k+1}^{\infty} \frac{\mathbf{z}_k^{2\boldsymbol{\beta}}}{\boldsymbol{\beta}!} = \sum_{s=L_k+1}^{\infty} \frac{|\mathbf{z}_k|^{2s}}{s!} \leqslant \frac{|\mathbf{z}_k|^{2(L_k+1)}}{(L_k+1)!}\, \mathrm{e}^{|\mathbf{z}_k|^2}$$

leads to

$$r \leqslant \left(\frac{D}{2}\right)^{n/2} \sum_{|\boldsymbol{\beta}|=L_k+1}^{\infty} |a_{\boldsymbol{\beta}}|^2 = \pi^{n/2} \left(\frac{D}{2}\right)^{n/2} \frac{|\mathbf{z}_k|^{2(L_k+1)}}{(L_k+1)!}. \qquad \square$$

5.4. Approximate partition of unity with Gaussians

Now we are in a position to prove the main result of this section. Suppose that the nodes $\{\mathbf{x}_j\}_{j\in J}$ are as described in Section 5.1 and let $G_1 \cup G_2$ be the associated piecewise uniform grid with stepsizes h_1 and h_2. Assign a finite set of nodes $\Sigma(g_\ell)$ to each grid point $g_\ell \in G_\ell$, $\ell = 1, 2$, fix a common degree L for all polynomials $\mathcal{P}_j$ in (5.1) and a positive number $D_0 < D$, and solve the linear system

$$\sum_{\mathbf{x}_j \in \Sigma(g_\ell)} \sum_{|\boldsymbol{\beta}|=0}^{L} \mathcal{C}_{\boldsymbol{\beta},\gamma}\left(\frac{\mathbf{x}_j - g_\ell}{h_\ell}, \frac{\mathbf{x}_k - g_\ell}{h_\ell}\right) c_{j,\boldsymbol{\beta}}(g_\ell) = \mathcal{C}_{\mathbf{0},\gamma}\left(\mathbf{0}, \frac{\mathbf{x}_k - g_\ell}{h_\ell}\right) \tag{5.26}$$

for all $\mathbf{x}_k \in \Sigma(g_\ell)$ and $0 \leqslant |\gamma| \leqslant L$. Following (5.19) define the polynomials

$$\begin{aligned} \mathcal{P}_j\left(\frac{\mathbf{x}-\mathbf{x}_j}{h_1\sqrt{D}}\right) &= \sum_{g_1 \in G(\mathbf{x}_j)} \sum_{|\boldsymbol{\beta}|=0}^{L} c_{j,\boldsymbol{\beta}}(g_1) \left(\frac{\mathbf{x}-\mathbf{x}_j}{h_1\sqrt{D}}\right)^{\boldsymbol{\beta}}, &\quad \mathbf{x}_j \in \mathbf{X}_1, \\ \mathcal{P}_k\left(\frac{\mathbf{x}-\mathbf{x}_k}{h_2\sqrt{D}}\right) &= \sum_{g_2 \in G(\mathbf{x}_k)} \sum_{|\boldsymbol{\beta}|=0}^{L} \tilde{a}_{g_2} c_{k,\boldsymbol{\beta}}(g_2) \left(\frac{\mathbf{x}-\mathbf{x}_k}{h_2\sqrt{D}}\right)^{\boldsymbol{\beta}}, &\quad \mathbf{x}_k \in \mathbf{X}_2. \end{aligned} \tag{5.27}$$

Recall that if $\mathbf{x}_j \in \mathbf{X}_1$ is an additional node $\mathbf{x}_j = h_1\mathbf{m} \notin \Omega_1$, $\mathbf{m} \in Z_1$, then $G(\mathbf{x}_j) = \mathbf{x}_j$ and the corresponding polynomial $\mathcal{P}_j = 1$.

Theorem 5.5. *Under Conditions* 5.1 *and* 5.2 *on the set of scattered nodes* $\mathbf{X}$ *for any* $\varepsilon > 0$ *there exist* $D > 0$ *and* L *such that the function* (5.1) *is an approximate partition of unity satisfying*

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \textit{for all } \mathbf{x} \in \Omega_1,$$

if the polynomials $\{\mathcal{P}_j\}$ *of degree* L *are generated via* (5.27) *by the solutions* $\{c_{j,\boldsymbol{\beta}}(g_\ell)\}$ *of the linear systems* (5.26) *for all* $g_\ell \in G_1 \cup G_2$.

Proof. From (5.9) we have to show that

$$\sup_{\mathbb{R}^n} \left(\sum_{g_1 \in G_1} \left| \omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right) \right| + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \left| \omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right) \right| \right) \leqslant \frac{\varepsilon}{2} \tag{5.28}$$

if L is sufficiently large. We start with estimating the first sum

$$\sum_{g_1 \in G_1} \left| \omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right) \right|,$$

where $g_1 = h_1\mathbf{m}$, $\mathbf{m} \in Z_1 \subset \mathbb{Z}^n$. Using (5.10) we can write

$$\omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right) = \omega\left(\frac{\mathbf{x}}{h_1} - \mathbf{m}\right),$$

where the points $\mathbf{y}_j$ in (5.10) are given by $\mathbf{y}_j = \mathbf{x}_j/h_1 - \mathbf{m}$, $\mathbf{x}_j \in \Sigma(g_1)$. By Lemmas 5.2 and 5.4 we have

$$\sum_{g_1 \in G_1} \left| \omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right) \right| \leqslant c_4 \sum_{\mathbf{m} \in Z_1} \mathrm{e}^{-\rho|\mathbf{x}/h_1 - \mathbf{m}|^2} \frac{D^{(L_{\mu_\mathbf{m}}+1+n/2)/2}}{D_0^{L_{\mu_\mathbf{m}}+1}\sqrt{(L_{\mu_\mathbf{m}}+1)!}} \left| \frac{\mathbf{x}_{\mu_\mathbf{m}}}{h_1} - \mathbf{m} \right|^{L_{\mu_\mathbf{m}}+1},$$

where $c_4 = c_3\,(\pi/2)^{n/4}$, $\mathbf{x}_{\mu_\mathbf{m}} \in \Sigma(g_1)$ is the node closest to $g_1 = h_1\mathbf{m}$ and $L_{\mu_\mathbf{m}}$ is the degree of the polynomial $\mathcal{P}_{\mu_\mathbf{m}, g_1}$. Since $|\mathbf{x}_{\mu_\mathbf{m}} - h_1\mathbf{m}| \leqslant \kappa_1 h_1$ by Condition 5.1 and $L_{\mu_\mathbf{m}} = L$ for all $\boldsymbol{\mu}_\mathbf{m}$ we conclude that

$$\sum_{g_1 \in G_1} \left| \omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right) \right| \leqslant c_4 \frac{D^{(L+1+n/2)/2}\,\kappa_1^{L+1}}{D_0^{L+1}\sqrt{(L+1)!}} \sup_{\mathbb{R}^n} \sum_{\mathbf{m} \in Z_1} \mathrm{e}^{-\rho|\mathbf{x}/h_1 - \mathbf{m}|^2}. \tag{5.29}$$

From

$$\rho = \frac{D - D_0}{(D - D_0)^2 + DD_0} \in (0, D) \quad \text{for any } D_0 \in (0, D),$$

we see, that for fixed D and D_0

$$\sum_{g_1\in G_1}\left|\omega_{g_1}\left(\frac{\mathbf{x}}{h_1}\right)\right| \to 0 \quad \text{if } L\to\infty. \tag{5.30}$$

We turn to

$$\sum_{g_2\in G_2}\tilde{a}_{g_2}\left|\omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right)\right|,$$

with $g_2 = h_1\mathbf{m}+h_2\mathbf{k},\ \mathbf{m}\in Z_2, \mathbf{k}\in S$. Using (5.10) we have

$$\omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right)=\omega\left(\frac{\mathbf{x}-\mathbf{m}h_1}{h_2}-\mathbf{k}\right),$$

and the points $\mathbf{y}_j$ in (5.10) are given by $\mathbf{y}_j=(\mathbf{x}_j-\mathbf{m}h_1)/h_2-\mathbf{k}$ with $\mathbf{x}_j\in\Sigma(g_2)$. Hence

$$\sum_{g_2\in G_2}\tilde{a}_{g_2}\left|\omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right)\right| = \sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S}a_{\mathbf{k}}\left|\omega\left(\frac{\mathbf{x}-\mathbf{m}h_1}{h_2}-\mathbf{k}\right)\right|$$
$$\leqslant c_4\sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S}a_{\mathbf{k}}\mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2}\frac{D^{(L_{\mu_{\mathbf{k}}}+1+n/2)/2}}{D_0^{L_{\mu_{\mathbf{k}}}+1}\sqrt{(L_{\mu_{\mathbf{k}}}+1)!}}\left|\frac{\mathbf{x}_{\mu_{\mathbf{k}}}-\mathbf{m}h_1}{h_2}-\mathbf{k}\right|^{L_{\mu_{\mathbf{k}}}+1}.$$

Here $\mathbf{x}_{\mu_{\mathbf{k}}}\in\Sigma(g_2)$ is the node closest to $g_2=h_1\mathbf{m}+h_2\mathbf{k}$ and $L_{\mu_{\mathbf{k}}}$ is the degree of the polynomial $\mathcal{P}_{\mu_{\mathbf{k}},g_2}$. By Condition 5.2 for fixed D and D_0

$$\frac{D^{(L+1+n/2)/2}}{D_0^{L+1}\sqrt{(L+1)!}}\left|\frac{\mathbf{x}_{\mu_{\mathbf{k}}}-\mathbf{m}h_1}{h_2}-\mathbf{k}\right|^{L+1}\leqslant\delta(L)\to 0 \quad \text{if } L\to\infty$$

uniformly for all $g_2\in G_2$. Hence we obtain

$$\sum_{g_2\in G_2}\tilde{a}_{g_2}\left|\omega_{g_2}\left(\frac{\mathbf{x}}{h_2}\right)\right|\leqslant C_1\delta(L)\sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S}a_{\mathbf{k}}\mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2} \tag{5.31}$$

because of $L_{\mu_{\mathbf{k}}}=L$ for all $\mu_{\mathbf{k}}$. The sum

$$\sum_{\mathbf{k}\in\mathbb{Z}^n}a_{\mathbf{k}}\mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2}$$
$$=\left(\frac{h_1^2}{\pi D(h_1^2-h_2^2)}\right)^{n/2}\sum_{\mathbf{k}\in\mathbb{Z}^n}\mathrm{e}^{-h_2^2|\mathbf{k}|^2/((h_1^2-h_2^2)D)}\,\mathrm{e}^{-\rho|\mathbf{x}-\mathbf{m}h_1-h_2\mathbf{k}|^2/h_2^2}$$

can be easily estimated by using Eq. (5.2). Setting

$$(h_1^2-h_2^2)D=h_1^2D_1-h_2^2/\rho$$

we derive

$$D_1=D+\frac{h_2^2}{h_1^2}\left(\frac{1}{\rho}-D\right)=D+H^2\,\frac{D_0^2}{D-D_0}.$$

and after some algebra

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathrm{e}^{-h_2^2|\mathbf{k}|^2/(h_1^2-h_2^2)D}\, \mathrm{e}^{-\rho|\mathbf{x}-h_2\mathbf{k}|^2/h_2^2}$$
$$= \left(\frac{\pi D(1-H^2)}{\rho D_1}\right)^{n/2} \mathrm{e}^{-|\mathbf{x}|^2/(h_1^2 D_1)} \left(1 + O(\mathrm{e}^{-\pi^2 D^2(1-H^2)/D_1})\right).$$

Therefore, we obtain

$$\sup_{\mathbb{R}^n} \sum_{\mathbf{m}\in Z_2} \sum_{\mathbf{k}\in S} a_{\mathbf{k}} \mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2} \leqslant C_2 \sup_{\mathbb{R}^n} \sum_{\mathbf{m}\in Z_2} \mathrm{e}^{-|\mathbf{x}-\mathbf{m}h_1|^2/(h_1^2 D_1)} \leqslant C_3,$$

with some constant C_3 depending on D, D_0 and the space dimension n. Now (5.28) follows immediately from (5.30) and (5.31). □

5.5. Numerical experiments

We have tested the construction (5.27), (5.26) in the one- and two-dimensional case for randomly chosen nodes with the parameters $D = 2$, $h = 1$, $\kappa_1 = 1/2$, $D_0 = 1$ and $D_0 = 3/2$. To see the dependence of the approximation error on the number of nodes in $\Sigma(m)$, $m \in \mathbb{Z}$, and the degree of polynomials we provide graphs of the difference to 1 for the following one-dimensional cases :

- $\Sigma(m)$ consists of 1 point, $L = 3$ and $L = 4$ (Fig. 1);
- $\Sigma(m)$ consists of 3 points, $L = 3$ and $L = 4$ (Fig. 2);
- $\Sigma(m)$ consists of 5 points, $L = 2$ and $L = 3$ (Fig. 3).

In all cases the choice $D_0 = 3/2$ gives better results as can be seen from Fig. 1. All other figures correspond to the parameter $D_0 = 3/2$.

As expected, the approximation becomes better with increasing degree L and more points in the subsets $\Sigma(m)$. The use of only one node in $\Sigma(m)$ reduces the approximation error by a factor 10^{-1} if L increases by 1. The cases of 3 and 5 points indicate, that enlarging the degree L of the polynomials by 1 gives a factor 10^{-2} for the approximation error.

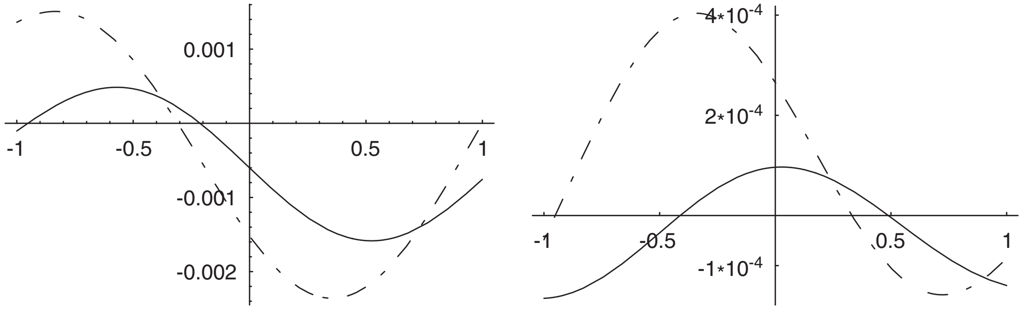

Fig. 1. The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 1 point, $D = 2$, $L = 3$ (on the left) and $L = 4$ (on the right). Solid and dot-dashed line correspond to $D_0 = 3/2$ and $D_0 = 1$, respectively.

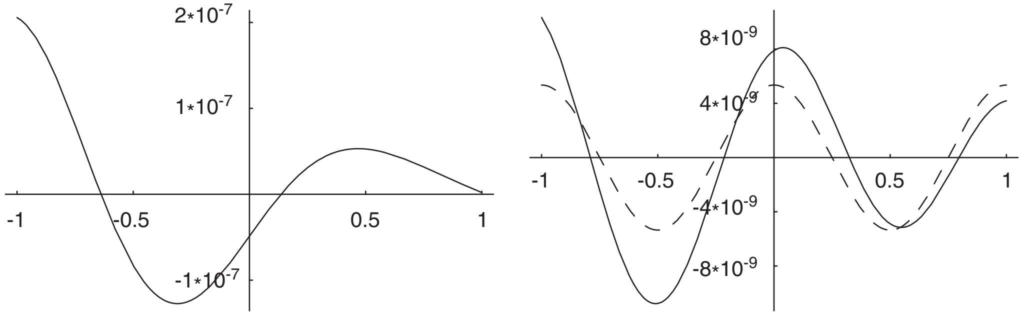

Fig. 2. The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 3 points, $D = 2$, $D_0 = 3/2$, $L = 3$ (on the left) and $L = 4$ (on the right). The saturation term obtained on uniform grid is depicted by dashed lines.

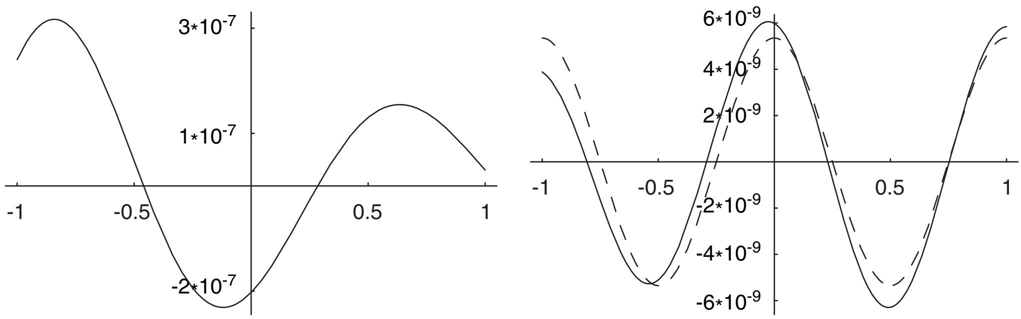

Fig. 3. The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 5 points, $D = 2$, $D_0 = 3/2$, $L = 2$ (on the left) and $L = 3$ (on the right). The saturation term obtained on uniform grid is depicted by dashed lines.

One should notice, that the plotted total error consists of two parts. Using (5.27, 5.26) we approximate the Θ-function

$$(2\pi)^{-1/2} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-(x-m)^2/2} = 1 + 2 \sum_{j=1}^{\infty} \mathrm{e}^{-2\pi^2 j^2} \cos 2\pi j x. \tag{5.32}$$

Hence, the plotted total error is the sum of the difference between (5.1) and (5.32) and the function

$$2 \sum_{j=1}^{\infty} \mathrm{e}^{-2\pi^2 j^2} \cos 2\pi j x, \tag{5.33}$$

which is the saturation term obtained on the uniform grid. The error plots on the right in Figs. 2 and 3 show that the total error is already majorized by the saturation term (5.33), which is shown by dashed lines.

In the following two Figs. 4 and 5 we depict the difference $Mu(x) - u(x)$ for the quasi–interpolation formula defined by (3.11) with Gaussian basis functions constructed via (5.27, 5.26) with $\Sigma(m)$ consisting of 5 points, and the approximation orders $N = 2$ and $N = 4$. For $N = 2$

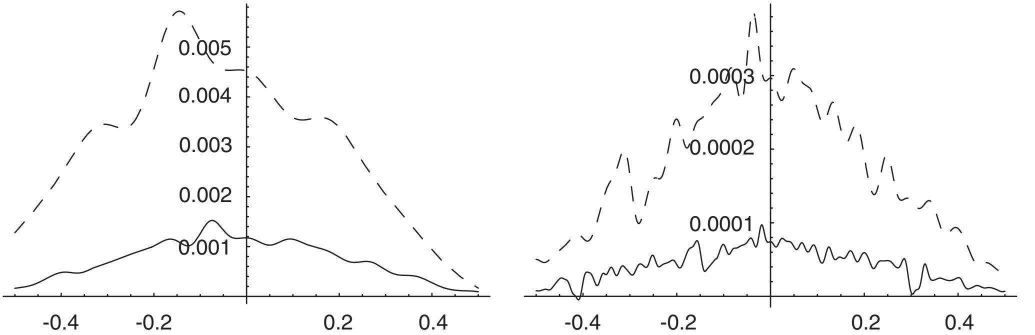

Fig. 4. The graph of $Mu(x) - u(x)$ with $N = 2$, $u(x) = (1 + x^2)^{-1}$. Dashed and solid lines correspond to $h = 1/16$ and $h = 1/32$ (on the left) and to $h = 1/64$ and $h = 1/128$ (on the right).

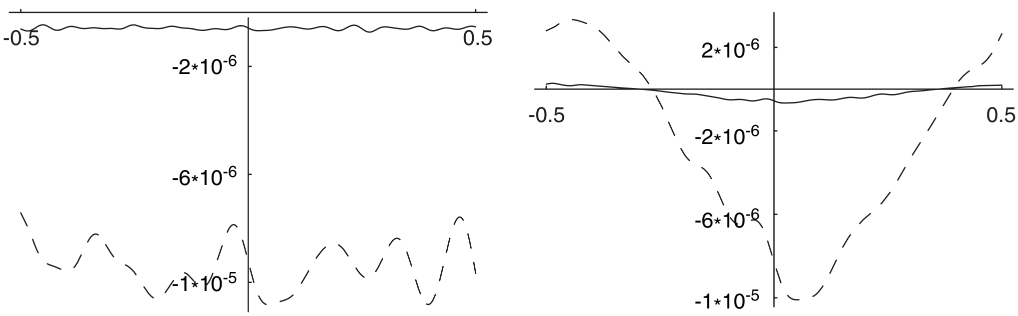

Fig. 5. The graph of $Mu(x) - u(x)$ with $N = 4$, $u(x) = x^4$ (on the right) and $u(x) = (1 + x^2)^{-1}$ (on the left). Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$, respectively.

Table 1
L_∞−approximation error for the function $u(x) = (1 + x^2)^{-1}$ in the interval $(-1/2, 1/2)$ using Mu with $N = 2$ (on the left) and $N = 4$ (on the right)

h	$N = 2$	$N = 4$
2^{-3}	1.89×10^{-2}	1.81×10^{-3}
2^{-4}	5.72×10^{-3}	1.38×10^{-4}
2^{-5}	1.51×10^{-3}	1.01×10^{-5}
2^{-6}	3.81×10^{-4}	6.65×10^{-7}
2^{-7}	9.65×10^{-5}	4.20×10^{-8}

we have used the parameters $L = 4$ (the degree of the polynomials $\mathcal{P}_j$), $D = 2$, $D_0 = 3/2$, and for $N = 4$ we have chosen $L = 6$, $D = 4$, $D_0 = 3$.

The h^N-convergence of these one-dimensional quasi-interpolants is confirmed in Table 1, which contains the uniform error of $Mu - u$ on the interval $(-1/2, 1/2)$ for the function $u(x) = (1 + x^2)^{-1}$ with different h.

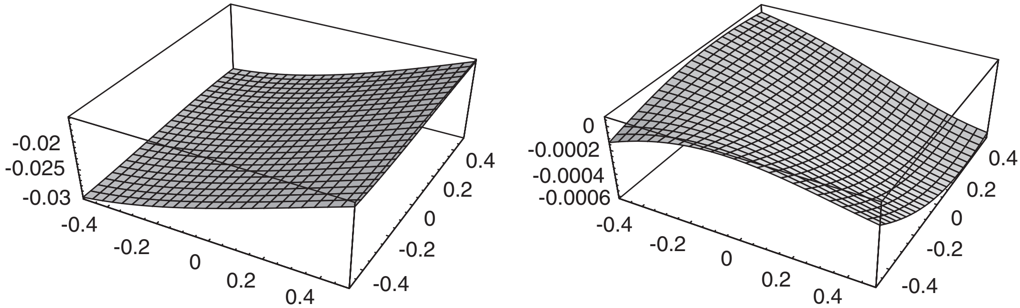

Fig. 6. The graph of $\Theta(\mathbf{x}) - 1$ when $L = 1$ and $\Sigma(\mathbf{m})$ consists of 1 point (on the left) and 5 points (on the right).

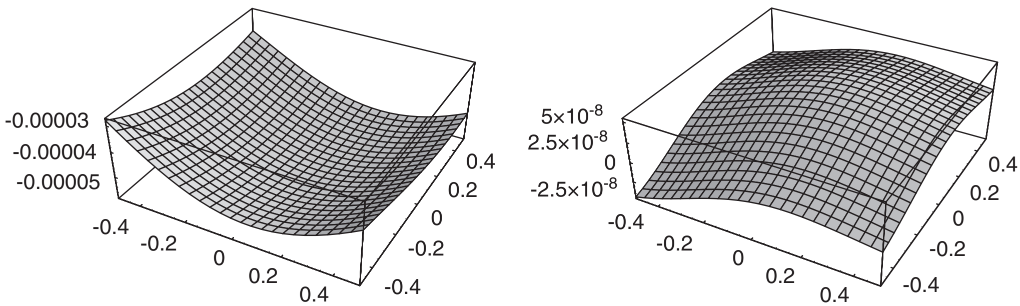

Fig. 7. The graph of $\Theta(\mathbf{x}) - 1$ when $L = 4$ and $\Sigma(\mathbf{m})$ consists of 1 point (on the left) and 5 points (on the right).

Similar experiments have been performed for the two-dimensional case. Here we provide graphs of

$$1 - \sum_{\mathbf{x}_j \in \mathbf{X}} \mathcal{P}_j \left(\frac{\mathbf{x} - \mathbf{x}_j}{\sqrt{D}} \right) \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/D}$$

for the following cases :

- deg $\mathcal{P}_j = 1$ and $\Sigma(\mathbf{m})$ consists of 1 or 5 points (Fig. 6);
- deg $\mathcal{P}_j = 4$ and $\Sigma(\mathbf{m})$ consists of 1 or 5 points (Fig. 7).

References

[1] I. Babuška, U. Banerjee, J.E. Osborn, On the approximability and the selection of particle shape functions, Numer. Math. 96 (2004) 601–640.

[2] M.D. Buhmann, N. Dyn, D. Levin, On quasi interpolation by radial basis functions with scattered data, Constr. Approx. 11 (1995) 239–254.

[3] C. de Boor, R.A. DeVore, A. Ron, Approximation from shift–invariant subspaces of $L_2(\mathbb{R}^d)$, Trans. AMS 341 (1994) 787–806.

[4] C. de Boor, A. Ron, Fourier analysis of the approximation power of principal shift-invariant spaces, Constr. Approx. 8 (1992) 427–462.

[5] N. Dyn, A. Ron, Radial basis function approximation: from gridded centers to scattered centers, Proc. London Math. Soc. 71 (1995) 76–108.

[6] R. Farwig, Multivariate interpolation of arbitrarily spaced data by moving least squares methods, J. Comput. Appl. Math. 16 (1986) 79–93.
[7] T. Ivanov, V. Maz'ya, G. Schmidt, Boundary layer approximate approximations for the cubature of potentials in domains, Adv. Comput. Math. 10 (1999) 311–342.
[8] K. Jetter, D.X. Zhou, Order of linear approximation from shift-invariant spaces, Constr. Approx. 11 (1995) 423–438.
[9] V. Karlin, V. Maz'ya, Time-marching algorithms for non local evolution equations based upon approximate approximations, SIAM J. Sci. Comput. 18 (1997) 736–752.
[10] D. Levin, The approximations power of moving least-squares, Math. Comput. 67 (1998) 1517–1531.
[11] V. Maz'ya, A new approximation method and its applications to the calculation of volume potentials. Boundary point method, in: 3. DFG-Kolloqium des DFG-Forschungsschwerpunktes Randelementmethoden, 1991.
[12] V. Maz'ya, Approximate approximations, in: J.R. Whiteman (Ed.), The Mathematics of Finite Elements and Applications. Highlights 1993, Wiley & Sons, Chichester, 1994.
[13] V. Maz'ya, V. Karlin, Semi-analytic time marching algorithms for semi-linear parabolic equations, BIT 34 (1994) 129–147.
[14] V. Maz'ya, G. Schmidt, Approximate approximations and the cubature of potentials, Rend. Mat. Acc. Lincei 6 (1995) 161–184.
[15] V. Maz'ya, G. Schmidt, On approximate approximation using Gaussian kernels, IMA J. Numer. Anal. 16 (1996) 13–29.
[16] V. Maz'ya, G. Schmidt, Approximate wavelets and the approximation of pseudodifferential operators, App. Comput. Harm. Anal. 6 (1999) 287–313.
[17] V. Maz'ya, G. Schmidt, On quasi-interpolation with non-uniformly distributed centers on domains and manifolds, J. Approx. Theory 110 (2001) 125–145.
[18] V. Maz'ya, G. Schmidt, W. Wendland, On the computation of multi-dimensional single layer harmonic potentials via approximate approximations, Calcolo 40 (2003) 33–53.
[19] H.N. Mhaskar, Approximation theory and neural networks, in: P.K. Jain et al. (Eds.), Wavelet and Allied Topics, Narosa Publishing House, New Delhi, 2001, pp. 247–289.
[20] H.N. Mhaskar, On the degree of approximation in multivariate weighted approximation, in: M.D. Buhmann, D.H. Mache (Eds.), Advanced Problems in Constructive Approximation, ISNM, vol. 142, Birkhäuser, Basel, 2002, pp. 129–141.
[21] M.J.D. Powell, The theory of radial basis functions in 1990, in: W. Light (Ed.), Advances in numerical analysis. vol. 2: Wavelets, Subdivision Algorithms, and Radial Basis Functions, Clarendon Press, Oxford, 1992, pp. 105–210.
[22] R. Schaback, Z.M. Wu, Construction techniques for highly accurate quasi-interpolation operators, J. Approx. Theory 91 (1997) 320–331.
[23] G. Schmidt, On approximate approximations and their applications, in: The Maz'ya anniversary collection, v.1, Operator Theory: Adv. Appl. 109 (1999) 111–138.
[24] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University press, Princeton, NJ, 1971.
[25] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1922.
[26] Z.M. Wu, J.P. Liu, Generalized Strang-Fix condition for scattered data quasi-interpolation, Adv. Comput. Math. 23 (2005) 201–214.