The rainbow number of matchings in regular bipartite graphs

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\textbf{A B S T R A C T}

Given a graph $G$ and a subgraph $H$ of $G$, let $rb(G, H)$ be the minimum number $r$ for which any edge-coloring of $G$ with $r$ colors has a rainbow subgraph $H$. The number $rb(G, H)$ is called the rainbow number of $H$ with respect to $G$. Denote as $mK_2$ a matching of size $m$ and as $B_{n,k}$ the set of all the $k$-regular bipartite graphs with bipartition $(X, Y)$ such that $|X| = |Y| = n$ and $k \leq n$. Let $k, m, n$ be given positive integers, where $k \geq 3$, $m \geq 2$ and $n > 3(m-1)$. We show that for every $G \in B_{n,k}$, $rb(G, mK_2) = k(m-2) + 2$. We also determine the rainbow numbers of matchings in paths and cycles.

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1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider simple, finite graphs only.

The Ramsey problem asks for the optimal total number of colors used on the edges of a graph without creating a monochromatic subgraph. In anti-Ramsey problems, we are interested in heterochromatic or rainbow subgraphs instead of monochromatic subgraphs in edge-colorings. Given a graph $G$ and a subgraph $H$ of $G$, if $G$ is edge-colored and $H$ contains no two edges of the same color, then $H$ is called a rainbow subgraph of $G$ and we say that $G$ contains the rainbow $H$. Let $f(G, H)$ denote the maximum number of colors in an edge-coloring of $G$ without any rainbow $H$. Define $rb(G, H)$, the minimum number of colors such that any edge-coloring of $G$ with at least $rb(G, H) = f(G, H) + 1$ colors contains a rainbow subgraph $H$. $rb(G, H)$ is called the rainbow number of $H$ with respect to $G$.

When $G = K_n$, $f(G, H)$ is called the anti-Ramsey number of $H$. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós in the 1970s. Let $P_k$ and $C_k$ denote the path and the cycle with $k$ edges, respectively. Simonovits and Sós [2] determined $f(K_n, P_k)$ for large enough $n$. Erdős et al. [3] conjectured that for every fixed $k \geq 3$, $f(K_n, C_k) = n \left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$, and proved it for $k = 3$ by showing that $f(K_n, C_4) = n - 1$. Alon [4] showed that $f(K_n, C_4) = \left\lceil \frac{4n}{3} \right\rceil - 1$, and the conjecture is thus proved for $k = 4$. Jiang and West [5] verified the conjecture for $k \leq 6$. Recently the conjecture was proved for all $k \geq 3$ by Montellano-Ballesteros and Neumann-Lara [6]. Axenovich, Jiang and Kündgen [7] determined $f(K_{m,n}, C_4k)$ for all $k \geq 2$.

In 2004, Schiermeyer [8] determined the rainbow numbers $rb(K_n, K_k)$ for all $n \geq k \geq 4$, and the rainbow numbers $rb(K_n, mK_2)$ for all $m \geq 2$ and $n \geq 3m + 3$, where $mK_2$ is a matching of size $m$. Li, Tu and Jin [9] proved that $rb(K_m, K_k) = m(p-2) + 2$ for all $m \geq n \geq p \geq 3$. Chen, Li and Tu [10] determined $rb(K_n, mK_2)$.

Let $B_{n,k}$ be the set of all the $k$-regular bipartite graphs with bipartition $(X, Y)$ such that $|X| = |Y| = n$ and $k \leq n$. In this paper we give an upper and a lower bound for $rb(G, mK_2)$, where $G \in B_{n,k}$. Let $k, m, n$ be given positive integers, where

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$k \geq 3$, $m \geq 2$ and $n > 3(m - 1)$. We show that for every $G \in B_{n,k}$, $rb(G, mK_2) = k(m - 2) + 2$. We also determine the rainbow numbers of matchings in paths and cycles.

2. Rainbow numbers of matchings in regular bipartite graphs

Denote by $mK_2$ a matching of size $m$ and by $B_{n,k}$ the set of all the $k$-regular bipartite graphs with bipartition $(X, Y)$ such that $|X| = |Y| = n$ and $k \leq n$. From a result of Li, Tu and Jin in [9] we know that if $n \geq 3$ and $2 \leq m \leq n$, then $rb(B_{n,k}, mK_2) = n(m - 2) + 2$. In this section we discuss the rainbow numbers of matchings in a $k$-regular bipartite graph $G \in B_{n,k}$.

A vertex cover of $G$ is a set $S$ of vertices such that $S$ contains at least one end-vertex of every edge of $G$. For any $U \subset V(G)$, denote by $N_G(U)$ the neighborhood of $U$ in $G$; we abbreviate it as $N(U)$ when there is no ambiguity.

**Lemma 2.1** ([1]). For any bipartite graph $G$, the size of a maximum matching equals the size of a minimum vertex cover. Let $P$ be a minimum vertex cover of $G$; then every maximum matching of $G$ saturates $P$.

Let $\text{ext}(G, H)$ denote the maximum number of edges that $G$ can have with no subgraph isomorphic to $H$.

**Theorem 2.2.** For any subgraph $H$ of a graph $G \in B_{n,k}$ if $|E(H)| > k(m - 1)$ and $2 \leq m \leq n$, then $mK_2 \subset H$. That is

$$\text{ext}(G, mK_2) = k(m - 1).$$

**Proof.** By contradiction. Suppose $H$ is a subgraph of $B_{n,k}$ with $|E(H)| > k(m - 1)$ and contains no $mK_2$. Then $H$ is bipartite and the maximum degree of the vertices in $H$ is $k$. By Lemma 2.1 $H$ has a vertex cover of size at most $m - 1$, which can cover at most $(m - 1)k$ edges, contrary to $|E(H)| > k(m - 1)$. □

**Theorem 2.3.** If $G \in B_{n,k}$ and $1 \leq m \leq n$, then

$$k(m - 2) + 2 \leq rb(G, mK_2) \leq k(m - 1) + 1.$$

**Proof.** The upper bound is obvious from Theorem 2.3. For the lower bound, let $G = (X, Y)$ and $Y_1 \subset Y$ with $|Y_1| = m - 1$; color the $k(m - 2)$ edges between $Y_1$ and $X$ with $k(m - 2)$ distinct colors and the remaining edges with one extra color. It is easy to check that $k(m - 2) + 1$ colors are used and there is no rainbow $mK_2$ in such a coloring. □

The following lemma may already exist. However, we cannot find it in the literature. For the convenience of the reader, we give a full proof of it.

**Lemma 2.4.** Let $G$ be a bipartite graph. Then there exists a maximum matching that saturates all the vertices of maximum degree.

**Proof.** Let $\Delta$ denote the maximum degree of $G$. Among all maximum matchings of $G$, let $M$ be one that saturates the largest number of vertices of degree $\Delta$. Suppose some vertex $v$ of degree $\Delta$ is not saturated by $M$; we derive a contradiction. Let $(X, Y)$ be a bipartition of $G$. Without loss of generality, suppose $v$ is in $X$. Let $S$ denote the set of all the vertices in $X$ reachable from $v$ by an $M$-alternating path and $T$ the set of vertices in $Y$ reachable from $v$ by an $M$-alternating path. If some vertex $w$ in $S$ has degree less than $\Delta$, then let $M'$ be obtained from $M$ by switching the $M$-edges along an $M$-alternating path from $v$ to $w$. We can check that $M'$ is a maximum matching in which $v$ is saturated instead of $w$, and $M$ and $M'$ saturate the set of vertices besides $v$ and $w$. This contradicts our choice of $M$.

So all the vertices in $S$ have degree $\Delta$. Since $M$ is a maximum matching in $G$, there is no $M$-augmented path in $G$, all the vertices in $S \cup T$ are $M$-saturated and there exists a natural bijection between $S$ and $T$ through $M$ (see for instance the proof of Hall’s Theorem in [11]). So $|S| = |T|$. Furthermore $N(\{v\} \cup S) = T$. But there are $\Delta|S| + |\Delta|$ edges from $\{v\} \cup S$ to $T$ while $T$ can be incident to at most $\Delta|T|$ edges in $G$, a contradiction. □

**Corollary 2.5.** If $G$ is a bipartite graph with maximum degree $k$, $|E(G)| \geq k(m - 2) + j$ with $1 \leq j \leq k$ and $G$ has no matching of size $m$, then $G$ contains $j$ pairwise edge disjoint matchings $M_1, M_2, \ldots, M_j$ of size $m - 1$. Furthermore, for any $1 \leq s \leq j$, the maximum degree of $G \setminus \cup_{i=1}^{s-1} M_i$ is $k - s$.

**Proof.** We prove by induction on $j$.

If $j = 1$ and $|E(G)| \geq k(m - 2) + 1$, since $G$ has no matching of size $m$, by Lemma 2.1 $G$ contains a maximum matching $M_1$ of size $m - 1$ which saturates all the vertices of degree $k$ and the maximum degree of $G \setminus M_1$ is $k - 1$. Suppose that when $j = t$ the result is true. Let $j = t + 1$ and $|E(G)| \geq k(m - 2) + t + 1$. By the induction hypothesis, $G$ has $t$ pairwise edge disjoint matchings $M_1, M_2, \ldots, M_t$ of size $m - 1$ and the maximum degree in $G \setminus \cup_{i=1}^{t} M_i$ is $k - t$. Now there are $k(m - 2) + t + 1 - (m - 1) = (k - t)(m - 2) + 1$ edges in $G \setminus \cup_{i=1}^{t} M_i$, by Lemmas 2.1 and 2.4, there is a matching $M_{t+1}$ of size $m - 1$ which saturates all the vertices of degree $k - t$ in $G \setminus \cup_{i=1}^{t} M_i$ and this completes the proof. □

The following theorem shows that for given $k$ and $m$, if $n$ is large enough, $rb(B_{n,k}, mK_2)$ will always be equal to the lower bound $k(m - 2) + 2$. 


Theorem 3.2. For all $m \geq 2$, $k \geq 3$, $n > 3(m-1)$, if $G$ is a $k$-regular bipartite graph with $n$ vertices in each partite set, then $rb(G, mk_2) = k(m-2) + 2$.

Proof. From Theorem 2.3 it suffices to show that for any $m \geq 2$, $k \geq 3$, if $n > 3(m-1)$, any coloring $c$ of $G$ with $k(m-2) + 2$ colors contains a rainbow $mk_2$. By contradiction, suppose there is no rainbow $mk_2$ in $G$. Let $H$ be a subgraph of $G$ formed by taking one edge of each color from $G$. We have $|E(H)| = k(m-2) + 2$ and there is no $mk_2$ in $H$. From Corollary 2.5, let $M$ and $M'$ be two edge disjoint matchings of size $m - 1$ in $H$.

Since $M$ and $M'$ are both maximum matchings in $H$, by Lemma 2.1 the edges in $M \cup M'$ are incident to at most $3(m-1)$ vertices, which can be incident to at most $3k(m-1)$ edges. If $n > 3(m-1)$, then $|E(G)| > 3k(m-1)$ and there is at least one edge, say $e$, in $G$ that is independent of $E(M) \cup E(G')$. Without loss of generality, suppose $c(e) \in C(M)$; then $M' \cup \{e\}$ is a rainbow $mk_2$ in $G$. \hfill \Box

3. Rainbow numbers of matchings in paths and cycles

In this section we suppose $n \geq 3$. Let $P_n$ be the path with $n$ edges with $V(P_n) = \{x_0, x_1, \ldots, x_n\}$ and $E(P_n) = \{e_i | e_i = x_{i-1}x_i, 1 \leq i \leq n\}$, and let $C_n$ be the cycle with $n$ edges.

Theorem 3.1. For any $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$,

$$2m - 2 \leq rb(P_n, mk_2) \leq 2m - 1.$$ 

Proof. For the upper bound, let $c$ be any coloring of $P_n$ with $2m - 1$ colors, and $G$ be the spanning subgraph formed by taking one edge of each color from $P_n$. Then $G$ is a bipartite graph, and so the size of its maximum matchings equals the size of its minimum vertex covers. Since one vertex can cover at most two edges in $G$, the size of a minimum vertex cover of $G$ is at least $m$, and so there is a matching of size $m$ in $G$ and hence there is a rainbow $mk_2$ in $P_n$.

To obtain the lower bound we need to show that there is a coloring $c$ of $P_n$ with $2m - 3$ colors without rainbow $mk_2$. Let $c(e_i) = i$ for $i = 1, \ldots, 2m - 4$ and color all the other edges with $2m - 3$. It is easy to see that there is no rainbow $mk_2$ in such a coloring. \hfill \Box

Let $G$ be a graph, $x', x'' \in V(G)$ with $N(x') \cap N(x'') = \emptyset$. Identify $x'$ and $x''$ into one vertex $x$ and let the resultant graph be $H$, that is $V(H) = V(G) \cup \{x\} \setminus \{x', x''\}$ and $E(H) = \{uv | u \in E(G) \text{ and } \{u, v\} \cap \{x', x''\} = \emptyset\} \cup \{xu | x'u \in E(G) \text{ or } x''u \in E(G)\}$. Let $rb(H, mk_2) = p$ and $c$ be any coloring of $G$ with $p$ colors. For each edge in $G$, color the corresponding edge in $H$ with the same color. Then there is a rainbow $mk_2$ in $H$. Since the corresponding edge set in $G$ of an independent edge set in $H$ is still independent, we have a rainbow $mk_2$ in $G$, and so $rb(G, mk_2) \leq rb(H, mk_2)$.

Notice that $C_n$ can be obtained from $P_n$ by identifying the two ends of $P_n$. Thus from above observation we have:

Theorem 3.2. $rb(P_n, mk_2) \leq rb(C_n, mk_2)$.

In Theorem 3.1, if we replace $P_n$ by $C_n$ and $m \leq \lfloor \frac{n}{2} \rfloor$ by $m \leq \lfloor \frac{n}{3} \rfloor$, then from Theorem 3.2 we get the following theorem.

Theorem 3.3. For any $1 \leq m \leq \lfloor \frac{n}{3} \rfloor$,

$$2m - 2 \leq rb(C_n, mk_2) \leq 2m - 1.$$ 

Theorem 3.4. For any $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$,

$$rb(P_n, mk_2) = \begin{cases} 2m - 1, & n \leq 3m - 3; \\ 2m - 2, & n > 3m - 3. \end{cases}$$

Proof. For $n \leq 3m - 3$, since $2m - 2 \leq rb(P_n, mk_2) \leq 2m - 1$, we can construct a coloring of $P_n$ with $2m - 2$ colors that contains no rainbow $mk_2$. In fact, let $p = n - (2m - 2)$, and for $1 \leq i \leq p$ let $c(e_{3i-2}) = c(e_{3i-1}) = 2i$ and $c(e_{3i-1}) = 2i - 1$, and for $1 \leq j \leq n - 2p$ let $c(e_{3j+p}) = 2p + j$. It is easy to check that for such a coloring, in any rainbow matching of $P_n$ only one color of $2i - 1$ and $2i$ ($1 \leq i \leq m - 1$) may appear, and so there is no rainbow $mk_2$ in $P_n$.

For $n > 3m - 3$, let $c$ be any coloring of $P_n$ with $2m - 2$ colors. We will prove that there is a rainbow $mk_2$ in $P_n$. By contradiction, suppose there is no rainbow $mk_2$ in $P_n$. Let $G$ be the spanning subgraph of $P_n$ formed by taking one edge of each color in $P_n$. $E(G) = \{e_{11}, e_{12}, \ldots, e_{2m-2}\}$, $1 \leq i_1 < i_2 < \cdots < i_{2m-2} \leq n$ with $c(e_j) = j$, $1 \leq j \leq 2m - 2$. There is no $mk_2$ in $G$. Notice that $G$ is bipartite, and so the size of maximum matchings equals the size of maximum vertex covers. Since one vertex of $G$ can cover at most two edges, there is a vertex cover of size $m - 1$ in $G$, and so $e_{i_{2m-1}}$ is adjacent to $e_{i_{2m-1}}, 1 \leq l \leq m - 1$.

Claim 1. Every edge $e \in P_n \setminus E(G)$ is adjacent to an edge in $E(G)$. Otherwise suppose there is an edge $e \in E(P_n) \setminus E(G)$ independent of $E(G)$. Notice that $M_1 = \{e_{i_1}, e_{i_3}, \ldots, e_{2m-3}\}$ and $M_2 = \{e_{i_2}, e_{i_4}, \ldots, e_{2m-2}\}$ are two disjoint matchings of
size $m - 1$ in $G$. Let $c(e) = c(e_{ij})$, and without loss of generality, let $e_{ij} \in M_1$. Then $M_2 \cup \{e\}$ is a rainbow $mk_2$ in $P_n$, a contradiction.

**Claim 2.** There is no subgraph isomorphic to $P_3$ in $P_n \setminus E(G)$. Otherwise the middle edge of $P_3$ is independent of $E(G)$, which is contrary to Claim 1.

From Claims 1 and 2 we know that every nontrivial component of $P_n \setminus E(G)$ is a single edge $P_1$ or a $P_2$. We consider three cases and each leads to a contradiction.

**Case 1.** All the nontrivial components of $P_n \setminus E(G)$ are single edges. From Claim 1 and $n > 3m - 3$, we can deduce that $n = 3m - 2$ and $E(G) = \{e_2, e_3, e_4, e_5, e_6\}$ with $c(e_2) = 2i - 1$, $c(e_3) = 2i$, $1 \leq i \leq m - 1$. Now $M_1 = \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_3|1 \leq i \leq m - 1\}$ have only $e_3$ in common and both are independent of $e_1$.

To avoid the existence of a rainbow $mk_2$ in $P_n$, we have $c(e_1) = c(e_5) = 2$. Similarly, $M_1 = \{e_1, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_6\} \cup \{e_3|1 \leq i \leq m - 1\}$ have only $e_5$ in common and both are independent of $e_4$, and $c(e_4) = c(e_6) = 4$. By the same method, we know that $c(e_3) = c(e_5) = 2i$, $1 \leq i \leq m - 1$. Then, $M_1 = \{e_1, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_6\} \cup \{e_3|1 \leq i \leq m - 1\}$ are disjoint and both are independent of $e_3$. Whatever color $e_{3m-2}$ receives, we will get a rainbow $mk_2$ in $P_n$, a contradiction.

Now at least one component of $P_n \setminus E(G)$ is isomorphic to $P_2$.

**Case 2.** At least one of the end edges of $P_n$ is in $P_n \setminus E(G)$, and there are at least two components in $P_n \setminus E(G)$ isomorphic to $P_2$. Without loss of generality, let $E(G) = \{e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. Now $M_1 = \{e_1, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_6\} \cup \{e_3|1 \leq i \leq m - 1\}$ have only $e_5$ in common and both are independent of $e_1$, and $c(e_1) = c(e_5) = 2$. Now $M_1 = \{e_1, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_6\} \cup \{e_3|1 \leq i \leq m - 1\}$ are disjoint and both are independent of $e_4$. Whatever color $e_4$ receives, we will get a rainbow $mk_2$ in $P_n$.

**Case 3.** Since none of the end edges of $P_n$ is in $P_n \setminus E(G)$, there are at least two components in $P_n \setminus E(G)$ isomorphic to $P_2$. Without loss of generality, let $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ with $c(e_2) = 1$, $c(e_3) = 2$, $c(e_4) = 3$, $c(e_5) = 4$, $c(e_3) = 2i - 1$, $c(e_3) = 2i$, $3 \leq i \leq m - 1$. Since $M_1 = \{e_1, e_4\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ have only $e_1$ in common and both are independent of $e_2$, we have $c(e_1) = c(e_5) = 1$. Now $M_1 = \{e_1, e_4\} \cup \{e_3|1 \leq i \leq m - 1\}$ and $M_2 = \{e_2, e_5\} \cup \{e_3|1 \leq i \leq m - 1\}$ are disjoint and both are independent of $e_7$. Whatever color $e_7$ receives, we will get a rainbow $mk_2$ in $P_n$.

From Theorems 3.2 and 3.4, we have $rb(C_n, mk_2) = 2m - 1$, $n \leq 3m - 3$. For $n > 3m - 3$, by a proof similar to that for Theorem 3.4, we have $rb(C_n, mk_2) = 2m - 2$. Thus we have:

**Theorem 3.5.** For any $m \leq \lfloor \frac{n}{2}\rfloor$,

$$rb(C_n, mk_2) = \begin{cases} 2m - 1, & n \leq 3m - 3; \\ 2m - 2, & n > 3m - 3. \end{cases}$$

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