Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra

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In this paper we introduce an algebra embedding $\iota: K\langle X \rangle \rightarrow S$ from the free associative algebra $K\langle X \rangle$ generated by a finite or countable set $X$ into the skew monoid ring $S = P \ast \Sigma$ defined by the commutative polynomial ring $P = K\langle X \times \mathbb{N}^* \rangle$ and by the monoid $\Sigma = \langle \sigma \rangle$ generated by a suitable endomorphism $\sigma: P \rightarrow P$. If $P = K\langle X \rangle$ is any ring of polynomials in a countable set of commuting variables, we present also a general Gröbner bases theory for graded two-sided ideals of the graded algebra $S = \bigoplus_i S_i$ with $S_i = P\sigma^i$ and $\sigma: P \rightarrow P$ an abstract endomorphism satisfying compatibility conditions with ordering and divisibility of the monomials of $P$. Moreover, using a suitable grading for the algebra $P$ compatible with the action of $\Sigma$, we obtain a bijective correspondence, preserving Gröbner bases, between graded $\Sigma$-invariant ideals of $P$ and a class of graded two-sided ideals of $S$. By means of the embedding $\iota$ this results in the unification, in the graded case, of the Gröbner bases theories for commutative and non-commutative polynomial rings. Finally, since the ring of ordinary difference polynomials $P = K\langle X \times \mathbb{N} \rangle$ fits the proposed theory one obtains that, with respect to a suitable grading, the Gröbner bases of finitely generated graded ordinary difference ideals can be computed also in the operators ring $S$ and in a finite number of steps up to some fixed degree.

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1. Introduction

Let $P$ be a $K$-algebra and let $\Sigma$ be a monoid of endomorphisms of $P$. If $I$ is an ideal of $P$ which is invariant under the maps in $\Sigma$ then it is possible to codify the action of $P$ and $\Sigma$ over $I$ as a single left module structure with respect to the skew monoid (or semigroup) ring $S = P \ast \Sigma$. The study of some properties of $I$, as for instance its finite $\Sigma$-generation, can be reduced hence to that of general properties of the operators ring $S$ as its Noetherianity (see McConnell and Robson, 2001; Lam, 2001). Ideals which are stable under the action of monoids of endomorphisms or groups of automorphisms are natural in many contexts as the representation theory (a classical reference is De Concini et al., 1980), or in the study of PI-algebras (Drensky, 2000; Giambruno and Zaicev, 2005) where $P$ is the free associative algebra and $\Sigma$ the complete monoid of endomorphisms of $P$. Another context of relevant interest is the study of so-called “difference ideals” (Levin, 2008) which are ideals invariant under shift operators in applications to combinatorics, (nonlinear) differential and difference equations. For the viewpoint of computing in the ring of (differential-difference) operators an important contribution is Mansfield and Szanto (2003).

To control in an effective way the structure of the left $S$-module $P/I$ one generally needs to compute a $K$-basis of it. If $P$ is a ring of polynomials in commutive or non-commutative variables and one fixes a suitable ordering for the monomials of $P$, then a $K$-linear basis of monomials for $P/I$ can be obtained by using the elements of a suitable generating set of $I$ as rewriting rules. Such generating set is usually called a “Gröbner basis” of $I$. Since $I$ is a $\Sigma$-invariant ideal, it is natural to consider $\Sigma$-bases of $I$ that is sets $G \subset I$ such that $I$ is the smallest $\Sigma$-ideal of $P$ containing $G$. In other words, $G$ is a generating set of $I$ as left $S$-module. It follows that one has to harmonize the notion of Gröbner basis with that of $\Sigma$-basis and attempts in this direction can be found for instance in Aschenbrenner and Hillar (2007), Brouwer and Draisma (2011) and also in Drensky and La Scala (2006), La Scala and Levandovskyy (2009). If the elements of $\Sigma$ are automorphisms, the main obstacle in the definition of a Gröbner $\Sigma$-basis is that their action on $P$ does not preserve the monomial ordering. Then, it has been shown in Brouwer and Draisma (2011) and before in La Scala and Levandovskyy (2009) that an appropriate setting to define Gröbner $\Sigma$-bases is that of a commutative polynomial ring $P = K[X]$ in an infinite number of variables and a monoid $\Sigma$ of monomial monomorphisms of infinite order which are compatible with the ordering and divisibility of monomials of $P$.

In this paper we propose a systematic study of the case when $\Sigma$ is generated by a single map $\sigma$. In this case the skew monoid ring $S$ coincides with the skew polynomial ring $P[s; \sigma]$ which is an Ore extension where $\sigma$-derivation is zero. The approach we follow is to consider an abstract map $\sigma$ satisfying compatibility conditions able to provide a “natural” Gröbner bases theory. Note that this generalizes in particular the results contained in Weispfenning (1992) where the map $\sigma : x_j \mapsto x_j^\epsilon$ with $\epsilon > 1$ has been studied. We choose to consider a single endomorphism essentially because a major application of our theory is the unification, in the graded case, of the Gröbner bases theory for non-commutative polynomials introduced in Green (1994), Mora (1986), Ufnarovski (1989) with the classical commutative theory based on the notion of $S$-polynomial (see for instance Greuel and Pfister, 2008). In Section 6 we show in fact that there exists an algebra embedding $i : K(X) \rightarrow S$ where $K(X)$ is the free associative algebra generated by the variables $x_i$ and $S$ is the skew polynomial ring defined by the algebra $P$ of commutative polynomials in double indexed variables $x_{id}(j)$ and the endomorphism $\sigma : P \rightarrow P$ such that $x_i(j) \mapsto x_{i(j+1)}$, for all $i, j$. This algebra embedding is a significant improvement of the linear map $i : K(X) \rightarrow P$ defined as $x_{id} \mapsto x_i(1)\cdots x_{ide}(d)$ and introduced by Feynman (1951), Dubilet et al. (1974) for the aims of physics and invariant/representation theory. In fact, the use of the map $i$ clarifies the phenomenon found in La Scala and Levandovskyy (2009) of a bijective correspondence between all graded two-sided ideals of $K(X)$ and some class of $\Sigma$-invariant ideals of $P$. Note that in the same paper, a competitive new algorithm for non-commutative homogeneous Gröbner bases based on this correspondence has been implemented and experimented in Singular (Decker et al., 2011).

In Section 2 one finds a brief account of the equivalence between the notion of $\Sigma$-invariant $P$-module and that of left $S$-module, together with the description of some properties of the generating sets of graded two-sided ideals of $S = \bigoplus S_i$ with $S_i = Ps^i$. A Gröbner basis theory for such ideals is introduced in Section 3 where we assume $P = K[x_0, x_1, \ldots]$. $\Sigma = \langle \sigma \rangle$ and $\sigma : P \rightarrow P$ be a...
monomorphism of infinite order sending monomials into monomials. Additional assumptions for \( \sigma \) are that \( \gcd(\sigma(x_i), \sigma(x_j)) = 1 \) for \( i \neq j \) and the monomial ordering of \( P \) is such that \( m < n \) implies that \( \sigma(m) < \sigma(n) \), for all monomials \( m, n \). Such conditions are quite natural in many contexts as the shift operators defining difference ideals (Levin, 2008) or the maps used in Brouwer and Draisma (2011). Note that the algorithms we introduce for the computation of homogeneous Gröbner bases in \( S \) are based on the free \( P \)-module structure of this ring and hence they appear as a variant of the classical module Buchberger algorithm where the number of \( S \)-polynomials to be considered is reduced owing to the symmetry defined by \( \Sigma \) on the graded ideals of the ring \( S \).

In Section 5 we analyze the notion of Gröbner \( \Sigma \)-basis for \( \Sigma \)-invariant ideals of \( P \). When \( P \) can be endowed with a suitable grading compatible with the action of \( \Sigma \), we describe a bijective correspondence between all graded \( \Sigma \)-invariant ideals of \( P \) and some class of graded two-sided ideals of \( S \). Such correspondence preserves Gröbner bases and gives rise to a “duality” between homogeneous algorithms in \( P \) and in \( S \). Note that for finitely generated ideals all these procedures admit termination when truncated at some degree. As we said earlier, in Section 6 the algebra embedding \( \iota: K(X) \to S \) is introduced and a bijective correspondence between the ideals of \( K(X) \) and suitable ideals of \( S \) is hence obtained by extension. The Gröbner bases are preserved by this correspondence and one obtains an alternative algorithm for computing non-commutative homogeneous Gröbner bases of \( K(X) \) in the free \( P \)-module \( S \). By means of the bijection of Section 5, we reobtain in Section 7 the ideal correspondence and related algorithms introduced in La Scala and Levandovskyy (2009) which provide the computation of non-commutative homogeneous Gröbner bases directly in \( P \). Therefore, the theory for such bases can be deduced by the classical Buchberger algorithm for commutative polynomial rings. In Section 8 we propose the explicit computation of a finite Gröbner basis of an ideal for such bases can be deduced by the classical Buchberger algorithm for commutative polynomial rings. In Section 8 we propose the explicit computation of a finite Gröbner basis of an ideal of ordinary difference polynomials that can be obtained as a special case by the algorithms we introduced. Moreover, in this section we provide some timings obtained by an improvement of the library freegb.lib of SINGULAR initially developed for La Scala and Levandovskyy (2009). Finally, in Section 9 we propose some conclusions and suggestions for future developments of the theory of Gröbner \( \Sigma \)-bases and its methods.

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2. Modules over skew monoid rings

Fix \( K \) any field and let \( P \) be a commutative \( K \)-algebra. Let now \( \Sigma \subset \text{End}_K(P) \) a submonoid of the monoid of \( K \)-algebra endomorphisms of \( P \). Denote \( S = P \triangleright \Sigma \) the skew monoid ring defined by \( \Sigma \) over \( P \) that is \( S \) is the free \( P \)-module with (left) basis \( \Sigma \) and the multiplication is defined by the identity \( \sigma f = \sigma(f) \sigma \), for all \( f \in P, \sigma \in \Sigma \). If \( \Sigma \neq \{id\} \) then \( S \) is a non-commutative \( K \)-algebra where the ring \( P \) and the monoid \( \Sigma \) are embedded. Note that if \( \Sigma = (\sigma) \) with \( \sigma : P \to P \) a map of infinite order one has that \( S \approx P[s; \sigma] \), the skew polynomial ring in the variable \( s \) and coefficients in \( P \) defined by the endomorphism \( \sigma \). Moreover, if \( P \) is a domain and all maps in \( \Sigma \) are injective then \( S \) is also a domain. To simplify notations, we denote \( f^\sigma = \sigma(f) \) for any \( f \in P, \sigma \in \Sigma \).

**Definition 2.1.** Let \( M \) be a \( P \)-module. We call \( M \) a \( \Sigma \)-invariant module if there is a monoid homomorphism \( \rho : \Sigma \to \text{End}_K(M) \) such that \( \rho(\sigma)(fx) = f^\sigma \rho(\sigma)(x) \), for all \( f \in P, x \in M \) and \( \sigma \in \Sigma \). We denote as usual \( \sigma \cdot x = \rho(\sigma)(x) \). If \( M, M' \) are \( \Sigma \)-invariant modules and \( \psi : M \to M' \) is a \( P \)-module homomorphism such that \( \psi(\sigma \cdot x) = \sigma \cdot \psi(x) \) for all \( x \in M, \sigma \in \Sigma \), then the map \( \psi \) is called a homomorphism of \( \Sigma \)-invariant modules.

**Proposition 2.2.** The category of \( \Sigma \)-invariant \( P \)-modules is equal to the category of left \( S \)-modules.

**Proof.** Let \( M \) be a left \( S \)-module. Then \( M \) is a \( P \)-module since \( P \subset S \). By restriction to \( \Sigma \subset S \), one has a monoid homomorphism \( \rho : \Sigma \to \text{End}_K(M) \). Moreover we have \( \sigma \cdot (fx) = (\sigma f) \cdot x = (f^\sigma \sigma) \cdot x = f^\sigma (\sigma \cdot x) \), for all \( f \in P, x \in M \) and \( \sigma \in \Sigma \). Let now \( M \) be a \( \Sigma \)-invariant \( P \)-module. We can define a left \( S \)-module structure by putting \((\sum_i f_i \sigma_i) \cdot x = \sum_i f_i(\sigma_i \cdot x) \) with \( f_i \in P, \sigma_i \in \Sigma \) and \( x \in M \). Consider
a homomorphism \( \varphi : M \to M' \) of \( \Sigma \)-invariant modules. Since \( \varphi \) is \( P \)-linear, one has \( \varphi((\sum_i f_i(\sigma_i) \cdot x)) = \sum_i f_i(\varphi(\sigma_i) \cdot x) = \sum_i f_i(\sigma_i \cdot \varphi(x)) = ((\sum_i f_i(\sigma_i)) \cdot \varphi(x). \right)

Let \( M \) be a left \( S \)-module and let \( G = \{g_i\} \subset M \) be a generating set of \( M \). Note that \( x \in M \) if and only if \( x = \sum_i f_i(\sigma_i \cdot g_i) \) with \( f_i \in P \) that is \( M \) is generated by \( \Sigma \cdot G = \{\sigma \cdot g_i\}, \sigma \) as \( P \)-module. We want now to describe homogeneous bases for graded two-sided ideals of \( S \). In fact, the algebra \( S \) has a natural grading over the monoid \( \Sigma \) that is \( S = \bigoplus_{\sigma \in \Sigma} S_{\sigma} \) and \( S_{\sigma} S_{\tau} \subset S_{\sigma \tau} \) where \( S_{\sigma} = P \sigma \). Note that \( S_{id} = P \), all \( S_{\sigma} \) are \( P \)-submodules of \( S \) and \( S_{\sigma \tau} = S_{\sigma \tau}. \right)

**Proposition 2.3.** Let \( J \subset S \) be a graded (two-sided) ideal and let \( G \subset J \) be a set of homogeneous elements. Then \( G \) is a generating set of \( J \) if and only if \( G \) is a basis of \( J \) as \( P \)-module.

**Proof.** Assume \( G = \{g_i\}_{\sigma_i} \) with \( g_i \in P, \sigma_i \in \Sigma \), for all \( i \). Let \( p_i, q_i \in S \) with \( q_i = \sum_{\sigma} q_{i, \sigma} \sigma \) and \( q_{i, \sigma} \in P \). It is sufficient to note that \( \sum_i p_i g_i \sigma_i q_i = \sum_{\sigma} \sum_i p_i g_i \sigma_i q_{i, \sigma} \sigma = \sum_{\sigma} \sum_i p_i g_i \sigma_i q_{i, \sigma} \sigma. \)

**Corollary 2.4.** Let \( f, g \in S \) and let \( g \) be a homogeneous element. Then, one has that \( f = pgq \) with \( p, q \in S \) if and only if \( f \) belongs to the (graded) left ideal generated by \( \{g_\sigma\}_{\sigma \in \Sigma}. \right)

3. Monomial orderings and Gröbner bases

Denote \( \mathbb{N} = \{0, 1, \ldots \} \) the set of non-negative integers and let \( X = \{x_0, x_1, \ldots \} \) be a countable set. From now on, we make the assumption that \( P = K[X] \) is a commutative polynomial ring in the variables set \( X \). Starting from Section 6 we will assume in particular that this set has the form \( X \times \mathbb{N} \). Moreover, we fix \( \sigma : P \to P \) an algebra homomorphism of infinite order and define the monoid \( \Sigma = \langle \sigma \rangle \cong \mathbb{N}. \right\)

Then, the skew monoid ring \( S = P \ast \Sigma \) is isomorphic to the skew polynomial ring \( P[s; \sigma] \) and we identify \( \Sigma = \{s^i\} \) with the monoid \( \{s^i\} \) of powers of the variable \( s \). Note that \( S \) is a free \( P \)-module of infinite rank. We denote \( f^{s^i} = f^{s^i} = s^{i}(f) \), for all \( f \in P, i \geq 0 \). Moreover, a homogeneous element \( f \in S_{l} = Ps^l \) for some \( i \) is also called \( s \)-homogeneous and we put \( \deg_s(f) = i \). Note finally that in the theory of difference ideals (Levin, 2008), the ring \( S \) is called ring of ordinary difference operators over \( P \).

Denote by \( \text{Mon}(P) \) the set of all monomials of \( P \) (including 1). Clearly, \( \text{Mon}(P) \) is a multiplicative \( K \)-basis of \( P \) that is \( mn \in \text{Mon}(P) \) for all \( m, n \in \text{Mon}(P) \). By definition of \( S \), a \( K \)-basis of such algebra is given by the elements \( ms^i \) where \( m \in \text{Mon}(P) \) and \( i \geq 0 \) is an integer. We call such elements the **monomials** of \( S \) and we denote the set of them as \( \text{Mon}(S) \). Clearly \( \text{Mon}(P) \subset \text{Mon}(S) \). Note that \( \text{Mon}(S) \) is in fact the “monomial basis” of \( S \) as a free \( P \)-module.

In what follows, we assume also that the endomorphism \( \sigma : P \to P \) is injective and **monomial** that is it stabilizes the set \( \text{Mon}(P) \). In other words, \( \{\sigma(x_i)\} \) is a set of algebraically independent monomials. Since \( P \) is a domain, it follows that \( S \) is also a domain and the \( K \)-basis \( \text{Mon}(S) \) is multiplicative since \( ms^i ms^j = mn^{i+j} \neq 0, \) for all \( m, n \in \text{Mon}(P) \) and \( i, j \geq 0 \).

We want to study now some divisibility relations in \( \text{Mon}(S) \). Let \( f, g \in S \). We say that \( f \) **left-divides** \( g \) if there is \( a \in S \) such that \( g = af \). Clearly, left divisibility is a partial ordering (up to units). Since \( \sigma \) is a monomial injective map, one has that if \( f, g \in \text{Mon}(S) \) then also \( a \in \text{Mon}(S) \).

**Proposition 3.1.** Let \( v = ms^j, w = ns^j \in \text{Mon}(S) \) with \( m, n \in \text{Mon}(P) \). Then \( v \) left-divides \( w \) if and only if \( i \leq j \) and \( m s^{j-i} | n. \)

**Proof.** Let \( a = ps^k \in \text{Mon}(S) \) with \( p \in \text{Mon}(P) \) such that \( ns^j = ps^k ms^i = pm^{i} s^{j+i}. \) Then, we have that \( j - i = k \geq 0 \) and \( m s^{i} | n. \)

Note that \( S \) has also a free \( P \)-module structure and so \( \text{Mon}(S) \) inherits another notion of divisibility. Precisely, let \( v, w \in \text{Mon}(S) \). We say that \( v \) **P-divides** \( w \) if \( \deg_p(v) = \deg_p(w) \) and there is \( a \in \text{Mon}(P) \) such that \( w = av \). Clearly \( P \)-divisibility is a partial ordering and one has the following result.
Proposition 3.2. Let \( v, w \in \text{Mon}(S) \). Then \( v \) left-divides \( w \) if and only if \( s^k v \ P \)-divides \( w \) for some \( k \geq 0 \).

Note that left divisibility coincides with \( P \)-divisibility when the monomials have the same \( s \)-degree. If there are \( v, w, a, b \in \text{Mon}(S) \) such that \( w = avb \) we say that \( v \) (two-sided) divides \( w \). It is easy to prove that such divisibility is also a partial ordering.

Proposition 3.3. Let \( v, w \in \text{Mon}(S) \). Then \( w \) is a multiple of \( v \) if and only if there is \( j \geq 0 \) such that \( w \) is a left multiple of \( vs^j \), that is \( w \) is a \( P \)-multiple of \( si vs^j \) for some \( i, j \geq 0 \).

Proof. Since monomials are \( s \)-homogeneous elements of \( S \), by applying Corollary 2.4 we obtain that \( w \) is a multiple of \( v \) if and only if \( w \) belongs to the (graded) left ideal generated by \( \{vs^j\}_{j \geq 0} \). Clearly, this happens when \( w \) is a left multiple of \( vs^j \) for some \( j \).

We start now considering monomial orderings.

Definition 3.4. Let \( \prec \) be a total ordering on the set \( \text{Mon}(S) \). We call \( \prec \) a monomial ordering of \( S \) if it satisfies the following conditions

(i) \( \prec \) is a well-ordering, that is every non-empty set of \( \text{Mon}(S) \) has a minimal element;
(ii) \( \prec \) is compatible with multiplication, that is if \( v \prec w \) then \( pvq \prec pwq \), for all \( v, w, p, q \in \text{Mon}(S) \).

It follows immediately that \( 1 \prec w \) for any \( w \in \text{Mon}(S) \) and if \( w = pvq \) with \( p \neq 1 \) or \( q \neq 1 \) then \( v \prec w \) for all \( v, w, p, q \in \text{Mon}(S) \). Note that the above conditions agree with general definitions of orderings on \( K \)-bases of associative algebras that provide a Gröbner basis theory (see for instance Green, 2000; Li, 2002). The same conditions define monomial orderings of the free algebras \( K(X) \) and \( K[X] \). Note that such algebras can be endowed with a monomial ordering even if the set of variables \( X \) is countable. This is provided by the Higman’s lemma (Higman, 1952) which implies that any multiplicatively compatible total ordering of the monomials such that \( 1 \prec x_0 \prec x_1 \prec \cdots \) is a monomial ordering. Recall that \( f^i \) stands for \( \sigma(f) \) for any \( f \in P \).

Definition 3.5. Let \( \prec \) be a monomial ordering on \( P \). We call \( \sigma \) compatible with \( \prec \) if \( \sigma \) is a strictly increasing map when restricted to \( \text{Mon}(P) \), that is \( m \prec n \) implies that \( m^i \prec n^i \) for all \( m, n \in \text{Mon}(P) \).

The following result is based essentially on Remark 3.2 in Brouwer and Draisma (2011).

Proposition 3.6. Assume \( \sigma \) be compatible with \( \prec \). Then \( \sigma \) is not an automorphism and \( m \preceq m^e \), for all \( m \in \text{Mon}(P) \).

Proof. Since \( \sigma \neq id \), there is \( m \in \text{Mon}(S) \) such that \( m \neq m^e \). If \( m > m^e \), by compatibility of \( \sigma \) one gets an infinite descending chain \( m > m^i > m^{i^2} > \cdots \) which contradicts the condition that \( \prec \) is a well-ordering. We conclude that \( m \prec m^i \). Assume that \( \sigma \) has the inverse \( \sigma^{-1} \). By applying \( \sigma \), from \( m^{i-1} \prec n^{i-1} \) it follows that \( m \prec n \). Since \( \sigma^{-1} \) is injective, we have therefore that \( m \prec n \) implies that \( m^{i-1} \prec n^{i-1} \). Now, by compatibility of \( \sigma^{-1} \) we obtain \( m \prec m^{i-1} \) which contradicts \( m \prec m^e \).

There are many endomorphisms \( \sigma \) with are compatible with usual monomial orderings on \( P \) like \( \text{lex}, \text{degrevlex}, \text{etc.} \). For instance, we have the following maps.

- \( \sigma(x_i) = x_{f(i)} \) for any \( i \), where \( f : \mathbb{N} \to \mathbb{N} \) is a strictly increasing map. Such maps have been considered in Brouwer and Draisma (2011). In particular, one may define the shift operator \( \sigma(x_i) = x_{i+1} \) which is used in difference algebra.
- \( \sigma(x_i) = x_i^e \) for any \( i \), with \( e > 1 \). This map has been considered in Weispfenning (1992).
Proposition 3.7. Let $\prec$ be a monomial ordering on $S$. Then $\sigma$ is compatible with the restriction of $\prec$ to $\text{Mon}(P)$. Moreover, for any $m, n \in \text{Mon}(P)$ and $i, j \geq 0$ one has that $ms^i \prec ns^j$ implies that $m \prec n$ or $i \prec j$.

Proof. Suppose $m \prec n$ with $m, n \in \text{Mon}(P)$. Then $sm \prec sn$ that is $m^ts \prec n^ts$. If $m^ts \succ n^ts$ then $m^ts \succ n^ts$ which is a contradiction. We conclude that $m^t \prec n^t$. Now, assume that $m \succ n$ and $i \succ j$. We have $m^s \succ m^j \succ ns^j$. □

Assume now $\sigma$ be compatible with a monomial ordering $\prec$ of $P$. We define a total ordering on $\text{Mon}(S)$ by putting $ms^i \prec' ns^j$ if and only if $i \prec j$, or $i = j$ and $m \prec n$, for all $m, n \in \text{Mon}(P)$ and $i, j \geq 0$. Clearly, the restriction of $\prec'$ to $\text{Mon}(P)$ is $\prec$.

Proposition 3.8. The ordering $\prec'$ is a monomial ordering on $S$ that extends $\prec$.

Proof. Clearly, an infinite descending sequence in $\text{Mon}(S)$ implies an infinite descending sequence in $\text{Mon}(P)$ which contradicts the condition that $\prec$ is a well-ordering. Let $ms^i, ns^j \in \text{Mon}(S)$ and suppose $ms^i \prec ns^j$ that is $i \prec j$, or $i = j$ and $m \prec n$. Let $qs^k \in \text{Mon}(S)$ and consider right multiplications $ms^qqs^k = mq^is^{i+k}$ and $ns^qqs^k = qn^js^{j+k}$. If $i \prec j$ then $i + k < j + k$. If $i = j$ and $m \prec n$ then $mq^i < qn^j = qn^j$. We conclude in both cases that $mq^i s^{i+k} < qn^j s^{j+k}$. For left multiplications $qs^km^s = qm^ks^{k+i}$ and $qs^kn^s = qn^ks^{k+j}$, note that $m \prec n$ implies that $m^s \prec n^s$. Then, one may argue in a similar way as for right multiplications. □

Clearly, a byproduct of Propositions 3.7 and 3.8 is that there exist monomial orderings on the skew polynomial ring $S$ if and only if $\sigma$ is compatible with a monomial ordering of $P$. Note that $\prec'$ is well known as module ordering when we consider $S$ as a free $P$-module. Moreover, by Proposition 3.7 it follows also that the monomial ordering of $S$ is uniquely defined by the one of $P$ when one compares monomials of the same $s$-degree.

From now on, we assume $S$ be endowed with a monomial ordering $\prec$.

Definition 3.9. Let $f \in S$, $f = \sum_i \alpha_i m_i s^i$ with $m_i \in \text{Mon}(P)$, $\alpha_i \in K^*$. Then, we denote $\text{lm}(f) = mk^s = \max \{m_i s^i\}$, $\text{lc}(f) = \alpha_k$ and $\text{lt}(f) = \text{lc}(f) \text{lm}(f)$. If $G \subset S$ we put $\text{Im}(G) = \{\text{lm}(f) \mid f \in G, f \neq 0\}$. We denote as $L(G)$ and $L(G)$ respectively the two-sided ideal and the left ideal of $S$ generated by $\text{Im}(G)$. Moreover, we denote by $L_P(G)$ the $P$-submodule of $S$ generated by $\text{Im}(G)$.

Proposition 3.10. Let $J$ be an ideal (respectively left ideal) of $S$. Then, the set $\{w + J \mid w \in \text{Mon}(S) \setminus L(M(J))\}$ (respectively $\{w + J \mid w \in \text{Mon}(S) \setminus L(M(J))\}$) is a $K$-basis of the space $S/J$. If $J \subset S$ is a $P$-submodule, in the same way one defines the $K$-basis $\{w + J \mid w \in \text{Mon}(S) \setminus L_P(J)\}$.

Proof. Let $w \in \text{Mon}(S)$. By induction on the monomial ordering of $S$, we can assume that for any monomial $v \in \text{Mon}(S)$ such that $v \prec w$ there is a polynomial $f \in S$ belonging to the span of $N = \text{Mon}(S) \setminus L(M(J))$ such that $v = f - w$. If $w \notin N$ then there is $g \in J$ such that $w = p \text{lm}(g)q$ with $p, q \in \text{Mon}(S)$. Therefore $f = w - (1/\text{lc}(g))pqg$ is such that $\text{lm}(f) \prec w$ and by induction $f - f' \in J$ for some $f' \in \langle N \rangle_K$. We conclude that $w - f' \in J$. Finally if $f \in N \cap J$ then necessarily $f = 0$. Mutatis mutandis one proves the remaining assertions. □

Definition 3.11. Let $J$ be an ideal (respectively left ideal) of $S$ and $G \subset J$. We call $G$ a Gröbner basis (respectively left basis) of $J$ if $L(G) = L(M(J))$ (respectively $L(M(G)) = L(M(J))$). As usual, if $J$ is a $P$-submodule of $S$ then $G$ is a Gröbner $P$-basis of $J$ when $L_P(G) = L_P(J)$.

Proposition 3.12. Let $J$ be an ideal (respectively left ideal) of $S$ and $G \subset J$. The following conditions are equivalent:

(i) $G$ is a Gröbner basis (respectively left basis) of $J$;
(ii) for any \( f \in J \), one has a Gröbner representation of \( f \) with respect to \( G \) that is \( f = \sum f_i g_i h_i \) (respectively \( f = \sum f_i g_i h_i \)) with \( \text{lm}(f) \supseteq \text{lm}(f_i) \text{lm}(g_i) \text{lm}(h_i) \) (respectively \( \text{lm}(f) \supseteq \text{lm}(f_i) \text{lm}(g_i) \)) and \( f_i, h_i \in S \), for all \( i \).

A similar characterization holds for Gröbner \( P \)-bases.

**Proof.** It follows immediately by the reduction process which is implicit in the proof of Proposition 3.10. \( \square \)

**Proposition 3.13.** Let \( J \) be a graded ideal of \( S \) and \( G \subset J \) be a subset of \( s \)-homogeneous elements. The following conditions are equivalent:

(i) \( G \) is a Gröbner basis of \( J \);

(ii) \( \Sigma G \) is a Gröbner left basis of \( J \);

(iii) \( \Sigma G \) is a Gröbner \( P \)-basis of \( J \).

**Proof.** Assume \( G = \{g_i\} \) is a Gröbner basis of \( J \) and put \( d_i = \text{deg}_s(g_i) \). If \( f \in J \) then one has \( f = \sum f_i g_i h_i \) where \( f_i, h_i \in S \) and \( \text{lm}(f) \supseteq \text{lm}(f_i) \text{lm}(g_i) \text{lm}(h_i) \), for all \( i \). Decompose \( h_i = \sum h_{ij} s_j^l \) with \( h_{ij} \in P \) for any \( i, j \). Then, we have \( \text{lm}(f) \supseteq \text{lm}(f_i) \text{lm}(g_i) \text{lm}(h_{ij}) s_j^l \), for all \( i, j \). Since \( \text{lm}(g_i) \) has \( s \)-degree \( d_i \), one obtains \( \text{lm}(f_i) \text{lm}(g_i) \text{lm}(h_{ij}) s_j^l = \text{lm}(f_i) \text{lm}(h_{ij}) g_i s_j^l \). Moreover, as in Proposition 2.3, we have \( f = \sum_{i,j} f_i g_i h_{ij} s_j^l = \sum_{i,j} f_i h_{ij}^{g_i} g_i s_j^l \). From \( \sigma \) compatible with \( \prec \) it follows that \( \text{lm}(h_{ij}^{g_i}) = \text{lm}(h_{ij}) g_i s_j^l \) and hence \( f \) has a left Gröbner representation with respect to \( G \Sigma \), that is this set is a left Gröbner basis of \( J \). The rest of the proof is straightforward. \( \square \)

**4. Buchberger algorithm**

After Proposition 3.13, in order to obtain a homogeneous Gröbner basis \( G \) of a (two-sided) graded ideal \( J \subset S \) one has to start with a homogeneous generating set \( H \) and consider the \( P \)-basis \( H' = \Sigma H \Sigma \). Then, one should transform \( H' \) into a homogeneous Gröbner \( P \)-basis \( G' \) of \( J \) and finally reduce \( G' \) as \( G' = \Sigma G \Sigma \) with \( G \subset J \). Apart with problems concerning termination of the module Buchberger algorithm (\( P \) is not Noetherian and \( S \) is a \( P \)-module of countable rank) that we will show solvable for the truncated algorithm up to some \( s \)-degree (see Proposition 4.7), it is more desirable to have a procedure able to compute \( G \) without actually considering all elements of \( G' \). To obtain this, we need an additional requirement for the endomorphism \( \sigma \).

Note that, since \( \sigma : P \to P \) is a ring homomorphism, such map is increasing with respect to the divisibility relation in \( P \), that is \( f | g \) implies that \( f^s | g^s \) and in this case \( (g/f)^s = g^s/f^s \) with \( f, g \in P \).

**Proposition 4.1.** The following conditions are equivalent:

(a) \( \gcd(x_i^s, x_j^s) = 1 \), for all \( i \neq j \);

(b) \( \gcd(m^s, n^s) = \gcd(m, n)^s \), for all \( m, n \in \text{Mon}(P) \).

Moreover, in this case one has \( m|n \) if and only if \( m^s|n^s \) and \( \text{lcm}(m^s, n^s) = \text{lcm}(m, n)^s \) with \( m, n \in P \). In other words, \( \sigma \) is a lattice homomorphism with respect to the divisibility relation in \( \text{Mon}(P) \).

**Proof.** Assume (a) and let \( m, n \in \text{Mon}(P) \) such that \( \gcd(m, n) = 1 \). If \( m = x_{i_1} \cdots x_{i_k} \) and \( n = x_{j_1} \cdots x_{j_l} \), then \( m^s = x_{i_1}^s \cdots x_{i_k}^s \) and \( n^s = x_{j_1}^s \cdots x_{j_l}^s \) with \( \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset \). Since \( \gcd(x_i^s, x_j^s) = 1 \) for all \( i \neq j \), we conclude that \( \gcd(m^s, n^s) = 1 \). Assume now \( \gcd(m, n) = u \) and hence \( \gcd(m/u, n/u) = 1 \). Then \( \gcd(m^s/u^s, n^s/u^s) = \gcd(m/u)^s, (n/u)^s) = 1 \) and therefore \( \gcd(m^s, n^s) = u^s \) that is (b) holds. Suppose \( m^s|n^s \) that is \( m^s = \gcd(m^s, n^s) = \gcd(m, n)^s \) since \( \sigma \) is injective we have that \( m = \gcd(m, n) \) that is \( m|n \). Moreover, one obtains \( \text{lcm}(m, n)^s = (mn/\gcd(m, n))^s = (mn)^s/\gcd(m, n)^s = m^s n^s/\gcd(m^s, n^s) = \text{lcm}(m^s, n^s) \) for all \( m, n \in \text{Mon}(P) \). \( \square \)
Definition 4.2. We say that \( \sigma \) is compatible with divisibility in \( \text{Mon}(P) \) if for all \( i \neq j \), one has \( \gcd(x_i^k, x_j^l) = 1 \) that is the variables occurring in the monomials \( x_i^k, x_j^l \) are disjoint.

Note that if a monomial endomorphism of \( P \) is compatible with divisibility then it is automatically injective since the monomials \( x_i^k \) are algebraically independent. Let \( | \) be the divisibility relation and \( \prec \) a monomial ordering on \( \text{Mon}(P) \). Throughout the rest of the paper, we make the assumption that the monomial endomorphism \( \sigma : P \to P \) is compatible both with \( | \) and with \( \prec \).

We recall now some basic results in the theory of module Gröbner bases by applying them to the free \( P \)-module \( S \) whose (left) free basis is \( \Sigma = \{ s_i \}_{i \geq 0} \). Consider \( f, g \in S \setminus \{0\} \) two elements whose leading monomials have the same \( s \)-degree (component), that is \( \text{lm}(f) = ms^l, \text{lm}(g) = ns^l \) with \( m, n \in \text{Mon}(P) \) and \( i \geq 0 \). If we put \( lc(f) = \alpha, lc(g) = \beta \) and \( l = \text{lcm}(m, n) \), one defines the \( S \)-polynomial \( \text{spoly}(f, g) = (l/\alpha m) f - (l/\beta n) g \). Clearly \( \text{spoly}(f, g) = -\text{spoly}(g, f) \) and \( \text{spoly}(f, f) = 0 \).

Proposition 4.3 (Buchberger criterion). Let \( G \) be a generating set of a \( P \)-submodule \( J \subset S \). Then \( G \) is a Gröbner basis of \( J \) if and only if for all \( f, g \in G \setminus \{0\} \) such that \( \deg_s(\text{lm}(f)) = \deg_s(\text{lm}(g)) \), the \( S \)-polynomial \( \text{spoly}(f, g) \) has a Gröbner representation with respect to \( G \).

Usually the above result, see for instance Eisenbud (1995), Greuel and Pfister (2008), is stated when \( P \) is a polynomial ring with a finite number of variables and \( S \) is a \( P \)-module of finite rank. In fact such assumptions are not needed since Noetherianity is not used in the proof, but only the existence of a monomial ordering for the ring \( P \) and the free module \( S \). See also the comprehensive Bergman’s paper Bergman (1978) where the “Diamond Lemma” is proved without any restriction on the finiteness of the variable set. In the following results we show how the Buchberger criterion, and hence the corresponding algorithm, can be reduced up to the symmetry defined by the monoid \( \Sigma \) on \( S \).

Lemma 4.4. Let \( f, g \in S \setminus \{0\} \) and let \( i \leq j \) such that \( \deg_s(\text{lm}(f)) + i = \deg_s(\text{lm}(g)) + j \). Then \( \text{spoly}(s_i f, s_j g) = s_i^j \text{spoly}(f, s_j s_i^{-j} g) \) and \( \text{spoly}(s_j f, s_i g) = \text{spoly}(f, s_j g s_i^{-j}) s_i^j \).

Proof. Let \( \text{lt}(f) = \alpha m^k, \text{lt}(g) = \beta n^l \) with \( \alpha, \beta \in \text{Mon}^* \) and \( m, n \in \text{Mon}(P) \). Then \( \text{lt}(s_i^j f) = \alpha m^{ij} s_i^j \), \( \text{lt}(s_j^i g) = \beta n^{il} s_j^i \) and \( \text{lt}(s_j^{-i} g) = \beta n^{il} s_j^{-i} \). By compatibility of \( \sigma \) with divisibility in \( \text{Mon}(P) \), if \( q = \text{lcm}(m, n^{i-j}) \) then \( \text{lcm}(m^i, n^{i-j}) = q^i \). Therefore \( h = \text{spoly}(f, s_j^{-i} g) = (q/\alpha m) f - (q/\beta n s_j^{-i}) s_j^{-i} g \) and hence we have \( s_i^j h = (q^j/\alpha m^{ij}) s_i^j f - (q^j/\beta n s_j^{-i}) s_j^{-i} g = \text{spoly}(s_i^j f, s_j^i g) \).

Note now that \( \text{lt}(s_j^i f) = \alpha m^{ij}, \text{lt}(s_i^j g) = \beta n^{il} \) and \( \text{lt}(s_j^{-i} g)ay = \beta n^{il} s_j^{-i} \). If \( q = \text{lcm}(m, n) \) and \( h = \text{spoly}(f, s_j^{-i} g) = (q/\alpha m) f - (q/\beta n) g s_j^{-i} \) we have simply that \( h s_i^j = (q/\alpha m) f s_i^j - (q/\beta n) g s_j^{-i} = \text{spoly}(f, s_j^i g) \).

Proposition 4.5 (Two-sided \( \Sigma \)-criterion). Let \( G \) be an \( S \)-homogeneous basis of a graded two-sided ideal \( J \subset S \). Then \( G \) is a Gröbner basis of \( J \) if and only if for all \( f, g \in G \setminus \{0\} \) and for any \( i, j \geq 0 \), the \( S \)-polynomials \( \text{spoly}(f, s_j^i g s_j^{-i}) (\deg_s(f) = \deg_s(g) + i + j) \) and \( \text{spoly}(f s_i^j, s_j g) \) (\( \deg_s(f) + i = \deg_s(g) + j \)) have a Gröbner representation with respect to \( \Sigma G \Sigma \).

Proof. By Proposition 3.13 we have to prove that \( G' = \Sigma G \Sigma \) is a Gröbner basis of \( J \) as \( P \)-module, that is \( G' \) is \( P \)-basis of \( J \) and the \( S \)-polynomials \( h = \text{spoly}(s_i^j f s_i^k, s_j^l g s_i^k) \) have a Gröbner representation with respect to \( G' \) for all \( f, g \in G \setminus \{0\} \) and for any \( i, j, k, l \geq 0 \) such that \( \deg_s(f) + i + k = \deg_s(g) + j + l \). Since \( G \) is a homogeneous basis of \( J \) as two-sided ideal, from Proposition 2.3 it follows that \( G' \) is a generating set of \( J \) as \( P \)-module. Consider now all possibilities \( i \leq j \) or \( i \geq j \) and \( k \leq l \) or \( k \geq l \) and apply Lemma 4.4. If \( i \leq j, k \leq l \) one has \( h = s_i^j \text{spoly}(f, s_j^{-i} g s_i^{-k}) s_k^l \), if \( i \leq j, k \geq l \) then \( h = s_i^j \text{spoly}(f s_i^{-k}, s_i^{-j} g s_i^{-l}) \), and so on. Then, assume that a \( S \)-polynomial \( h = \text{spoly}(f, g) \), with \( f, g \in G' \setminus \{0\} \), has a Gröbner representation with respect to \( G' \) as \( P \)-basis of \( J \), that is \( h = \sum_i f_i g_i \) with \( f_i \in P, g_i \in G' \) and \( \text{lm}(h) \geq \text{lm}(f_i \text{lm}(g_i)) \) for all \( i \). We have to prove that \( s_i^k h s_l^j \) has also a Gröbner representation with respect to \( G' \) for any \( k, l \geq 0 \). One has that \( s_k^l h s_i^j = \sum f_i^k s_i^l g_i s_i^j \) and \( \text{lm}(s_k^l h s_i^j) = \text{lm}(f_i^k s_i^l g_i s_i^j) \).
A criterion similar to Proposition 4.5 holds clearly for Gröbner left bases of left ideals of $S$ where no restrictions about the $s$-homogeneity of bases and ideals are needed.

**Proposition 4.6 (Left $\Sigma$-criterion).** Let $G$ be a basis of a left ideal $J \subset S$. Then $G$ is a Gröbner basis of $J$ if and only if for all elements $f, g \in G \setminus \{0\}$ such that $i = \deg_r(\text{lt}(f)) - \deg_r(\text{lt}(g)) \geq 0$, the $S$-polynomial $\text{spoly}(f, s^i g)$ has a Gröbner representation with respect to $\Sigma G$.

A standard procedure in the (module) Buchberger algorithm is the following.

**Algorithm 4.1 REDUCE**

**Input:** $f \in S$ and $G \subset S$.
**Output:** $h \in S$ such that $f - h \in (G)_P$ and $h = 0$ or $\text{lt}(h) \not\in \text{LM}_P(G)$.

1. $h := f$;
2. **while** $h \neq 0$ and $\text{lt}(h) \not\in \text{LM}_P(G)$ **do**
   1. choose $g \in G, g \neq 0$ such that $\text{lt}(g) P$-divides $\text{lt}(h)$;
   2. $h := h - (\text{lt}(h)/\text{lt}(g)) g$;
3. **end while**;
4. return $h$.

Note that if $\text{lt}(g) = \alpha ms^i$, $\text{lt}(h) = \beta ns^i$ with $\alpha, \beta \in K^*$ and $m, n \in \text{Mon}(P)$, by definition $\text{lt}(h)/\text{lt}(g) = (\alpha m)/(\beta n)$. Moreover, the termination of REDUCE is provided since $\prec$ is a well-ordering on $\text{Mon}(S)$. In particular, even if $G$ is an infinite set, there are only a finite number of elements $g \in G, g \neq 0$ such that $\text{lt}(g) P$-divides $\text{lt}(h)$ and hence $\text{lt}(g) \preceq \text{lt}(h)$.

It is well known that if REDUCE($f, G = h \neq 0$ then $f$ has a Gröbner representation with respect to $G$. Moreover, if REDUCE($f, G = h \neq 0$ then clearly we have REDUCE($f, G \cup \{h\} = 0$. Therefore, from Proposition 4.5 it follows immediately the correctness of the following algorithm.

**Algorithm 4.2 SkewGBasis**

**Input:** $H$, an $s$-homogeneous basis of a graded two-sided ideal $J \subset S$.
**Output:** $G$, an $s$-homogeneous Gröbner basis of $J$.

1. $G := H$;
2. $B := \{(f, g) \mid f, g \in G\}$;
3. **while** $B \neq \emptyset$ **do**
   1. choose $(f, g) \in B$;
   2. $B := B \setminus \{(f, g)\}$;
   3. **for all** $i, j \geq 0$ such that $i + j = \deg_r(f) - \deg_r(g)$ **do**
      1. $h := \text{Reduce}(\text{spoly}(f, s^i g^j), \Sigma G \Sigma)$;
      2. **if** $h \neq 0$ **then**
         1. $B := B \cup \{h, (h, k), (k, h) \mid k \in G\}$;
         2. $G := G \cup \{h\}$;
      **end if**;
   **end if**;
4. **end for**;
5. **for all** $i, j \geq 0$ such that $j - i = \deg_r(f) - \deg_r(g)$ **do**
   1. $h := \text{Reduce}(\text{spoly}(f^i s^j g), \Sigma G \Sigma)$;
   2. **if** $h \neq 0$ **then**
      1. $B := B \cup \{(h, k), (k, h) \mid k \in G\}$;
      2. $G := G \cup \{h\}$;
   **end if**;
5. **end for**;
6. **end while**;
7. return $G$.

Clearly, all well-known criteria (product criterion, chain criterion, etc.) can be applied to SkewGBasis to shorten the number of $S$-polynomials to be considered. In fact, this algorithm can be understood
as the usual (module) Buchberger procedure applied to the $P$-basis $\Sigma H \Sigma$, where an additional criterion to avoid “useless pairs” is provided by Proposition 4.5. Note that owing to Proposition 4.6, one has also a similar procedure for computing a Gröbner left basis of any left ideal of $S$. Since the set $\Sigma H \Sigma$ if infinite even if the basis $H$ is eventually finite ($S$ is a non-Noetherian ring) one has that SkewGBasis does not admit general termination. In particular, the cycle “for all $i, j \geq 0$ such that $j - i = \deg_s(f) - \deg_s(g)$ do” never stops unless one bounds the $s$-degree $\deg_s(f) + i = \deg_s(g) + j$. As for other non-Noetherian structures like the free associative algebra that in fact can be embedded in $S$ (see Section 6), the termination of homogeneous Gröbner bases computations can be obtained only under truncation.

**Proposition 4.7.** Let $J \subset S$ be a graded two-sided ideal and fix $d \geq 0$. Assume that $J$ has a $s$-homogeneous basis $H$ such that $H_d = \{ f \in H \mid \deg_s(f) \leq d \}$ is a finite set. Then, there exists an $s$-homogeneous Gröbner basis $G$ of $J$ such that $G_d$ is also finite. In other words, if we consider a selection strategy for the $S$-polynomials based on their $s$-degree, we obtain that the $d$-truncated version of the algorithm SkewGBasis terminates in a finite number of steps.

**Proof.** Denote $H'_d = \{ s^{i} f s^{j} \mid f \in H_d, i, j \geq 0, i + j + \deg_s(f) \leq d \}$. Since $H_d$ is finite one has that $H'_d$ is also finite. Then, consider $X_d$ the finite set of variables of $P$ occurring in the elements of $H'_d$ and define $P^{(d)} = K[X_d]$ and $S^{(d)} = \bigoplus_{i \leq d} P^{(d)} s^{i}$. In fact, the $d$-truncated algorithm SkewGBasis computes a subset of a Gröbner basis of the $P^{(d)}$-submodule $J^{(d)} \subset S^{(d)}$ generated by $H'_d$. By Noetherianity of the ring $P^{(d)}$ and the free $P^{(d)}$-module $S^{(d)}$ which has finite rank, we clearly obtain termination. $\Box$

Note that the above result implies algorithmic solution of the membership problem for graded ideals of $S$ which are finitely generated up to any degree.

### 5. $\Sigma$-invariant ideals of $P$

In this section we define Gröbner bases of $\Sigma$-invariant ideals $I \subset P$ which generates $I$ up to the action of $\Sigma$. Moreover, if $P$ can be endowed with a suitable grading, we show how such bases can be computed in the algebra $S$ for a class of graded $\Sigma$-invariant ideals. As usual, we fix a monomial endomorphism $\sigma : P \rightarrow P$ which is compatible both with the divisibility and a monomial ordering on $\text{Mon}(P)$ and we extend this to an ordering on $\text{Mon}(S)$. From Section 2 we know that $\Sigma$-invariant ideals of $P$ are just left $S$-submodules of $P$. Since we make use of identification $\Sigma = \{ s^{i} \}$, for all $f \in P \subset S$ and for any $i \geq 0$ one has that $s^{i} f = f s^{i} = \sigma^{i} (f)$ and $s^{i} f = (s^{i} \cdot f) s^{i}$.

**Definition 5.1.** Let $I \subset P$ be a $\Sigma$-invariant ideal and $G \subset I$. We say that $G$ is a $\Sigma$-basis of $I$ if $G$ is a basis of $I$ as left $S$-module. In other words, $\Sigma \cdot G$ is a basis of $I$ as $P$-ideal.

**Proposition 5.2.** Let $G \subset P$. Then $\text{Im}(\Sigma \cdot G) = \Sigma \cdot \text{Im}(G)$. In particular, if $I$ is a $\Sigma$-invariant $P$-ideal then $\text{LM}_P(I)$ is also $\Sigma$-invariant.

**Proof.** Since $\sigma$ is compatible with the monomial ordering of $P$, it is sufficient to note that $\text{Im}(s^{i} \cdot f) = s^{i} \cdot \text{Im}(f)$ for any $f \in P$ and $i \geq 0$. $\Box$

**Definition 5.3.** Let $I \subset P$ be a $\Sigma$-invariant ideal and $G \subset I$. We call $G$ a Gröbner $\Sigma$-basis of $I$ if $\text{Im}(G)$ is a basis of $\text{LM}_P(I)$ as left $S$-module. In other words, $\Sigma \cdot G$ is a Gröbner basis of $I$ as $P$-ideal.

The computation of Gröbner $\Sigma$-bases of $\Sigma$-invariant $P$-ideals is relevant, for instance, in applications to difference algebra (cf. Levin, 2008, Chapter 3). Such computations appear also in other contexts, see for instance Drensky and La Scala (2006) and Brouwer and Draisma (2011). Note that in the latter paper Gröbner $\Sigma$-bases are named “equivariant Gröbner bases”.

In analogy with Propositions 4.5 and 4.6, we present here a $\Sigma$-criterion that allows to reduce the number of $S$-polynomials to be checked to provide that a $\Sigma$-basis is of Gröbner type.
Proposition 5.4 ($\Sigma$-criterion in $P$). Let $G$ be a $\Sigma$-basis of a $\Sigma$-invariant ideal $I \subset P$. Then $G$ is a Gröbner $\Sigma$-basis of $I$ if and only if for all $f, g \in G \setminus \{0\}$ and for any $i \geq 0$, the $S$-polynomial $\text{spoly}(f, s^i \cdot g)$ has a Gröbner representation with respect to $\Sigma \cdot G$.

Proof. Consider any pair of elements $s^j \cdot f, s^i \cdot g \in \Sigma \cdot G$ ($f, g \in G$) and let $i \leq j$. By compatibility of $\sigma$ with divisibility in $\text{Mon}(P)$ (cf. Lemma 4.4), one has that $\text{spoly}(s^j \cdot f, s^i \cdot g) = s^i \cdot \text{spoly}(f, s^j \cdot g)$ with $k = j - i$. Assume that $\text{spoly}(f, s^k \cdot g) = h = \sum \sigma_i(f_i(s^i \cdot g_i))$ ($f_i \in P, g_i \in G$) is a Gröbner representation with respect to $\Sigma \cdot G$. Since the endomorphism $\sigma$ is compatible with the monomial ordering of $P$, we have also the Gröbner representation $\text{spoly}(s^j \cdot f, s^i \cdot g) = s^i \cdot h = \sum \sigma_i(s^i \cdot f_i)(s^{i+j} \cdot g_i)$. $\square$

Note that some version of this criterion can be found in Brouwer and Draisma (2011, Theorem 2.5), where it is called “equivariant Buchberger criterion”. Before than this, the same ideas have been used in La Scala and Levandovskyy (2009) for Proposition 3.11. From this criterion it follows immediately the correctness of the following algorithm.

Algorithm 5.1 SIGMA BASIS

Input: $H$, a $\Sigma$-basis of a $\Sigma$-invariant ideal $I \subset P$.
Output: $G$, a Gröbner $\Sigma$-basis of $I$.

$G := H$;

while $B \neq \emptyset$ do

choose $(f, g) \in B$;

$B := B \setminus \{(f, g)\}$;

for all $i \geq 0$ do

$h := \text{REDUCE}(\text{spoly}(f, s^i \cdot g), \Sigma \cdot G)$;

if $h \neq 0$ then

$B := B \cup \{(h, h), (h, k), (k, h) | k \in G\}$;

$G := G \cup \{h\}$;

end if;

end for;

end while;

return $G$.

As for the algorithm SKEWGBASIS, all criteria to avoid useless pairs can be added to SIGMA BASIS. Note that termination of this algorithm is not provided in general (note the infinite cycle “for all $i \geq 0$ do”) and this is, in fact, one of the main problems in applications to differential/difference algebra. Nevertheless, in what follows we describe some class of $\Sigma$-invariant ideals of $P$ where a truncated version of the algorithm SIGMA BASIS stops in a finite number of steps. Such ideals are in bijective correspondence with a class of graded (two-sided) ideals of $S$ which have truncated termination of SKEWGBASIS provided by Proposition 4.7.

Consider now the $P$-module homomorphism $\pi : S \rightarrow P$ such that $s^i \mapsto 1$, for all $i$. Clearly $\pi$ is a left $S$-module epimorphism whose kernel is the left ideal of $S$ generated by $s - 1$.

Definition 5.5. Let $J$ be a graded ideal of $S$ and put $J^P = \pi(J)$. Clearly $J^P$ is a $\Sigma$-invariant ideal of $P$.

Proposition 5.6. Let $J \subset S$ be a graded ideal. If $G$ is a homogeneous basis of $J$ then $G^P = \pi(G)$ is a $\Sigma$-basis of $J^P$.

Proof. Since the map $\pi$ is a left $S$-module homomorphism, it is sufficient to note that $G \Sigma$ is a left basis of $J$ and $\pi(G \Sigma) = \pi(G) = G^P$. $\square$

We introduce now a grading on the algebra $P$ which is compatible with action of $\Sigma$. We start extending the structure $(\mathbb{N}, \max, +)$ in the following way.
Definition 5.7. Let $-\infty$ be an element disjoint by $\mathbb{N}$ and put $\hat{\mathbb{N}} = \{-\infty\} \cup \mathbb{N}$. Then, we define a commutative idempotent monoid $(\hat{\mathbb{N}}, \max)$ with identity $-\infty$ that extends $(\mathbb{N}, \max)$ (with identity 0) by imposing that $\max(-\infty, x) = x$ for any $x \in \mathbb{N}$. Moreover, we define a commutative monoid $(\hat{\mathbb{N}}, +)$ with identity $-\infty$ extending the monoid $(\mathbb{N}, +)$ by putting $-\infty + x = -\infty$, for all $x \in \mathbb{N}$. Since + clearly distributes over max, one has that $(\hat{\mathbb{N}}, \max, +)$ is a commutative idempotent semiring, also known as commutative dioid or max-plus algebra (Gondran and Minoux, 2008).

Note that if $\sigma^{-\infty} \in \text{End}_K(P)$ is the map such that $x_i \mapsto 0$ for any $x_i \in X$, then $\hat{\Sigma} = \{\sigma^{-\infty}\} \cup \Sigma$ is a commutative monoid isomorphic to $(\hat{\mathbb{N}}, +)$. Denote now $M = \text{Mon}(P)$ the set of monomials of the polynomial ring $P$.

Definition 5.8. A mapping $w : M \to \hat{\mathbb{N}}$ such that for all $m, n \in M$ and $x_i \in X$ one has

(i) $w(1) = -\infty$;
(ii) $w(mn) = \max(w(m), w(n))$;
(iii) $w(s \cdot x_i) = 1 + w(x_i)$,

is called a weight function of $P$ endowed with $\sigma$.

Note that if $m = x_{i_1} \cdots x_{i_d} \neq 1$ with $w(x_{i_1}) \leq \cdots \leq w(x_{i_d})$ then $w(m) = w(x_{i_d})$. Moreover, the condition (iii) implies that $w(s^i \cdot m) = i + w(m)$ for all $i \in \mathbb{N}$, $m \in M$ and hence $s^i \cdot m = m$ if and only if $m = 1$ or $i = 0$. We put $M_i = \{m \in M \mid w(m) = i\}$ for all $i \in \mathbb{N}$ and define $P_i \subset P$ the subspace spanned by $M_i$. We have that $P_{-\infty} = K$. Clearly $P = \bigoplus_{i \in \mathbb{N}} P_i$ is a grading of the algebra $P$ defined by the monoid $(\hat{\mathbb{N}}, \max)$ by means of the function $w$. Then, an element $f \in P_i$ is said $w$-homogeneous of weight $i$.

In what follows, we assume that $P$ is endowed with a weight function. In fact, such functions are easily to define. Consider for instance the polynomial ring $P = K[X \times \mathbb{N}]$ and denote $x_i(j)$ each variable $(x_i, j) \in X \times \mathbb{N}$. Let $\sigma : P \to P$ be the algebra monomorphism of infinite order such that $\sigma(x_i(j)) = x_i(j + 1)$, for all $i, j$. Clearly $\sigma$ is a monomial map compatible with divisibility in Mon($P$) and many usual monomial orderings on $P$, like lex, degrevlex, etc., are compatible with $\sigma$. For the algebra $P$ endowed with the map $\sigma$ we can clearly define the weight function $w(x_i(j)) = j$. In Section 6 we show how to embed the free associative algebra $K(X)$ into the skew polynomial ring defined by $P$ and the monoid $\Sigma = (\sigma)$. Moreover, if we put $x_i(j) = \sigma^j(u_i)$ where $x_i(0) = u_i = u_i(t)$ is a set of (algebraically independent) univariate functions and $\sigma$ is the shift operator $u_i(t) \mapsto u_i(t + h)$ then $P = K[X \times \mathbb{N}]$ is by definition the ring of ordinary difference polynomials with constant coefficients in the field $K$ (see Levin, 2008). Such algebra is used to study systems of (ordinary) difference equations for applications in combinatorics or discretization of systems of differential equations.

Definition 5.9. Let $I$ be an ideal of $P$. We call $I$ w-graded if $I = \sum_i I_i$ with $I_i = I \cap P_i$ for any $i \in \hat{\mathbb{N}}$.

Define now the skew monoid ring $\hat{\Sigma} = P \ast \hat{\Sigma}$ extending $\Sigma = P \ast \Sigma$ and let $\hat{\pi} : \hat{\Sigma} \to P$ the left $\hat{\Sigma}$-module epimorphism extending $\pi$ that is $s^i \mapsto 1$, for all $i \in \hat{\mathbb{N}}$. The existence of a weight function for $P$ implies that one has also a mapping $\xi : P \to \hat{\Sigma}$ such that $\hat{\pi} \xi = id$.

Proposition 5.10. Define $\xi : P \to \hat{\Sigma}$ the homogeneous injective map such that $f \mapsto \sum_{i \in \mathbb{N}} f_i s^i$, for all $f = \sum_{i \in \mathbb{N}} f_i$. Then $\xi$ is a $\hat{\Sigma}$-equivariant map.

Proof. For all $i, j \in \hat{\mathbb{N}}$ and $f \in P_j$ one has that $s^i \cdot f \in P_{i+j}$ and therefore $\xi(s^i \cdot f) = (s^i \cdot f)s^{i+j} = s^i fs^j = s^i \xi(f)$. □

Let $I \subset P$ be a w-graded $\Sigma$-invariant ideal and consider $\xi(I) \subset \hat{\Sigma}$. Note that if $I \neq P$ then $I_{-\infty} = 0$ and the set $\xi(I)$ is in fact contained in $S$. Then, to get rid of the symbol $-\infty$ we restrict ourselves to ideals not containing constants.
Definition 5.11. Let $I \subseteq P$ be a $w$-graded $\Sigma$-invariant ideal of $S$. Denote by $I^S$ the graded (two-sided) ideal of $S$ generated by $\xi(I) \subseteq S$. In other words, if we put $G = \xi(\bigcup_{i \geq 0} I_i) = \{ f s^i \mid f \in I_i, i \geq 0 \}$ then $I^S$ is the left ideal generated by $G \Sigma = \{ f s^i \mid f \in I_i, j \geq i \geq 0 \}$ or equivalently $I^S$ has the basis $G \Sigma = \Sigma G \Sigma$ as $P$-submodule of $S$. We call $I^S$ the skew analogue of $I$.

Proposition 5.12. Let $I \subseteq P$ be a $w$-graded $\Sigma$-invariant ideal. Then $I^{SP} = I$, that is there is a bijective correspondence between all $w$-graded $\Sigma$-invariant ideals different from $P$ and their skew analogues in $S$.

Proof. Put $J = I^{SP} = \pi(I^S)$. For any $f \in I_i$ and $j \geq i$ we have clearly $\pi(f s^j) = f$. Since the elements $f s^j$ are a left basis of $I^S$, the ideal $I$ is $\Sigma$-invariant and $\pi$ is a left $S$-module homomorphism, we have that $J \subseteq I$. Moreover, the elements $f \in I_i$ are a basis of $I = \sum I_i$ and one has also that $I \subseteq J$. □

The next propositions need the following lemmas.

Lemma 5.13. If $s^k \cdot m$ divides $n$, with $m, n \in M$, then $w(n) - k \geq w(m)$.

Proof. Since $n = q(s^k \cdot m)$ with $q \in M$, we have $w(n) \geq w(s^k \cdot m) = k + w(m)$. □

Lemma 5.14. Let $m, n \in M$ and put $l = \text{lcm}(m, n)$. Then, one has that $w(l) = \max(w(m), w(n))$.

Proof. By property (ii) of Definition 5.8, it is sufficient to note that max is an idempotent operation and hence the weight of a monomial depends only on the variables occurring in the support. □

Proposition 5.15. Let $I \subseteq P$ be a $w$-graded $\Sigma$-invariant ideal, then $I^S$ is a graded ideal of $S$. Let $G = \bigcup_{i \geq 0} G_i$ be a $w$-homogeneous $\Sigma$-basis of $I$ that is $G_i \subseteq I_i$. Then $G^S = \xi(G) = \{ f s^i \mid f \in G_i, i \geq 0 \}$ is an s-homogeneous basis of $I^S$.

Proof. Consider the elements $f s^j$ with $f \in I_i$, $j \geq i$ which form a left basis of $I^S$. Since $G$ is a $\Sigma$-basis, one has $f = \sum_k f_k(s^k \cdot g_k)$ with $f_k \in P$, $g_k \in G_k$. From $w(f) = l_i$ by Lemma 5.13 we obtain that $i - k \geq i_k$. We have therefore that $f s^j = \sum_k f_k(s^k \cdot g_k)s^j = \sum_k f_k s^k g_k s^{j-k}$ with $j - k \geq i - k \geq i_k$ and hence $g_k s^{j-k} \in G^S \Sigma$, for all $k$. □

Note now that by Proposition 3.7 we have that $ms^j \ll ns^j$ if and only if $m < n$, for all $m, n \in M$ and for any $i \geq 0$. In other words, if $f s^j (f \in P)$ is an $s$-homogeneous element of $S$ then $\operatorname{Im}(f s^j) = \operatorname{Im}(fs^j)$.

Lemma 5.16. Let $I \subseteq P$ be a $w$-graded $\Sigma$-invariant ideal. If $G = \bigcup_{i \geq 0} I_i$, by definition $I^S$ is the graded ideal of $S$ generated by $G^S = \xi(G)$. Then $G^S$ is an s-homogeneous Gröbner basis of $I^S$.

Proof. Let $f s^i, g s^j \in G^S \Sigma$ that is the $w$-homogeneous elements $f, g \in G$ are such that $i \geq w(f), j \geq w(g)$. Assume $i \geq j$ and put $k = i - j$. By Proposition 4.6 we have to check for Gröbner representations of the $S$-polynomial $\text{spoly}(f s^i, s^k g s^j)$ with respect to $\Sigma G^S \Sigma$. Since $G$ is clearly a Gröbner $\Sigma$-basis of $I$, one has that the $S$-polynomial $\text{spoly}(f s^i, s^k g s^j)$ has a Gröbner representation with respect to $\Sigma \cdot G$, say $h = \text{spoly}(f s^i, s^k g s^j) = \sum f_1(s^i \cdot g_i)$ with $f_1 \in P$, $g_i \in G$. Note that $\text{spoly}(f s^i, s^k g s^j) = h s^j = \sum f_1(s^i \cdot g_i)s^j$. We have to prove now that $i \geq l + w(g_i)$ for any $l$, because in this case one has the Gröbner representation $h s^1 = \sum f_1(s^i \cdot g_i)s^{1-i}$. In fact, by Lemmas 5.13 and 5.14 we have that $\max(w(f), w(g)) = w(h) \geq i + w(g_i)$. Then, from $i \geq w(f)$ and $i \geq j \geq w(g)$ one obtains the claim. □

Proposition 5.17. Let $G \subseteq \bigcup_{i \geq 0} P_i$. Then $\operatorname{Im}(G)^S = \operatorname{Im}(G)$. Moreover, if $I \subseteq P$ is a $w$-graded $\Sigma$-invariant ideal then $\operatorname{LM}_P(I)^S = \operatorname{LM}(I^S)$. 


The skew letterplace embedding generated up to weight \( \Sigma \) is a \( \Sigma \)-basis of \( I \) such that \( Gd \) is also a finite set. In other words, if we consider for the algorithm \( \text{SkewGBasis} \) stops in a finite number of steps. We denote by \( \Sigma \text{GBasis} \) the set of variables. We obtain that \( \text{Im}(G)^S = \text{Im}(G^S) \). Consider now \( G = \bigcup_l I_l \). By definition \( I^S \) is the ideal of \( S \) generated by \( G^S \). Moreover, since \( I = \bigcup_l I_l \) one has that \( \text{Im}(G) = \text{Im}(I) \) and hence \( \text{LM}(I^S) \) is the ideal generated by \( \text{Im}(G)^S = \text{Im}(G^S) \). Finally, by Lemma 5.16 one has that \( \text{LM}(I^S) \) is the ideal of \( S \) generated by \( \text{Im}(G^S) \).

**Proof.** If \( f \in P_l \) is a \( w \)-homogeneous element then \( w(\text{Im}(f)) = w(f) = i \) and \( \text{Im}(f)s^i = \text{Im}(f^S) \). We obtain that \( \text{Im}(G)^S = \text{Im}(G^S) \). Consider now \( G = \bigcup_l I_l \). By definition \( I^S \) is the ideal of \( S \) generated by \( G^S \). Moreover, since \( I = \bigcup_l I_l \) one has that \( \text{Im}(G) = \text{Im}(I) \) and hence \( \text{LM}(I^S) \) is the ideal generated by \( \text{Im}(G)^S = \text{Im}(G^S) \). Finally, by Lemma 5.16 one has that \( \text{LM}(I^S) \) is the ideal of \( S \) generated by \( \text{Im}(G^S) \).

**Proposition 5.18.** Let \( I \subseteq P \) be a \( w \)-graded \( \Sigma \)-invariant ideal. Let \( G = \bigcup_l G_l \) be a \( w \)-homogeneous Gröbner \( \Sigma \)-basis of \( I \). Then, \( G^S = \xi(G) \) is an \( s \)-homogeneous Gröbner basis of \( I^S \).

**Proof.** By hypothesis \( \text{Im}(G) \) is a \( \Sigma \)-basis of \( \text{LM}(I) \). Then \( \text{Im}(G)^S = \text{Im}(G^S) \) is a basis of \( \text{LM}(I^S) \) that is \( G^S \) is a Gröbner basis of \( I^S \).

**Proposition 5.19.** Let \( I \subseteq P \) be a \( w \)-graded \( \Sigma \)-invariant ideal. If \( G \) is an \( s \)-homogeneous Gröbner basis of \( I^S \) then \( G^p = \pi(G) \) is a Gröbner \( \Sigma \)-basis of \( I \).

**Proof.** Let \( f \in I_l \) for some \( l \geq 0 \) and consider the element \( f s^i \in I^S \). Since \( G \) is an \( s \)-homogeneous Gröbner basis of \( I^S \), there is \( g s^j \in G \ (g \in P, k \geq 0) \) such that \( \text{Im}(f s^i) = q s^i \text{Im}(g s^j) s^i \) that is \( \text{Im}(f s^i) = q s^i \text{Im}(g) s^{i+j} \) is also a finite set, consider \( G_{\text{GBasis}} \) this set of variables. We denote by \( \pi(G) \) is a \( \Sigma \)-basis of \( I \). The following result provides algorithmic solution of the membership problem for a class of \( \Sigma \)-invariant ideals. Note that such kind of results are quite rare, for instance, in the theory of difference ideals.

**Proposition 5.20.** Let \( I \subseteq P \) be a \( w \)-graded \( \Sigma \)-invariant ideal and fix \( d \geq 0 \). Assume that \( I \) has a \( w \)-homogeneous basis \( H \) such that \( H_d = \{ f \in H \mid w(f) \leq d \} \) is a finite set. Then, there is a \( w \)-homogeneous Gröbner \( \Sigma \)-basis \( G \) of \( I \) such that \( G_d \) is also a finite set. In other words, if we consider for the algorithm SIGMAGBAs a selection strategy of the \( S \)-polynomials based on their weights, we obtain that the \( d \)-truncated version of SIGMAGBAs stops in a finite number of steps.

**Proof.** First of all, note that the algorithm SIGMAGBAs essentially computes a subset \( G \) of a Gröbner basis \( \Sigma \cdot G \) obtained by applying the Buchberger algorithm to the basis \( \Sigma \cdot H \) of \( I \). Moreover, by property (iii) of Definition 5.8 and Lemma 5.14 the elements of \( \Sigma \cdot H \) and hence of \( \Sigma \cdot G \) are all \( w \)-homogeneous. Denote \( H_d' = \{ s^i \cdot f \mid i \geq 0, f \in H_d, i + w(f) \leq d \} \). Since \( H_d' \) is also a finite set, consider \( X_d \) is the finite set of variables of \( P \) occurring in the elements of \( H_d' \) and define \( P(d) = K[X_d] \). In fact, the \( d \)-truncated algorithm SIGMAGBAs computes a subset of a Gröbner basis of the ideal \( I(d) \) of \( P(d) \) generated by \( H_d' \). By Noetherianity of the ring \( P(d) \), we clearly obtain termination.

Note that the above result can be obtained also by Proposition 4.7. In fact, if \( I \subseteq P \) is finitely \( \Sigma \)-generated up to weight \( d \) then \( I^S \) is a graded ideal of \( S \) which is finitely generated up to \( s \)-degree \( d \). Precisely, if \( H = \bigcup_l H_l \) is a \( w \)-homogeneous \( \Sigma \)-basis of \( I \) and the set \( \bigcup_{l \leq d} H_l \) is finite for all \( d \), then \( \{ f s^i \mid f \in H_i, i \leq d \} \) is also a finite set that generates \( I^S \) up to degree \( d \).

### 6. The skew letterplace embedding

Denote \( \mathbb{N}^* = \{1, 2, \ldots\} \) the set of positive integers and let \( X = \{x_1, x_2, \ldots\} \) be a finite or countable set of variables. We denote by \( x_i(j) \) each element \( (x_i, j) \) of the product set \( X \times \mathbb{N}^* \) and define \( P = \)
Consider the algebra monomorphism of infinite order \( \sigma : P \to P \) such that \( x_i(j) \mapsto x_i(j + 1) \) for all \( i, j \). Note that \( \sigma \) is a monomial map that is compatible with divisibility in \( \text{Mon}(P) \). Then, put \( S = P[s; \sigma] \) the skew polynomial ring in the variable \( s \) defined by \( P \) and \( \sigma \). Finally, let \( F = K(X) \) denote the free associative algebra generated by \( X \). We consider \( F \) as a graded algebra with respect to the total degree. Recall that \( S = \bigoplus_{i \in \mathbb{N}} S_i \) is also a graded algebra with \( S_i = P^i_s \).

**Definition 6.1.** Let \( A \subset S \) be a \( K \)-subalgebra. If \( A \) is spanned by a submonoid \( M \subset \text{Mon}(S) \) then we call \( A \) a monomial subalgebra of \( S \) and we denote \( \text{Mon}(A) = M \). In this case, a monomial ordering of \( S \) can be restricted to \( A \).

For instance, \( P \) is a monomial subalgebra of \( S \). We have now a result about the possibility to embed the free associative algebra \( F \) into the skew polynomial ring \( S \).

**Proposition 6.2.** The graded algebra homomorphism \( \iota : F \to S, x_i \mapsto x_i(1)s \) is injective. Then, the free associative algebra \( F \) is isomorphic to \( R = \text{Im} \iota \), a graded monomial subalgebra of \( S \).

**Proof.** It is sufficient to note that by the commutation rule of the variable \( s \) and the definition of the endomorphism \( \sigma \), any word \( x_{i_1} \cdots x_{i_n} \in \text{Mon}(F) \) maps into \( x_{i_1}(1) \cdots x_{i_n}(d)s^d \in \text{Mon}(S) \). \( \square \)

We call \( S \) the skew letterplace algebra and the algebra monomorphism \( \iota \) the skew letterplace embedding. In Section 7 we will give motivation for such names. Fix now a monomial ordering \( \prec \) on the algebra \( S \) that is \( \sigma \)-compatible with the restriction of \( \prec \) to \( \text{Mon}(P) \). It is easy to show that many usual monomial orderings on \( P \) (lex, degrevlex, etc.) satisfy such condition. Recall that the existence of monomial orderings for \( P \) is provided by the Higman’s lemma which implies the following result (see for instance Aschenbrenner and Hillar, 2007, Corollary 2.3, and remarks at beginning of page 5175).

**Proposition 6.3.** Let \( \prec \) be a total ordering on the set \( \text{Mon}(P) \) such that for all \( m, n, t \in \text{Mon}(P) \) one has \( 1 \prec m \) and if \( m \prec n \) then \( tm \prec tn \). Then \( \prec \) is also a well-ordering of \( \text{Mon}(P) \) that is a monomial ordering of \( P \) if and only if the restriction of \( \prec \) to the variables set \( X \times \mathbb{N}^n \) is a well-ordering.

Clearly, it is easy to assign well-orderings to the set \( X \times \mathbb{N}^n \) which is in bijective correspondence to \( \mathbb{N}^P \). Note that the algebra \( P \) has also a multigrading which is defined as follows. If \( m = x_{i_1}(j_1) \cdots x_{i_k}(j_k) \in \text{Mon}(P) \), then we denote \( \partial(m) = \mu = (\mu_k)_{k \in \mathbb{Z}^n} \) where \( \mu_k = \#(\alpha_j : j_k = k) \). If \( P_{\mu} \subset P \) is the subspace spanned by all monomials of multidegree \( \mu \) then \( P = \bigoplus_{\mu} P_{\mu} \) is clearly a multigrading of the algebra \( P \). If \( \mu = (\mu_k) \) is a multidegree, we denote \( i \cdot \mu = (\mu_{k-i})_{k \in \mathbb{N}^n} \) where we put \( \mu_{k-i} = 0 \) when \( k-i < 0 \). By definition of the map \( \sigma \), if we denote \( S_{\mu,i} = P_{\mu}^i \) one obtains that \( S = \bigoplus_{\mu,i} S_{\mu,i} \) and \( S_{\mu,i}S_{\nu,j} \subset S_{\mu+(i,j),i+j} \). The elements of each subspace \( S_{\mu,i} \subset S \) are said multihomogeneous. An ideal \( J \subset S \) is called multigraded if \( J = \bigoplus_{\mu,i} J_{\mu,i} \) with \( J_{\mu,i} = J \cap S_{\mu,i} \). In other words, the ideal \( J \) is generated by multi-homogeneous elements. For any integer \( i \geq 0 \) we denote by \( ^{1'i} \) the multidegree \( \mu = (\mu_k)_{k \in \mathbb{N}^n} \) such that \( \mu_k = 1 \) if \( k \leq i \) and \( \mu_k = 0 \) otherwise. Clearly, a homogeneous element \( f^S \in S \) (\( f \in P \)) belongs to the graded subalgebra \( R \) if and only if \( f \) is multi-homogeneous and \( \partial(f) = 1' \). In other words, \( R_l = R \cap S_{1_l} = S_{1^l,1} = P_{1^l}S^l \).

**Lemma 6.4.** Let \( f^S \in S \) with \( f \in P \) a multi-homogeneous element and consider \( f_{ij}s^l, g_{js}^l, h_{jk}s^k \in S \) where \( f_{ij}, g_{js}, h_{jk} \in P \) are multi-homogeneous elements such that \( f^S = \sum_{i+j+k} f_{ij}s^i g_{js}^j h_{jk}s^k \). Then, from \( f^S \in R \) it follows that \( f_{ij}s^l, g_{js}^l, h_{jk}s^k \in R \), for all \( i, j, k \).

**Proof.** Clearly we have \( f = \sum_{i+j+k} f_{ij}s^i g_{js}^j h_{jk}s^k \). Denote \( \mu = \partial(f_{ij}), \nu = \partial(g_{js}^j) \) and \( \rho = \partial(h_{jk}s^k) \) and put \( \alpha = \min\{k \mid \nu_k > 0\} \) and \( \beta = \min\{k \mid \rho_k > 0\} \). By definition of the map \( \sigma \), one has that \( \alpha \geq i + 1 \)
and $\beta \geq i + j + 1$. If we assume $f s^l \in R$ that is $1^l = \partial(f) = \mu + v + \rho$, then necessarily $\mu = 1^l$, $v = i \cdot 1^j$ and $\rho = (i + j) \cdot 1^k$ and hence $\partial(f_{ij}) = 1^l$, $\partial(g_j) = 1^j$, $\partial(h_{jk}) = 1^k$. □

**Proposition 6.5.** Let $I$ be a graded (two-sided) ideal of $R \subset S$ and let $J$ be the extension of $I$ to $S$ that is $J$ is the (multigraded) ideal generated by $I$ in $S$. If $G$ is a multi-homogeneous basis of $J$ then $G \cap R$ is a (homogeneous) basis of $I$. In particular, the contraction $J \cap R$ is equal to $I$, that is there is a bijective correspondence between all graded ideals of $R$ and their extensions to $S$.

**Proof.** Consider $f s^l \in I \subset R \ (f \in P)$ a homogeneous element and let $G = \{g_j s^l\}$ with $g_j \in P$, $g_j$ multi-homogeneous. Since $f$ is multi-homogeneous and $G$ is a basis of $J \cap I$, one has $f s^l = \sum_{i+j+k=1} f_{ij} s^l g_j s^l h_{jk} s^k$ with $f_{ij}, h_{jk} \in P$, $f_{ij}, h_{jk}$ multi-homogeneous. From Lemma 6.4 it follows immediately that all elements $f_{ij} s^l, g_j s^l, h_{jk} s^k \in R$ that is $G \cap R$ is a basis of $I$. □

**Proposition 6.6.** Let $I \subset R$ be a graded ideal and let $J \subset S$ be its extension. If $G \subset J$ is a multi-homogeneous Gröbner basis of $J$ then $G \cap R$ is a homogeneous Gröbner basis of $I$.

**Proof.** If $f s^l = \sum_{i+j+k=1} f_{ij} s^l g_j s^l h_{jk} s^k$ is a Gröbner representation in $S$ of a homogeneous element $f s^l \in I \subset J$ with respect to $G = \{g_j s^l\}$, then it is sufficient to use the same argument of Proposition 6.5 to obtain that $f s^l$ has a Gröbner representation in $R$ with respect to $G \cap R$. □

We obtain finally an algorithm to compute Gröbner bases of graded two-sided ideals of the subring $R \subset S$ which is isomorphic to the free associative algebra $F$ by the map $\iota$. Note that the considered monomial orderings on $F$ are obtained as the restriction of monomial orderings on $S$ to the monomial subalgebra $R$. By applying Proposition 6.6, the computation of homogeneous Gröbner bases in $R$ is obtained as a slight modification of the algorithm $\text{SkewGBasis}$ for the ideals of $S$. It is interesting to note that the latter procedure is in turn a variant of the Buchberger algorithm for modules over commutative polynomial rings. Thus, we may say that these computations in associative algebras are reduced to analogue ones over commutative rings via the notion of skew polynomial ring (see also Section 7). This reverses somehow the trivial fact that commutative algebras are just a subclass of the associative ones.

**Algorithm 6.1** $\text{FreeGBasis2}$

Input: $H$, a homogeneous basis of a graded two-sided ideal $I \subset R$.
Output: $G$, a homogeneous Gröbner basis of $I$.

$G := H$;

$B := \{(f, g) \mid f, g \in G\}$;

while $B \neq \emptyset$ do

choose $(f, g) \in B$;

$B := B \setminus \{(f, g)\}$;

for all $i, j \geq 0$, $i + j = \deg(f) - \deg(g)$ and $\text{spoly}(f, s^l g^s) \in R$ do

$h := \text{REDUCE}(\text{spoly}(f, s^l g^s), \Sigma \Sigma)$;

if $h \neq 0$ then

$B := B \cup \{(h, h), (h, k), (k, h) \mid k \in G\}$;

$G := G \cup \{h\}$;

end if;

end for;

for all $i, j \geq 0$, $j - i = \deg(f) - \deg(g)$ and $\text{spoly}(f s^l, s^l g) \in R$ do

$h := \text{REDUCE}(\text{spoly}(f s^l, s^l g), \Sigma \Sigma)$;

if $h \neq 0$ then

$B := B \cup \{(h, h), (h, k), (k, h) \mid k \in G\}$;

$G := G \cup \{h\}$;

end if;

end for;

end while;

return $G$. 

Note explicitly that conditions \( \text{spoly}(f, s^i g^j), \text{spoly}(f s^i, s^j g) \in R \) are equivalent to ask that such multi-homogeneous elements of \( S \) have multidegrees of type \((1^d, d)\), for some \( d \geq 0 \).

**Proposition 6.7.** The algorithm \textsc{FreeGBasis2} is correct.

**Proof.** Since \( G \) is multi-homogeneous implies that \( \Sigma G \Sigma \) is also multi-homogeneous, the procedure \textsc{Reduce} clearly preserves multi-homogeneity. Moreover, any element \( f \in G \) \((f \notin H)\) is obtained by reduction of a \( S \)-polynomial, say \( h \). Owing to Proposition 6.6 we are interested only in the elements \( f \in R \) and this holds if and only if \( h \in R \). \( \square \)

Assume now that the graded ideal \( I \subset R \) has a finite number of generators up to some degree \( d > 0 \). Note that the \( d \)-truncated algorithm \textsc{FreeGBasis2} has termination provided by termination of \textsc{SkewBasis} as stated in Proposition 4.7. This generalizes a well-known result about algorithmic solution of the word problem (membership problem) for finitely presented graded associative algebras.

**7. Letterplace in \( P \)**

As in Section 5, consider the \( P \)-linear map \( \pi : S \to P \) such that \( s^i \mapsto 1 \), for all \( i \). Note now that \( \iota' = \pi \iota : F \to P \) is an injective \( K \)-linear map such that \( x_{i_1} \cdots x_{i_d} \in \operatorname{Mon}(F) \mapsto x_{i_1}(1) \cdots x_{i_d}(d) \in \operatorname{Mon}(P) \). Recall that \( F = \bigoplus_i F_i \) is a graded algebra with respect to total degree. Moreover, consider the weight map \( w : \operatorname{Mon}(P) \to \hat{\mathbb{N}} \) such that \( w(x_i(j)) = j \) for all \( i, j \geq 1 \) and the corresponding grading \( P = \bigoplus_{i \in \mathbb{N}} P_i \) defined by the monoid \((\hat{\mathbb{N}}, \max)\). Then, we have that \( \iota' \) is a homogeneous map (note \( K = F_0 = P_{-\infty} \) and \( P_0 = 0 \)) and \( \iota = \xi \iota' \) which is an algebra homomorphism.

**Definition 7.1.** Let \( I \subseteq F \) be a graded (two-sided) ideal. Denote by \( I' \subseteq P \) the \( w \)-graded \( \Sigma \)-invariant ideal \( \Sigma \)-generated by \( \iota'(I) \). In other words, if \( G = \{ \iota'(f) \mid f \in I_i, i > 0 \} \) then \( I' \) is the ideal of \( P \) generated by \( \Sigma \cdot G \). We call \( I' \) the letterplace analogue of \( I \).

**Proposition 7.2.** Let \( I \subseteq F \) be a graded ideal and \( I' \subseteq P \) its letterplace analogue. Denote by \( J = I^{S} \) the skew analogue of \( I' \) and call \( J \) the skew letterplace analogue of \( I \). We have that \( J \) is the extension to \( S \) of the ideal \( \iota(I) \subset R \). Then, there is a bijective correspondence between all graded ideals of \( F \) and their (skew) letterplace analogues.

**Proof.** Let \( J' \) be the extension of \( \iota(I) \) to \( S \). By definition \( J' \) is the ideal generated by the elements \( \iota(f) = \iota'(f)s^i \), for all \( f \in I_i \). Since \( I' \) is \( \Sigma \)-generated by the \( w \)-homogeneous elements \( \iota'(f) \) of weight \( i \), we conclude that \( J = I^{S} = J' \). Moreover, the bijective correspondence between graded two-sided ideals of \( F \) and their letterplace analogues in \( P \) is obtained by composing the bijections contained in Propositions 5.12 and 6.5. \( \square \)

The bijection between graded ideals of \( F \) and their letterplace analogues has been introduced in La Scala and Levandovskyy (2009) and called “letterplace correspondence”. The motivation of such name is essentially historical since the linear map \( \iota' \) was first considered in Feynman (1951), Doubilet et al. (1974). Note that in these articles the endomorphism \( \sigma \) and the algebra embedding \( \iota \) were not introduced. The polynomial ring \( P \) was named there the “letterplace algebra” because in the monomial \( \iota'(x_{i_1} \cdots x_{i_d}) = x_{i_1}(1) \cdots x_{i_d}(d) \) the indices \( 1, \ldots, d \) play the role of the “places” where the “letters” \( x_{i_1}, \ldots, x_{i_d} \) occur in the word \( x_{i_1} \cdots x_{i_d} \in \operatorname{Mon}(F) \).

Fix now a monomial ordering \( \prec \) on the algebra \( S \) that is \( \sigma \) compatible with the restriction of \( \prec \) to \( \operatorname{Mon}(P) \). By restricting \( \prec \) to \( R \) one obtains a monomial ordering on \( F \). Denote by \( V \) the image of the map \( \iota' \) that is \( V = \bigoplus_i V_i \) is a graded subspace of \( P \) where \( V_i = P_{1i} \subset P_i \). Note that \( V \) is a left \( R \)-module isomorphic to \( R \approx F \). In fact, \( V = \pi(R) \) and the restriction \( \pi : R \to V \) has the restriction \( \xi : V \to R \) as its inverse. In La Scala and Levandovskyy (2009) one has the following result which is now a direct consequence of Propositions 5.18 and 6.6.
Proposition 7.3. Let $I \subseteq F$ be a graded ideal and denote by $J \subseteq P$ its letterplace analogue. Then $J$ is a multigraded (hence w-graded) $\Sigma$-invariant ideal of $P$. If $G$ is a multi-homogeneous (hence w-homogeneous) Gröbner $\Sigma$-basis of $J$ then $t^{-1}(G \cap V)$ is a homogeneous Gröbner basis of $I$.

From this result and algorithm SIGMABASIS one obtains the correctness of the following procedure which also has been introduced in La Scala and Levandovskyy (2009).

Algorithm 7.1 FreeGBasis

\begin{verbatim}
Input: $H$, a homogeneous basis of a graded two-sided ideal $I \subseteq F$.
Output: $t^{-1}(G)$, a homogeneous Gröbner basis of $I$.
$G := \{ \ell \}$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ do
    choose $(f, g) \in B$;
    for all $i \geq 0$ such that $spoly(f, s^i \cdot g) \in V$ do
        $h := \text{Reduce}(spoly(f, s^i \cdot g), \Sigma \cdot G)$;
        if $h \neq 0$ then
            $B := B \cup \{(h, h), (h, k), (k, h), | k \in G\}$;
            $G := G \cup \{h\}$;
        end if;
    end for;
end while;
return $t^{-1}(G)$.
\end{verbatim}

Assume finally that the graded ideal $I \subseteq F$ has a finite number of generators up to some degree $d > 0$. Note that the $d$-truncated algorithm FreeGBasis has now termination provided by Proposition 5.20.

8. Examples and timings

In this section we propose an explicit computation and some timings in order to provide some concrete experience with the algorithms we introduced.

Let $X = \{x\}$ and consider the ring of ordinary difference polynomials $P = K[X \times \mathbb{N}]$ that is $P$ is the polynomial ring in the variables $x^j$ which are the shifts of a single univariate function $x = x(0)$. Moreover, let $P$ be endowed with the lexicographic monomial ordering where $x(0) < x(1) < \cdots$. Denote by $J$ the difference ideal generated by the single difference polynomial $g_1 = x(2)x(0) - x(1)$.

This ideal has been considered in Grove and Ladas (2005) as an example of an ordinary difference ideal with respect to the product criterion, to compute a Gröbner basis of $J$ one should consider all the $\Sigma$-polynomials $\text{spoly}(\sigma^i \cdot g_1, \sigma^{i+2} \cdot g_1)$ for any $i \geq 0$. We are interested in fact in computing a Gröbner $\Sigma$-basis of $J$ and hence we can apply the $\Sigma$-criterion that kills all these $\Sigma$-polynomials except for $\text{spoly}(g_1, \sigma^2 \cdot g_1)$. The reduction of this element with respect to $\Sigma \cdot \{g_1\}$ leads to $g_2 = x(4)x(1) - x(3)x(0)$. Now the current $\Sigma$-basis of $J$ is $\{g_1, g_2\}$. The $\Sigma$-polynomials that survive to product and $\Sigma$-criterion are now

$\text{spoly}(g_2, \sigma^2 \cdot g_1), \text{spoly}(g_2, \sigma \cdot g_1), \text{spoly}(g_1, \sigma^2 \cdot g_1), \text{spoly}(g_1, \sigma \cdot g_2), \text{spoly}(g_2, \sigma^2 \cdot g_2)$. 


Then \( \text{spoly}(g_2, \sigma^2 \cdot g_1) \rightarrow 0 \) and \( \text{spoly}(g_2, \sigma \cdot g_1) \) reduces to \( g_3 \) with respect to \( \Sigma \cdot \{g_1, g_2\} \). The list of new \( S \)-polynomials arising from \( g_3 \) that pass product and \( \Sigma \)-criterion is

\[
\text{spoly}(g_3, g_1), \quad \text{spoly}(g_3, \sigma \cdot g_1), \quad \text{spoly}(g_3, \sigma^3 \cdot g_1), \quad \text{spoly}(g_3, \sigma^2 \cdot g_2), \\
\text{spoly}(g_2, \sigma \cdot g_3), \quad \text{spoly}(g_1, \sigma^2 \cdot g_3), \quad \text{spoly}(g_3, \sigma^3 \cdot g_3), \quad \text{spoly}(g_2, \sigma^4 \cdot g_3).
\]

We have now that \( \text{spoly}(g_3, \sigma \cdot g_1) \rightarrow 0 \), \( \text{spoly}(g_3, g_1) \rightarrow 0 \) and \( \text{spoly}(g_2, \sigma \cdot g_3) \rightarrow g_4 \). Up to all criteria, including chain criterion, the list of \( S \)-polynomials has to be updated with the following ones

\[
\text{spoly}(g_4, \sigma \cdot g_1), \quad \text{spoly}(g_4, \sigma^2 \cdot g_1), \quad \text{spoly}(g_4, \sigma^3 \cdot g_1), \quad \text{spoly}(g_4, \sigma^4 \cdot g_1), \\
\text{spoly}(g_4, \sigma^3 \cdot g_2), \quad \text{spoly}(g_2, \sigma \cdot g_4), \quad \text{spoly}(g_1, \sigma^2 \cdot g_4), \quad \text{spoly}(g_3, \sigma^3 \cdot g_4), \\
\text{spoly}(g_4, \sigma^3 \cdot g_4), \quad \text{spoly}(g_2, \sigma^4 \cdot g_4), \quad \text{spoly}(g_4, \sigma^4 \cdot g_4).
\]

Now, one has the following reductions: \( \text{spoly}(g_4, \sigma \cdot g_1) \rightarrow 0 \), \( \text{spoly}(g_4, \sigma^2 \cdot g_1) \rightarrow 0 \) and \( \text{spoly}(g_1, \sigma \cdot g_2) \rightarrow f = x(5)x(1) - x(3)x(0)^2 \). We will show that the element \( f \) of the Gröbner \( \Sigma \)-basis of \( J \) is in fact redundant because \( g_5 \) is also in this basis. The new \( S \)-polynomials arising from \( f \) are

\[
\text{spoly}(f, \sigma \cdot g_1), \quad \text{spoly}(f, \sigma^5 \cdot g_1), \quad \text{spoly}(f, g_2), \quad \text{spoly}(f, \sigma \cdot g_2), \\
\text{spoly}(f, \sigma^4 \cdot g_2), \quad \text{spoly}(g_1, \sigma \cdot f), \quad \text{spoly}(g_3, \sigma^2 \cdot f), \quad \text{spoly}(g_4, \sigma^2 \cdot f), \\
\text{spoly}(g_2, \sigma^3 \cdot f), \quad \text{spoly}(g_4, \sigma^3 \cdot f), \quad \text{spoly}(f, \sigma^4 \cdot f).
\]

Then, we start again with reductions: \( \text{spoly}(f, \sigma \cdot g_2) \rightarrow 0 \), \( \text{spoly}(f, \sigma \cdot g_1) \rightarrow 0 \), \( \text{spoly}(g_2, \sigma^3 \cdot g_1) \rightarrow 0 \), \( \text{spoly}(g_2, \sigma \cdot g_4) \rightarrow g_5 \) and therefore \( f \) is redundant. The last \( S \)-polynomials to be added are

\[
\text{spoly}(g_5, \sigma^3 \cdot g_1), \quad \text{spoly}(g_5, \sigma^5 \cdot g_1), \quad \text{spoly}(g_5, \sigma \cdot g_2), \quad \text{spoly}(g_5, \sigma^4 \cdot g_2), \\
\text{spoly}(g_5, f), \quad \text{spoly}(g_5, \sigma^4 \cdot f).
\]

If \( G = \{g_1, g_2, g_3, g_4, g_5\} \) then we have that all remaining \( S \)-polynomials to be considered reduce to zero with respect to \( \Sigma \cdot G \), that is \( G \) is a Gröbner \( \Sigma \)-basis (difference basis) of the \( \Sigma \)-ideal (difference ideal) \( J \).

We present now some timings obtained with an implementation of the algorithm FreeGBasis. This implementation, which is still under development, is an improvement of the one we presented in La Scala and Levandovskyy (2009). We decided not to start implementing also FreeGBasis2 until FreeGBasis will evolve to some final form. We propose here new comparisons with the system Magma that contains one of the most effective implementations of the classical algorithm (Mora, 1986; Green, 1994; Ufnarovski, 1989) for computing non-commutative Gröbner bases. Note that this implementation takes also advantage by the use of Faugère’s F4 approach. The tests were performed on a PC with four Intel Core i7 CPU 940 2.93 GHz processors with 12 GB RAM running Ubuntu Linux. We used Singular 3-1-3 with freegb.lib release 14203 and Magma version 2.17-8. We measured the time for real execution of the process (thus differently to the way we did comparisons in La Scala and Levandovskyy, 2009) in “min:sec” format. The number of generators in the input and in the output are given as well.

<table>
<thead>
<tr>
<th>Example</th>
<th>Magma</th>
<th>Singular</th>
<th>#In</th>
<th>#Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>G3-5-6-2d12</td>
<td>0:10</td>
<td>1:15</td>
<td>11</td>
<td>5885</td>
</tr>
<tr>
<td>G2-3-13-4d10</td>
<td>0:05</td>
<td>0:01</td>
<td>10</td>
<td>275</td>
</tr>
<tr>
<td>G3-8-13d8</td>
<td>0:05</td>
<td>0:04</td>
<td>18</td>
<td>1490</td>
</tr>
<tr>
<td>serf-g2d8</td>
<td>0:05</td>
<td>0:01</td>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td>cliff5d9</td>
<td>0:08</td>
<td>0:12</td>
<td>41</td>
<td>168</td>
</tr>
<tr>
<td>C41d6</td>
<td>0:05</td>
<td>0:10</td>
<td>6</td>
<td>50</td>
</tr>
<tr>
<td>C41x5d5</td>
<td>0:08</td>
<td>0:04</td>
<td>6</td>
<td>44</td>
</tr>
<tr>
<td>C41yd5</td>
<td>0:05</td>
<td>0:03</td>
<td>6</td>
<td>44</td>
</tr>
<tr>
<td>C41zd6</td>
<td>0:08</td>
<td>0:10</td>
<td>6</td>
<td>44</td>
</tr>
<tr>
<td>C41yd6</td>
<td>0:05</td>
<td>0:01</td>
<td>6</td>
<td>35</td>
</tr>
</tbody>
</table>
This table shows essentially that the letterplace approach to the computation of non-commutative Gröbner bases is comparable with the classical algorithms and hence it is feasible. From the viewpoint of implementations we record that MACA achieved significant improvements with respect to comparisons included in La Scala and Levandovskyy (2009) and this stimulate us to further optimize our code. In fact, there is an ongoing work to enhance freegb.lib in SINGULAR. We will make more extensive comparisons in future articles that will be concentrated on technical aspects of implementing the letterplace algorithms.

Here is a brief description of the examples we considered for testing. In all the examples the last integer indicates the total degree that bounds the computations. The examples G3−5−6−2, G2−3−13−4 refer to the class of presented groups \( G(l, m, n; q) = (r, s)^l, s^m, (rs)^n, [r, s]^l \), where \([r, s]\) denotes the commutator. As for the example G3−8−13, this is one from the class of groups \( G(m, n, p) = (a, b, c)^m, b^n, c^p, (ab)^2, (bc)^2, (ca)^2, (abc)^2 \). All these groups has been considered by Coxeter (1939) for the problem of determining their finiteness. For our computations we considered a homogenization of the ideal of the free associative algebra defining the group algebra of such groups. The example serf−g2 are modified full Serre relations built from the Cartan matrix \( G_2 \). The following non-commutative polynomials are explicitly the generators we considered for homogenization.

\[
\begin{align*}
\frac{f_1 f_2 f_2 - 2 f_2 f_1 f_2 + f_2 f_2 f_1}{e_1 e_2 e_2 - 2 e_2 e_1 e_2 + e_2 e_2 e_1}, \\
\frac{f_1 f_1 f_1 f_1 f_1 f_2 - 4 f_1 f_1 f_1 f_2 f_1 + 6 f_1 f_1 f_1 f_2 f_1 - 4 f_1 f_2 f_1 f_1 f_1 + f_2 f_1 f_1 f_1 f_1}{e_1 e_1 e_1 e_2 - 4 e_1 e_1 e_2 e_1 + 6 e_1 e_2 e_1 e_1 - 4 e_2 e_1 e_1 e_1 + e_2 e_1 e_1 e_1}, \\
\frac{f_2 e_1 - e_1 f_2}{f_1 e_1 - e_1 f_1 + h_1, \ f_2 e_2 - e_2 f_2 + h_2,} \\
\frac{h_1 h_2 - h_2 h_1}{h_1 e_1 - e_1 h_1 - 2 e_1, \ f_1 h_1 - h_1 f_1 - 2 f_1, \ h_1 e_2 - e_2 h_1 + e_2}, \\
\frac{f_2 h_1 - h_1 f_2 + f_2}{h_2 e_1 - e_1 h_2 + 3 e_1, \ f_1 h_2 - h_2 f_1 + 3 f_1, \ h_2 e_2 - e_2 h_2 - 2 e_2,} \\
\frac{f_2 h_2 - f_2 h_2 - 2 f_2}{f_2 h_2 - 2 f_2 - 2 f_2}.
\end{align*}
\]

Let \( F_3 = K(x_1, x_2, x_3, x_4, x_5) \) and define \( \Gamma \subset \text{End}_K(F_5) \) the submonoid of all algebra endomorphisms sending variables into variables. The example cliff5 is the ideal \( I \subset F_5 \) which is \( \Gamma \)-generated by the polynomials \( [x_1, x_2, x_3] = (x_1 x_2 + x_2 x_1) x_3 - x_3 (x_1 x_2 + x_2 x_1) \) and \( s_5 = \sum_{\pi \in S_5} \text{sgn}(\pi) x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)} x_{\pi(5)} \). The quotient ring \( F_5/I \) is the generic Clifford algebra in 5 variables of a 4-dimensional vector space. Finally, the family C41 of examples originates from random linear substitutions into the ideal of 6 generators, defining the non-cancellative monoid \( C(4, 1) \) (see Jespers and Okniński, 2007) and includes also variations of those. For instance, C41W is given by

\[
\begin{align*}
x_4 x_4 - 25 x_4 x_2 - x_1 x_4 - 6 x_1 x_3 - 9 x_1 x_2 + x_1 x_1, \\
x_4 x_3 + 13 x_4 x_2 + 12 x_4 x_1 - 9 x_3 x_4 + 4 x_3 x_2 + 41 x_3 x_1 - 7 x_1 x_4 - x_1 x_2, \\
x_3 x_3 - 9 x_3 x_2 + 2 x_1 x_4 + x_1 x_1, \quad 17 x_4 x_2 - 5 x_2 x_2 - 41 x_1 x_4, \\
x_2 x_2 - 13 x_2 x_1 - 4 x_1 x_3 + 2 x_1 x_2 - x_1 x_1, \quad x_2 x_1 + 4 x_1 x_2 - 3 x_1 x_1
\end{align*}
\]

while C41 is given by

\[
\begin{align*}
189 x_4 x_4 + 63 x_4 x_3 - 66 x_4 x_2 - 161 x_4 x_1 - 103 x_3 x_4 + 19 x_3 x_3 + 262 x_3 x_2 + 467 x_3 x_1 \\
- 360 x_2 x_4 - 144 x_2 x_3 + 24 x_2 x_2 + 136 x_2 x_1 + 175 x_1 x_4 + 35 x_1 x_3 - 160 x_1 x_2 - 315 x_1 x_1, \\
27 x_4 x_4 + 409 x_3 x_3 + 82 x_4 x_2 - 42 x_4 x_1 - 57 x_3 x_4 + 403 x_3 x_3 + 26 x_3 x_2 - 42 x_3 x_1 - 50 x_2 x_4 \\
- 434 x_2 x_3 - 12 x_2 x_2 - 14 x_2 x_1 + 45 x_1 x_4 + 435 x_1 x_3 + 30 x_1 x_2, \\
232 x_4 x_4 - 29 x_4 x_3 + 77 x_4 x_2 + 332 x_4 x_1 - 147 x_3 x_4 + 175 x_3 x_3 + 60 x_3 x_2 - 269 x_3 x_1 \\
- 107 x_2 x_4 + 184 x_2 x_3 + 83 x_2 x_2 - 217 x_2 x_1 + 28 x_1 x_4 - 217 x_1 x_3 - 139 x_1 x_2 + 120 x_1 x_1, \\
52 x_4 x_4 + 233 x_4 x_3 - 129 x_4 x_2 + 135 x_4 x_1 - 248 x_3 x_4 - 205 x_3 x_3 + 171 x_3 x_2 + 138 x_3 x_1
\end{align*}
\]
\begin{align*}
+ 100x_2x_4 - 58x_2x_3 - 177x_2x_1 + 84x_1x_4 + 39x_1x_3 - 43x_1x_2 - 73x_1x_1, \\
-225x_4x_4 - 150x_4x_3 - 179x_4x_2 - 262x_4x_1 + 91x_3x_4 - 94x_3x_3 + 225x_3x_2 + 74x_3x_1 \\
+ 214x_2x_4 + 224x_2x_3 + 90x_2x_2 + 266x_2x_1 - 175x_1x_4 - 50x_1x_3 - 205x_1x_2 - 190x_1x_1, \\
289x_4x_4 - 170x_4x_3 - 289x_4x_2 - 153x_4x_1 - 186x_3x_4 + 95x_3x_3 + 177x_3x_2 + 106x_3x_1 \\
- 231x_2x_4 + 35x_2x_3 + 168x_2x_2 + 175x_2x_1 + 241x_1x_4 + 60x_1x_3 - 115x_1x_2 - 233x_1x_1. 
\end{align*}

9. Conclusions and future directions

From the previous sections we can conclude that, owing to the notion of Gröbner \( \Sigma \)-basis and the skew letterplace embedding \( \iota \), the theory of non-commutative Gröbner bases developed for the free associative algebra \( F = K\langle X \rangle \) using the concepts of overlappings, tips or obstructions (Green, 1994; Mora, 1986; Ufnarovski, 1989) can be deduced from, unified to the classical Buchberger theory for commutative polynomial rings based on \( S \)-polynomials, at least in the graded case. From a practical point of view, one obtains the alternative algorithms \textsc{FreeGBasis} and \textsc{FreeGBasis2} which are implementable in any computer algebra system providing commutative Gröbner bases. The feasibility of such methods has been already shown in La Scala and Levandovskyy (2009) and confirmed by the new timings we have collected in Section 8.

Moreover, the general theory developed in this paper can be applied to any context where a monoid of endomorphisms \( \Sigma \) acts on the polynomial algebra \( P = K[X] \) in a way which is compatible with Gröbner bases theory. We propose not only an abstract definition of what this may mean contributing to a current research trend (see for instance Drensky and La Scala, 2006; Aschenbrenner and Hillar, 2007; Brouwer and Draisma, 2011), but also a method to transfer the related algorithms from \( P \) to the skew monoid ring \( S = P * \Sigma \) when a suitable grading is given for \( P \). This theory applies in particular to the shift operators and hence a stimulating field of applications are the rings of difference polynomials. The simple calculation proposed in Section 8 gives some feeling of this. In particular, we aim to extend the Gröbner \( \Sigma \)-bases theory to any finitely generated free commutative monoid \( \Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle \) in order to cover partial difference ideals and to extend the letterplace method for \( F \) to the non-graded case by means of suitable (de)homogenization techniques. An effective implementation of all proposed algorithms will be clearly important to understand the actual performance of the methods.

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