Let $\Pi = \Pi R_R$ be an arbitrary product of copies of $R_R$. We say $R$ is right $\Pi$-coherent if every finitely generated submodule of $\Pi$ is finitely presented. This notion is called strong coherence in [7], where it seems to have been invented.

Here we obtain a new characterization of $\Pi$-coherence, and study $\Pi$-coherence for polynomial rings. Using Goldie's Theorem, we show that if $R$ is a semiprime two-sided noetherian ring then $R[S]$ is right $\Pi$-coherent for any set of variables $S$ (Theorem 6). We also obtain the same result for two-sided noetherian rings $R$, not assumed to be semiprime, provided $R$ contains an uncountable field.

Call ring $R$ a left *-ring (star ring) if it has the property that $\text{Hom}_R(, R_R) = (\cdot)^*$ takes finitely generated left $R$-modules to finitely generated right $R$ modules.

The above ideas are connected by the following:

**Theorem 1.** For any ring $R$, the following are equivalent:

a. $R$ is right $\Pi$-coherent
b. $R$ is a left *-ring
c. For each $n \geq 1$, right annihilators of subsets of $R_n$ are finitely generated.

**Proof.** Let

$$0 \longrightarrow_{R} RK \longrightarrow_{R} RF \longrightarrow_{R} R M \longrightarrow_{R} 0,$$

where $RF$ is finitely generated and free. Then, $\text{Hom}(R M, R R)$ is finitely generated if and only if $\text{Hom}(R M, RF)$ is, since $RF = R^{(n)}$ for some integer $n$. 

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Therefore consider the exact sequence:

\[ \text{Hom}( R_K, R_F) \xleftarrow{a^*} \text{Hom}( R_F, R_F) \xleftarrow{b^*} \text{Hom}( R_M, R_F) \rightarrow 0. \]

We want \( \text{Im} b^* \) to be finitely generated, but this is just \( \ker a^* \).

If one thinks of \( a( R_K) \) as rows in \( R_F \), then

\[ \ker a^* = \{ f \in \text{End}( R_F, R_F) \mid f( a R K) = 0 \}. \]

If \( M_n(T) \) is the \( n \times n \) matrix ring then the right annihilator of a set of matrices in \( M_n(R) \) is the right annihilator of a collection of rows. Further, a right ideal \( A_R \) in \( M_n(R) \) is finitely generated as a right \( M_n(R) \) module if and only if \( A_R \) is finitely generated as a right \( R \) module. This is true, because if \( \{ A_i \} \) is a collection of generators, then, if \( E_{ab} \) is a set of matrix units, since any matrix is a linear combination of scalar multiples of the \( E_{ab} \), every matrix in \( A_R \) is a scalar combination of the \( \{ A_i E_{ab} \} \). This gives the equivalence of \( a \) and \( c \).

Now, let \( F \) be a finitely generated free module. Then, if \( \mathcal{P} R \) is an infinite product

\[ \text{Hom}( F, \mathcal{P} R) = \text{Hom}( F, \mathcal{P} F). \]

So, if \( f \in \text{Hom}( F, \mathcal{P} F) \), \( f \) can be represented by \( S = (s_1, \ldots, s_n) \), \( s_i \in M_n(R) \), where the \( s_i \) are matrices acting on the left.

Now, as before, we ask that \( \ker f \) be finitely generated. This says that \( r_F(\{ s_i \} \) is finitely generated as a right \( R \) module. As before, this means \( r_{M_n(R)}(\{ s_i \} \) is a finitely generated right \( M_n(R) \) ideal.

So, we have \( b \) and \( c \) are equivalent.

Unfortunately, R. E. Johnson [6] and Jeanne Kerr [8] have provided examples of rings \( R \) with the a.c.c. on annihilators whose \( 2 \times 2 \) matrix ring does not have the a.c.c. on annihilators.

A module \( M_R \) is coherent if every finitely generated submodule of \( M_R \) is finitely presented. (Bourbaki requires \( M_R \) to also be finitely generated, but we do not.) Then the previous says that products of copies of \( R \) are coherent. In this case, we say \( R \) is \( \mathcal{P} \)-coherent. If \( R \) is coherent then finite products of \( R \) are coherent [2]. (Outline for modules \( M_1 \) and \( M_2 \): if \( 0 \rightarrow K \rightarrow F \rightarrow M_1 \oplus M_2, \) by switching to direct sums of projections, without loss of generality, each projection is onto, so we have \( 0 \rightarrow K \oplus K \rightarrow F \oplus F \rightarrow M_1 \oplus M_2 \rightarrow 0 \) and the result follows from Schanuels' Lemma.)

Now if \( F \) is a finitely generated free module, and \( \mathcal{P} R \) is a finite product, any map from the former into the latter may be thought of as

\[ 0 \rightarrow K_R \rightarrow F \rightarrow \mathcal{P} F \]
by increasing the number of $R$'s. The projection maps in the right term are represented by $n \times n$ matrices, where $n$ is the dimension of $F$. If we regard $F$ as columns, we isolate this basically notational observation as:

**Proposition 2.** $R$ is right coherent if and only if right annihilators of finite subsets of $M_n(R)$ are finitely generated $M_n(R)$ modules, and $R$ is II-coherent if and only if right annihilators of subsets of $M_n(R)$ are finitely generated.

The generic example of coherent, non-Noetherian rings is polynomial rings, as presented in Chase [4]. We discuss the coherence and strong coherence of polynomial rings and since, as we see above, these properties may be restated in terms of annihilators, we use the following theorem, proved by the author and Robert Guralnik [3]:

**Proposition 3.** If $R$ has the a.c.c. on right annihilators and contains an uncountable field in its center then $R[S]$, the polynomial ring in any set of variables $S$, has the a.c.c. on right annihilators.

Now, it is well known that the d.c.c. on right annihilators is equivalent to the a.c.c. on left annihilators, and by a well-known theorem of Faith [5] a ring $R$ has the d.c.c. on right annihilators if and only if, for every subset $A \subseteq R$, we have $r(A) = r(B)$, where $B$ is a finite subset of $A$. So we have:

**Proposition 4.** If $R$ is right coherent and $M_n(R)$ has the d.c.c. on right annihilators for every $n$, then $R$ is right II-coherent.

**Proposition 5.** If $R$ is right Noetherian and $S$ is any set of variables, then $M_n[R[S]]$ has the property that the annihilator of any finite subset is finitely generated; i.e., $R$ is right coherent, by Proposition 2.

**Proof.** $M_n[R[S]] \approx M_n[R][S] = T$. Let $A$ be a finite subset of $T$. Then, $A$ involves only finitely many variables, say $\{x_1, \ldots, x_N\} = L$. View $T$ as polynomials in $\{x_{N+1}, x_{N+2}, \ldots\}$ with coefficients in $R[L]$. If $K$ is any ring and $W$ is any set of variables then the right annihilator of a subset $B$ of $K$ in $K[A]$ is clearly polynomials with coefficients in the right annihilator of $B$ in $K$.

Putting this together, the hypotheses give that $M_n[R]$ is noetherian, so the polynomial ring in a finite number of variables is noetherian. The remarks in the above paragraph complete the argument.

Now, the classic conditions on annihilators are preserved by extensions, so, if $R$ is a coherent ring that can be embedded in a ring $T$ with the
property that \( M_n(T) \) has the a.c.c. on left annihilators, then \( R \) is strongly right coherent.

P. Pillay [10], generalizing a theorem of Small, for the finite case, showed that if \( R \) is two-sided noetherian and has an artinian quotient rings, so does \( R[S] \) for any number of variables, this means that the following can be proved a bit more generally, but we present the following, short proof for a large class of rings:

**Theorem 6.** If \( R \) is a semiprime left and right noetherian ring, and \( S \) is any set of variables, then \( R[S] \) is right \( \Pi \)-coherent.

By Proposition 5, \( R[S] \) is right coherent, so by Proposition 4 it is enough to show that \( M_n[R[S]] \) has the d.c.c. on right annihilators. \( R \) can be embedded in a product of matrix rings over division rings. So \( M_n[R[S]] \) can be embedded in a product of matrix rings whose entries are polynomials in the variables in \( S \) with coefficients in division rings. Now if \( D \) is any of these division rings, \( D[L] \), where \( L \) is a finite set of variables, is noetherian, then \( D[S] \) satisfies the Ore condition. Therefore, each \( D[S] \) can be embedded in a division ring so that \( R[S] \) can be embedded in a left and right artinian ring; therefore \( M_n[R[S]] \) has the d.c.c. on left and right annihilators for every \( n \), so that by Proposition 5 if we can show that \( M_n[R[S]] \) is right coherent we are done, but by Proposition 3, we need only show right annihilators of finite subsets of matrix rings with entries in \( R[S] \) are finitely generated. This is the content of Proposition 5.

Note, this and Proposition 5 generalize the remark in Chase, that the polynomial ring in any number of variables over a field is coherent.

Now, we can eliminate the semiprime hypothesis if we add another.

**Theorem 7.** If \( R \) is left and right noetherian and contains an uncountable subfield then \( R[S] \) is right \( \Pi \)-coherent for any set of variables \( S \).

**Proof.** Since \( R \) is right noetherian, \( M_n[R] \) is right noetherian and so by Proposition 6, \( M_n[R][S] \) is coherent for any \( n \). But, by Proposition 4, \( M_n[R][S] \approx M_n[R[S]] \) has the a.c.c. on left annihilators, which is the d.c.c. on right annihilators. Therefore, by Proposition 7, \( R[S] \) is strongly coherent.

**Remarks.**

1. Anderson and Dobbs study coherence and infinite products of \( R \) in [1].

2. Note, if \( R \) is a commutative noetherian ring, D. D. Anderson points out that \( R[S] \) can be embedded in a noetherian ring by localizing
at polynomials of content 1; this then gives the a.c.c. on annihilators for appropriate matrix rings, since these will be subrings of matrix rings over noetherian rings.

3. The notion studied in this note is called strong coherence in [6]. There it is used in combination with other conditions to study $f$-projectivity and related matters. A module $M$ is $f$-projective if the inclusion map from every finitely generated submodule factors through a finitely generated projective.

4. The referee points out that the noninjective von Neumann regular Utumi (e.g., commutative) rings provide a class of coherent rings which are not $II$-coherent. This follows from a theorem of Kobayashi [9] who proved that a commutative regular ring $R$ is injective if and only if $R$ duals of finitely generated modules are finitely generated. Now apply our Theorem 1.

REFERENCES