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Local analysis of the normalizer problem

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Abstract

For a finite group G, and a commutative ring R, the automorphisms of G inducing an inner automorphism of the group ring RG form a group $\mathrm{Aut}_R(G)$. Let $\mathrm{Aut}_{\mathrm{int}}(G) = \mathrm{Aut}_A(G)$, where A is the ring of all algebraic integers in $\mathbb{C}$. It is shown how Clifford theory can be used to analyze $\mathrm{Aut}_{\mathrm{int}}(G)$. It is proved that $\mathrm{Aut}_{\mathrm{int}}(G)/\mathrm{Inn}(G)$ is an abelian group, and can indeed be any finite abelian group. It is an outstanding question whether $\mathrm{Aut}_{\mathbb{Z}}(G) = \mathrm{Inn}(G)$ if G has an abelian Sylow 2-subgroup. This is shown to be true in some special cases, but also a group G with abelian Sylow subgroups and $\mathrm{Aut}_{\mathrm{int}}(G) \neq \mathrm{Inn}(G)$ is given. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The normalizer of a finite group G in the units of its integral group ring $\mathbb{Z}G$ is an interesting object. Its study includes the study of central units, which is already a very difficult and broad subject. Moreover, there is an apparently "small" quotient of the normalizer, naturally isomorphic to a certain subgroup of $\mathrm{Out}(G)$, which measures the extend to which there are "non-obvious" units normalizing G. It has been studied already in [8,12,14,17,19,25].

The aim of this paper is to get a better understanding of when this quotient is non-trivial, since a close investigation of certain examples led to the construction of the first counterexample to the so-called "isomorphism problem for integral group rings" [8,9], and that is where our interest stems from. More generally group rings RG are

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considered, where R is a *G-adapted ring*, that is, an integral domain of characteristic 0 in which no prime divisor of $|G|$ is invertible. Typical examples of G-adapted rings are

- $R=\mathbb{Z}_\pi$, the semilocalization of $\mathbb{Z}$ at a finite set π of primes containing the set $\pi(G)$ of prime divisors of $|G|$, i.e. the intersection of the localizations $\mathbb{Z}_{(p)}=\{a/b\colon a,b\in\mathbb{Z},(p,b)=1\}$, with $p\in\pi$;
- $R=\mathcal{O}$, a ring of integers in an algebraic number field.

One reason for this is that it has been conventional wisdom in the community of researchers in this area that $\mathbb{Z}_\pi$ is as good as $\mathbb{Z}$ (cf. [27, p. 267]), while at the same time it is much easier to work with the semilocal ring $\mathbb{Z}_\pi G$.

To be more precise, the following notation is introduced. For a commutative ring R, let $\mathrm{Aut}_R(G)$ be the group of automorphisms of G which induce an inner automorphism of RG. The quotient $\mathrm{Out}_R(G)=\mathrm{Aut}_R(G)/\mathrm{Inn}(G)$ – changing 'A' to 'O' will always have this fixed meaning – has two natural interpretations:

- $\mathrm{Out}_R(G)$ is the kernel of the natural map $\mathrm{Out}(G)\to\mathrm{Out}(RG)$;
- $\mathrm{Out}_R(G)\cong \mathrm{N}_U(G)/G\cdot \mathrm{C}_U(G)$, where $U=\mathrm{U}(RG)$ is the group of units in RG.

In particular, $\mathrm{Out}_R(G)=1$ means that the units in RG normalizing G are the 'obvious' ones. Note that $\mathrm{Aut}_R(G)$ is always contained in $\mathrm{Aut}_{\mathbb{Q}}(G)=\mathrm{Aut}_c(G)$, the group of *class-preserving automorphisms* (cf. [12, Proposition 2.5]). Hence, from now on, it is assumed that R is G-adapted, aiming for a nice blending of group theory and ring theory. A group G is said to have the *normalizer property* (NP) if

$$\mathrm{Out}_{\mathbb{Z}}(G)=1,\quad\text{or, equivalently,}\quad \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)=G\cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G)).\qquad\text{(NP)}$$

This may as well be understood as the *normalizer problem*, i.e. the problem to 'determine' in some sense the groups G with (NP), a subject of research initiated by the questions [12, Question 3.7; 28, Problem 43].

By the way, there is an analog of the normalizer problem for so called *unitary subgroups* of integral group rings, which is discussed in [16], but there, 'counterexamples' are easily found.

Finding groups which does not satisfy (NP) is a difficult task. In positive direction, it has been shown that (NP) holds for groups with a normal Sylow 2-subgroup [12, Theorem 3.6], and for groups whose generalized Fitting subgroup is a p-group [8, 4.2.4]. In both cases, the proof relies on the following result, which is essentially due to Coleman [3].

Lemma (Coleman [3]). *Let P be a p-subgroup of G, and let S be a commutative ring with $pS\neq S$. Then* $\mathrm{N}_{\mathrm{U}(SG)}(P)=\mathrm{N}_G(P)\cdot \mathrm{C}_{\mathrm{U}(SG)}(P)$.

(In this form, it appears first in [26, Proposition 1.14]; see also [12, Proposition 2.3].) It means that a unit of the group ring SG normalizing P acts by conjugation on P like a group element $g\in G$.

The automorphisms of G whose restriction to any Sylow subgroup equals the restriction of some inner automorphism of G form a characteristic subgroup $\mathrm{Aut}_{\mathrm{Col}}(G)$ of G. Coleman's result shows that $\mathrm{Aut}_R(G) \leq \mathrm{Aut}_{\mathrm{Col}}(G)$ (recall that R is G-adapted), which may justify the notation (occasionally, automorphisms of $\mathrm{Aut}_{\mathrm{Col}}(G)$ are called *Coleman automorphisms*, for short). This important fact is recorded explicitly:

$$\mathrm{Aut}_R(G) \leq \mathrm{Aut}_{\mathrm{c}}(G) \cap \mathrm{Aut}_{\mathrm{Col}}(G). \qquad (*)$$

Roggenkamp and Zimmermann [25] realized that the question whether $\mathrm{Out}_R(G)$ is trivial or not can be attacked by using some kind of a general theory (an improved and simplified version is given by Proposition 2.6). They furnished examples of finite, three-step abelian groups G with $\mathrm{Out}_{\mathbb{Z}_{\pi(G)}}(G) \cong C_2$, and hence also with $\mathrm{Out}_{\mathcal{O}}(G) \cong C_2$ for some suitable chosen ring $\mathcal{O}$ of integers in an algebraic number field (cf. Theorem 2.1).

This result indicates that there 'should be' groups which do not satisfy (NP). Nevertheless, it took some more time until it was shown that $\mathrm{Out}_{\mathbb{Z}}(G) = 1$ for their groups G, and the following was proved [8, 4.5, Theorem A; 9]: *there are finite groups G with* $\mathrm{Out}_{\mathbb{Z}}(G) \neq 1$. *One example is a metabelian group of order* $2^{25} \cdot 97^2$, *with normal Sylow* 97-*subgroup.*

It should be remarked that if $\mathrm{Out}_{\mathbb{Z}}(G) \neq 1$, then G is necessarily of *even* order. This follows from the following result of J. Krempa (a proof is given in [12, 3.2 Theorem]), since each prime divisor of the order of $\mathrm{Aut}_{\mathrm{c}}(G)$ divides $|G|$ [11, I 4 Aufgabe 12].

Lemma (Krempa). $\mathrm{Out}_{\mathbb{Z}}(G)$ *is an elementary abelian* 2-*group.*

Krempa's result has recently been generalized by Mazur [19]. In particular, if $\mathcal{O}$ is a ring of integers in a number field K such that the complex conjugation is central in the Galois group of the normal closure of K, then $\mathrm{Out}_{\mathcal{O}}(G)$ is again an elementary abelian 2-group.

An old problem, posed first by G. Higman in 1940, is the "isomorphism problem for integral group rings". Given finite groups X and Y, is it true that $\mathbb{Z}X \cong \mathbb{Z}Y$ implies $X \cong Y$? Though exciting results in positive direction have been achieved, it is now known that in general the question has a negative answer: a counterexample has been given in [8,9]. It is not intended to go into details, but a few words may be in order. In this work, non-isomorphic groups X and Y of order $2^{21} \cdot 97^{28}$ are constructed which have isomorphic group rings over $\mathbb{Z}$. Though the main idea for the construction of X is based on the pioneering feat of Roggenkamp and Scott (cf. [15,22,27]), a substantial part of the construction involves a group $G < X$ with $\mathrm{Out}_{\mathbb{Z}}(G) \neq 1$. It follows that counterexamples which are constructed in that way must be necessarily of *even* order, by Krempa's result. Thus, the isomorphism problem for finite groups of odd order is still open.

It should be remarked that to that time, it has been known that a group G with $\mathrm{Out}_R(G) \neq 1$ gives rise to non-isomorphic (*infinite* polycyclic) groups $X = G \times \mathbb{Z}$ and Y with $RX \cong RY$. This observation of Mazur [18] has been refined in [8,9] insomuch that it is also applicable to finite groups (but then, one has to deal with $\mathrm{N}_{\mathrm{U}(RG)}(G)$ in more detail).

The isomorphism problem may be formulated just as well for group rings $\mathcal{O}X$, with $\mathcal{O}$ being the ring of integers in an algebraic number field, and then one may ask for counterexamples. In this context, the subgroup $\mathrm{Aut}_{\mathrm{int}}(G) \leq \mathrm{Aut}(G)$, consisting of those $\sigma \in \mathrm{Aut}(G)$ which induce an inner automorphism of some group ring $\mathcal{O}X$, with $\mathcal{O}$ as above, should play an important role.

In Section 2.2, it is proved that $\mathrm{Out}_{\mathrm{int}}(G)$ is contained in the center of $\mathrm{Out}_{\mathrm{c}}(G)$. In particular, $\mathrm{Out}_{\mathrm{int}}(G)$ is an abelian group, and examples are given showing that $\mathrm{Out}_{\mathrm{int}}(G)$ can be indeed any finite abelian group, in contrast to Krempa's result. Hence, further investigations may very well lead to counterexamples to the isomorphism problem for $\mathcal{O}G$, with G finite of *odd* order.

In Section 2, it is first recalled that $\mathrm{Out}_{\mathrm{int}}(G)$ is the intersection of all $\mathrm{Out}_{\mathbb{Z}_{(p)}}(G)$, with p ranging over the prime divisors of $|G|$. This enables one to use Clifford theory in the study of $\mathrm{Out}_{\mathrm{int}}(G)$, and a version for automorphisms is established (extending [25, Proposition 1]), which will be applied to produce various groups with $\mathrm{Out}_{\mathrm{int}}(G) \neq 1$.

To the best of our knowledge, for any known class of groups with (NP) this property can be verified entirely group-theoretically, i.e. by showing that in ($*$), the group on the right-hand side is a $2'$-group. An exception might be the class of groups with abelian Sylow 2-subgroups (for which it is not known whether (NP) holds), which will be examined in Section 4.

It seems that for groups G with $\mathrm{Out}_{\mathbb{Z}}(G) \neq 1$, the structure of the Sylow 2-subgroups is the most sensitive towards modifications. In [19, p. 176], Mazur conjectured, with frankly very little supporting evidence, that

$$\mathrm{Out}_{\mathbb{Z}}(G) = 1 \quad \text{if } G \text{ has abelian Sylow 2-subgroups.} \tag{Ab}$$

One reason for that might be seen in Lemma 4.1, which tells us that for a counterexample G of minimal order (if existing), $\mathrm{O}_2(G) = 1$, so a non-inner group automorphism $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$ cannot be a *central group automorphism*, i.e. cannot induce the identity on $G/\mathrm{Z}(G)$. However, in all so far known examples, $\mathrm{Out}_{\mathbb{Z}}(G)$ were covered by the group of central group automorphisms; a first example in a completely different direction is Example 4.12, where a group G with abelian Sylow subgroups and $\mathrm{Out}_{\mathrm{int}}(G) \neq 1$, but $\mathrm{Out}_{\mathbb{Z}}(G) = 1$, is given. Nevertheless, (Ab) still remains a challenge.

Mazur proved (Ab) when Sylow 2-subgroups are of order 2. In Section 4, this result is extended to groups G which have a normal 2-complement and which have a Sylow 2-subgroup which is either cyclic or abelian of exponent at most 4.

In this context, note that according to [19, p. 176], Roggenkamp and Marciniak produced a preprint where it is proved that (Ab) holds if in addition G is metabelian. However, this is a special case of what has been proved in [7,10]: If G is a metabelian group with abelian Sylow p-subgroups, then $\mathrm{Out}_{\mathrm{c}}(G)$ is a p'-group.

The notation used is mostly standard. For group elements x, y we set $x^y = y^{-1}xy$ and $[x, y] = x^{-1}x^y$. By $\mathrm{conj}(y)$ we denote any homomorphism of the form $x \mapsto x^y$. Recall that the following characteristic subgroups of $\mathrm{Aut}(G)$ have been introduced:

- $\mathrm{Aut_c}(G)$ is the group of class-preserving automorphisms;
- $\mathrm{Aut_{Col}}(G)$ consists of those $\alpha \in \mathrm{Aut}(G)$ such that for any Sylow subgroup P of G, there is $\gamma \in \mathrm{Inn}(G)$ with $\alpha|_P = \gamma|_P$ ('Coleman automorphisms' for short);
- $\mathrm{Aut}_R(G)$ consists of those $\alpha \in \mathrm{Aut}(G)$ which induce an inner automorphism of the group ring RG;
- $\mathrm{Aut_{int}}(G)$ consists of those $\alpha \in \mathrm{Aut}(G)$ which induce an inner automorphism of $\mathcal{O}G$, for some ring $\mathcal{O}$ of algebraic integers in an algebraic number field.

Also, the corresponding 'outer' automorphism groups have been introduced.

2. Some theoretical background

2.1. Local–global principle for class-preserving automorphisms

Let G be a finite group, $\alpha \in \mathrm{Aut}(G)$, and R an integral domain of characteristic 0. Then RG becomes an $R(G \times G)$-module, denoted by ${}_1RG_\alpha$, by letting $m \cdot (x, y) = x^{-1}m(y\alpha)$ for all $m \in RG$ and $x, y \in G$ (and linear extension of this operation). Note that ${}_1RG_\alpha \cong {}_1RG_1$ if and only if α induces an inner automorphism of RG. This interpretation of an automorphism as an invertible bimodule allows us to prove a local–global principle for class-preserving group automorphisms (see [25,6] for more details).

Theorem 2.1. *For any finite group G, there is a ring of integers $\mathcal{O}$ in an algebraic number field such that*

$$\mathrm{Out_{int}}(G) = \bigcap_{p \mid |G|} \mathrm{Out}_{\mathbb{Z}_{(p)}}(G) = \mathrm{Out}_{\mathbb{Z}_{\pi(G)}}(G) = \mathrm{Out}_{\mathcal{O}}(G).$$

Proof. Let $\alpha \in \mathrm{Aut}(G)$. Then the bimodules ${}_1\mathbb{Z}G_\alpha$ and ${}_1\mathbb{Z}G_1$ are in the same genus if and only if α belongs to $\bigcap_p \mathrm{Aut}_{\mathbb{Z}_{(p)}}(G)$, or, equivalently, to $\mathrm{Aut}_{\mathbb{Z}_{\pi(G)}}(G)$ (cf. also Remark 2.2). By a theorem of Jacobinski [13, Satz 7] (see also [4, 51.33]) there is some ring $\mathcal{O}$ of algebraic integers such that ${}_1\mathbb{Z}G_\alpha$ and ${}_1\mathbb{Z}G_1$ are in the same genus if and only if ${}_1\mathcal{O}G_\alpha$ and ${}_1\mathcal{O}G_1$ are isomorphic. It remains to show that $\mathrm{Aut_{int}}(G)$ is contained in $\mathrm{Aut}_{\mathbb{Z}_{(p)}}(G)$ for all p, which follows from a generalization of the Noether–Deuring theorem [21]. □

Remark 2.2. Let $\alpha \in \mathrm{Aut}(G)$ such that there are units u_p in $\mathbb{Z}_{(p)}G$ with $\alpha = \mathrm{conj}(u_p)$, for all $p \in \pi = \pi(G)$. Without lost of generality, each u_p is contained in $\mathbb{Z}G$. Let $m = \prod_{p\in\pi} p$. Then $u = \sum_{p\in\pi}(m/p)u_p$ is a unit in $\mathbb{Z}_\pi G$ with $\alpha = \mathrm{conj}(u)$. Indeed, $\mathrm{rad}(\mathbb{Z}_\pi)G \subseteq \mathrm{rad}(\mathbb{Z}_\pi G)$ and $\mathbb{Z}_\pi G/\mathrm{rad}(\mathbb{Z}_\pi)G \cong \bigoplus_{p\in\pi} \mathbb{F}_p G$, so u is a unit modulo the radical and hence a unit in $\mathbb{Z}_\pi G$ (see [4, 5.10]).

Remark 2.3. Roggenkamp pointed out where the obstruction for getting globally central automorphisms from local data lies [23, p. 82; 24]. Let R be a Dedekind ring of characteristic 0, and let $\mathrm{Cl}_{RG}(\mathrm{Z}(RG))$ be the subgroup of the locally free class group $\mathrm{Cl}(\mathrm{Z}(RG))$ consisting of those isomorphism classes of invertible ideals $\mathfrak{a}$ in $\mathrm{Z}(RG)$ so that $\mathfrak{a}RG$ is a principal ideal in RG. Then Fröhlich's localization sequence (cf. [4, 55.25, 55.26]) can be extended to the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Cl}(\mathrm{Z}(RG)) & \longrightarrow & \mathrm{Picent}(RG) & \longrightarrow & \prod_{P\in\max(R)} \mathrm{Picent}(R_PG) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow \cong & & \\ 1 & \longrightarrow & \mathrm{Cl}_{RG}(\mathrm{Z}(RG)) & \longrightarrow & \mathrm{Outcent}(RG) & \xrightarrow{\lambda} & \prod_{P\in\max(R)} \mathrm{Outcent}(R_PG) & & \end{array}.$$

In particular, the image of the natural map $\mathrm{Out}_{\mathrm{int}}(G) \to \mathrm{Outcent}(RG)$ is a subgroup of $\mathrm{Cl}_{RG}(\mathrm{Z}(RG))$.

It is an open problem (see [23, IX 1.13; 24]) whether λ is surjective, i.e. whether for any M in $\mathrm{Picent}(RG)$ there is an invertible bimodule in the same genus as M which is RG-free from the left (say). If R is a semilocal ring, then λ is an isomorphism (see [4, 55.26, 55.16]).

2.2. Clifford theory for automorphisms

Let N be a normal subgroup of a finite group G, and R a commutative ring. A classical result of Clifford describes the blocks of RG which do not have N in their kernels in terms of blocks of inertia groups of central idempotents of RN (see [2; 20, 6, Section 1 Lemma 1.7; 23, Part 1, XIII; 24], for a more thorough treatment of integral Clifford theory). This will yield criteria for when certain automorphisms of G induce inner automorphisms of these blocks.

Notation 2.4. For a finite group K, the following notation for idempotents is used:

$$\varepsilon_K = \frac{1}{|K|}\sum_{k\in K} k \quad \text{and} \quad \eta_K = 1 - \varepsilon_K.$$

Moreover, the following notation is fixed:

- G a finite group;
- N a normal subgroup of G;
- R a commutative ring;
- η an idempotent of RN, central in RG, with central primitive decomposition $\eta = e_1 + \cdots + e_s$ in RN;
- $\mathscr{E}$ a set of representatives of the orbits on $\{e_1, \ldots, e_s\}$ under the operation of G;
- $\mathsf{T}(e)$ inertia group $\{g \in G\colon e^g = e\}$, for all $e \in \mathscr{E}$.

Note that it is *not* required that the inertia groups $\mathsf{T}(e)$ are normal subgroups of G. It will be necessary to know how the isomorphism in the next (well-known) theorem takes place, so a proof is included for the convenience of the reader.

Theorem 2.5. *With notation as above*,

$$\eta \cdot RG = \prod_{e\in\mathscr{E}} \left(\bigoplus_{\mathsf{T}(e)g} \bigoplus_{\mathsf{T}(e)h} g^{-1} \cdot eR\mathsf{T}(e)e \cdot h \right) \cong \prod_{e\in\mathscr{E}} \mathrm{Mat}_{|G:\mathsf{T}(e)|}(eR\mathsf{T}(e)e).$$

Proof. Clearly, $\eta \cdot RG$ is the sum of its R-submodules

$$A_{e,g,k,h} = g^{-1}eg \cdot R\mathsf{T}(e) \cdot k \cdot h^{-1}eh,$$

where e ranges over $\mathscr{E}$ and the g, h, k are taken from a set of right coset representatives X of $\mathsf{T}(e)$ in G. Assume that $s = g^{-1}eg \cdot t \cdot k \cdot h^{-1}eh \neq 0$ for some $t \in \mathsf{T}(e)$. Then $t'=gtkh^{-1} \in \mathsf{T}(e)$, and $s=g^{-1}et'eh$. Conversely, given $t' \in \mathsf{T}(e)$, then $\mathsf{T}(e)g^{-1}t'h=\mathsf{T}(e)k$ for some $k \in X$, and it follows that $t=g^{-1}t'hk^{-1} \in \mathsf{T}(e)$, so $g^{-1}et'eh \in A_{e,g,k,h}$. Hence $\sum_{k\in X} A_{e,g,k,h} = g^{-1} \cdot eR\mathsf{T}(e)e \cdot h$, and it follows that

$$\eta \cdot RG = \prod_{e\in\mathscr{E}} \left(\bigoplus_{\mathsf{T}(e)g} \bigoplus_{\mathsf{T}(e)h} g^{-1} \cdot eR\mathsf{T}(e)e \cdot h \right).$$

Fix some $e \in \mathscr{E}$, and let $\{g_1, \ldots, g_n\}$ be a set of right coset representatives of $\mathsf{T}(e)$ in G. Then the direct factor of $\eta \cdot RG$ belonging to e can be written conveniently as

$$M_e = \begin{bmatrix} g_1^{-1} \cdot eR\mathsf{T}(e)e \cdot g_1 & \cdots & g_1^{-1} \cdot eR\mathsf{T}(e)e \cdot g_n \\ \vdots & & \vdots \\ g_n^{-1} \cdot eR\mathsf{T}(e)e \cdot g_1 & \cdots & g_n^{-1} \cdot eR\mathsf{T}(e)e \cdot g_n \end{bmatrix},$$

whence is isomorphic to the matrix ring $\mathrm{Mat}_{|G:\mathsf{T}(e)|}(eR\mathsf{T}(e)e)$. □

The theorem allows a proof of the following proposition, which provides a useful criteria of when certain automorphisms of G lie in $\mathrm{Aut}_R(G)$.

Proposition 2.6. *Assume that there is $\sigma \in \mathrm{Aut}(G)$ such that for all $e \in \mathscr{E}$, the automorphism σ stabilizes e and leaves every right coset of $\mathsf{T}(e)$ in G invariant. Then the following are equivalent*:

(i) *σ induces an inner automorphism of $eR\mathsf{T}(e)e$, say conjugation with u_e, for all $e \in \mathscr{E}$*;
(ii) *σ induces an inner automorphism of $\eta \cdot RG$.*

In either case, σ induces an inner automorphism of $\eta \cdot RG$, given by conjugation with

$$\sum_{e\in\mathscr{E}} \sum_{\mathsf{T}(e)g} e^g \cdot g^{-1}u_e(g\sigma) \in \eta \cdot RG.$$

Proof. Let $e \in \mathscr{E}$, and M_e the matrix ring defined above. If (i) holds, then the diagonal matrix

$$D = \begin{bmatrix} g_1^{-1} \cdot u_e \cdot g_1\sigma & & & \\ & g_2^{-1} \cdot u_e \cdot g_2\sigma & & 0 \\ & 0 & \ddots & \\ & & & g_n^{-1} \cdot u_e \cdot g_n\sigma \end{bmatrix}$$

lies in M_e, and σ is given on M_e by conjugation with D, so (ii) holds.

Note that σ fixes $g_i^{-1} \cdot eR\mathsf{T}(e)e \cdot g_j$, for all i, j. Hence if (ii) holds, then σ is given on M_e by conjugation with a diagonal matrix, whose $(1,1)$-entry is of the form $g_1^{-1} \cdot u_e \cdot g_1\sigma$, where u_e is a unit in $eR\mathsf{T}(e)e$ such that σ is given on $eR\mathsf{T}(e)e$ by conjugation with u_e, so (i) holds. The additional remark is just a reinterpretation of the matrix D. □

Remark 2.7. In particular, the proposition provides a simple method for constructing class-preserving automorphisms of groups. Namely, if $\sigma \in \mathrm{Aut}(G)$ stabilizes N, induces an inner automorphism of G/N and satisfies condition (i) with $\eta = \eta_N$ (for example, if σ induces inner automorphisms of all inertia groups), then $\sigma \in \mathrm{Aut}_R(G)$.

A precursor of the proposition is [25, Proposition 1], which has been used by Roggenkamp and Zimmermann to produce for the first time groups G with $\mathrm{Out}_{\mathrm{int}}(G) \neq 1$.

The following proposition is essentially a special case of Proposition 2.6 and will be applied in Example 4.12.

Proposition 2.8. *Assume that the finite group G is a semi-direct product $V \rtimes H$ with V an elementary abelian p-group, for some prime p. Let $\sigma \in \mathrm{Aut}(G)$ be defined by $h\sigma = h$ for all $h \in H$, and $v\sigma = v^m$ for all $v \in V$ and some fixed $m \in \mathbb{N}$. If U is a subgroup of index p in V, let $N_U = \mathrm{N}_G(U)/U$, and denote by σ_U the automorphism of N_U induced by σ. Let $R = \mathbb{Z}[p^{-1}]$. Then $\sigma \in \mathrm{Aut}_R(G)$ if and only if $\sigma_U \in \mathrm{Aut}_R(N_U)$ for all U.*

Proof. Let $U_1, \ldots, U_s$ be the subgroups of index p in V, and let $e_i = \eta_V \varepsilon_{U_i}$. Then $\{e_1, \ldots, e_s\}$ is a complete set of orthogonal primitive central idempotents in $\eta_V RV$, with $\mathsf{T}(e_i) = \mathrm{N}_G(U_i)$. Let $\sigma \in \mathrm{Aut}(G)$. By Proposition 2.6, $\sigma \in \mathrm{Aut}_R(G)$ if and only if σ induces an inner automorphism of $e_i R\mathsf{T}(e_i)e_i$, for all i. Fix some $e = e_i$, and let $U = U_i$. Since σ induces the identity on $\varepsilon_V RG = RG/V$, and $e \oplus \varepsilon_V = \varepsilon_U$, it follows that σ induces an inner automorphism of $\varepsilon_U R\mathsf{T}(e) = RN_U$ if and only if σ induces an inner automorphism of $eR\mathsf{T}(e)e$. This proves the proposition. □

With this knowledge at hand, it is not difficult to find groups G with $\mathrm{Out}_{\mathrm{int}}(G) \neq 1$. To the end of this section, a relatively small example is given, which may be compared with the group of order $2^7 \cdot p_1^2 \cdot p_2^2$ of derived length 3 given by Roggenkamp and Zimmermann [25].

Example 2.9. A supersolvable metabelian group G of order $2^7 \cdot 5^2 = 3200$ is constructed with $\mathrm{Out}_{\mathrm{int}}(G) \neq 1$. Let $P = \langle w : w^8 \rangle \rtimes (\langle b : b^2 \rangle \times \langle c : c^4 \rangle) \rtimes \langle s : s^2 \rangle$ with relations $w^b = w^{-1}$, $w^c = w^5$, $[w,s] = [b,s] = 1$ and $c^s = w^4 c$. The group G has normal subgroups M and N of order 5 and is a semi-direct product $G = (M \times N) \rtimes P$ with $\mathrm{C}_P(M) = \langle w, b, s \rangle$ and $\mathrm{C}_P(N) = \langle w, b, c^2 s \rangle$. An automorphism $\phi \in \mathrm{Aut}(G)$ is defined by $\phi|_{MN} = \mathrm{id}|_{MN}$ and $x\phi = x^s$ for all $x \in P$, so that ϕ restricted to MP, NP is given by conjugation with s, $c^2 s$ respectively. Assume that there is $g \in G$ with $\phi = \mathrm{conj}(g)$. Then it follows that $g \in P$, and $g \in \mathrm{Z}(P)s \cap \mathrm{C}_P(MN) = \langle w^4, c^2 \rangle s \cap \langle w, b \rangle = \emptyset$, a contradiction. Hence $\phi \notin \mathrm{Inn}(G)$.

Next, it is shown that $\phi \in \mathrm{Aut}_{\mathrm{int}}(G)$. Let $R = \mathbb{Z}_{(2)}[\zeta_5]$, where $\zeta_5 = \exp(2\pi i/5)$. Then Proposition 2.6 may be applied with the normal subgroup N and $\eta = \eta_N$ to get $\phi \in \mathrm{Aut}_R(G)$ (since each inertia group equals $MN\mathrm{C}_P(N)$). Interpreting ϕ as invertible bimodule, it follows from the Noether–Deuring theorem [21] that $\phi \in \mathrm{Aut}_{\mathbb{Z}_{(2)}}(G)$. Since ϕ is given by conjugation with the unit $u = \varepsilon_{\langle w^4 \rangle} + \eta_{\langle w^4 \rangle}(w + w^{-1})$ in $\mathbb{Z}\left[\frac{1}{2}\right] G$, it follows by Theorem 2.1 that $\phi \in \mathrm{Aut}_{\mathrm{int}}(G)$.

It should be pointed out that the automorphism ϕ is a *central group automorphism*, that is, ϕ induces the identity on $G/\mathrm{Z}(G)$. In Example 4.12, an automorphism $\sigma \in \mathrm{Aut}_{\mathrm{int}}(G)$ is presented which is not the product of an inner group automorphism and a central group automorphism.

3. Structure of $\mathrm{Out}_{\mathrm{int}}(G)$

A surprisingly simple proof for the following result is given.

Proposition 3.1. *The group* $\mathrm{Out}_{\mathrm{int}}(G)$ *is contained in the center of* $\mathrm{Out}_{\mathrm{c}}(G)$. *In particular*, $\mathrm{Out}_{\mathrm{int}}(G)$ *is an abelian group.*

Proof. For any commutative ring R, let $*$ be the usual anti-involution of RG associated with the group basis G. Then for any $u \in \mathrm{N}_{\mathrm{U}(RG)}(G)$ and $g \in G$,

$$guu^* = u(u^{-1}gu)u^* = u(u^{-1}g^{-1}u)^* u^* = uu^* g u^{-*} u^* = uu^* g,$$

showing that $uu^* \in \mathrm{Z}(RG)$. Therefore, for any $u, v \in \mathrm{N}_{\mathrm{U}(RG)}(G)$,

$$\begin{aligned}
[u,v][u,v]^* &= u^{-1}v^{-1}(u(vv^*))u^* v^{-*} u^{-*} \\
&= u^{-1}(v^*(uu^*))v^{-*}u^{-*} \quad (vv^* \text{ commutes with } u) \\
&= u^* v^* v^{-*} u^{-*} \quad (uu^* \text{ commutes with } v^*) \\
&= 1.
\end{aligned}$$

Let $\phi \in \mathrm{Aut}_{\mathrm{c}}(G)$ and $\alpha \in \mathrm{Aut}_{\mathrm{int}}(G)$; it has to shown that $[\phi, \alpha] \in \mathrm{Inn}(G)$. There is $u \in \mathrm{U}(\mathbb{Q}G)$ with $\phi = \mathrm{conj}(u)$ and $v \in \mathrm{U}(\mathcal{O}G)$ with $\alpha = \mathrm{conj}(v)$ (for some ring $\mathcal{O}$ of algebraic integers). It follows that

$$[u,v]=u^{-1}\underbrace{(v^{-1}uv)}_{\in\mathbb{Q}G}=\underbrace{(u^{-1}v^{-1}u)}_{\in\mathcal{O}G}v\in\mathbb{Q}G\cap\mathcal{O}G=\mathbb{Z}G.$$

As $[u,v][u,v]^*=1$, it follows that $[u,v]=\pm g$ for some $g\in G$ (just look at the 1-coefficient of $[u,v][u,v]^*$ – this argument goes back to S.D. Berman). Hence $[\phi,\alpha]=\operatorname{conj}(g)\in\operatorname{Inn}(G)$. □

The picture would not be complete without examples showing that indeed any abelian group occurs as some $\operatorname{Out}_{\rm int}(G)$. The groups given in the following proposition have a similar structure as the groups given by Dade in [5, Section 2].

Proposition 3.2. *Let q be a natural power of a prime p. Then there exists a finite metabelian group G such that $\operatorname{Out}_{\rm int}(G)$ is cyclic of order q.*

Proof. Let $S=\langle v,\tau,\sigma\colon v^q=\tau^q=\sigma^{q^2}=1,\ [v,\tau]=[v,\sigma]=1,\ \tau^\sigma=v\tau\rangle$, a group of order q^4. Let $K_1=\langle v,\sigma^q\tau^{-1}\rangle$ and $K_2=\langle v,\tau\rangle$, both normal subgroups of S with S/K_1 and S/K_2 cyclic of order q^2, generated by the image of σ.

Choose primes $p_1\neq p_2$ such that pq does not divide p_i-1. Put $r=q/p$. For $i=1,2$, let M_i be an indecomposable faithful $\mathbb{F}_{p_i}\langle\sigma^r\rangle$-module of finite dimension, and let P_i be the direct sum of r copies of M_i. Define a (faithful) operation of $\langle\sigma\rangle$ on P_i by letting $(a_1,a_2,\ldots,a_{r-1},a_r)^\sigma=(a_r\sigma^r,a_1,a_2,\ldots,a_{r-1})$, for all $a_j\in M_i$. Let $R_i=\mathbb{Z}[p_i^{-1}]$, and note that each central primitive idempotent of $\eta_{P_i}\cdot R_iP_i$ is of the form $\eta_{P_i}\varepsilon_U$ for some maximal subgroup U of P_i (recall Notation 2.4 for idempotents). Since the module $P_i|_{\langle\sigma^r\rangle}$ decomposes into a direct sum of copies of M_i, it follows from the Krull–Schmidt theorem that σ^r operates fixed-point free on the set of central primitive idempotents belonging to $\eta_{P_i}\cdot R_iP_i$.

Define the group G to be the semi-direct product $(P_1\times P_2)\rtimes S$ with $C_S(P_i)=K_i$ and the given action of σ.

Let e_i be a central primitive idempotent of $\eta_{P_i}\cdot R_iP_i$. Then for the inertia group $\mathsf{T}(e_i)$ of e_i in G, it follows that $[G,G]\leq P_1P_2\langle v\rangle\leq\mathsf{T}(e_i)\leq P_1P_2\langle v,\tau,\sigma^q\rangle$.

Let $\beta\in\operatorname{Aut}_{\rm int}(G)$. Seeking for a ‘canonical’ coset representative of β in $\operatorname{Inn}(G)\cdot\beta$, choose $g_i\in P_i$ with $C_S(g_i)=K_i$ (this can be done since S/K_i is cyclic). Then, as $\beta\in\operatorname{Aut}_{\rm c}(G)$, the automorphism β can be altered by an inner group automorphism so that $g_i\beta=g_i$. By Coleman’s lemma, there are $s_i\in S$ with $\sigma|_{P_i}=\operatorname{conj}(s_i)|_{P_i}$. Then $g_i=g_i\beta=g_i^{s_i}$, and it follows that $s_i\in K_i$. Hence β fixes P_1P_2 element-wise. Additionally, by Sylow’s theorem, β can be altered by an inner group automorphism so that β fixes S; then there is $x\in S$ with $\beta|_S=\operatorname{conj}(x)|_S$, again by Coleman’s lemma.

Clearly $e_i\beta=e_i$, and β induces the identity on $G/\mathsf{T}(e_i)$. Hence β induces an inner automorphism of $e_iR_i\mathsf{T}(e_i)e_i$, by Proposition 2.6. Note that $\mathsf{T}(e_2)=P_1P_2H$ with $K_2\leq H\leq S$, and that β induces an automorphism of the quotient $e_2R_2(P_2H)e_2$ of $e_2R_2\mathsf{T}(e_2)e_2$. Since $\tau\in Z(P_2H)$, it follows that $\tau\beta=\tau$ and $x\in\langle v,\tau,\sigma^q\rangle=Z(S)\langle\tau\rangle$. Thus without lost of generality, x is a power of τ.

Now define $\beta \in \mathrm{Aut}(G)$ by $g\beta = g$ for $g \in P_1P_2$, and $\beta|_S = \mathrm{conj}(\tau)|_S$. Then β induces the identity on the inertia group $\mathsf{T}(e_i)$, and on the abelian quotient $G/\mathsf{T}(e_i)$, so $\beta \in \mathrm{Aut}_{\mathrm{int}}(G)$ by Proposition 2.6 and Theorem 2.1. Since $\mathrm{Z}(S)\tau^r \cap \mathrm{C}_{P_1P_2}(S) = \langle v, \sigma^q \rangle \tau^r \cap \langle v \rangle = \emptyset$, it follows that the image of β in $\mathrm{Out}(G)$ has order q. This completes the proof of the proposition. □

Corollary 3.3. *For any finite abelian group A, there is a finite metabelian group G with* $\mathrm{Out}_{\mathrm{int}}(G) \cong A$.

Proof. Follows from Proposition 3.2 and Remark 4.3(1). □

Corollary 3.4. *For any finite abelian group A, there is a finite metabelian group G such that* $\mathrm{Cl}_{\mathbb{Z}G}(\mathrm{Z}(\mathbb{Z}G))$ *contains a copy of A* (*cf. Remark* 2.3).

Proof. Follows from Proposition 3.2 and Krempa's lemma. □

4. Groups with abelian Sylow 2-subgroups

Let G be a finite group. If G has an abelian Sylow 2-subgroup P, it is conjectured that $\mathrm{Out}_{\mathbb{Z}}(G) = 1$ [19, p. 176]. This is shown to be true if G has a normal 2-complement, and P is cyclic, or abelian of exponent at most 4. However, also an example is given showing that $\mathrm{Out}_{\mathrm{int}}(G)$ can be non-trivial (but $\mathrm{Out}_{\mathbb{Z}}(G) = 1$ for this example).

The following simple but important observation explains why it is difficult to disprove (NP) for groups with abelian Sylow 2-subgroups: for a counterexample G of minimal order, $\mathrm{O}_2(G) = 1$ holds, and thus $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$ cannot be a central group automorphism.

Lemma 4.1. *Assume that G has an abelian Sylow p-subgroup P, and that $\sigma \in \mathrm{Aut}(G)$ is of p-power order such that $\sigma|_P = \gamma|_P$ for some $\gamma \in \mathrm{Inn}(G)$. If σ induces an inner automorphism of $G/\mathrm{O}_p(G)$, then $\sigma \in \mathrm{Inn}(G)$.*

Proof. Without lost of generality, σ induces the identity on $G/\mathrm{O}_p(G)$. There is $x \in G$ with $\sigma|_P = \mathrm{conj}(x)|_P$, and without lost of generality, x is a p-element since the order of σ is a power of p, and P is fixed by σ. It follows that $x \in P$, and, as P is abelian, $\sigma|_P = \mathrm{id}|_P$. Now a common 1-cohomology argument [11, I 16.18] shows that σ is given by conjugation with some $g \in \mathrm{Z}(P)$. □

Next, some elementary facts about class-preserving and Coleman automorphisms are recorded which will be needed later.

Proposition 4.2. *The prime divisors of $|\mathrm{Aut}_{\mathrm{c}}(G)|$ and $|\mathrm{Aut}_{\mathrm{Col}}(G)|$ lie in $\pi(G)$, the set of prime divisors of $|G|$.*

Proof. It is known that prime divisors of the order of $\mathrm{Aut}_c(G)$ lie in $\pi(G)$ (see [11, I.4 Aufgabe 12]). Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, and assume that σ has order r, with $(r, |G|) = 1$; then $\sigma = \mathrm{id}$ has to be shown. Let $p \in \pi(G)$. Since the action of σ on G is coprime, there is a Sylow p-subgroup P of G which is fixed by σ. By assumption, there is $x \in G$ with $\sigma|_P = \mathrm{conj}(x)|_P$. So σ induces an automorphism of P whose order divides r and the order of x. It follows that $\sigma|_P = \mathrm{id}|_P$. As $p \in \pi(G)$ was chosen arbitrary, it follows that $\sigma = \mathrm{id}$. □

Remark 4.3. (1) If σ is a class-preserving or a Coleman automorphism of G, and N a normal subgroup of G, then σ fixes N, and induces a class-preserving or a Coleman automorphism of G/N, respectively. In particular, $\mathrm{Aut}_c(G \times H) \cong \mathrm{Aut}_c(G) \times \mathrm{Aut}_c(H)$ for finite groups G and H (and similar for $\mathrm{Aut}_{\mathrm{Col}}(-)$).

(2) If $\sigma \in \mathrm{Aut}(G)$ is of p-power order, and if there are $U \leq G$ and $x \in G$ with $\sigma|_U = \mathrm{conj}(x)|_U$, then there is $\gamma \in \mathrm{Inn}(G)$ such that $\sigma\gamma|_U = \mathrm{id}$, and the order of $\sigma\gamma$ is still a power of p (taking for $\sigma\gamma$ a suitable power of $\sigma\,\mathrm{conj}(x^{-1})$). In proofs, this fact will be used several times without any further comment, just indicated by a phrase like 'modifying σ (by an inner automorphism)'.

Lemma 4.4. *Let $\alpha \in \mathrm{Aut}(G)$ be of p-power order, for some prime p. Assume that there is $N \trianglelefteq G$ such that $N\alpha = N$, and α induces the identity on G/N. Let U be a subgroup of G. Then, if there is $h \in G$ such that $g\alpha = g^h$ for all $g \in U$, there is $n \in N$ with $g\alpha = g^{nh_p}$ for all $g \in U$, where h_p denotes the p-part of h.*

Proof. The proof consists of a straightforward calculation. Let q be a power of p such that $\alpha^q = \mathrm{id}$, and h^q is the p'-part of h. Then for all $g \in U$,

$$g = g\alpha^q = g^k \quad \text{with} \quad k = h(h\alpha)(h\alpha^2)\dots(h\alpha^{q-1}).$$

Since α induces the identity on G/N, there is $n \in N$ with $k = h^q n^{-1}$. It follows that $g\alpha = g^{h^q h_p} = g^{(h^q n^{-1})(nh_p)} = g^{k(nh_p)} = g^{nh_p}$ for all $g \in U$. □

The following proposition has been proved for G supersolvable of odd order [17, Theorem 5], and under additional assumptions in [17, Theorem 6].

Proposition 4.5. *Let G be a solvable group of odd order, and let ϕ, ψ be automorphisms of G such that*:

(i) *ϕ and ψ are of 2-power order*;
(ii) $\phi \cdot \psi = \psi \cdot \phi$;
(iii) *for all $x \in G$, x is conjugate in G to $x\phi$ or to $x\psi$.*

Then $\phi = \mathrm{id}$ or $\psi = \mathrm{id}$.

Proof. The hypothesis of the proposition is equally satisfied if ϕ is replaced by ϕ^2, or ψ by ψ^2, so without lost of generality, $\phi^2 = \mathrm{id}$ and $\psi^2 = \mathrm{id}$. Since G is solvable, there is a non-trivial, abelian normal subgroup A of G with $A\phi = A$ and $A\psi = A$, and whose order is minimal subject to this condition. The automorphisms ϕ and ψ induce automorphisms $\bar{\phi}$ and $\bar{\psi}$ of $\bar{G} = G/A$. By induction on the order of G, assume that $\bar{\psi} = \mathrm{id}$. Then $A = \mathrm{C}_A(\psi) \times [A, \psi]$ (coprime action since G is of odd order, cf. [1, 24.6]), and $\mathrm{C}_A(\psi)$ is a normal subgroup of G. Indeed, if $x \in \mathrm{C}_A(\psi)$ and $g \in G$, then $g\psi = yg$ for some $y \in A$, and $x^g\psi = x\psi^{g\psi} = x^{yg} = x^g$ since A is abelian. Moreover, $\mathrm{C}_A(\psi)$ is ϕ-invariant, so either $\mathrm{C}_A(\psi) = A$ or $\mathrm{C}_A(\psi) = 1$ by minimality of A. In the first case, $\psi = \mathrm{id}$ since G is of odd order, so assume that $\mathrm{C}_A(\psi) = 1$. Then $a\psi = a^{-1}$ for all $a \in A$. Note that if some $g \in G$ is conjugate to g^{-1} within G, then $g = 1$ since G is of odd order. Hence, it follows from (iii) that $a\phi$ is conjugate within G to a for all $a \in A$. Since $A = \mathrm{C}_A(\phi) \times [A, \phi]$ and ϕ inverts the elements of $[A, \phi]$, it follows that $[A, \phi] = 1$ and $A = \mathrm{C}_A(\phi)$. Put $W = \mathrm{C}_G(\psi) \leq G$. As ψ operates on each coset Ax, $x \in G$, with at least one fixed point, $G = A \rtimes W$. Since ϕ and ψ commute, $W\phi = W$. Take any $g \in G$. Then $a^g = a^g\phi = a^{g\phi}$ for all $a \in A$, so $c = (g\phi)g^{-1} \in \mathrm{C}_G(A)$. Write $g = wa$ with $w \in W$ and $a \in A$. Then $c = (w\phi)a \cdot a^{-1}w^{-1} = (w\phi)w^{-1} \in W$. Fix some $1 \neq b \in A$. Assume there is $h \in G$ with $(bc)^h = (bc)\psi = b^{-1}c$. As $bc \in \mathrm{C}_G(A)$, and $G = AW$, there is $h \in W$ with this property. But then $bb^h = cc^{-h} \in A \cap W = 1$ as $c \in W$, so h inverts b, a contradiction. Therefore, by (iii), bc and $(bc)\phi$ are conjugate in G. Now $c\phi = c^{-1}$ by definition of c, so $(bc)\phi = bc^{-1}$, and it follows that $\bar{c}$ and $\bar{c}^{-1}$ are conjugate in $\bar{G} = G/A$. Therefore, $\bar{c} = 1$ and $c \in A \cap W = 1$. This shows that $g\phi = g$, and, as g was arbitrary, $\phi = \mathrm{id}$. □

In [17], the following property of a finite group has been considered.

Property $\mathscr{W}$. *The group* $\mathrm{Out}_{\mathrm{c}}(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$ *is of odd order.*

In [17, Remarks, p. 6270], it is asked whether there exist results similar to Proposition 4.5 for the action of abelian 2-groups on groups of odd order, which would allow one to prove that groups with abelian Sylow 2-subgroups have property $\mathscr{W}$. Example 4.11 below shows that this is not the case.

Mazur has proved that a finite group G with a Sylow 2-subgroup of order 2 has property $\mathscr{W}$ [17, Theorem 7]. Note that such a group has a normal 2-complement, by Burnside's normal p-complement theorem (cf. [1, 39.1]). Proposition 4.5 allows us to prove the following generalizations.

Proposition 4.6. *Let G be a finite group with a Sylow* 2*-subgroup of order* 2*. Then* 2 *does not divide the order of* $\mathrm{Out}_{\mathrm{c}}(G)$.

Proof. Let $\sigma \in \mathrm{Aut}_{\mathrm{c}}(G)$, of order a power of 2; it has to be shown that $\sigma \in \mathrm{Inn}(G)$. Let $w \in G$ be an element of order 2, and H a normal 2-complement. Without lost of generality, $w\sigma = w$. Then $\tau = \sigma \cdot \mathrm{conj}(w) \in \mathrm{Aut}(G)$ is of 2-power order, and $\sigma\tau = \tau\sigma$. The automorphisms σ, τ induce automorphisms ϕ, ψ of H, respectively. Let $h \in H$. Then $h\sigma = h^g$ for some $g \in G = H \cup Hw$, and it follows that h is conjugate in H to $h\phi$ or $h\psi$. Hence, by Proposition 4.5 $\phi = \mathrm{id}$ or $\psi = \mathrm{id}$, which implies $\sigma = \mathrm{id}$ or $\sigma = \mathrm{conj}(w)$. □

Proposition 4.7. *Let G be a finite group with a cyclic Sylow 2-subgroup. Then G has Property $\mathscr{W}$.*

Proof. Take a minimal counterexample G, and $\sigma \in \mathrm{Aut}_{\mathrm{c}}(G) \cap \mathrm{Aut}_{\mathrm{Col}}(G)$, a non-inner automorphism of order a power of 2. The proof proceeds in a number of steps.

(1) $\mathrm{O}_2(G) = 1$.

This follows from Lemma 4.1 and the minimality of G.

(2) The Frattini subgroup $\Phi(G)$ of G is trivial, and the Fitting subgroup $\mathrm{F}(G)$ is the direct product of minimal normal subgroups of G.

Assume the contrary. By minimality of G, the automorphism σ can be modified by an inner automorphism so that it induces the identity on $G/\Phi(G)$. As $\Phi(G)$ is nilpotent (see [11, III 3.6]), it follows from (1) that $\Phi(G)$ is a $2'$-group, yielding the contradiction $\sigma = \mathrm{id}$ (see [11, III 3.18]). The second statement follows from the first one (see [11, III 4.4, 4.5]).

(3) G has more than one minimal normal subgroup (the trivial subgroup is not considered as being 'minimal').

Assume the contrary, and let A be the (unique) minimal normal subgroup of G. By Burnside's normal p-complement theorem (see [1, 39.1]), G has a normal 2-complement, so G is solvable, by the Odd Order Theorem. Therefore A is an abelian p-group for some prime p, and $F := \mathrm{F}(G) = \mathrm{O}_p(G)$. Hence σ can be altered by an inner automorphism to get $\sigma|_F = \mathrm{id}|_F$. As $\mathrm{C}_G(F) \subseteq F$ (see [11, III 4.2]), it follows that σ induces the identity on G/F. As $p \neq 2$ by (1), it follows that $\sigma = \mathrm{id}$, a contradiction.

(4) Any 2-element of G operates trivial or fixed-point free on each minimal normal subgroup of G.

To see this, let A be a minimal normal subgroup of G, and modify σ by an inner automorphism so that σ induces the identity on G/A. Then $\mathrm{C}_A(\sigma) \trianglelefteq G$, so either $A = \mathrm{C}_A(\sigma)$ or $A = [A, \sigma]$. In the first case, $\sigma = \mathrm{id}$ by (1), so $A = [A, \sigma]$. By Lemma 4.4, there is a 2-element $x \in G$ with $\sigma|_A = \mathrm{conj}(x)|_A$. It follows that $A = [A, x]$, and that $\mathrm{C}_A(y) \trianglelefteq G$ for all $y \in \langle x \rangle$. Since A is a minimal normal subgroup and Sylow 2-subgroups of G are cyclic, the assertion follows.

(5) Let S be a Sylow 2-subgroup of G. Then $\mathrm{C}_S(A) \neq 1$ for each minimal normal subgroup A of G.

Assume there is a minimal normal subgroup A of G with $\mathrm{C}_S(A) = 1$. By (3), there is a minimal normal subgroup B of G with $A \cap B = 1$. Without lost of generality, σ induces the identity on G/B. As in (4) it follows that $B = [B, \sigma]$, and clearly $\sigma|_A = \mathrm{id}_A$. Take any $a \in A \backslash \{1\}$ and $b \in B \backslash \{1\}$. There is $x \in G$ with $(ab)\sigma = a^x b^x$, and it is easily seen (cf. Lemma 4.4) that x can be chosen to be 2-element. It follows that $x \in \mathrm{C}_G(a)$ and $x \notin \mathrm{C}_G(b)$. So $x \in \mathrm{C}_G(A)$ by (4), and $x \neq 1$, which implies that $\mathrm{C}_S(A) \neq 1$.

Now the proof of the proposition is easily completed. As S is cyclic, it follows from (2), (5) and [11, III 4.2] that $1 \neq \mathrm{C}_S(\mathrm{F}(G)) \leq \mathrm{F}(G)$, contradicting (1). □

Corollary 4.8. *If G has cyclic Sylow 2-subgroups, then* $\mathrm{Out}_{\mathbb{Z}}(G) = 1$.

Proof. By Krempa's result, $\mathrm{Out}_{\mathbb{Z}}(G)$ is a 2-group, so the corollary follows from Coleman's result and the above proposition. □

Up to now, considerations were entirely group-theoretical. The proof of the following lemma indicates further possibilities to analyze the group $\mathrm{Out}_{\mathbb{Z}}(G)$.

Lemma 4.9. *Let G be a finite group with a normal 2-complement H. Let P be a Sylow 2-subgroup of G, and $S \trianglelefteq P$ such that P/S is abelian of exponent ≤ 4. Then for any $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$, there is $\gamma \in \mathrm{Inn}(G)$ such that $\sigma\gamma$ induces a Coleman automorphism of HS.*

Proof. By assumption, there is a unit u of $\mathbb{Z}G$ with $\sigma = \mathrm{conj}(u)$, and it does no harm if u is assumed to be augmented (i.e. the 'sum over the coefficients' is 1). Let $N = HS$. By a well-known result of G. Higman, u maps to an element of G/N under the natural map $\mathbb{Z}G \to \mathbb{Z}G/N$ (see [28, Theorem (2.7)]). Let $g_1, \dots, g_r$ be a set of coset representatives of N in G. Then $u = \sum y_i g_i$ for some $y_1, \dots, y_r \in \mathbb{Z}N$, and without lost of generality, y_i has augmentation 1 if $i = 1$ (and augmentation 0 otherwise). Note that G acts (from the right) on the coset Ng_1 via $ng_1 \cdot g := g^{-1}(ng_1)(g\sigma)$ (for all $n \in N$, $g \in G$). As $g^{-1}(y_1g_1)(g\sigma) = y_1g_1$, it follows that in y_1g_1, viewed as an integral linear combination of elements of Ng_1, the elements of an orbit all have the same coefficient. Let p be a prime, and P a Sylow p-subgroup. Since the augmentation of y_1g_1 is not divisible by p, there must be a fixed-point ng_1 $(n \in N)$ in the support of y_1g_1 under the operation of P. It follows that $\sigma|_P = \mathrm{conj}(ng_1)|_P$, and $\sigma \cdot \mathrm{conj}(g_1^{-1})$ induces a Coleman automorphism of N. □

Corollary 4.10. *Let G be a finite group with a normal 2-complement and an abelian Sylow 2-subgroup of exponent at most 4. Then $\mathrm{Out}_{\mathbb{Z}}(G) = 1$.*

Proof. Let H be the normal 2-complement, and $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$; then $\sigma \in \mathrm{Inn}(G)$ has to be shown. By Krempa's result, it can be assumed that the order of σ is a power of 2. By Lemma 4.9, σ can be modified by an inner automorphism such that σ induces a Coleman automorphism of H, so $\sigma|_H = \mathrm{id}|_H$ by Proposition 4.2. But then $\sigma = \mathrm{id}$. □

Finally, some illuminating examples of groups G with abelian Sylow subgroups are given. They include some Frobenius groups, which belong to a certain family of subgroups of affine semi-linear groups having non-inner class-preserving automorphisms; see [10] for further details on these groups.

Example 4.11. Consider the following matrices in $\mathrm{GL}(2,5)$:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $U = \mathbb{F}_5 \oplus \mathbb{F}_5$ be the underlying vector space on which these matrices act. It is easily shown that

- A, B and T_1 are of order 3,4 and 2, respectively, and $A^B = A^{-1}$, $A^{T_1} = A^{-1}$. The group $H = \langle A, B\rangle = \langle A\rangle \rtimes \langle B\rangle$ has order 12.
- The semi-direct product $F_1 = U \rtimes H$, i.e.

$$F_1 = \langle v, w, a, b : v^5 = w^5 = [v, w] = 1,\ a^3 = b^4 = 1,\ a^b = a^2,\\ v^a = v^2w,\ w^a = v^3w^2,\ v^b = v^3,\ w^b = w^2\rangle,$$

  is a Frobenius group.
- Each of the $6 = (5^2 - 1)/(5 - 1)$ matrices of order 4 in H has eigenvalues 2 and 3. In particular, for each $0 \neq u \in U$ there is $M \in H$ such that $uM = 2u$.
- The automorphism σ_1 of F_1, defined by $u\sigma_1 = 2u$ and $h\sigma_1 = h$ for all $u \in U$, $h \in H$, is a non-inner class-preserving automorphism.

Note that $\sigma_1 \notin \mathrm{Aut}_{\mathbb{Z}_{(5)}}(F_1)$, by Coleman's result.

On the other hand, let $R = \mathbb{Z}\left[\frac{1}{5}\right]$. Then $\sigma_1 \in \mathrm{Aut}_R(F_1)$. To see this, let $0 \neq u \in U$, and recall that there is $M \in H$ of order 4 with $uM = 3u$, and $u'M = 2u'$ for some $u' \in U$ with $U = \langle u, u'\rangle$. Then $N := \mathrm{N}_{F_1}(\langle u\rangle) = U\langle M\rangle$, and σ_1 induces on $\bar{N} = N/\langle u\rangle$ the inner automorphism $\mathrm{conj}(\bar{M})$. It follows from Proposition 2.8 that σ_1 induces an inner automorphism of $\mathbb{Z}\left[\frac{1}{5}\right]F_1$.

Let $N = \langle n\rangle \cong C_3$ and $T = \langle t\rangle \cong C_2$. Let t act on N by $n^t = n^{-1}$, and on $F_1 \leq U \rtimes \mathrm{GL}(2,5)$ by conjugation with the matrix T_1. Form the corresponding semi-direct product $G = (N \times F_1) \rtimes T$, and note that G has abelian Sylow subgroups.

The automorphism σ_1 extends to an automorphism σ of G with $n\sigma = n$ and $t\sigma = t$. Note that $\sigma \in \mathrm{Aut}_{\mathrm{col}}(G)$. As before, one can show that σ induces an inner automorphism of $\mathbb{Z}\left[\frac{1}{5}\right]G$. In particular, $\sigma \in \mathrm{Aut}_{\mathrm{c}}(G)$, and it follows that Property $\mathscr{W}$ need not hold for a group with abelian Sylow subgroups.

Let $\zeta = \exp(2\pi \mathrm{i}/3)$, and $R = \mathbb{Z}_{(5)}[\zeta]$. For the idempotent $e = \frac{1}{3}(1 + \zeta n + \zeta^2 n^2)$, it is easily seen that $eR\mathsf{T}(e)e = eR(N \times F_1)e \cong RF_1$. Hence condition (i) of Proposition 2.6 does not hold, and $\sigma \notin \mathrm{Aut}_{\mathbb{Z}_{(5)}}(G)$.

Example 4.12. A group of order $2^5 \cdot 3 \cdot 7 \cdot 5^2 \cdot 13^2 = 2839200$, with all Sylow subgroups being abelian, shall be constructed such that

$$\mathrm{Out}_{\mathrm{int}}(G) \neq 1 \quad \text{and} \quad \mathrm{Out}_{\mathbb{Z}}(G) = 1.$$

Let F_1 and T_1 be as in the previous example.

The following matrices are elements of $\mathrm{GL}(2,13)$:

$$C = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let V be the underlying vector space. One verifies the following:

- C, D and T_2 are of order 7, 4 and 2, respectively, and $C^D = C^{-1}$, $C^{T_2} = C^{-1}$. The group $K = \langle C, D\rangle = \langle C\rangle \rtimes \langle D\rangle$ has order 28.

- The semi-direct product $F_2 = V \rtimes K$ is a Frobenius group.
- Each of the $14 = (13^2 - 1)/(13 - 1)$ matrices of order 4 in K has eigenvalues 5 and 8. In particular, for each $0 \neq v \in V$ there is $M \in K$ such that $vM = 5v$.
- The automorphism σ_2 of F_2, defined by $v\sigma_2 = 5v$ and $k\sigma_2 = k$ for all $v \in V$, $k \in K$, is a non-inner class-preserving automorphism.

The group G is a semi-direct product

$$G = (F_1 \times F_2) \rtimes \langle t : t^2 \rangle,$$

where the involution t acts on $F_1 \leq U \rtimes \mathrm{GL}(2,5)$ by conjugation with the matrix T_1, and on $F_2 \leq U \rtimes \mathrm{GL}(2,13)$ by conjugation with T_2.

Let σ be the automorphism of G which fixes each element of $VHK\langle t \rangle$, and acts on U by multiplication with 2. Clearly, σ is a non-inner automorphism.

Let $0 \neq u \in U$. Recall that there is $M \in H$ of order 4 with $uM = 3u$, and $u'M = 2u'$ for some $u' \in U$ with $U = \langle u, u' \rangle$. If $M = A^{-i}B^jA^i$ (such $i, j \in \mathbb{N}$ exist), then $N := \mathrm{N}_G(\langle u \rangle) = UV\langle M, A^{-i}tA^i \rangle K$. Hence, σ induces on $\bar{N} = N/\langle u \rangle$ an inner automorphism, given by conjugation with $\bar{M}$. It follows from Proposition 2.8 that σ induces an inner automorphism of $\mathbb{Z}\left[\frac{1}{5}\right]G$.

Note that $\sigma \cdot \mathrm{conj}(B^{-1}tD)$ fixes each element of $UHK\langle t \rangle$, and acts on V by multiplication with 5. In the same way as before, it is verified that σ induces an inner automorphism of $\mathbb{Z}\left[\frac{1}{13}\right]G$.

Hence $\sigma \in \mathrm{Aut}_{\mathrm{int}}(G)$ by Theorem 2.1. It follows from Corollary 4.10 that $\mathrm{Out}_{\mathbb{Z}}(G) = 1$. Thus, everything is proved.

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