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Note

σ-Restricted Growth Functions and p,q-Stirling Numbers

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The restricted growth functions are known to encode set partitions. They are words whose subword of leftmost occurrences is the identity permutation. We generalize the notion of restricted growth function by considering words whose subword of leftmost occurrences is a fixed general permutation. We prove a natural generalization of results of Wachs and White which state that the enumerators for the joint distribution of two pairs of inversion like statistics on restricted growth functions are the p,q-Stirling numbers. © 1994 Academic Press, Inc.

The p, q-Stirling numbers of the second kind were defined recursively in [WW]

$$S_{p,q}(n,k) = \begin{cases} p^{k-1}S_{p,q}(n-1,k-1) + [k]_{p,q}S_{p,q}(n-1,k) & \text{if } 0 < k \le n \\ 1 & \text{if } n = k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where

$$[k]_{p,q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}.$$

When p is set equal to 1, Gould's q-Stirling numbers of the second kind [G] are obtained. There has been a considerable amount of recent interest in properties and combinatorial interpretations of the q-Stirling numbers, the p, q-Stirling numbers, and related numbers; see, e.g., [M1], [M2], [GR], [WW], [L], [Sa1], [Sa2], [Sa3], [Si], [B], [BDS], [W].

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It is shown in [WW] that $S_{p,q}(n,k)$ is the enumerator for the joint distribution of two pairs of inversion-like statistics on set partitions or, equivalently, on restricted growth functions (words whose subword of leftmost occurrences is the word $12 \cdots k$). In this paper, we generalize this result to words whose subword of leftmost occurrences is some fixed general permutation σ.

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$ and let $[k]^n$ denote the set of all words of length n over the alphabet $[k]$. For $i = 1, 2, \ldots, n$, the ith letter of a word $w \in [k]^n$ will be denoted by $w(i)$. A word $w \in [k]^n$ is called a *restricted growth function* or *RG function* if

$$w(1) = 1$$
$$w(i) \le \max_{1 \le j < i} w(j) + 1,$$

for all $i = 2, 3, \ldots, n$. Let $RG(n,k)$ be the set of restricted growth functions of length n and maximum k. There is a natural bijection between $RG(n,k)$ and the set of partitions of $\{1, 2, \ldots, n\}$ into k blocks (see [SW]). Hence the Stirling numbers of the second kind $S(n,k)$ enumerate $RG(n,k)$.

Let $\mathscr{S}_k$ denote the symmetric group on letters $1, 2, \ldots, k$. More generally, $\mathscr{S}_A$ denotes the set of all permutations of the letters in A, where A is any finite set of integers of size k. For $\sigma \in S_A$ the usual *inversion index*, $\mathrm{inv}(\sigma)$ is the number of pairs (i,j) such that $1 \le i < j \le k$ and $\sigma(i) > \sigma(j)$. For any word $w \in [k]^n$ and $i = 1, 2, \ldots, n$, let

$lb_i(w)$ = number of distinct letters to *left* of and *bigger* than $w(i)$,

$ls_i(w)$ = number of distinct letters to *left* of and *smaller* than $w(i)$,

$rb_i(w)$ = number of distinct letters to *right* of and *bigger* than $w(i)$,

$rs_i(w)$ = number of distinct letters to *right* of and *smaller* than $w(i)$.

The four inversion-like statistics are defined as

$$lb(w) = \sum_{i=1}^{n} lb_i(w),$$

$$ls(w) = \sum_{i=1}^{n} ls_i(w),$$

$$rb(w) = \sum_{i=1}^{n} rb_i(w),$$

$$rs(w) = \sum_{i=1}^{n} rs_i(w).$$

In [WW], it is proved that

$$S_{p,q}(n,k) = \sum_{w \in RG(n,k)} p^{ls(w)} q^{lb(w)} = \sum_{w \in RG(n,k)} p^{rb(w)} q^{rs(w)}. \qquad (2)$$

The first equality is easily proved by establishing the recurrence relation (1). The second equality is much more difficult to prove. It involves constructing a bijection on $RG(n, k)$ which takes *ls* to *rb* and *lb* to *rs*. The *lb* and *ls* statistics were first introduced by Milne [M2], who established the $p = 1$ and $q = 1$ cases of the first equation of (2). The *rs* statistic arose in the work of Ismail and Stanton [IS]. Stanton [St] discovered the $p = 1$ case of the second equation of (2). The *rb* statistic was introduced in [WW].

For $w \in [k]^n$, the set of leftmost occurrences of w is defined as

$$L(w) = \{i \in [n] | i \text{ is the position of the leftmost occurrence of the letter } w(i)\},$$

and the set of rightmost occurrences of w is defined as

$$R(w) = \{i \in [n] | i \text{ is the position of the rightmost occurrence of the letter } w(i)\}.$$

The *subword of leftmost occurrences* of w is defined as $w(i_1)w(i_2)\dots w(i_j)$, where $\{i_1 < i_2 < \cdots < i_j\} = L(w)$. For example, if $w = 52215141312$ then

$$L(w) = \{1, 2, 4, 7, 9\},$$
$$R(w) = \{5, 7, 9, 10, 11\},$$

and the subword of leftmost occurrences of w is 52143.

Note that w is a restricted growth function on the letters $1, 2, \dots, k$ if and only if its subword of leftmost occurrences is the permutation $12\dots k$. This leads to the following generalization of the notion of restricted growth function. Let σ be any permutation in $\mathscr{S}_A$. A word $w \in A^n$ shall be called a σ-restricted growth function if its subword of leftmost occurrences is σ. For example, the word 52215141312 is a 52143-restricted growth function. Let $RG(n, \sigma)$ be the set of σ-restricted growth functions of length n.

Since $|RG(n, \sigma)| = |RG(n, k)|$ for any $\sigma \in \mathscr{S}_k$, it is natural to ask whether some version of (2) holds with $RG(n, k)$ replaced by $RG(n, \sigma)$. It is easy to establish a generalization of the first equation of (2) for any $\sigma \in \mathscr{S}_k$. The second equation can be generalized only for a restricted class of permutations σ which we shall now define. A permutation $\sigma \in \mathscr{S}_A$ shall be called a *min–max* permutation if each letter of σ is either smaller or larger than all the letters that follow it. For example, 651243 is a min–max permutation. Note that $\sigma \in \mathscr{S}_A$ is a min–max permutation if and only if $\sigma(1) = \min A$ or $\sigma(1) = \max A$, and $\sigma(2)\sigma(3)\ldots\sigma(k)$ is a min–max permutation.

THEOREM 1. (a) *For any* $\sigma \in \mathscr{S}_k$,

$$\sum_{w \in RG(n,\sigma)} p^{ls(w)} q^{lb(w)} = \left(\frac{q}{p}\right)^{\mathrm{inv}(\sigma)} S_{p,q}(n,k).$$

(b) *For any* min–max *permutation* $\sigma \in \mathscr{S}_k$,

$$\sum_{w \in RG(n,\sigma)} p^{rb(w)} q^{rs(w)} = \left(\frac{q}{p}\right)^{\mathrm{inv}(\sigma)} S_{p,q}(n,k).$$

Proof of (a). We shall construct a simple bijection β: $RG(n,k) \to RG(n,\sigma)$ such that $ls(\beta(w)) = ls(w) - \mathrm{inv}(\sigma)$ and $lb(\beta(w)) = lb(w) + \mathrm{inv}(\sigma)$. From this it follows that

$$\sum_{w \in RG(n,\sigma)} p^{ls(w)} q^{lb(w)} = \left(\frac{q}{p}\right)^{\mathrm{inv}(\sigma)} \sum_{w \in RG(n,k)} p^{ls(w)} q^{lb(w)}.$$

The result is therefore a consequence of (2).

Let $w \in RG(n,k)$ with $L(w) = \{i_1 < i_2 < \cdots < i_k\}$. Define $\beta(w)$ as follows:

- Let $\beta(w)(i_\nu) = \sigma(\nu)$ for all $\nu = 1, 2, \ldots, k$.
- For each $i \notin L(w)$, let j be such that $i_j < i < i_{j+1}$ (where $i_{k+1} = n + 1$). Then set $\beta(w)(i)$ equal to the $w(i)$th smallest letter in $\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}$.

For example, if $n = 8$ and $\sigma = 231$ then $\beta(11212331) = 22323131$. It is easy to see that β is indeed a bijection from $RG(n,k)$ to $RG(n,\sigma)$ which satisfies

$$ls_i(w) = ls_i(\beta(w)) \qquad \text{and} \qquad lb_i(w) = lb_i(\beta(w))$$

for all $i \notin L(w)$;

$$\sum_{i \in L(w)} ls_i(\beta(w)) = \binom{k}{2} - \mathrm{inv}(\sigma) = \sum_{i \in L(w)} ls_i(w) - \mathrm{inv}(\sigma)$$

and

$$\sum_{i \in L(w)} lb_i(\beta(w)) = \text{inv}(\sigma) = \sum_{i \in L(w)} lb_i(w) + \text{inv}(\sigma).$$

It follows that β meets the desired specifications. ∎

Remark. It is also easy to prove (a) directly from the defining recurrence relation (1).

Proof of (b). We shall construct a bijection

$$\phi\colon RG(n,\sigma) \to RG(n,k)$$

such that

$$R(\phi(w)) = R(w), \tag{3a}$$

$$rs(\phi(w)) = rs(w) - \text{inv}(\sigma), \tag{3b}$$

$$rb(\phi(w)) = rb(w) + \text{inv}(\sigma). \tag{3c}$$

The result then follows from (2), (3b), and (3c).

The construction of ϕ is recursive and involves two involutions which we now define. First let $A = \{a_1 < a_2 < \cdots < a_k\}$. The simple involution $\gamma\colon A^n \to A^n$ is defined by letting $\gamma(w)$ be the word obtained from w by replacing each a_i, $i = 1, 2, \dots, k$, by a_{k-i+1}. Clearly, γ satisfies

$$R(\gamma(w)) = R(w), \tag{4a}$$

$$rs(\gamma(w)) = rb(w) \tag{4b}$$

$$rb(\gamma(w)) = rs(w). \tag{4c}$$

The second involution $\tau\colon RG(n,k) \to RG(n,k)$ is given by the following lemma.

LEMMA 2. *There exists an involution $\tau\colon RG(n,k) \to RG(n,k)$ satisfying*

$$R(\tau(w)) = R(w) \tag{5a}$$

$$rs(\tau(w)) = rb(w) - \binom{k}{2} \tag{5b}$$

$$rb(\tau(w)) = rs(w) + \binom{k}{2}. \tag{5c}$$

Proof. For any $S \subseteq [n]$ let $S^b = \{n + 1 - i \mid i \in S\}$. We shall use a bijection from $RG(n,k)$ to $RG(n,k)$ constructed at the end of Section 5 of [WW]. This bijection, denoted here by ρ, satisfies

$$L(\rho(w)) = R(w)^b$$
$$lb(\rho(w)) = rs(w)$$
$$ls(\rho(w)) = rb(w).$$

We also need the simple involution μ: $RG(n,k) \to RG(n,k)$ defined by letting

$$\mu(w)(i) = \begin{cases} w(i) & \text{if } i \in L(w) \\ \max_{j<i}\{w(j)\} - w(i) + 1 & \text{otherwise,} \end{cases}$$

for each $i = 1, 2, \ldots, n$. Clearly μ satisfies

$$L(\mu(w)) = L(w)$$
$$ls(\mu(w)) = lb(w) + \binom{k}{2}$$
$$lb(\mu(w)) = ls(w) - \binom{k}{2}.$$

Now let τ: $RG(n,k) \to RG(n,k)$ be the composition $\rho^{-1}\mu\rho$. It is easy to verify that τ is an involution satisfying (5a), (5b), and (5c). ∎

Proof of Theorem 1b *continued*. We are now ready to recursively construct the main bijection $\phi = \phi_{n,\sigma}$: $RG(n,\sigma) \to RG(n,k)$, where σ is a min–max permutation in $\mathscr{S}_A$, $A = \{a_1 < a_2 < \cdots < a_k\}$ and $n \geq k$. If $k = 1$ then $RG(n,\sigma) = \{a_1^n\}$ and $RG(n,k) = \{1^n\}$. So define ϕ by setting $\phi(a_1^n) = 1^n$.

Now suppose that $k > 1$ and that ϕ: $RG(n,\beta) \to RG(n,|B|)$ has been defined for all $n \geq |B|$ and all min–max $\beta \in \mathscr{S}_B$, where $B \subsetneq A$. We will use the notation $w|_B$ to denote the subword of $w \in A^n$ consisting of letters in B. For example, $23155143|_{\{3,4,5\}} = 35543$. We shall also let $w + 1$ denote the word obtained from w by adding 1 to each letter of w. For example, $23114243 + 1 = 34225354$.

Since σ is a min–max permutation, $\sigma(1) = a_1$ or $\sigma(1) = a_k$. If $\sigma(1) = a_1$ then $\phi(w)$ is obtained by setting

$$\phi(w)(i) = 1 \qquad \text{whenever } w(i) = a_1$$

and

$$\phi(w)|_{\{2,3,\ldots,k\}} = \phi(w|_{\{a_2,a_3,\ldots,a_k\}}) + 1.$$

If $\sigma(1) = a_k$ then set

$$\phi(w) = \tau(\phi(\gamma(w))).$$

We now prove, by induction on k, that ϕ: $RG(n, \sigma) \to RG(n, k)$ is a well defined bijection satisfying (3a), (3b), and (3c). For $k = 1$, the result holds trivially. Assume it holds for $k - 1$, where $k > 1$.

Case 1. Let $\sigma(1) = a_1$. Since $\sigma' = \sigma(2)\sigma(3)\dots\sigma(k)$ is a min–max permutation and $w|_{\{a_2, a_3, \dots, a_k\}} \in RG(n', \sigma')$, where $n' < n$, it follows by induction that $\phi(w|_{a_2, a_3, \dots, a_k\}})$ is defined. Therefore $\phi(w)$ is well defined. It is also easy to see how to invert ϕ and thereby conclude that ϕ is indeed a bijection.

To establish (3a), we first note that the rightmost 1 occupies the same position in $\phi(w)$ as does a_1 in w. By induction, we have

$$R(w|_{\{a_2, a_3, \dots, a_k\}}) = R(\phi(w|_{\{a_2, a_3, \dots, a_k\}})) = R(\phi(w)|_{\{2, 3, \dots, k\}}).$$

Since the subwords $w|_{\{a_2, a_3, \dots, a_k\}}$ and $\phi(w)|_{\{2, 3, \dots, k\}}$ occupy the same positions in their respective words, it follows that the set of rightmost occurrences of the letters $a_2, a_3, \dots, a_k$ in w is the same as the set of rightmost occurrences of the letters $2, 3, \dots, k$ in $\phi(w)$. Hence (3a) holds.

Next we establish (3b). For any $w \in A^n$, we let $m(w)$ be the position of the rightmost a_1 in w minus the number of times a_1 occurs in w. We clearly have

$$rs(w) = rs(w|_{\{a_2, a_3, \dots, a_k\}}) + m(w).$$

Since $\phi(w)(i) = 1$ whenever $w(i) = a_1$, we also have

$$m(w) = m(\phi(w)).$$

Hence by induction we have

$$\begin{aligned} rs(\phi(w)) &= rs(\phi(w)|_{\{2, 3, \dots, k\}}) + m(\phi(w)) \\ &= rs(\phi(w|_{\{a_2, a_3, \dots, a_k\}})) + m(w) \\ &= rs(w|_{\{a_2, a_3, \dots, a_k\}}) - \operatorname{inv}(\sigma(2)\sigma(3)\dots\sigma(k)) + m(w) \\ &= rs(w) - \operatorname{inv}(\sigma). \end{aligned}$$

To establish (3c), we define $m_j(w)$, for $w \in A^n$ and $j = 2, 3, \dots, k$, to be the number of times that a_1 occurs to the left of the rightmost occurrence of a_j in w (let $m_j(w) = 0$ if a_j does not occur in w). Clearly we have

$$rb(w) = rb\big(w|_{\{a_2, a_3, \dots, a_k\}}\big) + \sum_{j=2}^{k} m_j(w).$$

Since $\phi(w)(i) = 1$ whenever $w(i) = a_1$, and $R(w) = R(\phi(w))$ we have

$$\sum_{j=2}^{k} m_j(\phi(w)) = \sum_{j=2}^{k} m_j(w).$$

Hence, by induction,

$$\begin{aligned} rb(\phi(w)) &= rb\big(\phi(w)|_{\{2,3,\dots,k\}}\big) + \sum_{j=2}^{k} m_j(\phi(w)) \\ &= rb\big(\phi\big(w|_{\{a_2, a_3, \dots, a_k\}}\big)\big) + \sum_{j=2}^{k} m_j(w) \\ &= rb\big(w|_{\{a_2, a_3, \dots, a_k\}}\big) + \operatorname{inv}(\sigma(2)\sigma(3)\dots\sigma(k)) + \sum_{j=2}^{k} m_j(w) \\ &= rb(w) + \operatorname{inv}(\sigma). \end{aligned}$$

Case 2. Let $\sigma(1) = a_k$. Clearly the restriction γ: $RG(n, \sigma) \to RG(n, \gamma(\sigma))$ is a bijection. Since $\gamma(\sigma)$ is a min–max permutation of $\mathscr{S}_A$ and $\gamma(\sigma)(1) = a_1$, it follows that ϕ: $RG(n, \gamma(\sigma)) \to RG(n, k)$ is a bijection by Case 1. The composition, $\tau\phi\gamma$: $RG(n, \sigma) \to RG(n, k)$ is therefore also a bijection.

The fact that ϕ preserves $R(w)$ follows from (4a), (5a), and (3a) (Case 1). To prove (3b), we use (5b), (3c) (Case 1), and (4c) to obtain

$$\begin{aligned} rs(\phi(w)) &= rs(\tau\phi\gamma(w)) \\ &= rb(\phi(\gamma(w))) - \binom{k}{2} \\ &= rb(\gamma(w)) + \operatorname{inv}(\gamma(\sigma)) - \binom{k}{2} \\ &= rs(w) - \operatorname{inv}(\sigma). \end{aligned}$$

Similarly, to prove (3c) we use (5c), (3b) (Case 1), and (4b). ∎

Theorem 1b has the following converse.

THEOREM 3. *Let $n > k$. If $\sigma \in \mathscr{S}_k$ satisfies*

$$\sum_{w \in RG(n,\sigma)} p^{rb(w)} q^{rs(w)} = \left(\frac{q}{p}\right)^{\mathrm{inv}(\sigma)} S_{p,q}(n,k) \tag{6}$$

then σ is a min–max *permutation.*

We shall use Lemma 4 below to prove Theorem 3. A permutation $\sigma \in \mathscr{S}_k$ is said to *avoid the pattern* 231 if there is no triple $r < s < t \in [k]$ such that $\sigma(t) < \sigma(r) < \sigma(s)$. Similarly σ is said to avoid the pattern 213 if there is no triple $r < s < t \in [k]$ such that $\sigma(s) < \sigma(r) < \sigma(t)$. Note that σ is a min–max permutation if and only if it avoids both the patterns 231 and 213.

LEMMA 4. *Let $n > k$. If $\sigma \in \mathscr{S}_k$ satisfies*

$$\sum_{w \in RG(n,\sigma)} q^{rs(w)} = q^{\mathrm{inv}(\sigma)} S_{1,q}(n,k) \tag{7}$$

then σ avoids the pattern 231.

Proof. Suppose σ does not avoid the pattern 231. We will show that then the coefficient of $q^{\mathrm{inv}(\sigma)}$ on the left hand side of (7) is less than the coefficient on the right hand side.

Since $S_{1,q}(n,k) = \sum_{w \in RG(n,k)} q^{rs(w)}$, the coefficient of $q^{\mathrm{inv}(\sigma)}$ on the right hand side of (7) is the number of $w \in RG(n,k)$ such that $rs(w) = 0$. Clearly $rs(w) = 0$ if and only if w is of the form $1^{\nu_1} 2^{\nu_2} \dots k^{\nu_k}$, where $\nu_1 + \nu_2 + \cdots + \nu_k = n$ and $\nu_i \geq 1$ for all $i = 1, 2, \dots, k$. Hence the coefficient of $q^{\mathrm{inv}(\sigma)}$ on the right hand side of (7) is the number of k-compositions of n.

We claim that for each k-composition $\nu_1 + \nu_2 + \cdots + \nu_k = n$ there is at most one $w \in RG(n, \sigma)$ with ν_i occurrences of i for each i, such that $rs(w) = \mathrm{inv}(\sigma)$. Indeed, w is obtained by inserting $\nu_i - 1$ copies of i, $i = 1, 2, \dots, k$ into the permutation σ so that no additional inversions are created. To do this, each of the $\nu_i - 1$ copies of i must be inserted to the right of the rightmost letter of σ that is less than or equal to i. The letters inserted between two adjacent letters (or after the last letter) of σ must be arranged in weakly increasing order. Clearly there is at most one way to make this insertion.

Suppose the pattern 231 occurs at $r < s < t$. Then we claim that there is no way to make the insertion if $j = \sigma(r)$ and $\nu_j > 1$. Indeed, the $\nu_j - 1$ copies of j must be inserted to the right of $\sigma(t)$. But this will create an inversion with $\sigma(s)$. Hence, for those compositions $\nu_1 + \nu_2 + \cdots + \nu_k = n$

in which $\nu_j > 1$ there is no $w \in RG(n, \sigma)$ with ν_i occurrences of i, $i = 1, 2, \ldots, k$ and $rs(w) = \mathrm{inv}(\sigma)$. It follows that the coefficient of $q^{\mathrm{inv}(\sigma)}$ on the left hand side of (7) is less than the number of k-compositions of n which, as we have already seen, equals the coefficient of $q^{\mathrm{inv}(\sigma)}$ on the right hand side of (7). ∎

Proof of Theorem 3. Since (7) is obtained from (6) by setting $p = 1$, it follows from Lemma 4 that σ avoids the pattern 231. By setting $q = 1$ in (6), we obtain

$$\sum_{w \in RG(n,\sigma)} p^{rb(w)} = p^{-\mathrm{inv}(\sigma)} S_{p,1}(n,k).$$

It follows immediately from the definition of $S_{p,q}(n,k)$ given in (1) that

$$S_{p,1}(n,k) = p^{\binom{k}{2}} S_{1,p}(n,k).$$

Hence we have

$$\sum_{w \in RG(n,\sigma)} p^{rb(w)} = p^{\binom{k}{2} - \mathrm{inv}(\sigma)} S_{1,p}(n,k). \tag{8}$$

Recall the bijection γ: $RG(n, \sigma) \to RG(n, \gamma(\sigma))$ defined by letting $\gamma(w)$ be the word obtained from w by replacing each letter $i \in [k]$ by $k - i + 1$. It follows from (4b) that (8) is equivalent to

$$\sum_{w \in RG(n,\gamma(\sigma))} p^{rs(w)} = p^{\mathrm{inv}(\gamma(\sigma))} S_{1,p}(n,k).$$

It now follows from Lemma 4 that $\gamma(\sigma)$ avoids the pattern 231. This is equivalent to σ avoiding the pattern 213. Since σ avoids both patterns 231 and 213, σ is a min–max permutation. ∎

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Note added in proof. Since this work, White [W] has generalized results of [WW] in a different but far reaching direction by considering statistics which interpolate between the statistics considered here.

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