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K_2 of Cyclic Group Rings over λ-Rings

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The theory developed in previous papers of the author is used to compute the algebraic K_2 of group rings of cyclic p-groups with coefficients in an arbitrary λ-ring. © 1998 Academic Press

1. INTRODUCTION

This paper is a sequel to [5] and [6]. The purpose of this series of papers is to demonstrate a method which enables one, e.g., to make an explicit calculation of $K_1(\mathbf{Z}[\pi])$ for *any* finitely generated abelian group π. These algebraic K-groups of integral group rings occur in topology in the classification of homotopy equivalences $f\colon X \to Y$ up to simple homotopy, in the case that the fundamental groups of X and Y are equal to π. The results known before only apply to finite groups (see for example [12]) or other special groups or yield only partial information: the result of [4] says that if π is infinite then $K_1(\mathbf{Z}[\pi])$ is not finitely generated if it is not trivial.

To accomplish the above aim we view the group ring of π over $\mathbf{Z}$ as the group ring of G over A. Here G is the torsion subgroup of π, and A is the group ring over $\mathbf{Z}$ of a complementary free subgroup of π. In fact our method is applicable to any commutative ring A which has the structure of a λ-ring and has no integral torsion in its module of Kähler differentials.

The strategy is the same as in [1]: there is an exact sequence by means of which $K_1(A[G])$ can be expressed in terms of the group $K_2(A[G]/C)$ and the groups $K_2(A[\chi_i]/\chi_i(C))$. Here C is a suitable small ideal of $\mathbf{Z}[G]$, and the χ_i are the characters of G. In the present paper $K_2(A[G]/C)$ is evaluated using the theory explained in [5] and [6]; in a future paper the groups $K_2(A[\chi_i]/\chi_i(C))$ will be computed.

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Thus in this series we try to generalize techniques from [13] by allowing the base ring A to be an *arbitrary* λ-ring, without restrictions on its dimension.

The result of the computation is expressed in terms of the additive group of A and similar data, thus explaining the lack of finite generation. In order to clarify this functorial dependence on A and also to simplify the computations, we restrict ourselves here to the case that G is cyclic. However the same ideas should suffice to treat the general case.

In [5] and [6] the author defined a homomorphism L: $K_2^{\mathrm{pro}}(R, I) \to K_{2,L}^{\mathrm{pro}}(R, I)$ and gave a criterion for it to be an isomorphism. Here $K_n(R, I)$ denotes the nth group in the algebraic K-theory of the ring R and the ideal I. If $n = 2$ and I is nilpotent then this group has a presentation in terms of elements of R and I, due to [11] and [9]. The group $K_{2,L}(R, I)$ is defined by a similar presentation but is more accessible to computation because of the linear nature of the relations; in fact this group is strongly related (but not identical) to the cyclic homology group $HC_1(R, I)$. The superscripts $^{\mathrm{pro}}$ mean that we really consider an inverse system of such groups associated to quotient rings of R. Everything is done under the assumption that R is equipped with a structure of λ-ring, that I is a λ-ideal, and that λ-operations of high order map elements of I to zero in such a quotient ring. For the details about these assumptions and other background material about λ-rings we refer to [6], in particular Sects. 2, 4, and 5.

In the present paper we apply this theory to the case that R is the group ring $A[G]/C$ and I is the augmentation ideal. Since C is a λ-ideal and since I is nilpotent modulo C the superscripts $^{\mathrm{pro}}$ can be omitted here. The map L is not an isomorphism in this situation but the theory of [5] and [6] is used to determine its kernel and cokernel and these turn out to be quite small.

2. STATEMENT OF THE RESULTS

Let p be a prime and let G be a cyclic group, generated by an element T of order $q = p^e$. Let A be a λ-ring, and let I be the augmentation ideal of the group ring $A[G]$, which is generated by $z = T - 1$. We shall compute the pro-group $K_2^{\mathrm{pro}}(A[G], I)$ associated to the filtration on $A[G]$ by powers of I. This means that we shall compute groups $K_2(A[G]/I_M, I)$, where the I_M constitute a filtration which is cofinal with the powers of I. In Sect. 10 we describe a choice for these I_M which is convenient for our calculations.

In order to be able to use the theory of [6] we must use a λ-ring structure on $A[G]$ such that I is at least ψ-nilpotent. The most obvious

λ-ring structure on $A[G]$ is the one for which $\theta^k(T) = 0$ and thus $\psi^k(T) = T^k$ for every $k > 1$. However in this situation $\psi^k(z)$ depends only on k modulo q, so I is not ψ-nilpotent.

As an A-module one has $A[G]/I^n = A \oplus I/I^n$, where I/I^n is p-primary torsion. Therefore we can define an "exotic" λ-ring structure on $A[G]$ by putting $\theta^l(z) = z^l/l$ for each prime $l \neq p$; then $\psi^l(z) = 0$ for such l. We keep $\theta^p(T) = 0$, so that $\psi^p(z) = (z+1)^p - 1$. See Sect. 13 for details about this λ-ring structure.

For the exotic λ-ring structure on $A[G]$ the ideal I is θ-nilpotent and ψ-nilpotent, but not λ-nilpotent. Thus the theory of [6] tells us that the associated logarithmic map L_2: $K_2^{\mathrm{pro}}(A[G], I) \to K_{2,L}^{\mathrm{pro}}(A[G], I)$ is defined but not necessarily an isomorphism. In this paper we shall determine the kernel and cokernel of L_2.

In order to formulate the result we refer to the following diagram with exact rows:

$$\begin{array}{ccccccc}
K_2(A[G]/I_M, I^2) & \xrightarrow{j_{GM}} & K_2(A[G]/I_M, I) & \xrightarrow{p_{GM}} & K_2(A[G]/I^2, I) & \longrightarrow & 0 \\
\downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_2} & & \downarrow{\scriptstyle L_3} & & \\
K_{2,L}(A[G]/I_M, I^2) & \xrightarrow{j_{GL}} & K_{2,L}(A[G]/I_M, I) & \xrightarrow{p_{GL}} & K_{2,L}(A[G]/I^2, I) & \longrightarrow & 0.
\end{array}$$

The vertical maps L_i are logarithmic maps associated to the exotic λ-ring structure on $A[G]$. Thus the map L_1 is an isomorphism by [6]. By chasing this diagram one finds an isomorphism

$$p_{GL}^*: \operatorname{cok}(L_2) \cong \operatorname{cok}(L_3)$$

and an exact sequence

$$0 \to \ker(j_{GM}) \xrightarrow{L_1^*} \ker(j_{GL}) \to \ker(L_2) \xrightarrow{p_{GM}^*} \ker(L_3) \to 0.$$

Here p_{GL}^*, L_1^*, and p_{GM}^* are of course the maps induced by p_{GL}, L_1, and p_{GM}. In Sect. 3 we shall prove:

THEOREM 1. *If $p > 2$ then L_3 can be identified with the map*

$$1 - \phi^p: \Omega_A/q\Omega_A \to \Omega_A/q\Omega_A.$$

If $p = 2$ then L_3 can be identified with the direct sum of this same map with the map

$$1 - \psi^2: A/2A \to A/2A.$$

Here Ω_A denotes the group of absolute Kähler differentials of A and the ϕ^i: $\Omega_A \to \Omega_A$ are the natural additive operations which were introduced in [5].

The above theorem takes care of $\operatorname{cok}(L_2)$ and part of $\ker(L_2)$; in order to describe the remaining part of $\ker(L_2)$ we must describe the cokernel of L_1^*. This is done in the following theorem

THEOREM 2. *There is a commutative diagram with vertical maps surjective*

$$\begin{array}{ccc} {}^q\Omega_A \oplus (A' \oplus A) & \xrightarrow{\Phi \oplus \Psi} & {}^q\Omega_A \oplus (A' \oplus A) \\ {\scriptstyle \kappa_1^M + (\kappa_2^M + \kappa_3^M)} \downarrow & & \downarrow {\scriptstyle \kappa_1^G + (\kappa_2^G + \kappa_3^G)} \\ \ker(j_{GM}) & \xrightarrow{L_1^*} & \ker(j_{GL}), \end{array}$$

where $A' = \{a \in A;\, 2a \in qA\}$. *The map* Ψ *is surjective modulo the kernel of* $\kappa_2^G + \kappa_3^G$.

In fact Φ is the map $1 - \phi^p$ on the q-torsion of Ω_A and the map Ψ is given by

$$\Psi = \begin{pmatrix} 1 - \psi^p & 0 \\ \psi^p & 1 - p\psi^p \end{pmatrix}.$$

The theorem implies that the cokernel of L_1^* is isomorphic to a quotient of the cokernel of $1 - \phi^p$: ${}^q\Omega_A \to {}^q\Omega_A$. In particular the map L_1 is surjective in the case that A is a group ring over **Z** of a free abelian group, or a polynomial ring. In Sect. 5 the maps κ_i^G are described in detail. They involve only the additive structure of A and Ω_A; thus the determination of their kernel for other examples will not be too difficult.

The paper is organized as follows. In Sects. 4 and 5 we describe the target and domain of j_{GL} and we compute $\ker(j_{GL})$. In Sect. 6 we construct an epimorphism π^*: $\ker(j_{XL}) \to \ker(j_{GM})$ where j_{XL} is a map similar to j_{GL} but for a truncated polynomial ring instead of a group ring. In Sect. 7 we compute $\ker(j_{XL})$. Thus $\operatorname{cok}(L_1^*)$ can be computed from a description of the map $L_1^*\pi^*$: $\ker(j_{XL}) \to \ker(j_{GL})$. This description involves the maps Φ and Ψ. In Sect. 8 we determine Φ, and in Sect. 9 we determine Ψ and prove the surjectivity mentioned in Theorem 2.

The last four sections contain proofs of technical results needed for these calculations. In particular the results of Sects. 11 and 12 are needed for the determination of Φ.

3. THE MAP L_3

In order to study the kernel and cokernel of L_3 it is useful to view $A[G]$ as the homomorphic image of the polynomial ring $A[x]$.

The standard λ-ring structure on $A[x]$ extending the given one on A is the one for which $\theta^k(x) = 0$ for every $k > 1$. The standard filtration on $A[x]$ is the one by powers of (x). The λ-ideal (x) is λ-nilpotent, so the logarithmic map is an isomorphism in this case.

There is a ring homomorphism $\pi\colon A[x] \to A[G]$ defined by $\pi(x) = z$. The map π induces an isomorphism $A[x]/(x^2, qx) \to A[G]/I^2$. The standard λ-ring structure on $A[x]$ induces one on these groups. The associated logarithmic map

$$L_4\colon K_2(A[G]/I^2, I) \to K_{2,L}(A[G]/I^2, I)$$

is an isomorphism. This means that the kernel and cokernel of L_3 can be computed.

Proposition 1. *Let $R = A[G]/(z^2) = A[x]/(x^2, qx)$. Then*

$$K_{2,L}(R, xR) = \operatorname{cok}(\Delta\colon xR \otimes xR \to xR \otimes_R \Omega_R)$$

is isomorphic to

$$\left(\frac{\Omega_A}{q\Omega_A}\right) \oplus \left(\frac{A}{qA + 2A}\right).$$

Proof. One has

$$xR \otimes_R \Omega_R \cong \left(\frac{\Omega_A}{q\Omega_A}\right) \oplus \left(\frac{A}{qA}\right).$$

Under this isomorphism the element $x \otimes \alpha + x \otimes a\delta x$ corresponds to the element $([\alpha], [a])$. We must compute the image of $\Delta\colon xR \otimes_R xR \to xR \otimes_R \Omega_R$. The group xR is generated by the elements xa with $a \in A$. Therefore, the group $xR \otimes_R xR$ is additively generated by the elements $xa_1 \otimes xa_2 = x \otimes xa_1a_2$ for $a_1, a_2 \in A$. But $\Delta(x \otimes xa) = x \otimes 2a\delta x$. ∎

The kernel and cokernel of L_3 are isomorphic to those of $L_3(L_4)^{-1}$. Although the evaluation of L_3 and L_4 could be done in a straightforward way, we use this occasion to demonstrate a technique which will be needed in a more complicated situation in Sect. 8.

The map $L_3(L_4)^{-1}$ is additive and natural for λ-homomorphisms. In Sect. 12 it is shown that there must then exist $c_i^k \in \mathbf{Z}$ such that

$$L_3(L_4)^{-1}([\alpha], [a]) = \left(\left[\sum_i c_i^1 \phi^i(\alpha)\right] + \left[\sum_i c_i^2 \phi^i(\delta a)\right], \left[\sum_i c_i^3 \psi^i(a)\right]\right).$$

Moreover the c_i^1 and c_i^2 are unique modulo $q\mathbf{Z}$ and the c_i^3 are unique modulo $q\mathbf{Z} + 2\mathbf{Z}$. To determine the c_i^k it suffices to evaluate this map for $A = \mathbf{Z}[t]$ and $\alpha = \delta t$ and for $a = t$.

This is done in the next two propositions showing that

$$L_3(L_4)^{-1} = (1 - \phi^p) \oplus (1 - \psi^p),$$

thus proving Theorem 1.

PROPOSITION 2. *One has*

$$L_3\langle x, t\rangle = x \otimes \delta t - x \otimes t^{p-1}\delta t + \mathrm{im}(\Delta),$$
$$L_4\langle x, t\rangle = x \otimes \delta t + \mathrm{im}(\Delta).$$

In other words one can take $c_1^1 = 1$, $c_p^1 = -1$ *and* $c_i^1 = 0$, *otherwise*.

Proof. By definition of the logarithmic map one has

$$L\langle x, t\rangle = \sum_{m,k} \theta^m(x) \otimes \phi^m \delta\eta^k(x, t) + \mathrm{im}(\Delta).$$

In this formula we substitute the results of Sect. 13. For both λ-ring structures one has $\eta^k(x, t) = 0$ for $k > 1$, whereas $\eta^1(x, t) = t$.

First consider the standard λ-ring structure. Then one has $\theta^m(x) = 0$ for $m > 1$, whereas $\theta^1(x) = x$. So the only contribution is $x \otimes \delta t$.

Finally consider the exotic λ-ring structure. Then $\theta^m(x) = 0$ for $m \neq 1, p$. For $m = 1$ one gets the same contribution as in the standard case and for $m = p$ one gets $\theta^p(x) \otimes \phi^p(\delta t) = \theta^p(x) \otimes t^{p-1}\delta t \equiv -x \otimes t^{p-1}\delta t$. ∎

PROPOSITION 3. *One has*

$$L_3\langle x, xt\rangle = x \otimes t\delta x - x \otimes t^p\delta x + \mathrm{im}(\Delta),$$
$$L_4\langle x, xt\rangle = x \otimes t\delta x + \mathrm{im}(\Delta).$$

In other words one can take $c_i^2 = 0$ *for all* i, *and* $c_1^3 = 1$, $c_p^3 = -1$, *and* $c_i^3 = 0$, *otherwise*.

Proof. By definition of the logarithmic map one has

$$L\langle x, xt\rangle = \sum_{m,k} \theta^m(x) \otimes \phi^m \delta\eta^k(x, xt) + \mathrm{im}(\Delta).$$

In this we substitute the results of Sect. 13. For both λ-ring structures one has $\eta^k(x, xt) = 0$ for $k > 1$, $\eta^1(x, xt) = xt$.

Now consider the standard λ-ring structure. Then one has $\theta^m(x) = 0$ for $m > 1$, whereas $\theta^1(x) = x$. So the only contribution is

$$\theta^1(x) \otimes \phi^1\delta xt = x \otimes (x\delta t + t\delta x) \equiv x \otimes t\delta x.$$

Finally consider the exotic λ-ring structure. Again $\theta^m(x) = 0$ for $m \neq 1, p$. For $m = 1$ one gets the same contribution as in the standard case and for $m = p$ one gets

$$\begin{aligned}\theta^p(x) \otimes \phi^p(\delta xt) &= \theta^p(x) \otimes \phi^p(x\delta t + t\delta x)\\ &= \theta^p(x) \otimes \left(\psi^p(x)\phi^p(\delta t) + \psi^p(t)\phi^p(\delta x)\right)\\ &= \theta^p(x) \otimes \left(\left((x+1)^p - 1\right)t^{p-1}\delta t + t^p(x+1)^{p-1}\delta x\right)\\ &\equiv (-x) \otimes \left(pxt^{p-1}\delta t + t^p\delta x\right) \equiv -x \otimes t^p\delta x. \quad \blacksquare\end{aligned}$$

4. DOMAIN AND TARGET OF j_{GL}

Let $M \in \mathbf{N}$. Let I_M be the ideal of $A[G]$ generated by the elements $p^{M+e-i}z^{p^i}$ for $0 \le i < e$. We write $\epsilon(n) = M + e - i$ for $p^i \le n < p^{i+1}$ and $0 \le i < e$. We write $\beta(n) = \min(\epsilon(n), e)$.

PROPOSITION 4. *Let $B = \mathbf{Z}[G]/I_M$ and $R = B \otimes A = A[G]/I_M$. Then*

$$K_{2,L}(R, zR) = \operatorname{cok}(\Delta\colon zR \otimes_R zR \to zR \otimes_R \Omega_R)$$

is isomorphic to a direct sum $CF_{\text{low}} \oplus CF_2 \oplus \cdots \oplus CF_{q-1}$, where

$$CF_{\text{low}} = \frac{\left(\dfrac{A}{p^{\epsilon(1)}A} \otimes_A \Omega_A\right) \oplus \dfrac{A}{p^{\beta(q-1)}A}}{\{([1] \otimes q\delta a, 0)\}},$$

$$CF_n = \frac{\left(\dfrac{A}{p^{\epsilon(n)}A} \otimes_A \Omega_A\right) \oplus \dfrac{A}{p^{\beta(n-1)}A}}{\{([1] \otimes \delta a, [na])\}} \qquad \textit{for } n = 2, \ldots, q-1.$$

Proof. Similar to the proof of the next proposition. $\blacksquare$

PROPOSITION 5. *Again let $B = \mathbf{Z}[G]/I_M$ and $R = B \otimes A = A[G]/I_M$. Then*

$$K_{2,L}(R, z^2R) = \operatorname{cok}\left(\Delta\colon z^2R \otimes_R z^2R \to z^2R \otimes_R \Omega_R\right)$$

is isomorphic to a direct sum $CS_{\text{low}} \oplus CS_4 \oplus \cdots \oplus CS_{q-1}$, *where*

$$CS_n = \frac{\left(\frac{a}{p^{\epsilon(n)}A} \otimes_A \Omega_A\right) \oplus \frac{A}{p^{\beta(n-1)}A}}{\{([1] \otimes \delta a, [na])\}} \qquad \textit{for } n = 4, \ldots, q-1.$$

For $q \geq 4$ *the group* CS_{low} *is defined as the quotient of*

$$MS_{\text{low}} = \left(\frac{qA}{p^{\epsilon(1)}A} \otimes_A \Omega_A\right) \oplus \left(\frac{a}{p^{\epsilon(2)}A} \otimes_A \Omega_A\right) \oplus \left(\frac{A}{p^{\epsilon(3)}A} \otimes_A \Omega_A\right) \oplus \frac{qA}{p^{\min(2e,\, \epsilon(1))}A} \oplus \frac{A}{p^{\min(e,\, \epsilon(2))}A} \oplus \frac{A}{p^{\beta(q-1)}A}$$

by the subgroup generated by the following three types of elements:

$$D_1(a) = \left([q] \otimes \delta a, \binom{q}{2}[1] \otimes \delta a, \binom{q}{3}[1] \otimes \delta a, 0, 0, 0\right),$$

$$D_2(a) = \left(0, [1] \otimes q\delta a, \binom{q}{2}[1] \otimes \delta a, [qa], \binom{q}{2}[a], 0\right),$$

$$D_3(a) = (0, 0, q[1] \otimes \delta a, 0, 0, 0).$$

Proof. One has

$$\begin{aligned}
z^2R \otimes_R \Omega_R &\cong z^2R \otimes_R (B \otimes \Omega_A \oplus A \otimes \Omega_B) \\
&\cong z^2R \otimes_R (R \otimes_A \Omega_A \oplus R \otimes_B \Omega_B) \\
&\cong z^2R \otimes_A \Omega_A \oplus z^2R \otimes_B \Omega_B \\
&\cong \left(\frac{qA}{p^{M+e}A} \oplus \bigoplus_{n=2}^{q-1} \frac{A}{p^{\epsilon(n)}A}\right) \otimes_A \Omega_A \\
&\qquad \oplus \left(\frac{qA}{p^{M+e}A} \oplus \bigoplus_{n=2}^{q-1} \frac{A}{p^{\epsilon(n)}A}\right) \otimes \frac{\mathbf{Z}}{q\mathbf{Z}} \\
&\cong \frac{qA}{p^{M+e}A} \otimes_A \Omega_A \oplus \bigoplus_{n=2}^{q-1} \left(\frac{A}{p^{\epsilon(n)}A} \otimes_A \Omega_A\right) \\
&\qquad \oplus \frac{qA}{p^{\min(2e,\, M+e)}A} \oplus \bigoplus_{n=2}^{q-1} \left(\frac{A}{p^{\min(e,\, \epsilon(n))}A}\right).
\end{aligned}$$

Under this isomorphism the element

$$qz \otimes \alpha_1 + z^2 \otimes \alpha_2 + \cdots + z^{q-1} \otimes \alpha_{q-1} + qz \otimes a_2 \delta z + z^2 \otimes a_3 \delta z + \cdots + z^{q-1} \otimes a_q \delta z$$

corresponds to the element

$$\left([q] \otimes \alpha_1, [1] \otimes \alpha_2, \dots, [1] \otimes \alpha_{q-1}, [qa_2], [a_3], \dots, [a_q]\right).$$

We must compute the image of Δ: $z^2R \otimes_R z^2R \to z^2R \otimes_R \Omega_R$. The group $z^2R \otimes_R z^2R$ is additively generated by the elements $z^i a_1 \otimes z^j a_2 = z^2 \otimes z^{i+j-2} a_1 a_2$ for $a_1, a_2 \in A$ and $i, j \geq 2$. Moreover z^i is a combination of $z^2, \dots, z^q$ for $i \geq q+1$. Therefore it suffices to consider

$$\Delta(z^2 \otimes z^i a) = z^{i+2} \otimes \delta a + z^{i+1} \otimes (i+2) a \delta z,$$

for $2 \leq i \leq q$.

1. If $i \leq q-3$ then this expression is already in the desired form. Since $q\delta z = 0$ this implies that $z^k \otimes p^{e-f} \delta a$ is in the image of Δ if p^f divides k and $4 \leq k < q$.

2. If $i = q-2$ then the first term is

$$z^q \otimes \delta a = -\sum_{k=1}^{q-1} z^k \otimes \binom{q}{k} \delta a \equiv -qz \otimes \delta a - z^2 \otimes \binom{q}{2} \delta a - z^3 \otimes \binom{q}{3} \delta a,$$

modulo terms of type (1). The second term is $z^{q-1} \otimes qa\delta z = 0$.

3. If $i = q-1$ then one gets

$$\begin{aligned} z^{q+1} \otimes \delta a + z^q \otimes a\delta z &= -z^2 \otimes q\delta a - z^3 \otimes \binom{q}{2} \delta a - \sum_{k=3}^{q-1} z^{k+1} \otimes \binom{q}{k} \delta a \\ &\quad - qz \otimes a\delta z - z^2 \otimes \binom{q}{2} a\delta z - \sum_{k=3}^{q-1} z^k \otimes \binom{q}{k} a\delta z. \end{aligned}$$

However the two sums are together equivalent to

$$\sum_{k=3}^{q-1} z^k \otimes (k+1)\binom{q}{k} a\delta z - \sum_{k=3}^{q-1} z^k \otimes \binom{q}{k} a\delta z \equiv 0,$$

modulo terms of type (1) and (2).

4. If $i = q$ then one gets

$$z^{q+2} \otimes \delta a + z^{q+1} \otimes 2a\delta z = -z^3 \otimes q\delta a - \sum_{k=2}^{q-1} z^{k+2} \otimes \binom{q}{k} \delta a - z^2 \otimes 2qa\delta z - \sum_{k=2}^{q-1} z^{k+1} \otimes 2\binom{q}{k} a\delta z.$$

However the two sums are together equivalent to

$$\sum_{k=2}^{q-1} z^{k+1} \otimes (k+2)\binom{q}{k} a\delta z - \sum_{k=2}^{q-1} z^{k+1} \otimes 2\binom{q}{k} a\delta z \equiv 0,$$

modulo terms of type (1), (2), and (3).

Case (1) explains the definition of CS_n, and cases (2), (3), and (4) the definitions of D_1, D_2, D_3, respectively. ∎

We indicate the necessary changes for $q < 4$. If $q = 3$ then CS_{low} is the quotient of

$$MS_{\mathrm{low}} = \left(\frac{3A}{3^{\epsilon(1)}A} \otimes_A \Omega_A\right) \oplus \left(\frac{A}{3^{\epsilon(2)}A} \otimes_A \Omega_A\right) \oplus \frac{3A}{3^{\min(2,\,\epsilon(1))}A} \oplus \frac{A}{3A}$$

by the subgroup generated by the elements

$$D_2(a) = ([3] \otimes 3\delta a, [1] \otimes 6\delta a, [-3a], 0),$$
$$D_3(a) = (-[3] \otimes 6\delta a, -[1] \otimes 9\delta a, 0, 0).$$

The cases (1) and (2) do not apply, and in cases (3) and (4) one expresses z^3, z^4, z^5 in terms of z and z^2. This subgroup is also generated by the elements $D_2'(a) = ([3] \otimes 3\delta a, 0, 0, 0)$ and $D_3'(a) = (0, [1] \otimes 3\delta a, [3a], 0)$.

If $q = 2$ then CS_{low} is the quotient of

$$MS_{\text{low}} = \left(\frac{2A}{2^{\epsilon(1)}A} \otimes_A \Omega_A\right) \oplus \frac{2A}{2^{\min(2,\,\epsilon(1))}A}$$

by the subgroup generated by the elements

$$D_3(a) = (-[2] \otimes 4\delta a, [0]).$$

The cases (1), (2) and (3) do not apply, and in case (4) one expresses z^2, z^3 in terms of z.

5. THE KERNEL OF j_{GL}

We write $A' = \{a \in A;\ 2a \in qA\}$. We introduce the following maps

$$\begin{aligned}
\kappa^G &: \Omega_A \to K_{2,L}(A[G]/I_M, (z)) & \kappa^G(\alpha) &= z \otimes \alpha,\\
\kappa^G_\Omega &: \Omega_A \to K_{2,L}(A[G]/I_M, (z^2)) & \kappa^G_\Omega(\alpha) &= qz \otimes \alpha,\\
\kappa^G_2 &: A' \to K_{2,L}(A[G]/I_M, (z^2)) & \kappa^G_2(a) &= z^2 \otimes \delta a + 2az \otimes \delta z,\\
\kappa^G_3 &: A \to K_{2,L}(A[G]/I_M, (z^2)) & \kappa^G_3(a) &= z^3 \otimes \delta a + 3az^2 \otimes \delta z.
\end{aligned}$$

Then $j_{GL}(\kappa^G_\Omega(\alpha)) = \kappa^G(q\alpha)$. Let κ^G_1 be the restriction of κ^G_Ω to ${}^q\Omega_A$.

PROPOSITION 6. $\ker(j_{GL}) = \operatorname{im}(\kappa^G_{\text{tot}})$, *where* $\kappa^G_{\text{tot}} = \kappa^G_1 + \kappa^G_2 + \kappa^G_3$.

Proof. We assume that $q > 3$. First some notation. The maps κ^G factorize through maps κ with values in MS_{low} defined by

$$\begin{aligned}
\kappa_1(\alpha) &= ([q] \otimes \alpha, 0, 0, 0, 0, 0),\\
\kappa_2(a) &= (0, [1] \otimes \delta a, 0, [2a], 0, 0),\\
\kappa_3(a) &= (0, 0, [1] \otimes \delta a, 0, [3a], 0).
\end{aligned}$$

Combining Propositions 4 and 5 we find that $j_{GL} = j^G_{\text{low}} \oplus j^G_3 \oplus \cdots \oplus j^G_{q-1}$, where the map $j^G_{\text{low}}\colon CS_{\text{low}} \to CF_{\text{low}} \oplus CF_2 \oplus CF_3$ is given by

$$\begin{aligned}
&j^G_{\text{low}}\big([[q] \otimes \alpha_1, [1] \otimes \alpha_2, [1] \otimes \alpha_3, [qa_2], [a_3], [a_q]]\big)\\
&\qquad = \big([[1] \otimes q\alpha_1, [a_q]], [[1] \otimes \alpha_2, [qa_2]], [[1] \otimes \alpha_3, [a_3]]\big)
\end{aligned}$$

and where $j_n^G\colon CS_n \to CF_n$ is the obvious map and therefore an isomorphism for $n \geq 4$. If $w = ([q] \otimes \alpha_1, [1] \otimes \alpha_2, [1] \otimes \alpha_3, [qa_2], [a_3], [a_q])$ is such that its class is in $\ker(j_{\text{low}}^G)$ then one has

$$\begin{aligned} [1] \otimes q\alpha_1 &= [1] \otimes q\delta b_1, \qquad & [qa_2] &= [2b_2], \\ [1] \otimes \alpha_2 &= [1] \otimes \delta b_2, & [a_3] &= [3b_3], \\ [1] \otimes \alpha_3 &= [1] \otimes \delta b_3, & [aq] &= 0. \end{aligned}$$

Thus there are $\theta_1, \theta_2, \theta_3 \in \Omega_A$ and $g_2, g_3, g_q \in A$ such that

$$\begin{aligned} q\alpha_1 &= q\delta b_1 + p^{\epsilon(1)}\theta_1, \qquad & qa_2 &= 2b_2 + qg_2, \\ \alpha_2 &= \delta b_2 + p^{\epsilon(2)}\theta_2, & a_3 &= 3b_3 + p^{\beta(2)}g_3, \\ \alpha_3 &= \delta b_3 + p^{\epsilon(3)}\theta_3, & a_q &= p^{\beta(q-1)}g_q. \end{aligned}$$

From the first equation one finds $\alpha_1 = \delta b_1 + p^{\epsilon(1)-e}\theta_1 + \eta$ for certain $\eta \in {}^q\Omega_A$. From the fourth equation one finds that $b_2 \in A'$; in fact $2b_2 = qc$, where $c = a_2 - g_2$. Since the terms involving the $\theta_1, \theta_2, \theta_3$ and the g_3, g_q represent zero one has

$$w = ([q] \otimes \delta b_1 + [q] \otimes \eta, [1] \otimes \delta b_2, [1] \otimes \delta b_3, [qa_2], [3b_3], 0).$$

Thus one finds

$$\begin{aligned} &w - \kappa_1(\eta) - \kappa_3(b_3) = ([q] \otimes \delta b_1, [1] \otimes \delta b_2, 0, [qa_2], 0, 0) \\ &w - \kappa_1(\eta) - \kappa_3(b_3) - D_1(b_1) \\ &\qquad = \left(0, [1] \otimes \delta b_2 - \binom{q}{2}[1] \otimes \delta b_1, -\binom{q}{3}[1] \otimes \delta b_1, [qa_2], 0, 0\right) \\ &w - \kappa_1(\eta) - \kappa_3(b_3) - D_1(b_1) + \binom{q}{3}\kappa_3(b_1) \\ &\qquad = \left(0, [1] \otimes \delta b_2 - \binom{q}{2}[1] \otimes \delta b_1, 0, [qa_2], 3\binom{q}{3}[b_1], 0\right). \end{aligned}$$

In order to write this in the form $\kappa_2(f_2)+\kappa_3(f_3)+D_2(h_2)+D_3(h_3)$ it is sufficient to have

$$b_2-\binom{q}{2}b_1=f_2+qh_2,$$

$$0=f_3+\binom{q}{2}h_2+qh_3,$$

$$qa_2=2f_2+qh_2,$$

$$3\binom{q}{3}b_1\equiv 3f_3+\binom{q}{2}h_2 \qquad \text{modulo } p^e.$$

One can satisfy these conditions by choosing

$$\begin{aligned} f_2 &= qa_2-b_2+\binom{q}{2}b_1,\\ h_2 &= -a_2+c-(q-1)b_1,\\ f_3 &= -\binom{q}{2}h_2,\\ h_3 &= 0. \end{aligned}$$

For $q=3$ one has $z^3=-3z^2-3z$ and thus

$$\begin{aligned} \kappa_1(\alpha) &= ([3]\otimes\alpha,0,0,0),\\ \kappa_2(a) &= (0,[1]\otimes\delta a,[2a],0),\\ \kappa_3(a) &= (-[3]\otimes\delta a,-[1]\otimes 3\delta a,[0],[0]). \end{aligned}$$

If $w=([3]\otimes\alpha_1,[1]\otimes\alpha_2,[3a_2],[a_3])$ has its class in $\ker(j_{GL})$ then one can write

$$w-\kappa_1(\eta)=([3]\otimes\delta b_1,[1]\otimes\delta b_2,[3a_2],[0])$$

for certain $\eta\in{}^3\Omega_A$, $b_1,b_2\in A$; and $2b_2=3c$ for certain $c\in A$. One can write this as $\kappa_2(f_2)+\kappa_3(f_3)+D_2'(h_2)+D_3'(h_3)$ by choosing

$$\begin{aligned} f_2 &= 3a_2-b_2+3b_1,\\ f_3 &= -b_1,\\ h_2 &= 0,\\ h_3 &= c-2b_1-a_2. \end{aligned}$$

For $q = 2$ one has $z^2 = -2z$ and thus

$$\kappa_1(\alpha) = ([2] \otimes \alpha, 0),$$
$$\kappa_2(a) = (-[2] \otimes \delta a, [2a]),$$
$$\kappa_3(a) = ([2] \otimes 2\delta a, [-6a]).$$

If $w = ([2] \otimes \alpha_1, [2a_2])$ has its class in $\ker(j_{GL})$ then one can write

$$w - \kappa_1(\eta) = ([2] \otimes \delta b_1, [2a_2])$$

for certain $\eta \in {}^2\Omega_A$, $b_1 \in A$. One can write this as $\kappa_2(f_2) + \kappa_3(f_3) + D_3(h_3)$ by choosing

$$f_2 = -2a_2 - 3b_1,$$
$$f_3 = -a_2 - b_1,$$
$$h_3 = 0. \quad \blacksquare$$

We list some elements in the kernel of κ^G_{tot}.

PROPOSITION 7. *One has*

$$\kappa_3^G(qa) = 0,$$
$$\kappa_2^G(q^2a) = 0,$$
$$\kappa_2^G\big(q\psi^{p^M}(a)\big) = 0.$$

For $p = 2$ the third statement can be strengthened to

$$\kappa_2^G\big(q'\psi^{2^M}(a)\big) + \binom{q}{2}\kappa_3^G\big(\psi^{2^M}(a)\big) = 0.$$

Proof. We leave the cases $q \le 3$ to the reader. If $q > 3$ then one has

$$\kappa_2(qa) = D_3(a),$$
$$\kappa_2(q^2a) = qD_2(a) - \binom{q}{2}D_3(a),$$
$$\kappa_2\big(q\psi^{p^M}(a)\big) = D_2\big(2\psi^{p^M}(a)\big)$$

and for $p = 2$ one has

$$\kappa_2\big(q'\psi^{2^M}(a)\big) + \binom{q}{2}\kappa_3\big(\psi^{2^M}(a)\big) = D_2\big(\psi^{2^M}(a)\big).$$

Here we have used the fact $q'\delta\psi^{2^M}(a) = q'2^M\phi^{2^M}\delta a$ is a multiple of $2^{\epsilon(2)}$. ∎

In fact $\kappa_2^G(qb) = 0$ if b is of the form $\sum_{i=0}^{M} p^i a_i^{p^{M-i}}$ with $a_i \in A$, and this is the case for $b = \psi^{p^M}(a)$.

6. THE MAP j_{GM}

Let $N \geq M + e$, with $M \geq 0$. Let J_N be the ideal of $A[x]$ generated by the elements $p^{N-i}x^{p^i}$ for $0 \leq i \leq N$. The map $\pi: A[x] \to A[G]$ induces a map $A[x]/J_N \to A[G]/I_M$.

According to Sect. 2 we must determine the image of $\ker(j_{GM})$ under L_1. To determine $\ker(j_{GM})$ we shall use the following diagram with exact rows

$$\begin{tikzcd}
K_3(A[x]/(x^2, qx),(x)) \arrow[r, "\partial_X"] \arrow[d, "\pi_1"', "\cong"] & K_2(A[x]/J_N,(x^2,qx)) \arrow[r, "j_{XM}"] \arrow[d, "\pi_2"'] & K_2(A[x]/J_N,(x)) \arrow[d, "\pi_3"'] \\
K_3(A[G]/(z^2),(z)) \arrow[r, "\partial_G"] & K_2(A[G]/I_M,(z^2)) \arrow[r, "j_{GM}"] & K_2(A[G]/I_M,(z)).
\end{tikzcd}$$

From this diagram it follows that $\ker(j_{GM}) = \pi_2(\ker(j_{XM}))$. To compute the kernel of j_{XM} we use the following diagram

$$\begin{tikzcd}
K_2(A[x]/J_N,(x^2,qx)) \arrow[r, "j_{XM}"] \arrow[d, "L_5"', "\cong"] & K_2(A[x]/J_N,(x)) \arrow[d, "L_6"', "\cong"] \\
K_{2,L}(A[x]/J_N,(x^2,qx)) \arrow[r, "j_{XL}"] & K_{2,L}(A[x]/J_N,(x)).
\end{tikzcd}$$

Here L_5 and L_6 are the logarithmic maps associated to the standard λ-ring structure on $A[x]$; since (x) is λ-nilpotent they are isomorphisms. Note that (x^2, qx) is a λ-ideal of $A[x]$, although $((x+1)^q - 1)$ is not. From this diagram it follows that $\ker(j_{XM})) = L_5^{-1}(\ker(j_{XL}))$.

Thus $L_1(\ker(j_{GM})) = L_1\pi_2 L_5^{-1}(\ker(j_{XL}))$. To compute the composition $L_1\pi_2 L_5^{-1}$ we use the following diagram

$$\begin{tikzcd}
K_2(A[x]/J_N,(x^2,qx)) \arrow[rr, "j_{XM}"] \arrow[dr, "\pi_2"] \arrow[dd, "L_7"'] & & K_2(A[x]/J_N,(x)) \arrow[dr, "\pi_3"] \arrow[dd, "L_8"' near start] & \\
& K_2(A[G]/I_M,(z^2)) \arrow[rr, "j_{GM}" near end] \arrow[dd, "L_1"' near start] & & K_2(A[G]/I_M,(z)) \arrow[dd, "L_2"'] \\
K_{2,L}(A[x]/J_N,(x^2,qx)) \arrow[rr, "j_{XL}" near end] \arrow[dr, "\pi_2"] & & K_{2,L}(A[x]/J_N,(x)) \arrow[dr, "\pi_3"] & \\
& K_{2,L}(A[G]/I_M,(z^2)) \arrow[rr, "j_{GL}"] & & K_{2,L}(A[G]/I_M,(z)).
\end{tikzcd}$$

Here L_7 and L_8 are the logarithmic maps associated to the "exotic" λ-ring structure on $A[x]/J_N$, which has $\psi^l(x) = 0$ for primes $l \neq p$ and $\psi^p(x) = (x+1)^p - 1$; see Sect. 13 for the details. For this λ-ring structure the ideals (x) and (x^2, qx) are θ-nilpotent and ψ-nilpotent, but not λ-nilpotent; thus L_7 and L_8 are defined but not necessarily isomorphisms.

The diagram commutes since the map $\pi\colon A[x]/J_N \to A[G]/I_M$ is a λ-map, and the logarithmic map is natural for λ-maps. In particular one has $L_1\pi_2 = \pi_2 L_7$, and thus $L_1(\ker(j_{GM})) = \pi_2 L_7 (L_5)^{-1} \ker(j_{XL})$. In the next sections we shall compute $\ker(j_{XL})$ and determine

$$L_7(L_5)^{-1}\colon \ker(j_{XL}) \to \ker(j_{XL}).$$

7. THE KERNEL OF j_{XL}

We write $\gamma(n) = N - i$ for $p^i \leq n < p^{i+1}$ and $0 \leq i \leq N$. We write $Q = p^N$.

PROPOSITION 8. *Let $R = A[x]/J_N$. Then*

$$K_{2,L}(R, xR) = \operatorname{cok}(\Delta\colon xR \otimes_R xR \to xR \otimes_R \Omega_R)$$

is isomorphic to a direct sum $XF_1 \oplus XF_2 \oplus \cdots \oplus XF_Q$, *where*

$$XF_1 = \frac{A}{QA} \otimes_A \Omega_A,$$

$$XF_n = \frac{\left(\dfrac{A}{p^{\gamma(n)}A} \otimes_A \Omega_A\right) \oplus \dfrac{A}{p^{\gamma(n-1)}A}}{\{([1] \otimes \delta a, [na])\}} \qquad \textit{for } n = 2, \dots, Q-1,$$

$$XF_Q = \frac{A}{pA}.$$

Proof. Similar to the proof of the next proposition. ∎

PROPOSITION 9. *If* $Q > 3$ *then*

$$K_{2,L}(R, x^2R + qxR)$$

is isomorphic to a direct sum $XS_1 \oplus XS_2 \oplus \cdots \oplus XS_Q$, *where*

$$XS_1 = \frac{qA}{QA} \otimes_A \Omega_A,$$

$$XS_2 = \frac{\left(\dfrac{A}{p^{\gamma(2)}A} \otimes_A \Omega_A\right) \oplus \dfrac{qA}{p^{\gamma(1)}A}}{\{([1] \otimes q^2\delta a, [2q^2a])\}},$$

$$XS_3 = \frac{\left(\dfrac{A}{p^{\gamma(3)}A} \otimes_A \Omega_A\right) \oplus \dfrac{A}{p^{\gamma(2)}A}}{\{([1] \otimes q\delta a, [3qa])\}},$$

$$XS_n = \frac{\left(\dfrac{A}{p^{\gamma(n)}A} \otimes_A \Omega_A\right) \oplus \dfrac{A}{p^{\gamma(n-1)}A}}{\{([1] \otimes \delta a, na)\}} \qquad \textit{for } n = 4, \dots, Q-1,$$

$$XS_Q = \frac{A}{pA}.$$

Proof. With $C = \mathbf{Z}[x]/J_N$ one has

$$\begin{aligned}
\left(x^2R + qxR\right) \otimes_R \Omega_R &\cong \left(x^2R + qxR\right) \otimes_R (C \otimes \Omega_A \oplus A \otimes \Omega_C) \\
&\cong \left(x^2R + qxR\right) \otimes_R (R \otimes_A \Omega_A \oplus R \otimes_C \Omega_C) \\
&\cong \left(x^2R + qxR\right) \otimes_A \Omega_A \oplus \left(x^2R + qxR\right) \otimes_C \Omega_C \\
&\cong \left(\frac{qA}{QA} \oplus \bigoplus_{n=2}^{Q-1} \frac{A}{p^{\gamma(n)}A}\right) \otimes_A \Omega_A \\
&\quad \oplus \left(\frac{qA}{QA} \oplus \bigoplus_{n=2}^{Q-1} \frac{A}{p^{\gamma(n)}A}\right) \otimes \frac{\mathbf{Z}}{Q\mathbf{Z}} \\
&\cong \frac{qA}{QA} \otimes_A \Omega_A \oplus \bigoplus_{n=2}^{Q-1} \left(\frac{A}{p^{\gamma(n)}A} \otimes_A \Omega_A\right) \\
&\quad \oplus \frac{qA}{QA} \oplus \bigoplus_{n=2}^{Q-1} \left(\frac{A}{p^{\gamma(n)}A}\right).
\end{aligned}$$

Under this isomorphism the element

$$\begin{aligned}
&qx \otimes \alpha_1 + x^2 \otimes \alpha_2 + \cdots + x^{Q-1} \otimes \alpha_{Q-1} \\
&\qquad + qx \otimes a_2 \delta x + x^2 \otimes a_3 \delta x + \cdots + x^{Q-1} \otimes a_Q \delta x
\end{aligned}$$

corresponds to the element

$$\left([q] \otimes \alpha_1, [1] \otimes \alpha_2, \ldots, [1] \otimes \alpha_{Q-1}, [qa_2], [a_3], \ldots, [a_Q]\right).$$

We must compute the image of

$$\Delta\colon \left(x^2R + qxR\right) \otimes_R \left(x^2R + qxR\right) \to \left(x^2R + qxR\right) \otimes_R \Omega_R.$$

The group $x^2R + qxR$ is additively generated by the elements qxa and the elements $x^i a$ with $a \in A$ and $i \geq 2$. Therefore the group $(x^2R + qxR) \otimes_R (x^2R + qxR)$ is additively generated by the elements $qx \otimes qxa$, $qx \otimes x^i a$ and $x^2 \otimes x^i a$ for $a \in A$ and $i \geq 2$. Thus it suffices to consider

$$\begin{aligned}
\Delta(qx \otimes qxa) &= x^2 \otimes q^2 \delta a + qx \otimes 2qa\,\delta x, \\
\Delta\left(qx \otimes x^i a\right) &= x^{i+1} \otimes q\,\delta a + x^i \otimes (i+1) qa\,\delta x, \\
\Delta(x^2 \otimes x^i a) &= x^{i+2} \otimes \delta a + x^{i+1} \otimes (i+2) a\,\delta x. \quad \blacksquare
\end{aligned}$$

We introduce maps

$$\begin{aligned}
\kappa^X\colon \Omega_A &\to K_{2,L}(A[x]/J_N,(x)), && \kappa^X(\alpha)=x\otimes\alpha,\\
\kappa^X_\Omega\colon \Omega_A &\to K_{2,L}(A[x]/J_N,(x^2,qx)), && \kappa^X_\Omega(\alpha)=qx\otimes\alpha,\\
\kappa^X_2\colon A' &\to K_{2,L}(A[x]/J_N,(x^2,qx)), && \kappa^X_2(a)=x^2\otimes\delta a+2ax\otimes\delta x,\\
\kappa^X_3\colon A &\to K_{2,L}(A[x]/J_N,(x^2,qx)), && \kappa^X_3(a)=x^3\otimes\delta a+3ax^2\otimes\delta x.
\end{aligned}$$

Then $j_{XL}(\kappa^X_\Omega(\alpha))=\kappa^X(q\alpha)$. Let κ^X_1 be the restriction of κ^X_Ω to ${}^q\Omega_A$.

PROPOSITION 10. $\ker(j_{XL})=\operatorname{im}(\kappa^X_{\text{tot}})$, *where* $\kappa^X_{\text{tot}}=\kappa^X_1+\kappa^X_2+\kappa^X_3$.

Proof. First assume $Q>3$. Combining Propositions 8 and 9 we find that j_{XL} is the direct sum $j^X_1\oplus j^X_2\oplus\cdots j^X_Q$, where $j^X_n\colon XS_n\to XF_n$ is the obvious map for each n. In particular it is an isomorphism for $n\geq 4$. The behaviour for $n=1,2,3$ is easily analyzed in the same spirit as in Proposition 6.

For $Q=q=3$ one has

$$XF_1=\frac{A}{3A}\otimes_A\Omega_A,\qquad XF_2=\frac{\left(\frac{A}{3A}\otimes_A\Omega_A\right)\oplus\frac{A}{3A}}{\{([1]\otimes\delta a,[2a])\}},\qquad XF_3=\frac{A}{3A},$$
$$XS_1=0,\qquad XS_2=\frac{A}{3A}\otimes_A\Omega_A,\qquad XS_3=\frac{A}{3A}.$$

The map j_{XL} is injective and the maps κ^X_i vanish.

For $Q=q=2$ one has

$$XF_1=\frac{A}{2A}\otimes_A\Omega_A,\qquad XF_2=\frac{A}{2A},$$
$$XS_1=0,\qquad XS_2=0.$$

The map j_{XL} is injective and the maps κ^X_i vanish. ∎

From the definitions of κ^X_2 and κ^X_3 and from the structure of XS_2 and XS_3 it is clear that κ^X_2 vanishes on q^2A and κ^X_3 vanishes on qA. This is related to the analogous properties of the κ^G_i by the fact that $\pi_2\kappa^X_{\text{tot}}=\kappa^G_{\text{tot}}$.

The maps κ^M_i which are mentioned in Theorem 2 are now defined as $\pi_2(L_5)^{-1}\kappa^X_i$.

8. THE MAP Φ

As mentioned in Sect. 6 we must compute the map

$$L_7(L_5)^{-1}: \ker(j_{XL}) \to \ker(j_{XL}).$$

In Proposition 10 we have seen that κ_{tot}^X is a surjection onto $\ker(j_{XL})$. Thus it is sufficient to compute $L_7(L_5)^{-1}\kappa_{\mathrm{tot}}^X$. For κ_1^X this is done in this section, for κ_2^X and κ_3^X in the next section.

$$\begin{tikzcd}
{}^q\Omega_A \arrow[r, "\kappa_1^X"] \arrow[d, "\Phi"] & K_{2,L}(A[x]/J_N, (x^2, qx)) \arrow[d, "L_7(L_5)^{-1}"] \\
{}^q\Omega_A \arrow[r, "\kappa_1^X"] & K_{2,L}(A[x]/J_N, (x^2, qx)),
\end{tikzcd}$$

PROPOSITION 11. *There are $f_i \in \mathbf{Z}$ such that the following diagram commutes*:
where $\Phi = \sum_i f_i \phi^i$.

Proof. We may assume $Q > 3$. The map $L_7(L_5)^{-1}\kappa_\Omega^X$ is a natural additive map; we shall write ξ_n^S for its component in XS_n for $4 \le n \le Q$. According to Sect. 12 there must be $c_i^k \in \mathbf{Z}$ such that $\alpha \in \Omega_A$ maps to

$$\left[[q] \otimes \sum c_i^1 \phi^i(\alpha)\right] \oplus \left[[1] \otimes \sum c_i^2 \phi^i(\alpha), 0\right] \oplus \left[[1] \otimes \sum c_i^3 \phi^i(\alpha), 0\right]$$
$$\oplus\ \xi_4^S \oplus \cdots \oplus \xi_Q^S \in XS_1 \oplus XS_2 \oplus \cdots \oplus XS_Q.$$

If one applies j_{XL} to this one gets

$$\left[[1] \otimes \sum q c_i^1 \phi^i(\alpha)\right] \oplus \left[[1] \otimes \sum c_i^2 \phi^i(\alpha), 0\right] \oplus \left[[1] \otimes \sum c_i^3 \phi^i(\alpha), 0\right]$$
$$\oplus\ \xi_4^S \oplus \cdots \oplus \xi_Q^S \in XF_1 \oplus XF_2 \oplus \cdots \oplus XF_Q.$$

The map $L_8(L_6)^{-1}\kappa^X$ is also a natural additive map. We shall write ξ_n^F for its component in XF_n for $4 \le n \le Q$. Again there must be $d_k^i \in \mathbf{Z}$ such that $\alpha \in \Omega_A$ maps to

$$\left[[1]\otimes\sum d_i^1\phi^i(\alpha)\right]\oplus\left[[1]\otimes\sum d_i^2\phi^i(\alpha),0\right]\oplus\left[[1]\otimes\sum d_i^3\phi^i(\alpha),0\right]$$
$$\oplus\ \xi_4^F\oplus\cdots\oplus\xi_Q^F\in XF_1\oplus XF_2\oplus\cdots\oplus XF_Q.$$

Since $\kappa^X(q\alpha)=j_{XL}(\kappa_\Omega^X(\alpha))$ the value of this map on $q\alpha$ must be the same as the result of the former map on α. According to Sect. 12 this means that

$$\begin{aligned} qc_i^1 &\equiv qd_i^1 && \text{modulo } p^{\gamma(1)}\mathbf{Z},\\ c_i^2 &\equiv qd_i^2 && \text{modulo } p^{\gamma(2)}\mathbf{Z},\\ c_i^3 &\equiv qd_i^3 && \text{modulo } p^{\gamma(3)}\mathbf{Z},\\ \xi_n^F(\alpha) &= \xi_n^S(q\alpha) && \text{for } n=4,\dots,q-1. \end{aligned}$$

Thus if $\gamma(2)\geq e$ this says that $c_i^2\in q\mathbf{Z}$, so $q\alpha=0$ implies that $c_i^2\alpha=0$. If on the other hand $\gamma(2)\leq e$ this says that $c_i^2\in p^{\gamma(2)}\mathbf{Z}$, so $c_i^2\alpha$ represents zero in XS_2 anyway. Therefore if $q\alpha=0$ one has

$$\begin{aligned} L_7(L_5)^{-1}\kappa_\Omega^X(\alpha) &= \left[[q]\otimes\sum c_i^1\phi^i(\alpha)\right]\oplus 0\oplus\cdots\oplus 0\\ &= \kappa_\Omega^X\left(\sum c_i^1\phi^i(\alpha)\right). \end{aligned}$$

Thus one can take $f_i=c_i^1$. ∎

It is clear from the proof that the coefficients f_i can be determined by evaluating the XS_1-part of $L_7(L_5)^{-1}\kappa_\Omega$ on a well chosen example. For this we take the ring $A=\mathbf{Z}[t]$ with the λ-ring structure given by $\psi^i(t)=t^i$, and the element $\delta t\in\Omega_A$.

PROPOSITION 12. *Let $A=\mathbf{Z}[t]$. Let*

$$y=t^{-1}\left(1-(1-tx)^q\right)=\sum_{i=1}^{q}\binom{q}{i}(-1)^{i-1}t^{i-1}x^i\in A[x]/J_N.$$

Then $L_7\langle y,t\rangle=qx\otimes\delta t$ and the XS_1 component of $L_7\langle y,t\rangle$ is

$$qx\otimes\delta t-qx\otimes t^{p-1}\delta t=qx\otimes(1-\phi^p)(\delta t).$$

Proof. Let L denote L_5 or L_7. One has

$$\begin{aligned} L\langle y,t\rangle &= \sum_{m,k}\theta^m(y)\otimes\phi^m\delta\eta^k(y,t)=\sum_m\theta^m(y)\otimes\phi^m\delta t\\ &= \sum_m\theta^m(y)\otimes t^{m-1}\delta t=\sum_m t^{m-1}\theta^m(y)\otimes\delta t, \end{aligned}$$

since $\eta^k(y,t)=0$ for $k>1$. However $\theta=\sum_m\theta^m$ has the property $\theta(a)+$

$\theta(b) = \theta(a + b - ab)$, so

$$\sum_m t^m \theta^m(y) = \sum_m \theta^m(ty) = q \sum_m \theta^m(tx) = q \sum_m t^m \theta^m(x).$$

Therefore

$$L\langle y, t\rangle = \sum q\theta^m(x) \otimes t^{m-1}\delta t.$$

In computing $L_5\langle y, t\rangle$ one must use the standard λ-ring structure on $A[x]/J_N$. Then $\theta^m(x) = 0$ for $m > 1$ and the only nonvanishing term is the one for $m = 1$.

In computing $L_7\langle y, t\rangle$ one must use the exotic λ-ring structure. If one is only interested in the XS_1-component then terms of degree > 1 in x can be neglected. Therefore one gets the following contributions

$$q\theta^1(x) \otimes \delta t = qx \otimes \delta t,$$
$$q\theta^p(x) \otimes t^{p-1}\delta t \equiv -qx \otimes t^{p-1}\delta t,$$

using the identities in Sect. 13. ∎

This proposition shows that one can take $\Phi = 1 - \phi^p$.

9. THE MAP Ψ

In this section we evaluate the compositions $L_7(L_5)^{-1}\kappa_2^X$ and $L_7(L_5)^{-1}\kappa_3^X$.

PROPOSITION 13. *The formulas*

$$\kappa_2^{XM}(a) = \langle x^2, a\rangle + \langle 2ax - a^2x^3, x\rangle,$$
$$\kappa_3^{XM}(a) = \langle x^3, a\rangle + 3\langle x^2a, x\rangle$$

define homomorphisms $\kappa_2^{XM}\colon A' \to \ker(j_{XM})$ *and* $\kappa_3^{XM}\colon A \to \ker(j_{XM})$.

Proof. The second formula clearly defines a map

$$\kappa_3^{XM}\colon A[x] \to K_2\big(A[x]/J_N, (x^2, qx)\big).$$

Computing in $K_2(A[x]/J_N, (x))$ one finds

$$\langle x^3, a\rangle = \langle x, x^2a\rangle - \langle xa, x^2\rangle = -\langle x^2a, x\rangle - 2\langle x^2a, x\rangle,$$

so the map lands in $\ker(j_{XM})$. For every $b \in A[x]$ one has

$$\begin{aligned}\langle x^3, bx\rangle &= \langle x^2, bx^2\rangle - \langle bx^3, x\rangle = -\langle bx^2, x^2\rangle - \langle bx^3, x\rangle \\ &= -2\langle bx^3, x\rangle - \langle bx^3, x\rangle,\end{aligned}$$

so the map vanishes on $xA[x]$. Finally one has

$$\kappa_3^{XM}(a_1 + a_2) - \kappa_3^{XM}(a_1) - \kappa_3^{XM}(a_2) = \kappa_3^{XM}(bx^3) = 0$$

if $b \in A[x]$ is chosen such that

$$\big(1 - (a_1 + a_2)x^3\big) = \big(1 - a_1x^3\big)\big(1 - a_2x^3\big)(1 - bx^6)$$

in $A[x]/x^{Q+3}$. Thus κ_3^{XM} is a homomorphism.

In the same way one checks that κ_2^{XM} defines a map $A'[x] \to \ker(j_{XM})$ and that $\kappa_{XM}^2(bx) = \kappa_{XM}^3(b)$ so that $\kappa_{XM}^2(bx^2) = 0$. Thus again

$$\kappa_2^{XM}(a_1 + a_2) - \kappa_2^{XM}(a_1) - \kappa_2^{XM}(a_2) = \kappa_2^{XM}(bx^2) = 0$$

if $b \in A[x]$ is chosen such that

$$\big(1 - (a_1 + a_2)x^2\big) = \big(1 - a_1x^2\big)\big(1 - a_2x^2\big)(1 - bx^4)$$

in $A[x]/x^{Q+2}$. ∎

PROPOSITION 14. *One has*

$$\begin{aligned}L_5\kappa_3^{XM}(a) &= \kappa_3^X(a), \\ L_7\kappa_3^{XM}(a) &= \kappa_3^X(a - p\psi^p(a)).\end{aligned}$$

Proof. Consider the exotic λ-structure on $R = A[x]/J_N$. One has

$$\begin{aligned}L_7\kappa_3^{XM}(t) = &\sum_{m,k} \theta^m(x^3) \otimes \phi^m\delta\eta^k(x^3, a) \\ &+ 3\sum_{m,k} \theta^m(ax^2) \otimes \phi^m\delta\eta^k(ax^2, x).\end{aligned}$$

But $L_7\kappa_3^{XM}(a)$ is in $\ker(j_{XL})$ by naturality, so its components in XS_n vanish for $n \geq 4$. Thus any term of degree ≥ 4 in x can be neglected.

The first sum can be evaluated using the following formulas from Sect. 13:

$$\begin{aligned} \theta^m(x^3) &\in x^3R && \text{for all } m, \\ \theta^m(x^3) &\in x^4R && \text{unless } m = 1, p, \\ \theta^1(x^3) &= x^3, && \\ \theta^p(x^3) &= -p^2x^3 && \text{modulo } x^4R, \\ \eta^k(x^3, a) &\in x^3R && \text{unless } k = 1, \\ \eta^1(x^3, a) &= a. && \end{aligned}$$

The second sum can be evaluated using the following formulas

$$\begin{aligned} \theta^m(x^2a) &\in x^2R && \text{for all } m, \\ \theta^m(x^2a) &\in x^3R && \text{unless } m = 1, p, \\ \theta^1(x^2a) &= x^2a, && \\ \theta^p(x^2a) &= -px^2\psi^p(a) && \text{modulo } x^3R, \\ \eta^k(x^2a, x) &\in xR && \text{for all } k, \\ \eta^k(x^2a, x) &\in x^2R && \text{unless } k = 1, \\ \eta^1(x^2a, x) &= x, && \\ \phi^p(\delta x) &= \delta x && \text{modulo } xR\delta x. \end{aligned}$$

Altogether one gets

$$\begin{aligned} L_7\kappa_3^{XM}(a) &= x^3 \otimes \delta a + 3x^2a \otimes \delta x - p^2x^3 \otimes \phi^p(\delta a) \\ &\qquad - 3px^2\psi^p(a) \otimes \delta a \\ &= x^3 \otimes \delta a + 3x^2a \otimes \delta x - px^3 \otimes \delta\psi^p(a) - 3px^2\psi^p(a) \otimes \delta a \\ &= \kappa_3^X(a - p\psi^p(a)). \end{aligned}$$

The computation for L_5 and the standard λ-structure is similar. ∎

PROPOSITION 15. *One has*

$$\begin{aligned} L_5\kappa_2^{XM}(a) &= \kappa_2^X(a), \\ L_7\kappa_2^{XM}(a) &= \kappa_2^X(a - \psi^p(a)) + \kappa_3^X(\psi^p(a)). \end{aligned}$$

Proof. Again we concentrate on the exotic λ-structure on R, and again we may neglect terms of degree ≥ 4 in x. However

$$\langle 2ax - a^2x^3, x\rangle = \langle 2ax, x\rangle + \langle bx^3, x\rangle,$$

where $b = -a^2\sum_i(2ax^2)^i$ gives rise to only such terms. Therefore it suffices to compute L_7 of $\langle x^2, a\rangle + \langle 2ax, x\rangle$.

First we consider $L_7(\langle x^2, a\rangle)$. By Sect. 13 we have

$$\begin{aligned}
\theta^m(x^2) &\in x^2R && \text{for all } m,\\
\theta^m(x^2) &\in x^4R && \text{unless } m = 1, p,\\
\theta^1(x^2) &= x^2,\\
\theta^p(x^2) &= -px^2 - p(p-1)x^3 && \text{modulo } x^4R,\\
\eta^k(x^2, a) &\in x^2R && \text{unless } k = 1,\\
\eta^1(x^2, a) &= a.
\end{aligned}$$

Next we consider $L_7(\langle 2ax, x\rangle)$ for $p > 2$. We have

$$\begin{aligned}
\theta^m(2ax) &\in xR && \text{for all } m,\\
\theta^m(2ax) &\in x^3R && \text{unless } m = 1, 2, p, 2p,\\
\theta^1(2ax) &= 2ax,\\
\theta^2(2ax) &\in x^2R,\\
\phi^2\delta(xR) &\subseteq \psi^2(x)\Omega_R + R\phi^2(\delta x)\\
&\subseteq x^2\Omega_R + xR\delta x,\\
\theta^p(2ax) &= \big(-2x - (p-1)x^2\big)\psi^p(a) && \text{modulo } x^3R,\\
\eta^k(2ax, x) &\in xR && \text{for all } k,\\
\eta^k(2ax, x) &\in x^3R && \text{unless } k = 1, 2,\\
\eta^1(2ax, x) &= x,\\
\eta^2(2ax, x) &= 2ax\theta^2(x) = ax^3,\\
\phi^p(\delta x) &= (1 + (p-1)x)\delta x && \text{modulo } x^2R\delta x.
\end{aligned}$$

Altogether one gets for $p > 2$:

$$\begin{aligned}
L_7\kappa_2^{XM}(a) &= x^2 \otimes \delta a + \bigl(-px^2 - p(p-1)x^3\bigr) \otimes \phi^p(\delta a) \\
&\quad + 2ax \otimes \delta x \\
&\quad + \bigl(-2x - (p-1)x^2\bigr)\psi^p(a) \otimes \bigl(1 + (p-1)x\bigr)\delta x \\
&= x^2 \otimes \delta a + 2ax \otimes \delta x \\
&\quad - x^2 \otimes \delta\psi^p(a) - 2\psi^p(a)x \otimes \delta x \\
&\quad - (p-1)x^3 \otimes \delta\psi^p(a) - 3(p-1)\psi^p(a)x^2 \otimes \delta x \\
&= \kappa_2^X(a) - \kappa_2^X(\psi^p(a)) - (p-1)\kappa_3^X(\psi^p(a)).
\end{aligned}$$

Here the κ_3^X term vanishes because $a \in A' = qA$ where κ_3^X vanishes.

Next we consider $L_7(\langle 2ax, x\rangle)$ for $p = 2$. We have

$$\begin{aligned}
\theta^m(2ax) &\in xR && \text{for all } m, \\
\theta^m(2ax) &\in x^3R && \text{unless } m = 1, 2, 4, \\
\theta^1(2ax) &= 2ax, \\
\theta^2(2ax) &= \bigl(2a^2 - \psi^2(a)\bigr)x^2 - 2\psi^2(a)x, \\
\theta^4(2ax) &= 4a^4x^4 - \psi^2(a^2x^2) \\
&= -4\psi^2(a)^2x^2 && \text{modulo } x^3R, \\
\eta^k(2ax, x) &\in xR && \text{for all } k, \\
\eta^k(2ax, x) &\in x^3R && \text{unless } k = 1, 2, \\
\eta^1(2ax, x) &= x, \\
\eta^2(2ax, x) &= 2ax\theta^2(x) = -2ax^2, \\
\psi^2(x) &= x^2 + 2x, \\
\phi^2(\delta x) &= (1 + x)\delta x.
\end{aligned}$$

Altogether one gets for $p = 2$

$$\begin{aligned}
L_7\kappa_2^{XM}(a) &= x^2 \otimes \delta a + (-2x^2 - 2x^3) \otimes \phi^2\delta a \\
&\quad + 2ax \otimes \delta x + 2ax \otimes \delta(-2ax^2) \\
&\quad + \bigl((2a^2 - \psi^2(a))x^2 - 2\psi^2(a)x\bigr) \otimes \phi^2\delta x \\
&\quad + \bigl(-2\psi^2(a)x\bigr) \otimes \phi^2\delta(-2ax^2) \\
&\quad - 4\psi^2(a)^2x^2 \otimes \phi^4\delta x
\end{aligned}$$

which can be evaluated and rearranged to

$$\begin{aligned}
&x^2 \otimes \delta a + 2ax \otimes \delta x \\
&\quad - x^2 \otimes \delta\psi^2(a) - 2\psi^2(a)x \otimes \delta x \\
&\quad - x^3 \otimes \delta\psi^2(a) - 3\psi^2(a)x^2 \otimes \delta x \\
&\quad - 2x^3 \otimes \delta(a^2) - 6a^2x^2 \otimes \delta x \\
&\quad + 4x^4 \otimes \delta\psi^2(a)^2 + 12\psi^2(a)^2x^2 \otimes \delta x \\
&\qquad\quad = \kappa_2^X\big(a - \psi^2(a)\big) + \kappa_3^X\big(-\psi^2(a) - 2a^2 + 4\psi^2(a)^2\big).
\end{aligned}$$

Here the part $\kappa_3^X(-2a^2 + 4\psi^2(a)^2)$ vanishes because $2a \in qA$, where κ_3^X vanishes. ∎

From the last two propositions it follows that one can take

$$\Psi = \begin{pmatrix} 1 - \psi^p & 0 \\ \psi^p & 1 - p\psi^p \end{pmatrix}.$$

Finally we prove a property of Ψ mentioned in Theorem 2.

PROPOSITION 16. *The map Ψ is surjective modulo the kernel of*

$$\kappa_2^G + \kappa_3^G = \pi_2\big(\kappa_2^X + \kappa_3^X\big).$$

Proof. Let $(a', a) \in A' \oplus A$. If $p > 2$ then $a' = qa_1$ for some $a_1 \in A$. Thus we can write

$$a' = (1 - \psi^p)(b_1) + \psi^{p^M}(qb_2),$$
$$a = \psi^p(b_1) + (1 - p\psi^p)(b_3) + qb_4$$

by choosing

$$b_1 = \sum_{i=0}^{M-1} \psi^{p^i}(qa_1), \qquad b_2 = a_1,$$

$$b_3 = \sum_{i=0}^{e-1} p^i\psi^{p^i}(c), \qquad b_4 = \psi^{p^e}(c), \qquad c = a - \psi^p(b_1).$$

If $p = 2$ then $a' = q'a_1 + a_2$ for some $a_1, a_2 \in A$ with $2a_2 = 0$. Thus we can write

$$a' = (1 - \psi^p)(b_1) + \psi^{p^M}(q'b_2),$$
$$a = \psi^p(b_1) + (1 - p\psi^p)(b_3) + \binom{q}{2}\psi^{p^M}(b_2) + qb_4$$

by choosing

$$b_1 = a_2 + \sum_{i=0}^{M-1} \psi^{p^i}(q'a_1), \qquad b_2 = a_1$$

$$b_3 = \sum_{i=0}^{e-1} p^i\psi^{p^i}(c), \qquad b_4 = \psi^{p^e}(c), \qquad c = a - \psi^p(b_1) - \binom{q}{2}\psi^{p^M}(a_1).$$

Here we used the fact that $\psi^2(a_2) = 2a_2^2 - \theta^2(2a_2) = 0$.

In both cases we have written (a', a) as $\Psi(b_1, b_3)$ modulo elements which are in the kernel of $\kappa_2^G + \kappa_3^G$ by Proposition 7. ∎

10. A CONVENIENT SYSTEM OF IDEALS IN $\mathbf{Z}[G]$

Let p be a prime, and let G be the cyclic group generated by an element T of order $q = p^e$. For $M \in \mathbf{N}$ let I_M be the ideal of $\mathbf{Z}[G]$ generated by the elements $p^{M+e-i}z^{p^i}$ with $0 \le i < e$. In this section we will show that this system of ideals has some nice properties. In preparation we consider ideals in $\mathbf{Z}[x]$.

For $N \in \mathbf{N}$ let J_N be the ideal of $\mathbf{Z}[x]$ generated by the $p^{N-i}x^{p^i}$ with $0 \le i \le N$; these ideals are considered in Sect. 6. Also for $M \in \mathbf{N}$ let E_M be the ideal of $\mathbf{Z}[x]$ generated by the elements

$$\begin{aligned} &p^{M+e-i}x^{p^i} && \text{for } 0 \le i \le e, \\ &p^{M-j}x^{q+jr} && \text{for } 1 \le j \le M, \end{aligned}$$

where $r = p^{e-1}(p-1)$.

PROPOSITION 17. *One has $p^{M-j}z^{q+jr} \in I_M$ for $0 \le j \le M$. This implies that $\pi(E_M) = I_M = \pi(J_{M+e})$.*

Proof. First note that $z^q = -\sum_{j=1}^{q-1}\binom{q}{j}z^j$ and that $\binom{q}{j}z^j$ is a multiple of $p^{e-i}z^{p^i}$, where $i = v_p(j)$, the p-valuation applied to j. Thus z^q is in the ideal generated by the $p^{e-i}z^{p^i}$ for $0 \le i < e$.

If we multiply this with p^M we get the first statement for $j = 0$. If we multiply this with $p^{M-j}z^{jr}$ then we find that $p^{M-j}z^{q+jr}$ is in the ideal generated by the $p^{M-j+e-i}z^{jr+p^i}$ for $0 \le i < e$. Thus it suffices to show that $w = p^{M-j+f}z^{jr+p^{e-f}} \in I_M$ for $1 \le f \le e$.

If $f \le j$ then $jr + p^{e-f} \ge (j-f)r + p^e$ which means that w is a multiple of $p^{M-j+f}z^{(j-f)r+q}$ which is in I_M by the induction hypothesis applied to $j - f$. The inequality is equivalent to $fr \ge p^e - p^{e-f}$ which is an equality for $f = 1$, whereas for $f \ge 2$ it follows from the fact that $2r \ge p^e$.

If $f-1 \geq j \geq 1$ then $e-1 \geq j$ and thus $jr + p^{e-f} \geq r \geq p^{e-1} \geq p^j$ so w is a multiple of $p^{M-j+f} z^{p^j}$ which is in I_M by definition of the latter. This finishes the proof of the first statement.

Thus $\pi(E_M) \subseteq I_M$. One has also $J_{M+e} \subseteq E_M$ since $p^i \geq q + jr$ if $j = i - e \geq 0$. Finally it is clear from the definitions that $I_M \subseteq \pi(J_{M+e})$. ∎

In the next Proposition we write $S = \mathbf{Z}[x]/E_M$ and we write ξ for the class of x in S, and $\tilde{q}$ for the class of $x^{-1}((1+x)^q - 1)$ in S.

PROPOSITION 18. *Let f: $\xi S \to \xi S$ be the multiplication by $\tilde{q}$. Then* $\operatorname{cok}(f)$ *splits as a direct sum of the cyclic subgroups generated by the ξ^n with $1 \leq n < q$.*

Proof. For $1 \leq n < q + Mr$ we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \xi^{n+1}S \arrow[r] \arrow[d, "f_{n+1}"] & \xi^n S \arrow[r] \arrow[d, "f_n"] & \xi^n S/\xi^{n+1}S \arrow[r] \arrow[d, "f'_n"] & 0 \\
0 \arrow[r] & \xi^{n+1}S \arrow[r] & \xi^n S \arrow[r] & \xi^n S/\xi^{n+1}S \arrow[r] & 0,
\end{tikzcd}$$

where the vertical maps are induced by f. In such a situation there is an exact sequence

$$\ker(f'_n) \xrightarrow{\partial} \operatorname{cok}(f_{n+1}) \to \operatorname{cok}(f_n) \to \operatorname{cok}(f'_n) \to 0$$

and thus the cardinalities satisfy

$$\#\operatorname{cok}(f_n) = (\#\operatorname{im}(\partial))^{-1}(\#\operatorname{cok}(f_{n+1}))(\#\operatorname{cok}(f'_n)).$$

The group $\xi^n S/\xi^{n+1}S$ is cyclic of order $p^{\epsilon(n)}$, where

$$\epsilon(n) = \begin{cases} M + e - i & \text{if } p^i \leq n < p^{i+1} \text{ and } 0 \leq i < e, \\ M - j & \text{if } q + jr \leq n < q + (j+1)r \text{ and } 0 \leq j < M, \\ 0 & \text{if } q + Mr \leq n. \end{cases}$$

The map f_n' is multiplication by $q = p^e$. Thus on the one hand

$$\#\mathrm{cok}(f_n') = \begin{cases} p^e & \text{if } \epsilon(n) \geq e, \\ p^{\epsilon(n)} & \text{if } \epsilon(n) < e \end{cases}$$

and on the other hand $\ker(f_n')$ is generated by

$$\begin{array}{ll} p^{\epsilon(n)-e}\xi^n + \xi^{n+1}S & \text{if } \epsilon(n) \geq e, \\ \xi^n + \xi^{n+1}S & \text{if } \epsilon(n) < e. \end{array}$$

A calculation as in the proof of Proposition 17 shows that for $0 \leq j < M$ one has $p^{M-j}x^{jr}((1+x)^{p^e} - 1) \in E_M$; in other words $f(p^{M-j}\xi^{jr+1}) = 0$ and therefore

$$\partial\big(p^{\alpha(n)}\xi^n + \xi^{n+1}S\big) = f\big(p^{\alpha(n)}\xi^n\big) = 0,$$

where α is defined for $1 \leq n \leq Mr$ by

$$\alpha(n) = M - j \text{ for } jr + 1 \leq n \leq (j+1)r \quad \text{and} \quad 0 \leq j < M.$$

Thus only part of $\ker(f_n')$ survives in $\mathrm{im}(\partial)$ and one has

$$\#\mathrm{im}(\partial) \leq \begin{cases} p^{\alpha(n) - \epsilon(n) + e} & \text{if } \epsilon(n) \geq e, \\ p^{\alpha(n)} & \text{if } \epsilon(n) < e. \end{cases}$$

From this one gets in both cases

$$\#\mathrm{cok}(f_n) \geq p^{\epsilon(n) - \alpha(n)}\#\mathrm{cok}(f_{n+1}).$$

Thus by induction one has

$$\#\mathrm{cok}(f) \geq \prod_n p^{\epsilon(n) - \alpha(n)}.$$

Consider the group $D = \bigoplus_{1 \leq n < q} D_n$ where $D_n = \mathbf{Z}/p^{\epsilon(n)}\mathbf{Z}$ and consider the homomorphism ρ: $D \to \mathrm{cok}(f)$ mapping $1 + p^{\epsilon(n)}\mathbf{Z} \in D_n$ to $\xi^n + \mathrm{im}(f)$. Then ρ is obviously a surjection, so one has

$$\#D \geq \#\mathrm{cok}(f).$$

However $\#D$ and $\prod_n p^{\epsilon(n) - \alpha(n)}$ are equal, so all inequalities are in fact equalities, and ρ is an isomorphism. ∎

Let π: $\mathbf{Z}[x] \to \mathbf{Z}[G]$ be the homomorphism defined by $\pi(x) = z$. Let $B = \mathbf{Z}[B]/\pi(E_M)$. Then π induces an isomorphism $\mathrm{cok}(f) \to zB$. Thus Proposition 18 yields a splitting of zB as the direct sum of cyclic subgroups generated by (the classes of) the z^n.

PROPOSITION 19. *For $i \geq 0$ one has*

$$p^{e+i}z \in \sum_{f=0}^{e-1} p^{e-f-1} z^{i(p-1)+p^{f+1}} B.$$

This implies that the ideals I_M are cofinal with the powers of zB.

Proof. Let $u_k = \sum p^{-e+k} \binom{p^e}{jp^k} z^{(j-1)p^k}$, where the sum is over all j prime to p such that $1 \leq j \leq p^{e-k}$. Then $\sum_{k=0}^{e} p^{e-k} z^{p^k} u_k = 0$. The first statement for $i = 0$ follows by multiplying this identity with some $v_0 \in B$ such that $v_0 u_0 - 1 \in z^q B$.

If the first statement is true for some $i = j$ and for $i = 0$ then

$$\begin{aligned} p^{e+j+1}z &\in p \sum_{f=0}^{e-1} p^{e-f-1} z^{j(p-1)+p^{f+1}} B \\ &\subseteq p^e z^{j(p-1)+p} B + \sum_{f=1}^{e-1} p^{e-f} z^{j(p-1)+p^{f+1}} B \\ &\subseteq \sum_{f=0}^{e-1} p^{e-f-1} z^{(j+1)(p-1)+p^{f+1}} B + \sum_{f=0}^{e-2} p^{e-f-1} z^{j(p-1)+p^{f+2}} B \\ &\subseteq \sum_{f=0}^{e-1} p^{e-f-1} z^{(j+1)(p-1)+p^{f+1}} B, \end{aligned}$$

so it is true for $i = j + 1$.

Now we show that $p^{M+e-i} z^{p^i} \in z^{M(p-1)+p} B$ for $0 \leq i \leq e$. The first statement shows that $p^{M+e-i} z^{p^i} \in z^{(M-i)(p-1)+p} z^{p^i - 1} B$ for $i \leq M$. If $e \leq M$ then this is sufficient. If $e \geq M + 1$ and $i \geq M + 1$ then $p^{M+e-i} z^{p^i} \in z^{p^{M+1}} B$ which is also sufficient.

Together with Proposition 17 this shows that

$$z^{q+Mr} B \subseteq I_M \subseteq z^{p+M(p-1)} B,$$

which proves the cofinality. ∎

PROPOSITION 20. *The map $B/qB \to \Omega_B$ mapping $b + qB$ to $b\delta z$ is an isomorphism.*

Proof. The ring B can be viewed as the quotient of $\mathbf{Z}[x]$ by the ideal generated by $(x+1)^q - 1$ and the $p^{M+e-i} x^{p^i}$ for $0 \leq i \leq e$. Therefore there is an isomorphism $B/D \to \Omega_B$ where D is the ideal generated by the classes of the derivatives of these polynomials: the $q(z+1)^{q-1}$ and the $p^{M+e}z^{p^i-1}$ for $0 \le i \le e$. Therefore $q \in D$ since $z+1$ is a unit in B. Furthermore, the elements of the second kind vanish modulo q. This means that $D = qB$. ∎

11. A SPLITTING PRINCIPLE

The results of this section are needed in the proofs in Sect. 12.

Given a commutative ring A and $\mathscr{M} = (M_1, \ldots, M_6) \in \mathbf{Z}^6$ we shall write

$$W(\mathscr{M}; A) = \frac{\left(\dfrac{M_5 A}{M_5 M_1 A} \otimes_A \Omega_A\right) \oplus \dfrac{M_6 A}{M_6 M_2 A}}{\{([M_5 M_3] \otimes \delta a, [M_6 M_4 a])\}}.$$

If $M_6 = 0$ then we want to discard the associated $W(\mathscr{M}, A)$ summand. Therefore we will assume that $M_2 = 1$ in this case. Similarly if $M_5 = 0$ we will assume that $M_1 = 1$.

Consider the ring $P(n) = \mathbf{Z}[v_1, v_2, \ldots, v_n]$ and consider the homomorphism $\pi\colon P(n) \to P(n)$ such that $\pi(v_j)$ is the elementary symmetrical polynomial $\sigma_j(v_1, v_2, \ldots, v_n)$.

PROPOSITION 21. *The induced map*

$$\pi_*\colon W(\mathscr{M}; P(n)) \to W(\mathscr{M}; P(n))$$

is split injective.

Proof. Assume that $M_5 \neq 0$ and $M_6 \neq 0$. Both $P(n)$ and $\Omega_{P(n)}$ can be written as direct sums of subgroups associated to each multidegree $I = (i_1, i_2, \ldots, i_n)$. Moreover the map $\delta\colon P(n) \to \Omega_{P(n)}$ is homogeneous. Therefore $W(\mathscr{M}, P(n))$ is a direct sum of subgroups associated to each multidegree I. We shall make this more precise.

If $I = (i_1, \ldots, i_n)$ and G is an abelian group then we write $G^{(I)}$ for the subgroup of G^n consisting of the $(g_1, \ldots, g_n)$ such that $g_j = 0$ if $i_j = 0$. Let

$$\delta_I\colon \mathbf{Z} \to \left(\frac{\mathbf{Z}}{M_1 \mathbf{Z}}\right)^{(I)} \oplus \left(\frac{\mathbf{Z}}{M_2 \mathbf{Z}}\right)$$

be the map defined by

$$\delta_I(k) = ([M_3 i_1 k], \ldots, [M_3 i_n k]) \oplus [M_4 k].$$

Then there is map τ_I: $\mathrm{cok}(\delta_I) \to W(\mathscr{M}, P(n))$ defined by

$$\begin{aligned}&\tau_I(([k_1],\ldots,[k_n]) \oplus [l]) \\ &\quad = \left[\left([M_5] \otimes \sum k_j v_1^{i_1} \cdots v_j^{i_j-1} \cdots v_n^{i_n} \delta v_j\right) \oplus \left[M_6 l v_1^{i_1} \cdots v_n^{i_n}\right]\right],\end{aligned}$$

where the first sum is over all j such that $i_j > 0$. Each τ_I is injective and $W(\mathscr{M}, P(n))$ is the direct sum of their images; we shall write p_I for the associated projections. We define

$$\epsilon_I: \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^{(I)} \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right) \to \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^n \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right)$$

as the obvious inclusion. It is split by a map q_I defined as "filling in the zeros." Thus the induced injection ϵ_I: $\mathrm{cok}(\delta_I) \to \mathrm{cok}(\epsilon_I \delta_I)$ is split by the map induced by q_I.

For any multidegree $I = (i_1, i_2, \ldots, i_n)$ we define a new multidegree $I+ = (i_1^+, i_2^+, \ldots, i_n^+)$ by the formula $i_j^+ = \sum_{k=j}^n i_k$. We define

$$\rho_I: \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^n \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right) \to \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^n \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right),$$

$$\rho_I(([m_1],\ldots,[m_n]) \oplus [l]) = (([m_1^+],\ldots,[m_n^+]) \oplus [l]),$$

where $m_j^+ = \sum_{k=j}^n m_k$. Obviously $m_j^+ = 0$ if $i_j^+ = 0$; thus ρ_I induces a map

$$\rho_{(I)}: \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^{(I)} \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right) \to \left(\frac{\mathbf{Z}}{M_1\mathbf{Z}}\right)^{(I+)} \oplus \left(\frac{\mathbf{Z}}{M_2\mathbf{Z}}\right).$$

Moreover $\rho_{(I)}\delta_I = \delta_{I+}$. Therefore it induces a map ρ_I': $\mathrm{cok}(\delta_I) \to \mathrm{cok}(\delta_{I+})$. It also induces a map ρ_I'': $\mathrm{cok}(\epsilon_I \delta_I) \to \mathrm{cok}(\epsilon_{I+}\delta_{I+})$ which is an isomorphism since ρ_I is. The situation is summarized in the diagram

$$\begin{tikzcd}
W(\mathcal{M}; P(n)) \arrow[r, "\pi_*"] \arrow[d, dashed, shift right, "p_I"'] & W(\mathcal{M}; P(n)) \arrow[d, "p_{I+}"] \\
\mathrm{cok}(\delta_I) \arrow[r, "\rho_I'"] \arrow[u, shift right, "\tau_I"'] \arrow[d, shift left, "\epsilon_I"] & \mathrm{cok}(\delta_{I+}) \arrow[d, "\epsilon_{I+}"] \\
\mathrm{cok}(\epsilon_I\delta_I) \arrow[r, "\rho_I''"] \arrow[u, dashed, shift left, "q_I"] & \mathrm{cok}(\epsilon_{I+}\delta_{I+}).
\end{tikzcd}$$

In this diagram one has $p_{I+}\pi_*\tau_i = \rho_I'$ since

$$\pi(v_1)^{i_1}\pi(v_2)^{i_2},\ldots,\pi(v_n)^{i_n} = \sigma_1(v_1,\ldots,v_n)^{i_1},\ldots,\sigma_n(v_1,\ldots,v_n)^{i_n}$$

is a sum of

$$v_1^{i_1}(v_1v_2)^{i_2},\ldots,(v_1v_2\cdots v_n)^{i_n} = v_1^{i_1^+}v_2^{i_2^+},\ldots,v_n^{i_n^+}$$

(which is detected by p_{I+}), and terms which are lexicographically of lower degree (and which are mapped to zero by p_{I+}).

Therefore

$$q_I(\rho_I'')^{-1}\epsilon_{I+}p_{I+}\pi_*\tau_I = q_I(\rho_I'')^{-1}\epsilon_{I+}\rho_I' = q_I(\rho_I'')^{-1}\rho_I''\epsilon_I = q_I\epsilon_I = 1.$$

Thus π_* is split by $q_I(\rho_I'')^{-1}\epsilon_{I+}p_{I+}$ on the summand $\mathrm{im}(\tau_I)$. ∎

PROPOSITION 22. *The map*

$$\pi_2 = \pi\otimes\pi\colon P(n')\otimes P(n')\to P(n')\otimes P(n')$$

also induces an injection on $W(\mathscr{M}; A)$.

Proof. One can identify $P(n')\otimes P(n')$ with $P(2n')$ by mapping $v_j\otimes 1$ to v_j and $1\otimes v_j$ to $v_{n'+j}$; thus π_2 can be viewed as a map $P(n)\to P(n)$ where $n = 2n'$. The same proof as for the Proposition 21 applies, with only the following changes:

$$i_j^+ = \begin{cases}\sum_{k=j}^{n'} i_k & \text{if } j\le n'\\ \sum_{k=j}^{2n'} i_k & \text{if } j > n',\end{cases}\qquad m_j^+ = \begin{cases}\sum_{k=j}^{n'} m_k & \text{if } j\le n'\\ \sum_{k=j}^{2n'} m_k & \text{if } j > n'.\end{cases}$$ ∎

Consider the ring $P(\infty) = \mathbf{Z}[v_1, v_2,\ldots]$ and consider the homomorphisms $\rho_n\colon P(\infty)\to P(n)$ such that $\rho_n(v_i) = v_i$ for $i\le n$, $\rho_n(v_i) = 0$ for $i > n$.

PROPOSITION 23. *The map $W(\mathscr{M}, P(\infty))\to \varprojlim W(\mathscr{M}, P(n))$ induced by ρ_n is injective.*

Proof. An easy exercise. ∎

12. NATURAL ADDITIVE MAPS

PROPOSITION 24. *Suppose there is given a natural additive map*

$$\xi : \Omega_A \to W(\mathscr{M}; A).$$

Then there are $C_i \in \mathbf{Z}$, *such that*

$$\xi(\alpha) = \left[[M_5] \otimes \sum_i C_i \phi^i \alpha, 0 \right],$$

for any λ*-ring* A *and element* $\alpha \in \Omega_A$.

Proof. First some notations. The ring $P(n) \otimes P(n)$ contains $s_i = v_i \otimes 1$ and $t_i = 1 \otimes v_i$. Let U be the universal λ-ring and $u \in U$ the universal element. Then $U \otimes U$ contains $u_1 = u \otimes 1$ and $u_2 = 1 \otimes u$. Write $U(n)$ for the subring of U generated by the $\lambda^i(u)$ with $i \leq n$.

First consider ξ on the ring $P(1) \otimes P(1) = \mathbf{Z}[s, t]$. There is an $n \in \mathbf{N}$ and there are $c^s_{i,j}, c^t_{i,j}, d_{i,j} \in \mathbf{Z}$ such that

$$\xi(s\delta t) = \left[[M_5] \otimes \left(\sum_{i,j}^{n} c^s_{i,j} s^{i-1} t^j \delta s + \sum_{i,j}^{n} c^t_{i,j} s^i t^{j-1} \delta t \right) \oplus \left[M_6 \sum_{i,j}^{n} d_{i,j} s^i t^j \right] \right].$$

In the ring $\mathbf{Z}[x, y, z]$ one has by additivity

$$\xi(x\delta yz) = \xi(xy\delta z) + \xi(xz\delta y).$$

All three terms can be evaluated by using naturality for the three obvious λ-ring homomorphisms $\mathbf{Z}[s, t] \to \mathbf{Z}[x, y, z]$. From this one finds that there must be $f_{i,j,k}, g^x_{i,j,k}, g^y_{i,j,k}, g^z_{i,j,k}, h_{i,j,k} \in \mathbf{Z}$ such that

$$\begin{aligned}
M_5 \Bigg(& \sum_{i,j} c^s_{i,j} x^{i-1} y^i z^j \delta x + \sum_{i,j} c^s_{i,j} x^i y^{i-1} z^j \delta y + \sum_{i,j} c^t_{i,j} x^i y^i z^{j-1} \delta z \\
& + \sum_{i,j} c^s_{i,j} x^{i-1} y^j z^i \delta x + \sum_{i,j} c^t_{i,j} x^i y^{j-1} z^i \delta y + \sum_{i,j} c^s_{i,j} x^i y^j z^{i-1} \delta z \\
& - \sum_{i,j} c^s_{i,j} x^{i-1} y^j z^j \delta x - \sum_{i,j} c^t_{i,j} x^i y^{j-1} z^j \delta y - \sum_{i,j} c^t_{i,j} x^i y^j z^{j-1} \delta z \Bigg) \\
& = M_5 M_3 \delta \Bigg(\sum_{i,j,k} f_{i,j,k} x^i y^j z^k \Bigg) + M_5 M_1 \Bigg(\sum_{i,j,k} g^x_{i,j,k} x^{i-1} y^j z^k \delta x \\
& \qquad + \sum_{i,j,k} g^y_{i,j,k} x^i y^{j-1} z^k \delta y + \sum_{i,j,k} g^z_{i,j,k} x^i y^j z^{k-1} \delta z \Bigg),
\end{aligned}$$

and

$$M_6\left(\sum_{i,j} d_{i,j}x^iy^iz^j + \sum_{i,j} d_{i,j}x^iy^jz^i - \sum_{i,j} d_{i,j}x^iy^jz^j\right)$$
$$= M_6M_4\left(\sum_{i,j,k} f_{i,j,k}x^iy^jz^k\right) + M_6M_2\left(\sum_{i,j,k} h_{i,j,k}x^iy^jz^k\right).$$

Comparing coefficients and writing $f_{i,j} = f_{i,j,i}$, $g^x_{i,j} = g^x_{i,j,i}$, $g^y_{i,j} = g^y_{i,j,i}$, $h_{i,j} = h_{i,j,i}$ and $C_i = c^t_{i,i} - iM_3f_{i,i,i} - M_1g^y_{i,i,i}$ one finds that for all i, j

$$c^s_{i,j} = iM_3f_{i,j} + M_1g^x_{i,j},$$
$$c^t_{i,j} = jM_3f_{i,j} + M_1g^y_{i,j} + \delta_{i,j}C_i,$$
$$d_{i,j} = M_4f_{i,j} + M_2h_{i,j}$$

(at least if $M_5 \neq 0$ and $M_6 \neq 0$). Substituting this in the formula for $\xi(s\delta t)$ one gets an expression which can be simplified to

$$\xi(s\delta t) = \left[[M_5] \otimes \left(\sum_i C_is^it^{i-1}\delta t\right) \oplus [0]\right].$$

Now consider the map $\pi_2\colon U(n) \otimes U(n) \to P(n) \otimes P(n)$ which is the restriction of the unique λ-homomorphism such that $\pi_2(u_1) = \sum_{k=1}^d s_k$ and $\pi_2(u_2) = \sum_{m=1}^d t_m$. By naturality of ξ and additivity of ξ one has

$$(\pi_2)_*\xi(u_1\delta u_2) = \xi((\pi_2)_*(u_1\delta u_2)) = \xi\left(\left(\sum_{k=1}^d s_k\right)\delta\left(\sum_{m=1}^d t_m\right)\right)$$
$$= \left[[M_5] \otimes \left(\sum_i \sum_{k=1}^d \sum_{m=1}^d C_is_k^it_m^{i-1}\delta t_m\right) \oplus [0]\right]$$
$$= \left[[M_5] \otimes \left(\sum_i C_i\phi^i\left(\sum_{k=1}^d s_k\right)\delta\left(\sum_{m=1}^d t_m\right)\right) \oplus [0]\right]$$
$$= (\pi_2)_*\left[[M_5] \otimes \left(\sum_i C_i\phi^i(u_1\delta u_2)\right) \oplus [0]\right].$$

But the map $(\pi_2)_*\colon W(\mathscr{M}; U(n) \otimes U(n)) \to W(\mathscr{M}; P(n) \otimes P(n))$ is injective according to Proposition 22. Therefore

$$\xi(a_1\delta a_2) = \left[[M_5] \otimes \left(\sum_i C_i\phi^i(a_1\delta a_2)\right) \oplus [0]\right],$$

for $a_1 = u_1$ and $a_2 = u_2$ in $U(n)$. By Proposition 23 the same is true for $a_1 = u_1$ and $a_2 = u_2$ in $U(\infty)$. By naturality the same identity holds for any λ-ring A and elements $a_1, a_2 \in A$. The stated formula now follows by additivity. ∎

PROPOSITION 25. *The C_i are unique modulo M_1.*

Proof. Suppose that $\xi(\alpha) = 0$ for all A and all $\alpha \in \Omega_A$. Then it is in particular the case for $A = \mathbf{Z}[s, t]$ and $\alpha = s\delta t$. So

$$\left[[M_5] \otimes \left(\sum_i^n C_i s^i t^{i-1} \delta t \right) \oplus [0] \right] = 0.$$

This means that there are $f_{i,j}, g^s_{i,j}, g^t_{i,j}, h_{i,j}$ such that

$$M_5\left(\sum_i^n C_i s^i t^{i-1} \delta t \right)\Bigg) = M_5 M_3 \delta\left(\sum_{i,j} f_{i,j} s^i t^j \right)$$
$$+ M_5 M_1\left(\sum_{i,j} g^s_{i,j} s^{i-1} t^j \delta s + \sum_{i,j} g^t_{i,j} s^i t^{j-1} \delta t \right),$$
$$0 = M_6 M_4\left(\sum_{i,j} f_{i,j} s^i t^j \right) + M_6 M_2\left(\sum_{i,j} h_{i,j} s^i t^j \right).$$

From the formulas that one gets by comparing coefficients one finds that $C_i = M_1(g^t_{i,i} - g^s_{i,i})$. ∎

PROPOSITION 26. *Suppose there is given a natural additive map*

$$\xi\colon A \to W(\mathscr{M}; A).$$

Then there are $c_i, d_i \in \mathbf{Z}$, such that

$$\xi(a) = \left[[M_5] \otimes \left(\sum_i c_i \phi^i \delta a \right) \oplus \left[M_6 \sum_i d_i \psi^i a \right] \right],$$

for any λ-ring A and element $a \in A$.

Proof. This is analogous to the proof of Proposition 24, and uses the fact that the map $\pi_*\colon W(\mathscr{M}; U(n)) \to W(\mathscr{M}; P(n))$ is injective. ∎

PROPOSITION 27. *If c_i, d_i yield the zero map then there are f_i, g_i, h_i such that $c_i = M_1 g_i + M_3 i f_i$ and $d_i = M_2 h_i + M_4 f_i$.*

Proof. Suppose that $\xi(a) = 0$ for all A and all $a \in A$. Then it is in particular the case for $A = \mathbf{Z}[t]$ and $a = t$. So

$$\left[[M_5] \otimes \left(\sum_i c_i t^{i-1}\delta t\right) \oplus \left[M_6 \sum_i d_i t^i\right]\right] = 0.$$

This means that there are f_i, g_i, h_i such that

$$M_5\left(\sum_i c_i t^{i-1}\delta t\right) = M_5 M_3 \delta\left(\sum_i f_i t^i\right) + M_5 M_1\left(\sum_i g_i t^{i-1}\delta t\right),$$

$$M_6\left(\sum_i d_i t^i\right) = M_6 M_4\left(\sum_i f_i t^i\right) + M_6 M_2\left(\sum_i h_i t^i\right).$$

By comparing the coefficients one gets the result. ∎

13. SOME λ-RING CALCULATIONS

In the paper the ring $A[x]/J_N$ is considered with two different λ-ring structures. In this section we discuss the details of these structures. See Sect. 1 of [5] and Sect. 4 and 9 of [6] for some of the terminology used.

Let $\mathbf{Z}_p[x]$ denote the polynomial ring $\mathbf{Z}[x]$ localized at p, and let $\mathbf{Z}[x]^0$ be its subring consisting of polynomials with constant term in $\mathbf{Z}$. As before we write J_N for the ideal generated by the $p^{N-i}x^{p^i}$ for $0 \leq i \leq N$, either in $\mathbf{Z}[x]$ or in $\mathbf{Z}[x]^0$. We shall identify $\mathbf{Z}[x]/J_N$ with $\mathbf{Z}[x]^0/J_N$ using the obvious map.

Because $\mathbf{Z}[x]^0$ has no $\mathbf{Z}$-torsion a λ-ring structure is characterized by giving the associated Adams operations ψ^m; see Proposition 1.9 of [5]. The standard structure has $\psi^m(x) = x^m$ for all m. The exotic structure has $\psi^l(x) = 0$ for primes $l \neq p$, and $\psi^p(x) = (x+1)^p - 1$.

PROPOSITION 28. *The ideal J_N of $\mathbf{Z}[x]^0$ is a λ-ideal for both λ-ring structures.*

Proof. Let $J_{N,k}$ denote the ideal of $\mathbf{Z}[x]^0$ generated by the $p^{N-i}x^{p^i}$ or $N-k \leq i \leq N$. We shall prove by induction on k that $J_{N,k}$ is a λ-ideal.

From the above description of the $\psi^m(x)$ we see that $\psi^m(x) \in (x)$ for all $m > 0$. By the Newton formula

$$\begin{aligned}\psi^m(x) - \lambda^1(x)\psi^{m-1}(x) + \cdots + (-1)^{m-1}\lambda^{m-1}(x)\psi^1(x)\\ = (-1)^{m-1} m\lambda^m(x),\end{aligned}$$

this implies that $m\lambda^m(x) \in (x)$ for $m > 0$. Since $\mathbf{Z}[x]^0/(x)$ has no torsion this implies that $\lambda^m(x) \in (x)$. By the formulas for $\lambda(ax)$ and $\lambda^m(a+b)$ this says that (x) is a λ-ideal. Thus (x^n) is also a λ-ideal for any n, since a product of λ-ideals is again a λ-ideal by Proposition 1.12 of [5]. In particular $J_{N,0}$ is a λ-ideal.

Now suppose that $J_{N,k-1}$ is a λ-ideal. Since $J_{N,k}$ is generated by $p^k x^{p^{N-k}}$ and $J_{N,k-1}$ it is sufficient to prove that $\lambda^n(p^k x^{p^{N-k}}) \in J_{N,k}$ for every $n > 0$.

If R is any λ-ring and $a \in R$ then we shall write $\lambda_t(a)$ for the formal power series $\sum_{n=0}^{\infty} t^n \lambda^n(a)$. One has

$$\begin{aligned}
&\sum_{n=0}^{\infty} t^n \lambda^n\left(p^k x^{p^{N-k}}\right) \\
&\quad = \lambda_t\left(p^k x^{p^{N-k}}\right) = \left(\sum_{n=0}^{\infty} t^n \lambda^n\left(x^{p^{N-k}}\right)\right)^{p^k} \\
&\quad = \sum \binom{p^k}{i_1, \ldots, i_m} \left(t^{n_1}\lambda^{n_1}\left(x^{p^{N-k}}\right)\right)^{i_1} \cdots \left(t^{n_m}\lambda^{n_m}\left(x^{p^{N-k}}\right)\right)^{i_m},
\end{aligned}$$

where the sum is over all $m \in \mathbf{N}$, $(n_1, \ldots, n_m) \in \mathbf{N}^m$, and $(i_1, \ldots, i_m) \in \mathbf{N}^m$ satisfying $i_1 + \cdots + i_m = p^k$.

Consider the term associated to $(i_1, \ldots, i_m)$ and let p^f be the highest power of p which divides all i_j. Thus there is some j such that p^{f+1} does not divide i_j, which implies that p^{k-f} divides the binomial symbol $\binom{p^k}{i_j}$. Therefore p^{k-f} divides the multinomial symbol $\binom{p^k}{i_1, \ldots, i_m} = p^k!/(i_1! \cdots i_m!)$ which is a multiple of $\binom{p^k}{i_j}$. On the other hand the term contains a factor $(\lambda^{n_j}(x^{p^{N-k}}))^{i_j}$ which is a multiple of $(x^{p^{N-k}})^{p^f}$; so the term is in $J_{N,k-f}$. This proves the statement. ∎

This proposition shows that $\mathbf{Z}[x]/J_N$ has an induced λ-ring structure, and thus also $A[x]/J_N = A \otimes \mathbf{Z}[x]/J_N$.

PROPOSITION 29. *For the exotic λ-ring structure on $A[x]/J_N$ the ideal (x) is ψ-nilpotent and θ-nilpotent.*

Proof. In $\mathbf{Z}[x]^0$ one has $\psi^n(x) = 0$ if $n > 1$ is not a power of p. On the other hand $\psi^{p^N}(x) \in J_N$. Thus in $\mathbf{Z}[x]/J_N$ one has $\psi^n(x) \equiv 0$ for large n. Thus if $a \in (x)$ then $\psi^n(a) \equiv 0$ for large n since ψ^n is multiplicative.

If m is not divisible by p and $a \in (x)$ then $\theta^m(a) = m^{-1}a^m$ which vanishes for $m \geq p^N$. If k is of the form $k = p^f$ and $a \in (x)$ then according to the proof of Proposition 5.2 of [5] one has

$$\theta^k(a) = \sum p^{l-f} \binom{p^{f-1}}{l} (\psi^p(a))^{p^{f-1}-l} (\theta^p(a))^l,$$

which is in $(x^{p^{f-1}})$ and therefore vanishes if $f - 1 \geq N$.

Now consider $n = km$ where k is of the form p^f and m is not divisible by p; one can choose r, s such that $rk + sm = 1$. According to the proof of Proposition 2.1 of [5] one has

$$\theta^n(a) = r \sum \mu(j) \psi^j \theta^m(a^{k/j}) + s \sum \mu(i) \psi^i \theta^k(a^{m/i}),$$

where μ denotes the Möbius function. If n is large enough then in the first sum either j or m or k/j is large; in all three cases the associated term vanishes for $a \in (x)$ according to the above remarks. The same applies to the second sum.

Finally consider an expression of the form

$$\psi^{n_1}(\theta^{n'_1}(a)) \cdots \psi^{n_g}(\theta^{n'_g}(a)),$$

with $a \in (x)$. If the degree $n_1 n'_1 + \cdots + n_g n'_g$ is large enough then either g or one of the n_i or one of n'_i is large enough; in all three cases the expression vanishes. ∎

It follows from Proposition 17 that the ideal (z) in $A[G]/I_M$ has the same properties.

In the remainder of this section we shall prove the properties of the operations θ^m, η^k etc. which we have used in Sects. 3 and 9. The ring $\mathbf{Z}[t]$ is equipped with the unique λ-ring structure such that $\psi^n(t) = t^n$ for all n.

We start with some remarks which apply to both λ-ring structures on $A[x]/J_N$. First the operations θ^m and η^k. From Proposition 2.4 in [5] it is clear that

$$\theta^m(ab) = 0 \qquad \text{for } m > 1$$

if $\theta^i(a) = 0 = \theta^i(b)$ for $i > 1$.

From Definition 3.1 in [5] it is clear that

$$\eta^1(a, b) = b,$$

$$\eta^2(a, b) = a\theta^2(b),$$

$$\eta^k(a, b) = 0 \qquad \text{for } k > 1 \text{ if } \theta^i(b) = 0 \text{ for } i > 1.$$

In particular $\eta^k(a,t) = 0$ for $k > 1$. From Proposition 5.1 in [5] it is clear that

$$\eta^k(a,b) \in (x^{i+j}) \qquad \text{if } a \in (x^i),\, b \in (x^j) \text{ and } k > 1;$$
$$\eta^k(a,b) \in (x^{2i+j}) \qquad \text{if } a \in (x^i),\, b \in (x^j) \text{ and } k > 2.$$

Next we discuss the operations ϕ^m.

PROPOSITION 30. *If l is prime then $\phi^l(\delta a) = a^{l-1}\delta a - \delta\theta^l(a)$.*

Proof. Let U be the universal λ-ring and $u \in U$ the universal element. Then

$$l\phi^l(\delta u) = \psi^l(\delta u) = \delta\psi^l(u) = \delta\big(u^l - l\theta^l(u)\big) = l\big(u^{l-1}\delta u - \delta\theta^l(u)\big).$$

But Ω_U has no torsion so $\phi^l(\delta u) = u^{l-1}\delta u - \delta\theta^l(u)$. The statement now follows by naturality. ∎

In particular $\phi^l(\delta a) = a^{l-1}\delta a$ if $\theta^l(a) = 0$. For example $\phi^m(\delta t) = t^{m-1}\delta t$ for all m.

Finally we discuss some properties of the exotic λ-ring structure on $A[x]/J_N$.

PROPOSITION 31. *If $a \in (x)$ then $\theta^n(x) \in (x^3)$ for $n \neq 1, 2, p, 2p$. If $a \in (x^2)$ then $\theta^n(x) \in (x^4)$ for $n \neq 1, p$.*

Proof. We write $n = km$ where k is of the form p^f and m is not divisible by p; one can choose r, s such that $rk + sm = 1$, with $s = 0$ if $f = 0$. One has

$$\theta^n(a) = r\sum \mu(j)\psi^j\theta^m(a^{k/j}) + s\sum \mu(i)\psi^i\theta^k(a^{m/i}).$$

Consider the first sum. For $b \in (x)$ one has $\theta^m(b) = m^{-1}b^m$ for such m; thus a term is in (x^3) unless $mk/j = 1, 2$. On the other hand $\mu(j)$ vanishes unless $j = 1, p$.

Consider the second sum, assuming $f > 0$. For $b \in (x)$ one has $\psi^i(b) = 0$ for such $i > 1$ and $\theta^k(b) \in (x^{p^{f-1}})$ for such f. Thus a term is in (x^3) unless $i = 1$ and either $m/i = 2$ and $p^{f-1} = 1$ or $m/i = 1$ and $p^{f-1} \leq 2$. In other words either $m = 2$, $k = p$ or $m = 1$, $k = p$ or $m = 1$, $k = 4$.

The other statement is proved similarly. ∎

In $\mathbf{Z}[x]^0$ one has

$$\theta^p(x) = \frac{x^p - \psi^p(x)}{p} = \frac{x^p - (x+1)^p + 1}{p} = -\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i.$$

Thus the same formula applies in $A[x]/J_N$. In particular one has

$$\theta^2(x) = -x \qquad \text{for } p = 2,$$
$$\theta^p(x) \equiv -x - \frac{1}{2}(p-1)x^2 \text{ modulo } x^3 \qquad \text{for } p > 2.$$

In the same way one checks that

$$\theta^4(x) = -x^2 - x^3,$$
$$\theta^p(x^2) \equiv -px^2 - 2\binom{p}{2}x^3 \text{ modulo } (x^4),$$
$$\theta^p(x^3) \equiv -p^2x^3 - 3p\binom{p}{2}x^4 \text{ modulo } (x^5).$$

The general formula

$$\theta^p(uv) = \theta^p(u)v^p + \psi^p(u)\theta^p(v)$$

can be used to show that for example

$$\theta^p(x^2a) = \theta^p(x^2)\psi^p(a) \text{ modulo } (x^4).$$

From the general remarks it follows that

$$\phi^p(\delta x) = (1+x)^{p-1}\delta(x+1) = (1+x)^{p-1}\delta x,$$
$$\phi^l(\delta x) = 0 \qquad \text{for primes } l \neq p.$$

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