Distributional Analog of a Functional Equation

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Abstract—in this paper, we shall apply an operator method for casting and solving the distributional analog of functional equations. In particular, the method will be employed to solve \( f_1(x + y) + f_2(x - y) + f_3(xy) = 0 \). © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper deals with the study of the distributional analog of the functional equation

\[
f_1(x + y) + f_2(x - y) + f_3(xy) = 0. \tag{1.1}
\]

This equation was studied in [1]. The method of solving functional equations by reduction to differential equations requires that the unknown functions satisfy smoothness properties that may not be natural. Jarai [2] proposed regularity theorems of the form "measurability implies differentiability" that alleviate such assumptions. Another approach we propose in this paper is along the lines of [3,4], which is based on distribution theory. Our rationale is that the corresponding distributional analog of (1.1) will obviate the justification for differentiating the underlying unknown functions.

Our method is based on defining operators on proper function spaces that mirror addition and multiplication of functions.

In Section 2, we provide notations and preliminaries, while in Section 3, we define the proper linear operators needed to cast equation (1.1) in distributions. In Section 4, we write equation (1.1) in distributions, and show that for locally integrable functions, we obtain the classical case. Finally, in Section 5, we provide the solution to the distributional analog of equation (1.1).

2. NOTATIONS

The following is a brief description of the notation that will be used in this paper. Let \( I \) represent the open interval \((0,1)\), and \( I^2 = I \times I \subset \mathbb{R}^2 \). We denote \( D(I) \) and \( D(I^2) \) to be

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the spaces of infinitely differentiable functions on $I$ and $I^2$ with compact support, respectively. $D'(I)$ and $D'(I^2)$ are the duals of $D(I)$ and $D(I^2)$. Likewise, $E(I)$ and $E(I^2)$ are the spaces of infinitely differentiable functions on $I$ and $I^2$, respectively. $E'(I)$ and $E'(I^2)$ are the duals of $E(I)$ and $E(I^2)$. We note the relation among these spaces $E(I) \subset E'(I) \subset D'(I)$ (see [5]).

The spaces $L_{\text{Loc}}(I)$ and $L_{\text{Loc}}(I^2)$ designate the spaces of equivalence classes of locally integrable functions on $I$ and $I^2$, respectively. Let $\lambda_f$ be the regular distribution corresponding to a locally integrable function $f \in L_{\text{Loc}}(I)$. We have

$$\langle \lambda_f, \phi \rangle = \int_I f(x)\phi(x) \, dx,$$

for any $\phi \in D(I)$. Finally, let $D$ denote the differentiation operator on $D'(I)$, whereas $D_1$ and $D_2$ are the partial differentiation operators on $D'(I^2)$ with respect to the first and second variable from $I^2$, respectively. These symbols will also be used to denote the differentiation operators on $D(I)$ and $D(I^2)$, the subspaces of $D'(I)$ and $D'(I^2)$.

In the next section, we will introduce and review some of the background material on linear operators on $D$.

## 3. SOME LINEAR OPERATORS ON $D$

In order to write equation (1.1) in distributions, we need to define the following operators and their adjoints. These operators are intended to be a generalization of the functions given in the functional equation (1.1). These operators were studied in [6,7]. We present them here for completeness.

1. Let $Q_+$ and $Q_-$ be operators from $D(I^2)$ into $D(I)$ defined by

$$Q_+ \phi(x) = \int_R \phi(x - y, y) \, dy = \int_I \phi(y, x - y) \, dy$$

and

$$Q_- \phi(x) = \int_R \phi(x + y, y) \, dy = \int_I \phi(y, y - x) \, dy.$$  \hspace{1cm} (3.1)

These operators are well defined since $\phi(x, y)$ has compact support. We note that $Q_{\pm}$ belong to $L[D(I^2); D(I)]$. The adjoints of $Q_{\pm}$ are $Q_{\pm}^*$ and $Q_{\pm}^*$, respectively, and they are defined from $D'(I)$ into $D'(I^2)$ by

$$\langle Q_{\pm}^* [T], \phi \rangle = \langle T, Q_{\pm} \phi \rangle = \langle T(x), Q_{\pm} \phi(x) \rangle,$$  \hspace{1cm} (3.3)

for any $\phi \in D(I^2)$ and $T \in D'(I)$. In Proposition 3.1, we will show that $Q_+^* [\lambda_f] = f(x + y)$, and $Q_-^* [\lambda_f] = f(x - y)$ if $T$ is a regular distribution $\lambda_f$.

2. For any $\phi \in D(I^2)$, we define the operator $R : D(I^2) \longrightarrow D(I)$ by

$$R \phi(x) = \int_I \phi \left( \frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) \frac{1}{y} \, dy.$$  \hspace{1cm} (3.4)

Its adjoint, denoted by $R^*$, is defined by

$$\langle R^* [T], \phi \rangle = \langle T, R \phi \rangle,$$  \hspace{1cm} (3.5)

for any $\phi \in D(I^2)$ and $T \in D'(I)$. In Proposition 3.1, we will show that $R^* [\lambda_f] = f(xy)$, when $T$ is a regular distribution $\lambda_f$.

We note that the operators $Q_+$, $Q_-$, and $R$ are all in $L[D(I^2); D(I)]$, and their adjoints are in the space $L[D'(I); D'(I^2)]$.

The next proposition describes the operators on regular distributions $\lambda_f$, where $f$ is a locally integrable function, i.e., $f \in L_{\text{Loc}}(I)$. 


PROPOSITION 3.1. Suppose \( f \in L_{\text{Loc}}(I) \). Then

1. \( Q^*_+ [\lambda_f] \in L_{\text{Loc}}(I^2) \), and \( Q^*_+ [\lambda_f] = f(x + y) \);
2. \( Q^*_- [\lambda_f] \in L_{\text{Loc}}(I^2) \), and \( Q^*_- [\lambda_f] = f(x - y) \);
3. \( R^* [\lambda_f] \in L_{\text{Loc}}(I^2) \), and \( R^* [\lambda_f] = f(xy) \).

The reader can find the proof of (1) and (2) in [7], and the proof of (3) in [6].

We note that the properties discussed in Proposition 3.1 are responsible for casting equation (1.1) in distributions.

We next describe the effect of differentiation.

PROPOSITION 3.2. If \( T \in \mathcal{D}'(I) \), then

1. \( D_1 Q^*_+[T] = D_2 Q^*_+[T] = Q^*_+[DT] \);
2. \( D_1 Q^*_-[T] = Q^*_-[DT] \) and \( D_2 Q^*_-[T] = -Q^*_-[DT] \);
3. \( D_1 R^*[T] = E_2(\Omega)R^*[DT] \) and \( D_2 R^*[T] = E_1(\Omega)R^*[DT] \), where \( \Omega = t \in \mathcal{E}(I) \).

We refer the reader to [6] for a proof of the results involving \( D_1 R^*[T] \) and to [7] for the proof of Parts (1) and (2).

4. THE DISTRIBUTIONAL ANALOG OF EQUATION (1.1)

In this section, we will use the operators \( R^* \) and \( Q_i \) to reformulate equation (1.1) in the domain of distributions.

Let \( T_i \) (\( i = 1, 2, 3 \)) be in \( \mathcal{D}'(I^2) \). Proposition 3.1 suggests that the distributional analog of equations (1.1) is

\[
Q^*_+[T_1] + Q^*_+[T_2] + R^*[T_3] = 0.
\]

(4.1)

The next proposition shows that (4.1) reduces to (1.1) in case \( T_i \) (\( i = 1, 2, 3 \)) are regular distributions.

PROPOSITION 4.1. If \( T_i \) (\( i = 1, 2, 3 \)) is a regular distribution, that is, there is a function \( f_i \in L_{\text{Loc}}(I) \) (\( i = 1, 2, 3 \)) such that \( \lambda_{f_i} = T_i \), then equation (4.1) reduces to (1.1).

PROOF. If equation (4.1) holds true for regular distribution \( T_i = \lambda_{f_i} \) (\( i = 1, 2, 3 \)), that is,

\[
Q^*_+[\lambda_{f_1}] + Q^*_+[\lambda_{f_2}] + R^*[\lambda_{f_3}] = 0,
\]

then for any \( \phi \in \mathcal{D}(I^2) \),

\[
\langle Q^*_+[\lambda_{f_1}] + Q^*_+[\lambda_{f_2}] + R^*[\lambda_{f_3}], \phi \rangle = 0.
\]

(4.2)

It follows from Propositions 3.1 and equation (4.2) that

\[
\int_{I^2} f_1(x + y) + f_2(x - y) + f_3(xy) \phi(x, y) \, dx \, dy = 0.
\]

We thus conclude that

\[
f_1(x + y) + f_2(x - y) + f_3(xy) = 0
\]

almost everywhere.

5. SOLVING DISTRIBUTIONAL EQUATION (4.1)

Apply \( D_1^2 \) and \( D_2^2 \) to equation (4.1) to yield

\[
Q^*_+[D^2T_1] + Q^*_+[D^2T_2] + y^2 R^*[D^2T_3] = 0
\]

(5.1)

and

\[
Q^*_+[D^2T_1] + Q^*_+[D^2T_3] + x^2 R^*[D^2T_3] = 0.
\]

(5.2)
respectively. Subtracting (5.2) from (5.1), we have
\[(y^2 - x^2) R^* [D^2T_3] = 0\]
or
\[R^* [D^2T_3] = 0, \quad (5.3)\]
because 0 < y < x. This equation implies \(D^2T_3 = 0\). Indeed, since \(R^*\) is a surjective map, for any \(\phi \in \mathcal{D}(I^2)\),
\[0 = \langle R^* [D^2T_3], \phi \rangle = \langle D^2T_3, R[\phi] \rangle = \langle D^2T_3, \psi \rangle.
\]
for every \(\psi \in \mathcal{D}(I^2)\).

If \(T_3\) is generated by a regular distribution, i.e., \(T_3 = \lambda f_3\) for \(f_3 \in L_{\text{loc}}(I)\), then we have \(f_3''(x) = 0\), or
\[f_3(x) = a + bx, \quad (5.4)\]
for some constant \(a\) and \(b\).

We apply \(D_1\) and \(D_2\) to (4.1) to get
\[Q^*_1[D_1] + Q^*_2[D_2] - yR^*[DT_3] = 0, \quad (5.5)\]
\[Q^*_1[D_1] - Q^*_2[D_2] + xR^*[DT_3] = 0, \quad (5.6)\]
respectively. Adding equations (5.5) and (5.6) yields
\[2Q^*_1[D_1] + (x + y)R^*[DT_3] = 0, \quad (5.7)\]
Applying \(D_1\) to equation (5.7) again and using the fact that \(R^*[D^2T_3] = 0\) in equation (5.3), we obtain
\[2Q^*_1[D^2T_1] + R^*[DT_3] + (x + y)R^*[D^2T_3] = 0, \quad \text{or} \quad 2Q^*_1[D^2T_1] + R^*[DT_3] = 0. \quad (5.8)\]
If \(T_1 = \lambda f_1\) and \(T_3 = \lambda f_3\), equations (5.8) and (5.4) imply that there is a constant \(\alpha_0\) such that
\[2f_1''(x) + \alpha_0 = 0, \quad (5.9)\]
which implies that
\[f_1(x) = -\frac{\alpha_0}{4} x^2 + \alpha_1 x + \alpha_2. \quad (5.9)\]
To find \(T_2\), we subtract equation (5.6) from (5.5)
\[2Q^*_1[D_1] - (x - y)R^*[DT_3] = 0. \quad (5.10)\]
Applying \(D_1\) to equation (5.10) and using the fact that \(R^*[D^2T_3] = 0\), we have
\[2Q^*_1[D^2T_2] - R^*[DT_3] + (y - x)R^*[D^2T_3] = 0, \quad \text{or} \quad 2Q^*_1[D^2T_2] - R^*[DT_3] = 0. \quad (5.11)\]
Similarly, if \(T_2 = \lambda f_2\) and \(T_3 = \lambda f_3\), equations (5.11) and (5.4) imply that there is a constant \(\beta_0\) such that
\[2f_2''(x) + \beta_0 = 0, \quad \text{or} \quad f_2(x) = -\frac{\beta_0}{4} x^2 + \beta_1 x + \beta_2. \quad (5.12)\]
We substitute equations (5.4), (5.9), and (5.12) into the original equation (1.1) to find a relation between the parameters.
\[-\frac{\alpha_0}{4} (x + y)^2 + \alpha_1 (x + y) + \alpha_2 + \frac{\beta_0}{4} (x - y)^2 + \beta_1 (x - y) + \beta_2 + a + b(xy) = 0.
After comparison of the coefficients, we obtain the following equations:

\[
\begin{align*}
  x^2 : & \quad -\frac{\alpha_0}{4} + \frac{\beta_0}{4} = 0, \\
  xy : & \quad -\frac{\alpha_0}{2} - \frac{\beta_0}{2} + b = 0, \\
  y^2 : & \quad -\frac{\alpha_0}{4} + \frac{\beta_0}{4} = 0, \\
  x : & \quad \alpha_1 + \beta_1 = 0, \\
  y : & \quad \alpha_1 - \beta_1 = 0, \\
  1 : & \quad \alpha_2 + \beta_2 + a - 0.
\end{align*}
\]

The solutions of these equations are

\[
\alpha_0 = \beta_0 = b, \quad \alpha_1 = \beta_1 = 0, \quad \alpha_2 + \beta_2 + a = 0.
\]

We have thus shown that

\[
\begin{align*}
  f_1(x) &= -\frac{b}{4} x^2 + \alpha_2, \\
  f_2(x) &= \frac{b}{4} x^2 + \beta_2, \\
  f_3(x) &= bx + a,
\end{align*}
\]

with \(\alpha_2 + \beta_2 + a = 0\). If we let \(b = 4\gamma\), then we have the following theorem.

**Theorem 5.1.** If \(T_i \in \mathcal{D}'(\mathcal{I})\) \((i = 1, 2, 3)\) satisfy equation (4.1), then there exist real constants \(a, \gamma, \alpha_2,\) and \(\beta_2\) such that \(T_i = \lambda f_i\), \((i = 1, 2, 3)\), where \(f_i(x) \in L_{loc}(\mathcal{I})\) is given by

\[
\begin{align*}
  f_1(x) &= -\gamma x^2 + \alpha_2, \\
  f_2(x) &= \gamma x^2 + \beta_2, \\
  f_3(x) &= 4\gamma x + a,
\end{align*}
\]

with \(\alpha_2 + \beta_2 + a = 0\).

Many other functional equations can be solved using this operator approach provided one can define the appropriate operators that will cast the underlying equations in distributions.

**REFERENCES**